Unfocussed Nash Equilibria*

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Abstract

A rectangular set of strategy profiles may be called an unfocussed Nash equilibrium if it contains no Nash equilibrium, but no profitable individual deviation from a point in the set produces a point outside it. As David Kreps (A Course in Microeconomic Theory. Princeton University Press, 1990, pp. 416-417) has noticed, it may happen that all the players would prefer an unfocussed equilibrium to any of singleton ones. This paper shows that such an unpleasant occurrence is impossible under the most widely used sufficient conditions for the existence of Nash equilibrium, although just a step aside brings us into a dangerous area. *Journal of Economic Literature* Classification Number: C 72.

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1 Introduction

A view seems commonly accepted that the concept of Nash equilibrium gives an adequate formal expression to the idea of a self-policing agreement between the participants of a strategic game. We start with an observation that for an agreement to be self-policing and to make sense for the players, it need not be comprehensive, i.e., include a complete specification of all actions of the players. Sometimes an agreement not to choose certain strategies is the best thing the players can reasonably hope for. Let us consider an example essentially due to Kreps (1990, pp. 416-417).

Example 1.1. Two players choose a natural number each. Utility levels associated with the strategy profiles fill the following "matrix:"

$$\begin{matrix} (1,1) & (0,0) & (0,0) & (0,0) & (0,0) & \dots \\ (0,0) & (2,2) & (3,4) & (3,4) & (3,4) & \dots \\ (0,0) & (4,3) & (2,2) & (3,4) & (3,4) & \dots \\ (0,0) & (4,3) & (4,3) & (2,2) & (3,4) & \dots \\ (0,0) & (4,3) & (4,3) & (4,3) & (2,2) & \dots \\ \dots & \dots & \dots & \dots & \dots & \ddots \end{matrix}$$

There is a unique Nash equilibrium in the game with utility levels $(1,1)$. Meanwhile, if both players do *not* choose their unique equilibrium strategies, their utility levels are strictly greater than that, $(2,2)$ at least. Besides, the consequences of a coordination failure are the same in both cases. It seems hardly reasonable to expect (to say nothing of recommending) the unique equilibrium to be chosen.

Let us formulate the behavioral assumptions implicit in our analysis of the example. First, each player has an ordinal utility function describing his preferences over possible outcomes, i.e., strategy profiles; the description of preferences with utility functions inflicts some loss of generality, but the simplicity of presentation is more important for us here. The players need not be able to compare lotteries on outcomes, or even understand what a probability is; nonetheless, cardinal utilities are easily incorporated if we consider profiles of mixed strategies as outcomes (concepts devoid of straightforward ordinal analogues, such as correlated equilibrium, for instance, will not appear in the paper).

Second, we make an assumption about each player's attitude towards situations where his choice does not determine a unique outcome of the game: if every outcome possible in one situation (after taking into account all information available or assumed at the moment of decision) is better than every outcome possible in another, then the player prefers the first situation to the second — a "sure-thing principle". If two situations involving uncertainty do not satisfy the condition, the player may have any preferences, or be unable to compare them at all.

Very formally speaking, one does not have to accept the sure-thing principle: ordinal preferences ensure the ability to compare *outcomes* and not necessarily anything else. Still, the principle is intuitively appealing and is regarded as a must in other branches of decision theory. It holds, in particular, if the players evaluate situations involving uncertainty with the worst possible outcome or with expectations under a subjective probability distribution.

Throughout the paper we assume that ordinal preferences imply the sure-thing principle too.

Now it is easy to see that an agreement not to choose the unique equilibrium strategies in Example 1.1 is self-policing in the same sense as an agreement to choose the equilibrium: if I expect my partner to honour the agreement, I will prefer any strategy permitted to me by the agreement to any (in this case, the only one) prohibited strategy. Besides, both players strictly prefer the first, incomplete, agreement to the second.

