An algorithm to solve a bi-level programming problem with integer upper level variables

S. Dempe*, V. V. Kalashnikov†, N. I. Kalashnykova‡, A. Arévalo Franco‡

*TU Bergakademie Freiberg, Germany, dempe@tu-freiberg.de

†CEMI, Moscow, slavkamx@mail.ru ‡UANL, Monterrey, Mexico, nkalash@einstein.fcfm.uanl.mx

We consider an hierarchical optimization problem in two levels; decision making at the upper level is governed by constraints that are defined in part by a second parametric optimization problem. Let this second problem be defined as follows:

$$\min_x \{f(x,y) \mid g(x,y) \leq 0, h(x,y) = 0\}, \tag{1}$$

where $f : R^n \times R^m \to R, g : R^n \times R^m \to R^p$ and $h : R^n \times R^m \to R^q$ with $g(x,y) = (g_1(x,y), \cdots, g_p(x,y))^T$ and $h(x,y) = (h_1(x,y), \cdots, h_q(x,y))^T$. This problem is called the lower level, or the follower's problem. Let $\psi(y)$ denote the solution set of problem (1) for a fixed parameter $y \in R^m$.

Now one we can formulate the bilevel problem as:

$$\min_{x,y} \{F(x,y) \mid y \in Y, x \in \psi(y)\}, \tag{2}$$

where $F : R^n \times R^m \to R$, and Y is a closed subset of R^m. The bilevel programming problem is called the upper level, or the leader's problem.

In order to assure that the bilevel programming problem is well-defined, we assume the following:

1) The set $M = \{(x,y) \,|g(x,y) \leq 0, h(x,y) = 0\}$ is nonempty.
2) Both $F(x,y)$ and $f(x,y)$ are bounded from below on M.
3) Both $F(x,y)$ and $f(x,y)$ are linear functions.

Definition 1 *A pair (x,y) is said to be feasible to the linear bilevel programming problem if $x \in \psi(y)$.*

Definition 2 *A feasible pair (x', y') is called an optimal solution to the linear bilevel programming problem if $F(x', y') \leq F(x,y)$ for all the feasible solutions.*

The mixed-integer bi-level linear programming problem with a parameter in the righthand side at the lower level is formulated as follows:

$$\min_{x,y} \left\{\langle a, x\rangle + \langle b, y\rangle \mid Gy = d, x \in \psi(y), y \in Z_+^m\right\}, \tag{3}$$

which represents the upper level where $a, x \in R^n$, $b, y \in R^m$, G is an $r \times m$ matrix, $d \in R^r$. Here $\psi(y)$ is defined as follows:

$$\psi(y) = Arg\min_x \{\langle c, x\rangle \mid Ax = y, x \geq 0\}, \tag{4}$$

which describes the feasible region of the lower level decision maker (the set of rational reactions). Here $c, x \in R^n$, A is an $m \times n$ matrix with $m \leq n$.

Let us rewrite the lower level problem as follows:

$$\varphi(y) = \min_x \{\langle c, x\rangle \mid Ax = y, x \geq 0\}. \tag{5}$$

We will also call $\varphi(y)$ the *lower level optimal value*. We suppose that the feasible set (4) is non-empty.

In this paper, we consider a reformulation of (3)–(5) as a classical optimization problem, based upon an approach reported in the literature (*see* [1], [5]). If we take into account the lower level optimal value function (5), then problem (3)–(5) can be replaced by:

$$\begin{aligned}\min_{x,y} \{\langle a, x\rangle + \langle b, y\rangle \mid Gy = d, \langle c, x\rangle \leq \varphi(y),\\ Ax = y,\ x \geq 0, y \in Z_+^m\}\end{aligned} \tag{6}$$

Our work is concentrated on the lower level objective value function (5). For this reason, we show some important characteristics (*see* [4] or [2]) that will be helpful for solving problem (6).

Consider the parametric linear programming problem (5)

$$\varphi(y) = \min_x \{\langle c, x\rangle \mid Ax = y, x \geq 0\}.$$

In order to solve this problem, we use the dual simplex algorithm, like in [2]. For that, let B be a corresponding basic matrix, i.e. a quadratic submatrix of A having the same rank as A, and such that $x^* = (x_B^*, x_N^*)^T$, with $x_B^* = B^{-1}y$ and $x_N^* = 0$. Moreover, let us fix $y = y^*$. Then we can say that $x^*(y^*) = B^{-1}y^*$ is an optimal basis solution of problem (5) for a fixed parameter y^*. And if the following inequality holds:

$$B^{-1}y \geq 0,$$

then $x^*(y) = B^{-1}y$ is also optimal for the parameter vector y.

