Metrics for scheduling problems*

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1 Introduction

In this paper, we propose an approach for obtaining metrics for a variety of scheduling problems.

We consider a scheduling problem $\mathcal{A}$, that is identified with the following:

- a set of numerical parameters $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$;
- a set of constraints for feasible values of parameters;
- a set of feasible solutions (schedules);
- a cost function that has to be minimized.

Any instance of problem $\mathcal{A}$ is identified by its size m and values of parameters from Ω. Let ω^A be the value of parameter $\omega \in \Omega$ for instance A. The cost value of a schedule π for instance A is denoted as $F^A(\pi)$.

An instance A of problem $\mathcal{A}$ is considered as a point in the m-dimentional space with coordinates $(\omega_1^A, \omega_2^A, \ldots, \omega_m^A)$. Our approach allows to obtain an upper bound for the optimal cost value (absolute error) that is computed based on two instances of the problem.

This upper bound is expressed as a function of $2m$ parameters (the first m parameters refer to the first instance, and the last m parameters – to the last one), and this function can be considered as a metrics in the m-dimensional space. Based on this metrics, we are able to define a distance between these instances as a natural difference of cost values.

2 Metrics

Let consider function $f : \underbrace{\mathbb{R} \times \mathbb{R} \ldots \times \mathbb{R}}_{2m} \to \mathbb{R}$, where the first m parameters determine instance A, and the last m parameters – instance B, i.e.,

$$f = f(\omega_1^A, \omega_2^A, \ldots, \omega_m^A, \omega_1^B, \omega_1^B, \ldots, \omega_m^B).$$

To simplify notation, we write $f = f(A, B)$.

Now, let there exist a function $f(A, B)$ such that for any schedule π and any instances A and B, we have

$$F^A(\pi) - F^B(\pi) \leq f(A, B).$$

Then, the following theorem holds:

Theorem 1 *For any instances A and B we have*

$$0 \leq F^A(\pi^B) - F^A(\pi^A) \leq \rho(A, B),$$

where π^A and π^B are optimal schedules for respective instances, and $\rho(A, B) = f(A, B) + f(B, A)$.

Theorem 1 has the following practical application. Suppose we cannot solve an instance A due to some reasons (high complexity, for example). Consider an instance B that we can solve (B is from a polynomially solvable area). Using Theorem 1 we can estimate the optimal value of the cost function for A based on the solution of B. Moreover, for some problems we can find (in polynomial time) such instance B from a polynomially solvable area that minimizes the absolute error.

Function $\rho(A, B)$ from Theorem 1 has the following properties:

1. $\rho(A, B) = \rho(B, A)$;
2. $\rho(A, A) = 0$ if and only if $f(A, A) = 0$;
3. If for any instances A, B, and C we have $f(A, C) \leq f(A, B) + f(B, C)$, than $\rho(A, C) \leq \rho(A, B) + \rho(B, C)$.

If Property 3 holds then we consider function $\rho(A, B)$ as *a metric* in m-dimensional space.

For some scheduling problems we can strengthen Theorem 1.

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Let us consider some partition of set Ω into subsets $\Omega_1, \Omega_2 \ldots, \Omega_k$, i.e., $\Omega_i \bigcap \Omega_j = \emptyset$ and $\bigcup_{i=1}^{k} \Omega_i = \Omega$. Suppose for each Ω_i there exists such a function $f_{\Omega \setminus \Omega_i}$, that for any schedule π and any instances $A \sim_{\Omega \setminus \Omega_i} B$, we have

$$F^A(\pi) - F^B(\pi) \leq f_{\Omega \setminus \Omega_i}(A, B).$$

Then the following theorem holds (to simplify notation we write $f_i = f_{\Omega \setminus \Omega_i}$).

Theorem 2 *For any instances A and B we have*

$$0 \leq F^A(\pi^B) - F^A(\pi^A) \leq \rho_1(A, B) + \ldots + \rho_k(A, B),$$

where π^A and π^B are optimal schedules for respective instances and $\rho_i(A, B) = f_i(A, B) + f_i(B, A)$.

Theorem 2 helps us to construct a metric for the problem in the case when we cannot obtain function $f(A, B)$ from Theorem 1 for any instances A and B. Based on Theorem 2 we can construct functions $f_i(A, B)$ for instances A and B that differ only in some subset parameters Ω_i.

3 Applications for some scheduling problems

The approach listed above allows us to easily obtain metrics and optimal cost value estimations for a variety of scheduling problems. In this section we give metrics for two classical scheduling problems: *minimization of maximum lateness* $(1 \mid r_j \mid L_{\max})$ and *minimization of total tardiness* $(1 \mid\mid \sum T_j)$ on a single machine.

For both problems we have n jobs that have to be processed on a single machine without preemption and idle time of the machine. Each job has a processing time $p_j > 0$ and a due date d_j. In the $1 \mid r_j \mid L_{\max}$ problem, release dates r_j are given for jobs, before this time the execution of a job cannot be started.

A schedule π for these problems is represented as a permutation of set $\{1, 2, \ldots, n\}$. The order of jobs in a schedule π allows us to compute the value of the completion time C_j of job j. The objective is to find an optimal schedule that minimizes:

- *maximum lateness* $L_{\max}(\pi) = \max_j \{C_j - d_j\}$;
- *total tardiness* $T(\pi) = \sum_j \max\{0, C_j - d_j\}$;

Maximum lateness on a single machine.

$$\begin{aligned} F^A(\pi^B) \quad - \quad & F^A(\pi^A) \leq \\ & (\max\{r_j^A - r_j^B\} - \min\{r_j^A - r_j^B\}) + \\ & (\max\{p_j^A - p_j^B\} - \min\{p_j^A - p_j^B\}) + \\ & (\max\{d_j^A - d_j^B\} - \min\{d_j^A - d_j^B\}). \end{aligned}$$

Total tardiness on a single machine

$$F^A(\pi^B) - F^A(\pi^A) \leq \sum_{j=1}^{n} |d_j^A - d_j^B| + n \sum_{j=1}^{n} |p_j^A - p_j^B|.$$

Based on the above estimations, we can minimize these estimations by choosing of appropriate instance B from a polynomially solvable area.

For example, consider the polynomial solvable case for the $1 \mid\mid \sum T_j$ problem when due dates of all jobs are constant d. In this case the SPT (Shortest Processing Time first) schedule is optimal. To minimize the estimation we need to find such a value d^* that minimizes the value $\sum_{j=1}^{n} |d_j^A - d^*|$.

References

[1] A.A. Lazarev, *Estimates of the Absolute Error and a Scheme for an Approximate Solution to Scheduling Problems.* Computational Mathematics and Mathematical Phisics, 2009, Vol. 49, No. 2, pp. 373–386.