Approximate method for solving scheduling problems with minimax criteria *

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We are given a set $N = \{1, \dots, n\}$ of n jobs that must be processed on m machines $M = \{1, \dots, m\}$. Preemption of the jobs is not allowed. Each machine can handle only one job at a time. For each job j we have: r_j – release date; $0 \le p_{ji} \le +\infty$ – processing time job j on the machine i (if $p_{ji} = +\infty$, then job j can not process on the machine i); d_j – due date. Between jobs ratios of a precedence in the form of an acyclic oriented graph $G \subset N \times N$ are set. Through π_i we will definete the schedule of the jobs processeded on the machine $i, i = 1, \dots, m$. Naturally, admissible schedules without artificial idle times of the machines, satisfying the graph are considered only.

In this paper, we consider the approach finding of the approximate solution with the guaranteed absolute error for the problems minimizing maximum lateness. The idea of the approach consists in construction to a initial instance A such instance B (with the same number of jobs) with minimum of estimation of absolute error that $0 \le L^A_{max}(\pi^B) - L^A_{max}(\pi^A) \le \rho_d(A,B) + \rho_r(A,B) + \rho_p(A,B)$, where

$$\rho_d(A,B) = \max_{j \in N}\{d^A_j - d^B_j\} - \min_{j \in N}\{d^A_j - d^B_j\},$$

$$\rho_r(A,B) = \max_{j \in N}\{r^A_j - r^B_j\} - \min_{j \in N}\{r^A_j - r^B_j\}$$

and

$$\rho_p(A,B) = \sum_{j \in N} |p^A_j - p^B_j|,$$

and π^A, π^B – optimal schedules for instances A and B, respectively. Besides $\rho(A,B) = \rho_d(A,B) + \rho_r(A,B) + \rho_p(A,B)$ satisfies to properties of the metrics in $3n$-dimensional space $\{(r_j, p_j, d_j) \,|\, j \in N\}$. A schedule π is uniquely determined by a permutation of the elements of N, which consists of m schedules π_i for each machine $i, i = 1, \dots, m, \pi = \bigcup_{i=1}^{m} \pi_i$. The objective function is maximum lateness $L_{max}(\pi) = \max_{j \in N} L_j(\pi)$, where $L_j(\pi) = C_j(\pi) - d_j$, and $C_j(\pi)$ is complete time job $j \in N$ in schedule π. Estimation of an absolute error for the NP-hard problem minimizing maximum lateness for single machine $1|r_j|L_{max}$ has been considered in Lazarev (2009), Lazarev et. al. (2006). We denote by $L^A_j(\pi)$ and $C^A_j(\pi)$ lateness and complete time of job j in schedule π for instance A with parameters $\{G^A, (r^A_j, p^A_j, d^A_j) |\, j \in N\}$. And, accordingly, $L^A_{max}(\pi) = \max_{j \in N} L^A_j(\pi)$ and π^A– optimal schedule for instance A. For two any instances A and B we'll define following functions:

$$\begin{cases} \rho_d(A,B) = \max_{j \in N}\{d^A_j - d^B_j\} - \min_{j \in N}\{d^A_j - d^B_j\}, \\ \rho_r(A,B) = \max_{j \in N}\{r^A_j - r^B_j\} - \min_{j \in N}\{r^A_j - r^B_j\}, \\ \rho_p(A,B) = \sum\limits_{j \in N} \left(\max\limits_{i \in M}(p^A_{ji} - p^B_{ji})_+ + \max\limits_{i \in M}(p^A_{ji} - p^B_{ji})_- \right), \\ \rho(A,B) = \rho_d(A,B) + \rho_r(A,B) + \rho_p(A,B), \end{cases}$$

where $(x)_+ = \begin{cases} x, x>0, \\ 0, x \le 0; \end{cases}$; $(x)_- = \begin{cases} -x, x<0, \\ 0, x \ge 0; \end{cases}$; $|x| = (x)_+ + (x)_-$. The "processing part" $\rho_p(A,B)$ can be written down on another: $\rho_p(A,B) =$

$$\sum_{j \in N} \left(\max_{i \in M}\{(p^A_{ji} - p^B_{ji}), 0\} - \min_{i \in M}\{(p^A_{ji} - p^B_{ji}), 0\} \right).$$

Theorem 1 *Let $A = \{G, (r^A_j, p^A_j, d^A_j) |\, j \in N\}$ and $B = \{G, (r^B_j, p^B_j, d^B_j) |\, j \in N\}$ (with identical graph preceding G) be two any instances then*

$$0 \le L^A_{max}(\pi^B) - L^A_{max}(\pi^A) \le \rho(A,B).$$

From "symmetric" the instances A and B holds $0 \le L^B_{max}(\pi^A) - L^B_{max}(\pi^B) \le \rho(A,B) = \rho(B,A)$.

The idea finding approximated solution of the problem consists of two stages. On the first step to the initial instance $A = \{G, (r^A_j, p^A_j, d^A_j) |\, j \in N\}$ is such change of its parameters r_j, p_j and d_j that the obtained instance $B = \{G, (r^B_j, p^B_j, d^B_j) |\, j \in N\}$ belongs to a set polynomially solvable instances of the initial problem. On the next step we'll find optimal schedule to instance B. According to Theorem 1 the schedule π^B to initial instance A have $0 \le L^A_{max}(\pi^B) - L^A_{max}(\pi^A) \le \rho(A,B)$.