It is possible to perturb y^* so that B remains a basic optimal matrix [4]. We denote by $\Re(B)$ a set that we call the *stability region* of B, which is defined as

$$\Re(B) = \left\{ y \mid B^{-1}y \geq 0 \right\}.$$

This region is nonempty because $y^* \in \Re(B)$. Furthermore, it is closed, but not necessarily bounded. If $\Re(B)$ and $\Re(B')$ are two different stability regions with $B \neq B'$, then only one of the following cases is possible:

1. $\Re(B) \cap \Re(B') = \{0\}$;
2. $\Re(B) \cap \Re(B')$ contains the common border of the regions $\Re(B)$ and $\Re(B')$;
3. $\Re(B) = \Re(B')$.

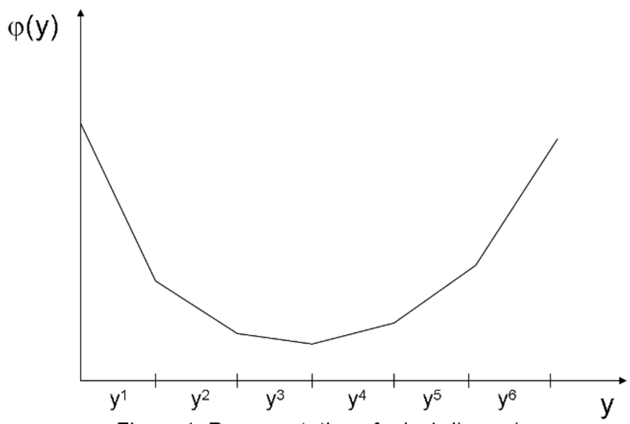

Figure 1. Representation of φ in 1 dimension

Moreover, $\Re(B)$ is a convex polyhedral set, on which the lower level optimal value function is a finite and linear function. Let $y^1, \cdots, y^q$ represent all the extreme points of the feasible set of problem (3). We denote by $B_1, \cdots, B_q$, resp., the corresponding basic optimal matrices. It can be shown that φ is a finite piecewise linear and convex function defined over the set:

$$\Re(B_1) \cup \cdots \cup \Re(B_q),$$

by

$$\varphi(y) = min \left\{ \left\langle c, x^*(y^1) \right\rangle, \left\langle c, x^*(y^2) \right\rangle, \ldots, \left\langle c, x^*(y^q) \right\rangle \right\}.$$

As we can see in Figure 1, the stability regions are represented by the segments on the y-axis. The function φ is nonsmooth, which makes this kind of problems hard to solve. Consulting the literature [5] we find that this function is also partially calm, which yields a new reformulation of our problem (6).

Theorem 1 *Let (x^*, y^*) solve problem (3)–(5), then (6) is partially calm at (x^*, y^*).*

The difficulty in the work with the objective value function (5) is due to the simple fact that we do not have it in an explicit form. Also (5) is not differentiable: *cf.* [3], [5] working with subdifferential calculus based upon the non-smooth Mangasarian-Fromovitz condition.

The tools that we use in this paper are mainly based on the fact that (5) is piecewise-linear and convex. As the objective and constraint functions are linear at both levels, the proposed algorithm is based upon an approximation of the optimal value function using the branch-and-bound method. Therefore, in every node of this structure, we apply a new branch-and-bound technique to process the integrality condition.

Acknowledgements: The second author acknowledges financial support from the R&D Department (Cátedra de Investigación) CAT-174 at the Tecnológico de Monterrey, whereas the research activity of the third and the fourth authors were supported by the Mexican National Council for Science and Technology (CONACYT), grant No. 89210. Finally, the second and the third authors are grateful to the Russian Humanitarian Research Foundation (RGNF) for the support within the project RGNF 08-02-00271.

References

[1] **Dempe, S.** *Foundations of Bilevel Programming*, Dordrecht/London/Boston: Kluwer Academic Publishers, 2002.

[2] **Dempe, S. and Schreier, H.** *Operations Research – Deterministische Modelle und Methoden*, Teubner Verlag, Wiesbaden, 2006.

[3] **Dempe, S. and Zemkoho, A.B.** *A Bilevel Approach to Optimal Toll Setting in Capacited Networks*, Preprint, TU Bergakademie Freiberg, 2008.

[4] **Grygarova, L.** Qualitative Untersuchung des I. Optimierungsproblems in mehrparametrischer Programmierung, *Aplications of Mathematics*, Vol. 15, No. 4, pp. 276 – 295, 1970.

[5] **Ye, J.J. and Zhu, D.L.** Optimality conditions for bilevel programming problems, *Optimization*, Vol. 33, pp. 9 – 27, 1995.