Approximation and visualization of Pareto frontier: Interactive Decision Maps technique

A. V. Lotov*, *Computing Center RAS, Lotov09@ccas.ru

An effective approach to decision support in multi-criteria decision making (MCDM) problems characterized by three to eight decision criteria is described. The approach is based on approximating the feasible set in the criterion space (or a broader criterion set, which has the same Pareto frontier) and visualization of the Pareto frontier by interactive displaying bi-criterion slices of this set (see [1]).

Mathematically, in MCDA problems, one starts with the given set of feasible decisions (in short, *feasible set*) X, elements of which can be of any nature, say, vectors of a finite dimensional space or functions of time, etc. The values of decision criteria are given by the mapping $f : X \to R^m$, where $m > 1$. We consider the feasible set in criterion space R^m (in short, *feasible criterion set*) $Z = f(X)$. Assuming that maximization of the criterion values is desirable, we say that the vector $z^* \in Z$ is the optimal criterion vector if does not exists such vector $z^{**} \in Z$ that $z_j^{**} \geq z_j^{**}$ for $j = 1, ..., m$ and $z^* \neq z^{**}$. The optimal criterion vectors form the non-dominated (Pareto-optimal) frontier of the feasible criterion set. It is denoted by $P(Z)$ and is known as the *Pareto frontier*. The efficient (Pareto-optimal) decision set denoted by $P(X)$ is comprised of decisions $x \in X$ that $f(x) \in P(Z)$.

To select a unique decision from the set $P(X)$ of mathematically equivalent decisions, the notion of the Decision Maker (DM) is introduced: the DM is a person whose preferences are used to select the unique $x^* \in P(X)$. The existing approaches to MCDM problems differ in the ways of solving the problem how the preferences of the DM, which are not given in advance, can be used for selecting the most preferred decision. A fast developing class of the MCDM methods (the Pareto frontier generation, or *a posteriori* methods) is based on the approximating the Pareto frontier and subsequent informing the DM concerning it. Then, the DM has to identify a decision by specifying the preferred point of the Pareto frontier.

Our approach named *Interactive Decision Maps* (IDM) technique, belongs to the Pareto frontier generation methods. In contrast to different techniques of this kind, it applies visualization of the Pareto frontier. The importance of application of the computer visualization is discussed in [2]. The IDM technique starts with approximating the set $H(Z) = Z + R^m_-$ where R^m_- is the non-positive orthant of R^m. The set $H(Z)$ is known as the Edgeworth-Pareto Hull of the set Z. It is the maximal set that has the same Pareto frontier as the set Z. Bi-criterion slices of $H(Z)$ are used in the IDM technique for informing the DM concerning the Pareto frontier by interactive computer visualization.

If the set $H(Y)$ is convex, a polyhedral approximation of this set is constructed as the solution set of a finite number of linear inequalities $Az \leq a$, where A is a matrix and a is a vector. These matrix and vector can be constructed by the iterative approximation techniques based on computing the values of the support function of the set $H(Z)$. In the multidimensional case ($m > 2$) the main problem is how to select the directions, for which the support function of $H(Z)$ is computed, since the way how the directions are selected influences the complexity of the approximating system and the computing time. The concept of *optimal* convergence of the approximating polyhedra has been developed in the framework of the theory of polyhedral approximation of convex sets. We have proposed iterative methods for polyhedral approximation of convex sets that are based on adaptive selecting of directions for the support function and have optimal rate of convergence (see Chapter 8 of the book [1], where the approximation theory is outlined and the proposed methods are described).

In the case of non-linear models, the set $H(Z)$ is usually not convex. In this case, one of two groups of methods can be used for approximating it, depending on feasibility of information concerning the Lipschitz constants. If one can estimate the Lipschitz constants for the mapping f, methods for covering the feasible set X can be used, which result in constructing inner and outer approximations of $H(Z)$ (see [3]).

We have developed the hybrid methods for approximating $H(Z)$ for non-linear models with unknown Lipschitz constants. The set $H(Z)$ is approximated by a collection of cones, which vertices are located in some feasible criterion points that are close to the Pareto frontier. The hybrid methods we have proposed include random search, adaptive local optimization, squeezing the search region and genetic techniques (see details in [4]). These methods proved to be an effective tool for approximating the set $H(Z)$. The set $H(Z)$ has been approximated and the Pareto frontier has been visualized in non-linear MCDM problems with several hundreds of decision variables (see [5]).

