Duality in unconstrained convex quadratic optimization problems

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Formally the problems of unconstrained minimization have no Lagrange function. Hence it is impossible to directly formulate a dual problem. Nevertheless it is possible to introduce artificial constrains with the help of additional variables thus obtaining an equivalent nonlinear programming problem, for which a conventional method can be used in order to define a dual problem. There exists a class of certain optimization problems where mutually dual problems are those of unconstrained optimization and the solution of either of these two problems can be expressed by that of the other one. These are referred to as the problems of nonlinear programming appearing in the regularization of linear equality systems and/or inequality systems. Due to the fact that mutually dual systems differ in their dimension it is reasonable to solve a problem of unconstrained optimization having a minor dimension. Let us consider a typical result appearing in regularization of linear equality system with nonnegative variables:

Theorem 1 *For any $\varepsilon > 0$ the solution $x(\varepsilon)$ of problem*

$$F(x(\varepsilon)) = \min_{x \in R^n_+} \frac{1}{2}\{\|b - Ax\|^2 + \varepsilon\|x\|^2\} \tag{1}$$

and the solution $u(\varepsilon)$ of problem

$$H(u(\varepsilon)) = \max_{u \in R^m} \{b^\top u - \frac{1}{2\varepsilon}\|(A^\top u)_+\|^2 - \frac{1}{2}\|u\|^2\} \tag{2}$$

are interrelated as follows:

$$x(\varepsilon) = \tfrac{1}{2}(A^\top u(\varepsilon))_+,$$
$$u(\varepsilon) = b - Ax(\varepsilon)$$

and $F(x(\varepsilon)) = H(u(\varepsilon))$ takes place.

It follows from the theorem that in case the matrix A of dimension $n \times m$ has $m < n$, it is expedient to solve a dual problem representing concave piecewise-quadratic unconstrained maximization problem (1) instead of the minimization problem (2). To solve problem (2) the generalized Newton method is quite effective. O.Mangasarian established finite termination of a generalized Newton method for minimizing a strongly convex, piecewise quadratic function on the n-dimensional real space R^n. Such a problem is a fundamental one in generating a linear or nonlinear kernel classifier for data mining and machine learning [1].

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References

[1] O. Mangasarian, *A Finite Newton Method for Classification Problems* //Optimization Methods and Software, vol. 17, 913-929, (2002)