Supermodular programming on finite lattices

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The basic concepts, definitions and optimization problems settings are given. The problems of minimization of supermodular functions on the different types of lattices: Boolean lattices, lattices with relative supplements (division lattices, lattices of vector subspaces of finite-dimensional vector space, geometrical lattices), lattices equal to Cartesian product of chains and Lattices of Cubes, are considered. The previously obtained theoretical results, on the basis of which the problems of minimization of supermodular functions on these lattices have been solved, are described. Its noted, that these results have been extended to the distributive lattices.

A theory of maximization of supermodular functions on Boolean lattices is described. The relation between the global minimum and maximum of supermodular functions for the main types of lattices is established.

A new type of lattices, lattice of Cubes, is defined and described. The problems of minimization and maximization of supermodular functions are considered on it. Particular examples of such functions are given. Optimization algorithms and the possibilities of setting and solving a new class of problems on the lattices of Cubes are discussed.

A general approach to the optimization on lattices with use of atomic lattices is described. It is proposed to map the atomic lattice into the corresponding Boolean lattice and then perform the optimization on this more ample Boolean lattice. If the properties of supermodularity of the function, defined on the atomic lattice, are obeyed on the Boolean lattice, then for the optimization it is possible to use all the theoretical results described above.

The original combinatorial algorithms of automated representation of high-dimensional Hyper-Cubes (Booleans) on a plane in the form of different projections and diagrams, keeping the properties of Boolean as a partially ordered set of its vertexes are worked out. This gives us the ample opportunities for construction of various schemes of looking through the elements of atomic lattices and for visualization of the optimization process.

The examples of various types of applied problems that have been solved using the elaborated optimization methods are given. Although the majority of these problems are NP-hard, solution of a great amount of applied tasks demonstrated the high practical effectiveness of the elaborated methods and algorithms. (Could be seen the analogy with the simplex-method).

Thus, the obtained theoretical results and a great amount of optimization problems for lattices with concrete types of supermodular functions allow to consider the methods and algorithms for solving the problems of optimization of supermodular functions on lattices as a new field of mathematical programming supermodular programming.

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