A bi-criterion slice of the set $H(Z)$ (both for convex and non-convex cases) is defined as follows. Let (z_1, z_2) designate two criteria, the so-called "axis" criteria, and y denote the remaining criteria, which we shall fix at y^*. A bi-criterion slice of $H(Z)$, parallel to the criterion plane (z_1, z_2) and related to y^* is defined as

$$G(H(Z), y^*) = \{(z_1, z_2) : (z_1, z_2, y^*) \in H(Z)\}.$$

Note that a slice of $H(Z)$ contains all feasible combinations of values for the specified criteria when the values of the remaining criteria are not worse than y^*.

The slices of $H(Z)$ are used in the IDM technique to display the decision maps, which are collections of slices arranged in a special way. To specify a particular decision map, one has to specify a "third", or color-associated, criterion. Then, a decision map is a collection of superimposed differently colored bi-criterion slices, for which the values of the color-associated criterion change, while the values of the remaining criteria are fixed. If one compares two slices of $H(Z)$ for two different values of the color-associated criterion, the slice for the worst value of this criterion encloses the slice for its better value. For this reason, frontiers of slices (tradeoff or compromise curves) provided at a decision map do not intersect.

After studying the Pareto frontier, the decision maker identifies the preferred criterion point z^* directly on the computer display. Then, a related decision x^* is computed. This method named the Feasible Goals Method develops the idea of usual goal methods. Thanks to their simplicity, goal methods have found broad applications. However, the goal methods have a disadvantage – if the feasibility information is unknown, the goals may be too ambitious or too pessimistic. We use the IDM technique to solve the problem by informing the DM concerning the Pareto frontier. This information aids the DM in applying goal-based methods deliberately.

Various methodological and real-life applications of the IDM technique are given in ([1]). Applications by real-life decision makers include environmental river management in the Oka River basin (Russia), search for acid rain abatement strategies between Finland and Russia, planning of energy sector in Israel (all described in [1]) as well as water management in Germany ([6]). Current applications include water management in the basin of the Lake Baikal (Russia) and in small rivers of Catalonia (Spain), analysis of the Russian banking system, developing efficient strategies of oil production in Mexico, etc.

Writing of this paper was partially supported by the grant of the President of Russian Federation for State Support of Leading Scientific Schools (project no. NSh-2982.2008.1) and by the Russian Foundation for Basic Research (projects no. 04-01-00662 and 09-01-12098).

References

[1] A.V. Lotov, V.A. Bushenkov and G.K. Kamenev, *Interactive Decision Maps. Approximation and Visualization of Pareto Frontier*, Kluwer, (2004).

[2] A.V. Lotov, and K.Miettinen, Visualizing the Pareto Frontier, in: *J.Branke, K.Deb, K.Miettinen, R.Slowinski (eds.) Multiobjective Optimization. Interactive and Evolutionary Approaches, Lecture Notes in Computer Science*, V. 5252, Springer, Berlin-Heidelberg, (2008), pp.213-244.

[3] Ju.G.Evtushenko, M.A.Posypkin, Parallel methods for solving global optimization problems, in: *Proc. of the 4th Int. Conf. on Parallel Computation and Control Problems (PACO 2008), Moscow, 27-29 October 2008*, Moscow, (2008), pp. 18-39.

[4] V. E. Berezkin, and G. K. Kamenev, and A. V. Lotov, Hybrid Adaptive Methods for Approximating a Nonconvex Multidimensional Pareto Frontier, *Computational Mathematics and Mathematical Physics*, 46(11), (2006), pp. 1918-1931.

[5] A.Lotov, G.Kamenev, V. Berezkin V., and K. Miettinen, Optimal Control of Cooling Process in Continuous Casting of Steel Using a Visualization-Based Multi-Criteria Approach, *Applied Mathematical Modelling*, 29(7), (2005), pp. 653-672.

[6] J.Dietrich, A.H.Schumann, A.V.Lotov, Workflow oriented participatory decision support for integrated river basin planning, in: *A. Castelletti and R. Soncini Sessa (eds.) Topics on System Analysis and Integrated Water Resource Management*, Elsevier, (2007), pp. 207-221.