To clarify the logic of our analysis further, let us replace the matrix in Example 1.1 with the following one:

$$\begin{array}{ccc} (1,1) & (0,0) & (0,0) \\ (0,0) & \underline{(3,2)} & \underline{(2,3)} \\ (0,0) & \underline{(2,3)} & \underline{(3,2)} \end{array}$$

Again, both players would prefer an agreement not to choose their (unique) equilibrium strategies to an agreement to choose the equilibrium, but this time it seems possible to argue that the former agreement is indistinguishable from an equilibrium where each player chooses the mixed strategy $\langle 0, 1/2, 1/2 \rangle$ (no alternative mixed equilibrium existed in Example 1.1). This argument is not accepted here: our players have ordinal preferences and they do not have to be able to compare probability distributions on the set of outcomes, so a mixed equilibrium, generally, makes no sense for them. Still, the above matrix admits a subtler argument: if, say, player 1 uses his mixed equilibrium strategy, player 2 gets utility level 3 with probability 1/2 and utility level 2 with probability 1/2 whichever of the last two columns he chooses, so there is no need for cardinal utilities, i.e., for the ability to compare arbitrary lotteries, to find that the two mixed strategies really form an equilibrium. The argument need not be accepted: as has already been mentioned, our players do not have to attach any meaning to the "probability 1/2;" besides, the symmetry generating that "ordinal mixed equilibrium" can, and will, easily be avoided in the examples to follow.

Pareto dominance is not the only reason why the players might prefer an incomplete agreement to a complete (Nash equilibrium) one.

Example 1.2. Consider the following bimatrix game:

$$\begin{array}{cccc} (5,5) & (0,2) & (0,2) & (0,1) \\ (2,0) & \underline{(5,3)} & \underline{(3,4)} & (2,1) \\ (2,0) & \underline{(3,5)} & \underline{(4,3)} & (2,1) \\ (1,0) & (1,2) & (1,2) & (3,3) \end{array}$$

There are two Nash equilibria with utility levels (5,5) and (3,3), respectively. One of them is Pareto better, but the other is less risky. If the players believe that the latter property outweighs the former, they may find an agreement not to choose either of the equilibrium strategies even more attractive: the agreement is self-enforcing, promises each player the utility level 3 at least, and is even less risky.

Example 1.3. Consider the following bimatrix game:

$$\begin{array}{cccc} (1,5) & (0,0) & (0,0) & (0,0) \\ (0,0) & \underline{(4,2)} & \underline{(3,4)} & (0,0) \\ (0,0) & \underline{(2,4)} & \underline{(4,3)} & (0,0) \\ (0,0) & (0,0) & (0,0) & (5,1) \end{array}$$

There are two Nash equilibria here [with utility levels (1,5) and (5,1)], and both are Pareto optimal; however, either of them treats the players so unequally that it may prove a rather hard task for the players to reach a deal. On the other hand, an agreement for each player not to choose either of equilibrium strategies is self-policing as well. From an egalitarian viewpoint as expressed by the leximin criterion (assuming that both players use the same ordinal scale), every outcome possible under the agreement is strictly preferred to either equilibrium. It seems thus not unreasonable to assert that this non-equilibrium agreement has good chances to be signed.

To summarize, Nash (or even strong) equilibria may exist but be rejected by the players, who would prefer a way of behavior leaving the final outcome unpredictable. As long as this happens in isolated examples, there is no need to worry: an arbitrary game need not have an equilibrium in the first place. However, if all equilibria in a game covered by a general existence theorem turn out to be irrelevant to the decision problem of the players, one cannot help feeling some dissatisfaction with the theorem in question.

In this paper we look for conditions ensuring not only existence, but also impossibility of this kind of irrelevance, of equilibria. Naturally, we address conditions already discovered in the literature on equilibrium existence.