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Let's consider a case when class of polynomially solvable instances of the problem is defined by system of k linear inequalities

$$\mathbf{X} * \mathbf{R} + \mathbf{Y} * \mathbf{P} + \mathbf{Z} * \mathbf{D} \leq \mathbf{H}, \tag{1}$$

(s. t. $p_j \geq 0, \forall j \in N$), where $R = (r_1, \ldots, r_n)^T$, $P = (p_1, \ldots, p_n)^T$, $D = (d_1^C, \ldots, d_n)^T$, and $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ – matrixes of dimension $k \times n$, and $H = (h_1, \ldots, h_k)^T$ – k-dimensional vector (the up index ^{T} designates transposition). Then in this class of instances (1) we find instance B with minimum "distance" $\rho(A, B)$ (to the initial instance A),

$$\left\{ \begin{array}{l} \min\ (x^d - y^d + x^r - y^r) + \sum_{j \in N} x_j^p \\ y^d \leq d_j^A - d_j^B \leq x^d, \quad \forall j \in N, \\ y^r \leq r_j^A - r_j^B \leq x^r, \quad \forall j \in N, \\ -x_j^p \leq p_j^A - p_j^B \leq x_j^p, \quad \forall j \in N, \\ 0 \leq x_j^p, \quad \forall j \in N, \\ \mathbf{X} * \mathbf{R^B} + \mathbf{Y} * \mathbf{P^B} + \mathbf{Z} * \mathbf{D^B} \leq \mathbf{H}. \end{array} \right. \tag{2}$$

Linear programming problem (2) with $3n+4+n$ variables ($r_j^B, p_j^B, d_j^B, j = 1, \ldots, n$, and x^d, y^d, x^r, y^r, and x_j^p, $j = 1, \ldots, n$) and $7n + k$ inequalities can be sometimes solved in polynomial time considering specificity of linear restrictions. For example, for the problem $1|r_j|L_{max}$ two cases have been allocated polynomially solvable instances:

$$\left\{ \begin{array}{l} d_1 \leq \ldots \leq d_n, \\ d_1 - r_1 - p_1 \geq \ldots \geq d_n - r_n - p_n \end{array} \right. \tag{3}$$

and

$$\max_{k \in N} \{d_k - r_k - p_k\} \leq d_j - r_j, \forall j \in N. \tag{4}$$

In the case (3) of the problem $1|r_j|L_{max}$ can be solved in polynomial time – $O(n^3 \log n)$ operations Lazarev (2009). And the task of linear programming (2) has been can be solved in polynomial time – $O(n \log n)$ operations Lazarev et. al. (2006).

In the case (4) the problem can be solved in $O(n^2 \log n)$ operations Hoogeveen (1996). As well as in case of (3) the minimum of absolute error of maximum lateness can be solved in polynomial time – in $O(n)$ operations Lazarev et. al. (2006).

On the set of nonequivalent instances, $\forall A$, we will define function

$$\varphi(A) = \max_{j \in N} r_j^A - \min_{j \in N} r_j^A + \max_{j \in N} d_j^A - \min_{j \in N} d_j^A + \sum_{j \in N} |p_j^A| \geq 0.$$

The function satisfies to properties of metrics:

$$\left\{ \begin{array}{l} \varphi(A) = 0 \Longleftrightarrow A \equiv 0; \\ \varphi(\alpha A) = \alpha \varphi(A); \\ \varphi(A + B) \leq \varphi(A) + \varphi(B). \end{array} \right. \tag{5}$$

So $||A|| = \varphi(A)$ and $\rho(A, B) = ||A - B||$.

Let's need to find optimal schedule for some instance A to the problem $\alpha|\beta|L_{\max}$ and we know that the corresponding problem $\alpha|\beta|C_{\max}$ is polynomially solvable. Then all parameters except d_j ($d_j^B = 0$), $\forall j \in N$, of instance B will be the analogous. Thus

$$0 \leq L_{\max}^A(\pi^B) - L_{\max}^A(\pi^A) \leq \rho(A, B) = \max_{j \in N} d_j^A - \min_{j \in N} d_j^A.$$

Then let's need to find optimal schedule for instance A to the problem $\alpha|\beta|L_{\max}$ and we know that the corresponding problem $\alpha|\beta, p_j = p|L_{\max}$ is polynomially solvable. All parameters except $p_j, \forall j \in N$, of instance B will be the same. Thus we should solve optimization problem

$$\rho(A, B) = \sum_{j \in N} |p_j - p| \to \min_p.$$

Solution of the task is $p^* = p_{[\frac{n+1}{2}]}$ (if $p_1 \leq \ldots \leq p_n$). So we draw $L_{\max}^A(\pi^B) - L_{\max}^A(\pi^A) \leq \sum_{j \in N} \left| p_j - p_{[\frac{n+1}{2}]} \right|$.

Let's our problem is $R|\beta|L_{\max}$ or $Q|\beta|L_{\max}$ then for the initial instance A of the problem we consider the problem $P|\beta|L_{\max}$ that is polynomially solvable. All parameters except $p_{ji}, \forall j \in N, \forall i \in M$, of instance B will be the same. Thus we should solve next optimization task

$$\sum_{j \in N} (\max_{i \in M} \{(p_{ji}^A - p_j^B), 0\} - \min_{i \in M} \{(p_{ji}^A - p_j^B), 0\}) \to \min_{p_j}.$$

So we can take any value for p_j^B from interval $[\min_{i \in M} p_{ji}^A, \max_{i \in M} p_{ji}^A], \forall j \in N$, and

$$L_{\max}^A(\pi^B) - L_{\max}^A(\pi^A) \leq \sum_{j \in N} \left(\max_{i \in M} p_{ji}^A - \min_{i \in M} p_{ji}^A \right).$$

References

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