On Monotonicity of a Mapping Related to the Non-Cooperative Many-Player Game

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Let X_i and X_i^0 be non-empty subsets of a Euclidean space E_i with $X_i \subset X_i^0$, and let $\widehat{X}_i$ be direct products of X_j where $1 \le j \le k$, $j \ne i$. Next, define $X = X_i \times \widehat{X}_i$, and let f_i be a real-valued function defined over X, $1 \le i \le k$; finally, let $k \ge 2$ be an integer number. We introduce a non-cooperative game Γ with k players by specifying, for each player i, her strategy set X_i and payoff function f_i. Denote by $X_\Gamma^* \subset X$ the set of Nash equilibrium points of the game Γ. We say that a game Γ satisfies condition Q_1 if X is a convex compact set, the function f_i is continuous on X and concave by $x_i \in X_i$ for each (fixed) $\widehat{x}_i \in \widehat{X}_i$. Condition Q_1 guarantees that the set X_Γ^* is non-empty. Now in addition to condition Q_1, we demand that the set X_i^0 be open and convex, the function f_i be defined on the set $X_i^0 \times \widehat{X}_i$ and concave (for each fixed $\widehat{x}_i \in \widehat{X}_i$) with respect to $x_i \in X_i^0$, $1 \le i \le k$. All these restrictions will be referred to as condition Q_2. When condition Q_2 holds for a game Γ we can define a point-to-set mapping T_Γ, which associates points of X with non-empty convex compact subsets of the Euclidean space $E = E_1 \times \cdots \times E_k$. The mapping T_Γ is determined by the relationship $T_\Gamma(x) = \{t = (t_1, \ldots, t_k)\colon\ -t_i \in \partial_{x_i} f_i(x),\ 1 \le i \le k\}$, $x \in X$, where $\partial_{x_i} f(x)$ is the subdifferential of f_i at the point x with respect to x_i.

Any point-to-set mapping T associating points of a set $X \subset E$ with non-empty subsets of E, generates a variational inequality problem $t \in T(x)$, $\langle t, x' - x\rangle \ge 0\ \forall x' \in X$. Denote by $X^*(T)$ the set of all solutions to the variational inequality problem generated by the mapping T. Taking into account that $X_\Gamma^* = X^*(T_\Gamma)$, one can find Nash equilibrium points of the non-cooperative game Γ by solving the variational inequality problem induced by the mapping T_Γ.

Now assume that a mapping T satisfies the following two requirements:

(A) the sets X and $T(x)$ for any $x \in X$ are convex and compact, the mapping T is upper semi-continuous on X;

(B) the mapping T is monotone.

In [1], we describe a quite efficient numerical method solving variational inequality problems, which is based upon an extension of the well-known method of levels. Under assumptions A and B, we also establishe that a sequence of points generated by this algorithm converges to the set $X^*(T)$. Now if requirement Q_2 holds for a game Γ, then mapping T_Γ satisfies condition A.

Consider a finite non-cooperative k-person game, in which each player i has n_i strategies, and her payoff function is determined by a k-dimesional table $A_i = (a^{(i)}_{s_1\ldots s_k})$, where $a^{(i)}_{s_1\ldots s_k}$ is the i-th player's payoff when player α chooses strategy s_α, $1 \le \alpha \le k$. If we extend the strategy set and introduce mixed strategies, we come to a k-person game Γ, in which

$$
\begin{aligned}
X_i = \Big\{ x_i = (x_{i1}, \ldots, x_{in_i}) :& \\
\sum_{j=1}^{n_i} x_{ij} = 1,\quad & x_{ij} \ge 0,\ 0 \le j \le n_i \Big\}, \\
f_i(x) = \sum_{s_1,\ldots,s_k} a^{(i)}_{s_1\ldots s_k} & x_{1s_1} \cdots x_{ks_k},\ 1 \le i \le k,
\end{aligned}
\tag{1}
$$

where $x = (x_1, \ldots, x_k) \in X$.

Theorem 1 *Let Γ be a game determined by relationships (1). The mapping T_Γ is monotone if, and only if it is possible to represent the tables A_i as follows: $A_i = \sum_{j=1}^k A_{ij}$, $1 \le i \le k$, where entries of a k-dimesional table A_{ij} depend only upon indices s_i and s_j when $i \ne j$, and do not depend on the index s_i whenever $i = j$. Besides, for any $i \ne j$, all entries of tables $A_{ij} + A_{ji}$ must equal zero.*

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References

[1] E. G. Gol'stein. *A Method to Solve Variational Inequalities Defined by Monotone Mappings.* Journal of Comp. Math. and Mathem. Physics, 2002, Vol. 42, No. 7. (*In Russian*).