First of all, we have to formalize the problem. A rectangular set of strategy profiles is called a generalized Nash equilibrium if no profitable individual deviation from a point in the set produces a point outside it. An agreement to restrict the choices of all players to a generalized Nash equilibrium is self-policing in the same sense as an agreement to choose a Nash equilibrium. Singleton generalized Nash equilibria are exactly usual Nash equilibria (so to speak, well focussed ones). The unpleasant feature of the above examples was the presence of "unfocussed equilibria," i.e., generalized Nash equilibria containing no singleton Nash equilibria (which happened to be more attractive for the players).

A strategic game is said to possess enough Nash equilibria if every generalized Nash equilibrium contains a Nash equilibrium. Theorem 3.1 below states that a strategic game possesses enough Nash equilibria if every strategy profile is connected to a Nash equilibrium with a finite improvement path (at each step of which the current profile of strategies is replaced with the result of an individual profitable deviation). The latter property is ensured, in particular, by Monderer and Shapley's (1996) FIP or by Milchtaich's (1996) FBRP properties, but is much weaker.

An infinite (topological) game may possess enough Nash equilibria (e.g., this holds for mixed extensions of finite games), but a weaker property appears more natural: A strategic game possesses almost enough Nash equilibria if every closed generalized Nash equilibrium contains a Nash equilibrium. Theorem 3.4 states that a strategic game has the property if every strategy profile is connected to a Nash equilibrium with a transfinite improvement path, combining individual profitable deviations with the picking of limit points. In this paper, transfinite paths play a purely technical rôle; accordingly, there is no need to worry about their interpretation.

Similar treatment can be given to other solution concepts for strategic games which are defined as maximizers for appropriate binary relations; in this paper, strong Nash equilibrium is considered (Example 1.3 shows that this concept may give rise to the same problem).

The paper is organized as follows. Section 2 contains basic formal definitions and some preliminary results; in particular, it is shown that the conditions of Gurvich's (1988) existence result do not ensure that the game possesses enough Nash equilibria. In Section 3, improvement paths are introduced and basic results involving them proven.

Section 4 considers convex-concave games; Theorem 4.2 states that such games possess almost enough Nash equilibria; however, the mere fact that the Kakutani (or even Brouwer) theorem applies to the best responses does not ensure the relevance of singleton equilibria.

Section 5 deals with games with strategic complementarities; Theorem 5.2 states that such games possess almost enough Nash equilibria. It should be noted that the fact does not follow from the results about best response dynamics in the literature (Topkis, 1979; Vives, 1990; Milgrom and Roberts, 1990); actually, a strategy profile in a supermodular game need not be connected to a Nash equilibrium with a best response improvement path.

2 Basic Notions

Let Γ be a strategic game defined by a finite set of players N, and strategy sets X_i and ordinal utility functions u_i on $X = \prod_{i \in N} X_i$ for all $i \in N$. When considering infinite games, we will assume that each X_i, hence X too, is a separable metric space.

Remark. It is sufficient to assume each X_i a Hausdorf topological space with a countable base of open sets, but we again prefer the simplicity of presentation.

We introduce a number of binary relations on X ($y, x \in X$, $i \in N$, $I \subseteq N$, $I \neq \emptyset$):

$$y \vartriangleright_i x \iff [y_{-i} = x_{-i} \,\&\, u_i(y) > u_i(x)];$$

$$y \vartriangleright x \iff \exists i \in N \, [y \vartriangleright_i x];$$

$$y \vartriangleright_I^* x \iff [y_{-I} = x_{-I} \,\&\, \forall i \in I \, (u_i(y) > u_i(x))];$$

$$y \vartriangleright^* x \iff \exists I \subseteq N \, [y \vartriangleright_I^* x];$$

$$y \vartriangleright_I^{**} x \iff [y_{-I} = x_{-I} \,\&\, \forall i \in I \, (u_i(y) \geq u_i(x)) \,\&\, \exists i \in I \, (u_i(y) > u_i(x))];$$

$$y \vartriangleright^{**} x \iff \exists I \subseteq N \, [y \vartriangleright_I^{**} x].$$

A strategy profile $x \in X$ is a Nash equilibrium if and only if x is a maximizer for $\vartriangleright$, i.e., if $y \vartriangleright x$ is impossible for any $y \in X$. Similarly, maximizers for the relation $\vartriangleright^*$ are called strong equilibria, and for $\vartriangleright^{**}$, very strong equilibria. Actually, both relations are used in the literature to define a strong equilibrium; for our purposes, however, the concept defined with $\vartriangleright^*$ is much more convenient.

