Transition to balanced growth in a decentralized economy

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We consider two schemes for a multisectoral economy to reach a balanced growth in the absence of central planning. In our case, the economy is closed and consists of n sectors. The output vector $x(t) \in R^n$ satisfies the inequality $Yx(t) \le x(t-1)$ at step t, $t = 1, 2, \ldots$, where $Y = \{y_{ij}\}$ is a technological matrix.

As is shown in [1], if the matrix Y is indecomposable and primitive, then the economic system can have a balanced growth $x(t) = \gamma^t x(0)$ at all steps $t \ge 1$ if and only if $x(0)$ is the Frobenius vector of Y. In this case we have $\gamma = 1/\lambda_Y$ where λ_Y is the Frobenius eigenvalue of Y. The balanced growth is also called a turnpike regime.

Further, we consider the case when the vector $x(0)$ is not Frobenius of Y and the system is decentralized. Let us introduce some notation.
By $y_{ji}(t)$ denote the volume of product j that may be used by sector i at step t,
by $x_i^p(t)$ denote the plan of output for sector i at step t (this plan must be fix by the sector before the start of step t) ,
by $x_i^d(t)$ denote the total demand for the product i that was produced at step t,
by $x_i^s(t)$ denote the total sale of the product i produced at step t.

Variables $x_i^d(t)$ and $x_j^p(t+1)$ are related as

$$x_i^d(t) = \sum_{j=1}^{n} y_{ij} x_j^p(t+1), \quad i = 1, \ldots, n.$$

Further we consider two schemes for a decentralized planning.

Scheme 1. The planning is based on a volume of sales.

Two assumptions given below are basic for the suggested scheme.

First, assume that the plan of output of any sector at step t is determined unambiguously by the total sale of products produced at step $t-2$:

$$x_i^p(t) = k x_i^s(t-2), \quad t \ge 2, \quad i = 1, \ldots, n, \qquad (1)$$

where coefficient k is the same for all sectors at all steps.

Second, the produced products are distributed according to the following procedure. Sector i having received orders from consumers calculates coefficient

$$\eta_i(t) = x_i^d(t-1) \,/ x_i(t-1)$$

characterizing endowment of the production plans with the resource that it has produced. Then these coefficients are made known to all sectors of the system. On the basis of the obtained data, each sector (or a certain information center) calculates the maximum value of these indicators:

$$\eta_{\max}(t) = \max_i \eta_i(t).$$

If it turns out that $\eta_{\max}(t) \le 1$, the demand of each sector for resources is satisfied fully, and volumes of deliveries are enough to fulfill the outlined plans:

$$y_{ij}(t) = y_{ij} x_j^p(t), \quad i, j = 1, \ldots, n.$$

In this case the volumes of production equal the plans: $x_i(t+1) = x_i^p(t+1)$. But if $\eta_{\max}(t) > 1$, then the full fulfillment of plans becomes impossible. In this case all sectors decrease the plan of output using this parameter:

$$x_i^p(t) := x_i^p(t) \,/ \eta_{\max}(t) \,, \quad i = 1, \ldots, n.$$

The demand for all resources decreases accordingly $\eta_{\max}(t)$ times. In this case the corrected plans are fully provided with resources, and the newly calculated indicator $\eta_{\max}(t)$ equals unity.

A plan allowable by resources determines unambiguously the total sale of product i produced at the previous step:

$$x_i^s(t-1) = \sum_{j=1}^{n} y_{ij} x_j^p(t), \quad i = 1, \ldots, n.$$

Then the production cycle begins at step t, after which by (1) the vector $x^p(t+1)$ is determined, etc.

Using induction, it is easy to show that the economic system will function here in the turnpike regime if vectors $x(0)$, $x^p(1)$ are Frobenius ones, $x^p(1) \geq \gamma x(0)$, and $\sqrt{k} \geq \gamma$.

To sum, a necessary condition for functioning of this scheme is the determination of vector $x^p(1)$ and parameter k which characterizes the assessment by sectors of the rate of economic growth.

