Opposite linear optimization

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Consider the following matrix game. Let X, Y be the unit simplexes in R^m and R^n respectively; $A = (a_{ij})_{m\times n}$ is a gain matrix for the "player" X. The value

$$v = \max_X \min_Y (x, Ay)$$

is a maximal guaranteed value of game for X. The problem

$$w = \min_X \max_Y \, (x, Ay)$$

is said to be opposite for X (the minimal risk problem). In the paper we study conditions ensuring inequality $v \le w$.

For linear programming problem

$$v^+ = \max\{(c, x) \ \mid Ax \le b\} \qquad (1)$$

we define the opposite problem as follows

$$v^- = \min\{(c, x) \mid Ax \ge b\}.$$

Theorem 1 *If both values v^+ and v^- are finite then $v^- \ge v^+$.*

For problem (1) consider the following problem

$$\bar{v} = \min_{k>0} \max\{(c, x) + kt \mid Ax + te^m \le b\}, \qquad (2)$$

where $e^m = (1 \ \dots \ 1)^\Gamma \in R^m$. Define the opposite problems

$$\max\{(c, x) \mid Ax \le e^m\} = \underline{k},$$

$$\min\{(c, x) \mid Ax \ge e^m\} = \bar{k}.$$

Denote as $f(k)$ the value of internal (max) problem in (2).

Theorem 2 *The inequality $\bar{k} \ge \underline{k}$ holds and $f(k)$ is finite for $\underline{k} \le k \le \bar{k}$. If v^+ is finite, then $\bar{v} = v^+ = f(k)$ for $k = \sum\limits_{i=1}^{n} \bar{u}_i$, where $(\bar{u}_1 \ \dots \ \bar{u}_m)$ is an arbitrary solution of dual problem for (1).*

For a system of affine functions $(a_{i\bullet,x}) - b_i, \quad i \in \overline{1, m}$ define two opposite problems

$$v^+ = \max_x \min_i \{(a_{i\bullet}, x) - b_i\},$$

$$v^- = \min_x \max_i \{(a_{i\bullet}, x) - b_i\}$$

which are obviously equivalent to the following opposite problems

$$v^+ = \max\{t \mid Ax - b \ge te^m\},$$

and

$$v^- = \min\{t \mid Ax - b \le te^m\}$$

respectively. The conditions guaranteeing the fulfilment of inequality $v^- \ge v^+$ are studied.

Consider the problem

$$v = \max_\alpha \min_\lambda \max_x \left\{ \left(\sum_j^s \lambda_j c_{j\bullet}, x \right) \middle| Ax \le \sum_j^k b_{\bullet j}, \alpha \right\}.$$

Here and next $\sum\limits_j^k \alpha_j = 1 = \sum\limits_j^s \lambda_j, \quad \alpha_j \ge 0, \quad \lambda_j \ge 0$. This problem may be reduced to the following LP problem

$$v = \max \left\{ x_0 | \, Ax \le \sum_j^k b_{\bullet j}, \alpha_j, \ x_0 \le (c_{j\bullet}, x) \ (j \in \overline{1, s}) \right\}.$$

It is shown that the opposite problem

$$v = \min_\alpha \max_\lambda \min_x \left\{ \left(\sum_j^s \lambda_j c_{j\bullet}, x \right) \middle| Ax \le \sum_j^k b_{\bullet j}, \alpha \right\},$$

is equivalent to the opposite one for the last LP problem.

The opposite problems admit economic interpretation as the choice of worst solutions with respect to goals and costs (for example minimal and maximal cost transportations in transport problem).