Equivalently, Nash equilibrium can be defined by conditions $x_i \in R_i(x_{-i})$ for all $i \in N$, where $R_i(\cdot)$ is the best response correspondence,

$$R_i(x_{-i}) = \{y_i \in X_i | \; \forall z_i \in X_i \, [u_i(y_i, x_{-i}) \geq u_i(z_i, x_{-i})]\}.$$

A generalized Nash equilibrium is a rectangular subset of X, $X' = \prod_{i \in N} X'_i$, such that, for each $i \in N$, each $x' \in X'$, and each $x_i \notin X'_i$,

$$u_i(x') \geq u_i(x'_{-i}, x_i). \tag{2.1}$$

Remark. If X' is a singleton, we have the usual definition of a Nash equilibrium. Moreover, x is a Nash equilibrium if and only if $\{x\}$ is a generalized Nash equilibrium.

Theorem 2.1. *A rectangular subset $X' \subseteq X$ is a generalized Nash equilibrium if and only if $x \in X'$ and $y \triangleright x$ imply $y \in X'$.*

Proof. Immediately follows from (2.1). □

Remark. The concept of an undominated set in choice theory immediately comes to mind; note, however, that in our context X' must be rectangular.

A rectangular subset $X' \subseteq X$ is a *generalized strong equilibrium* if and only if $x \in X'$ and $y \triangleright^* x$ imply $y \in X'$.

A rectangular subset $X' \subseteq X$ is a *generalized very strong equilibrium* if and only if $x \in X'$ and $y \triangleright^{**} x$ imply $y \in X'$.

A strategic game *possesses enough Nash equilibria* if every generalized Nash equilibrium contains a Nash equilibrium.

A strategic game *possesses enough strong equilibria* if every generalized strong equilibrium contains a strong equilibrium.

A strategic game *possesses enough very strong equilibria* if every generalized very strong equilibrium contains a very strong equilibrium.

A strategic game *possesses almost enough Nash equilibria* if every generalized Nash equilibrium which is closed as a subset of X contains a Nash equilibrium.

A strategic game *possesses almost enough strong equilibria* if every generalized strong equilibrium which is closed as a subset of X contains a strong equilibrium.

Remark. Since X itself is always a generalized very strong equilibrium, a game possessing (almost) enough Nash, or (very) strong, equilibria actually has an equilibrium.

Example 2.1. Let us consider a two person game with $X_1 = X_2 = [0, 1]$, $u_1(x_1, x_2) = \min\{l_1(x_1), l_2(x_1, x_2)\}$, where $l_1(x_1) = x_1$ and $l_2(x_1, x_2) = x_1 \cdot x_2/2(x_2 - 2) + (x_2^2 - 4x_2)/4(x_2 - 2)$, and $u_2(x_1, x_2) = u_1(x_2, x_1)$. It is easily checked that $l_1(x_1)$ increases, and $l_2(x_1, x_2)$ decreases, in x_1 and that $l_1(x_1) = l_2(x_1, x_2)$ iff $x_1 = x_2/2$, which, therefore, is the best response function for player 1; similarly, $x_2 = x_1/2$ is the best response function for player 2. The unique Nash equilibrium is $(0, 0)$, providing each player with a utility level 0. Meanwhile, whenever $x_1 > 0$ and $x_2 > 0$, both players receive strictly positive utility levels; therefore, the conditions define a generalized Nash (actually, even very strong) equilibrium, Pareto dominating the unique singleton equilibrium. It is easy to see that there is no other generalized Nash equilibrium (closed or not): If X'_i contains a strategy $x'_i > 0$, then the unique optimal response, $x'_i/2$, must be in the partner's X'_j, then $x'_i/4 \in X'_i$, etc.; against a fixed strategy of the partner, say, $x'_i/2$, small strategies $x'_i/4^k \in X'_i$ produce infinitesimal utility levels, hence every strictly positive strategy must be included in X'_i. To summarize, this game possesses almost enough Nash equilibria, but not enough.