Theorem 1 *If matrix Y is indecomposable and primitive, vectors $x(0)$ and $x^p(1)$ are strictly positive and inequalities $\sqrt{k} \geq \gamma > 1$ hold, then this scheme of planning either brings the system asymptotically to the turnpike, or keeps the turnpike regime at appropriate values $x(0)$ and $x^p(1)$.*

Proof. Introduce a parameter $\beta(t)$ showing what part of initial plans of output is realized at step t. It is obvious that

$$\beta(t) = \begin{cases} 1, & \eta_{\max}(t) \leq 1; \\ 1/\eta_{\max}(t), & \eta_{\max}(t) > 1. \end{cases}$$

Using induction, write the relationship between vectors $x(t)$ and $x^p(1)$:

$$x(t) = \beta(t)\beta(t-1)\cdots\beta(1)\,(kY)^{t-1}\,x^p(1).$$

Using Fobenius eigenvalue λ_Y of matrix Y, write the latter equality as:

$$x(t) = \beta(t)\beta(t-1)\cdots\beta(1)\,(\lambda_Y k)^{t-1}\left(\frac{Y}{\lambda_Y}\right)^{t-1} x^p(1).$$

As matrix Y is indecomposable and primitive, then (see [2]) sequence $(Y/\lambda_Y)^t\, x^p(1)$ as $t \to \infty$ converges to limit μx_Y where $\mu = \|x^p(1)\|\|x_Y\|^{-1}$, and x_Y is the Fobenius vector of Y. This means that the sequence of normalized vectors $x(t)/\|x(t)\|$ as $t \to \infty$ converges to limit $x_Y/\|x_Y\|$, i.e., to normalized Fobenius vector of Y. It follows from this that equality $x(t+1) = \gamma x(t)$ holds as exactly as it is needed at large enough t.

Scheme 2. The planning is based on a joint demand.

Here we assume that the plan of output of any sector at step t is determined unambiguously by the joint demand for products produced at step $t-2$:

$$x^p(t) = R(t)x^d(t-2), \quad t \geq 2, \tag{2}$$

where $R(t) = \{r_{ij}(t)\}$ is a diagonal matrix, $r_{ii}(t) > 0$. Further, the procedures for scheme 1 and scheme 2 are identical.

Using induction, write the relationship between vectors $x(t)$ and $x^p(1)$:

$$x(t) = \beta(t)R(t)YR(t-1)Y\cdots R(2)Yx^p(1). \tag{3}$$

If $R(t) \equiv R$ at all t, then using (3), we get

$$x(t) = \beta(t)(RY)^{t-1}x^p(1).$$

Since matrix Y is primitive and all diagonal elements of R are positive, it follows that RY is a primitive matrix. Denote by $\lambda_{(RY)}$ the Frobenius eigenvalue of RY. Arguing as above, we see that sequence $\left(RY/\lambda_{(RY)}\right)^{t-1} x^p(1)$ as $t \to \infty$ converges to limit $\mu x_{(RY)}$ where $\mu = \|x^p(1)\|\|x_{(RY)}\|^{-1}$, and $x_{(RY)}$ is the Frobenius vector of RY. It is clear that if all diagonal elements of matrix R are identical and equal to r, then matrixes Y and RY have identical Frobenius vectors. In this case the cheme of planning brings the system asymptotically to the turnpike if $r \geq \gamma^2$.

Now suppose that diagonal elements of matrix R are not identical. Further assume that $r_{ii} \geq \gamma^2 > 1$, $i = 1, \dots, n$. In this case we have $Y \leq RY$. This implies (see [2]) that Frobenius egenvalues of matrixes Y and RY satisfy the inequality $\lambda_Y \leq \lambda_{(RY)}$. This means that the rate of growth $\gamma_{(RY)} \equiv 1/\lambda_{(RY)}$ does not exceed γ. Specifically, if matrix Y is strictly positive, then $Y < RY$ and $\lambda_Y < \lambda_{(RY)}$ (see [2]). In this case we have $\gamma_{(RY)} < \gamma$.

Let us remark that scheme 1 has a preference with respect to scheme 2. The latter has not a relation between the planning and real production. In this case the plan numbers may multiple exceed the real output. In addition, this imbalance is not bounded above. On the other hand, scheme 1 has a feedback between the real output and the plan numbers. In this case, the disparity between these variables is relatively small if the parameter k is specificed in a reasonable way.

The considered schemes show the theoretical possibility of asymptotic entry of a multisectoral economy in the turnpike in the absence of central planning and control. Notwithstanding, in making plans, the proposed schemes require certain coordination of the tasks of the sectors. The purpose of the coordination is determination of constant k in (1) or r in (2). These indicators should equal the squared growth rate on the turnpike or exceed it to a measure of rationality. If all sectors plan their activity proceeding from a single growth rate, then the economic system will asymptomatically reach the turnpike of balanced growth.

References

[1] H. Nikaido, *Convex Structures and Economic Theory.* Academic Press, 1968.

[2] R. A. Horn and Ch. R. Johnson, *Matrix Analysis.* Cambridge University Press, 1986.