It is worth noting that the singleton equilibrium in the example belongs to the closure of the generalized one, so the ensured gain from switching to the latter is infinitesimal. The notion of "almost enough equilibria" deserves attention exactly because of this connection, which holds under relatively mild topological assumptions.

Theorem 2.2. *If each u_i is upper semicontinuous in x and continuous in x_{-i}, then the closure of a generalized Nash equilibrium is a generalized Nash equilibrium.*

The proof is a straightforward simplification of that of the next theorem.

Theorem 2.3. *If each u_i is upper semicontinuous in x and continuous in x_{-i}, then the closure of a generalized strong equilibrium is a generalized strong equilibrium.*

Proof. We start with a technical lemma.

Lemma 2.3.1. *If, under the same assumptions on u_i's, $x^* \triangleright_I^* x^0$, then there exists an open neighbourhood $U \ni x^0$ such that $(x_I^*, x_{-I}) \triangleright_I^* x$ for all $x \in U$.*

Proof. For each $i \in I$, denoting $u_i(x^*) - u_i(x^0) = 2\varepsilon_i > 0$, we pick open neighbourhoods $V_i' \ni x^0$ and $V_i'' \ni x_{-I}^0$ such that $u_i(x^*) > u_i(x) + \varepsilon_i$ for all $x \in V_i'$ and $u_i(x_I^*, x_{-I}) > u_i(x^*) - \varepsilon_i\,[= u_i(x_I^*, x_{-I}^0) - \varepsilon_i]$ for all $x_{-I} \in V_i''$. Then we denote $U = \bigcap_{i \in I}[V_i' \cap (V_i'' \times X_I)] \subseteq X$. Now $u_i(x_I^*, x_{-I}) > u_i(x^*) - \varepsilon_i > u_i(x)$ for all $i \in I$ and all $x \in U$. □

Let X' be a generalized strong equilibrium. Suppose there exist $x' \in \operatorname{cl} X'$, $x^* \in X \setminus \operatorname{cl} X'$, and $I \subseteq N$ such that $x^* \triangleright_I^* x'$; $x^* \notin \operatorname{cl} X'$ and $x_{-I}^* = x_{-I}'$ imply $x_j^* \notin X_j'$ for some $j \in I$. Lemma 2.3.1 (applied to x^* and $x' = x^0$) implies the existence of an appropriate neighbourhood U. Since $x' \in \operatorname{cl} X'$, there exists $x'' \in U \cap X'$. Now $(x_I^*, x_{-I}'') \triangleright_I^* x''$, and $(x_I^*, x_{-I}'') \notin X'$ because $x_j^* \notin X_j'$; this plainly contradicts our assumption that X' is a generalized strong equilibrium. □

An analogue of Theorem 2.3 for very strong equilibrium is wrong.

Example 2.2. Let $N = \{1, 2\}$ and $X_1 = X_2 = [0, 1] \cup \{2\}$; on $[0, 1] \times [0, 1]$, the utilities are the same as in Example 2.1, $u(2, 2) = \langle 2, -1 \rangle$, $u(2, x_2) = \langle -1, -2 \rangle$ for $0 \leq x_2 \leq 1$, and $u(x_1, 2) = \langle 1, 0 \rangle$ for $0 \leq x_1 \leq 1$. Now $X' =]0, 1[\times]0, 1[$ is a generalized very strong equilibrium, but its closure, $[0, 1] \times [0, 1]$, is not: $(0, 2) \triangleright^{**} (0, 0)$.

We complete the section with an important negative example. Gurvich (1988) showed that a two person game form possesses a Nash equilibrium for whatever preferences of the players if and only if the form is dense. (The latter property means that, whenever one player cannot ensure $x \in A$ with a choice of his strategy, the partner can ensure $x \notin A$, where A is an arbitrary set of outcomes). The theorem is justly regarded as an important equilibrium existence result.

Example 2.3. Let us consider the following two person game form (with three strategy for either player and five outcomes):

a	a	a
a	b	c
a	d	e

It is easy to see that the form is dense and, indeed, has a Nash equilibrium for whatever preferences of the players: if they choose the first row and first column, respectively, then neither player can change the outcome, to say nothing of improving his utility level. Suppose, however, that the utilities of the players satisfy these inequalities: $u_1(b) > u_1(e) > u_1(d) > u_1(c) > u_1(a)$ and $u_2(c) > u_2(d) > u_2(b) > u_2(e) > u_2(a)$. Now the game has a unique singleton equilibrium, but there is also a generalized Nash equilibrium, "do not choose your unique equilibrium strategy," ensuring a strictly better result for either player. It seems reasonable to expect that the players will ignore the unique Nash equilibrium, preferring to play a (sub)game without an equilibrium but with higher utility levels. In other words, the Gurvich theorem provides no grounds to believe that the participants of a game generated from a dense game form will choose a Nash equilibrium (which fact does not compromise its formal validity as an existence result).

Most likely, the Gurvich theorem shares this unpleasant feature with many other results about game forms, although the question is not investigated here. Danilov and Sotskov (2002) admit that equilibria constructed in the theory of mechanism design often look unnatural. Note, however, that our current concern is not whether the players are able to choose an equilibrium, but whether it is in their interests to do so.

3 Improvement Paths

A *finite (individual) improvement path* is a sequence $\{x^k\}_{k=0,1,\ldots,m}$ such that $x^{k+1} \triangleright x^k$ whenever $0 \leq k \leq m-1$.

Theorem 3.1. *A strategic game possesses enough Nash equilibria if every strategy profile is connected to a Nash equilibrium with a finite improvement path.*

Proof. Let X' be a generalized Nash equilibrium; pick $x^0 \in X'$. By our assumption, there exists a finite improvement path connecting x^0 to a Nash equilibrium x^m. If $x^m \in X'$, we are home; otherwise, we pick the minimal k for which $x^k \notin X'$ (note that $k > 0$). Now we have $x^{k-1} \in X'$, $x^k \notin X'$, and $x^k \triangleright_i x^{k-1}$; but this obviously contradicts Theorem 2.1. □

Remark. Young (1993) called a stronger version of the condition of the theorem (he only considered best response improvement paths) "weak acyclicity." Since the condition does not prohibit any kind of improvement cycles (even though is implied by acyclicity), a term like "weak von Neumann–Morgenstern stability" might be more appropriate.

A *finite (weak) coalition improvement path* is a sequence $\{x^k\}_{k=0,1,\ldots,m}$ such that $x^{k+1} \triangleright^* x^k$ ($x^{k+1} \triangleright^{**} x^k$) whenever $0 \leq k \leq m-1$.

Theorem 3.2. *A strategic game possesses enough strong equilibria if every strategy profile is connected to a strong equilibrium with a finite coalition improvement path.*

Theorem 3.3. *A strategic game possesses enough very strong equilibria if every strategy profile is connected to a very strong equilibrium with a finite weak coalition improvement path.*

The proofs are virtually identical with that of Theorem 3.1.

When considering infinite games, it is often useful to consider improvement paths parameterized by countable ordinal numbers. The concept was developed in Kukushkin (2000); the mathematical background can be found, e.g., in Natanson (1974, Chapter XIV). Here I provide just a sketch of formal definitions.

A partially ordered set is well ordered if every subset contains a least point. Ordinal numbers, or just ordinals, are types of well ordered sets. The set of all countable ordinals, denoted K, is well ordered (but uncountable) itself. We denote $\langle 0, \alpha \rangle = \{\beta \in K | \quad \beta < \alpha\}$; note that $\alpha \notin \langle 0, \alpha \rangle$ (actually, α is the type of $\langle 0, \alpha \rangle$). For each $\alpha \in K$, its successor, denoted $\alpha + 1$, is uniquely defined. An ordinal $\alpha \in K \setminus \{0\}$ is called isolated if $\alpha = \beta + 1$; otherwise, α is called a limit ordinal number. The least limit ordinal is ω: the type of the chain of all natural numbers. It is sometimes convenient to consider a partial function $\alpha - 1$ defined by the equality $\alpha = (\alpha - 1) + 1$ for isolated α and not defined at all for limit ordinals. Every countable subset of K has a least upper bound in K (Theorem 2, Section 5, Chapter XIV of Natanson). Every limit ordinal $\alpha \in K$ is the least upper bound of a strictly increasing infinite sequence in K (Theorem 4, Section 5, Chapter XIV of Natanson).

A *countable (individual) improvement path* is a mapping $\pi : \langle 0, \mu \rangle \to X$, where $\mu \in K$, satisfying these two conditions:

1. $\pi(\alpha + 1) \triangleright \pi(\alpha)$ whenever $\alpha + 1 \in \langle 0, \mu \rangle$;

2. if $\alpha \in \langle 0, \mu \rangle$ and α is a limit ordinal, there exists an infinite sequence $\{\beta^k\}_{k=0,1,\ldots}$ for which $\beta^{k+1} > \beta^k$ for all k, $\alpha = \sup_k \beta^k$, and $\pi(\alpha) = \lim_{k \to \infty} \pi(\beta^k)$.

Remark. Obviously, a finite improvement path is a particular case of a countable improvement path.

Theorem 3.4. *A strategic game possesses almost enough Nash equilibria if every strategy profile is connected to a Nash equilibrium with a countable improvement path.*

Proof. Let X' be a generalized Nash equilibrium; pick $x^0 \in X'$. By our assumption, there exists a countable improvement path π connecting x^0 to a Nash equilibrium x^*. If $x^* \in X'$, we are home; otherwise, we pick the first $\alpha \in K$ for which $\pi(\alpha) \notin X'$ ($\alpha > 0$). If α is a limit ordinal, we have a contradiction with the closedness of X'. If $\alpha = \beta + 1$, we have a contradiction with Theorem 2.1 exactly in the same way as in Theorem 3.1. □

Remark. The game in Example 2.1 satisfies the conditions of the theorem, so the "almost" in the formulation cannot be dropped.

A *countable coalition improvement path* is a mapping $\pi : \langle 0, \mu \rangle \to X$, where $\mu \in K$, satisfying these two conditions:

1. $\pi(\alpha + 1) \triangleright^* \pi(\alpha)$ whenever $\alpha + 1 \in \langle 0, \mu \rangle$;

2. if $\alpha \in \langle 0, \mu \rangle$ and α is a limit ordinal, there exists an infinite sequence $\{\beta^k\}_{k=0,1,\ldots}$ for which $\beta^{k+1} > \beta^k$ for all k, $\alpha = \sup_k \beta^k$, and $\pi(\alpha) = \lim_{k \to \infty} \pi(\beta^k)$.

Theorem 3.5. *A strategic game possesses almost enough strong equilibria if every strategy profile is connected to a strong equilibrium with a countable coalition improvement path.*

The proof is virtually identical with that of the previous theorem.

Theorems 3.1–3.5 are almost tautological; still, they are applicable to a rather wide variety of strategic game models. First, finite games with an acyclic relation $\triangleright$ satisfy the condition of Theorem 3.1: Rosenthal's (1973) congestion games and other examples of finite potential games (Monderer and Shapley, 1996) should be mentioned. The same condition holds when $\triangleright$ is not acyclic, but it is possible to describe a way of extending improvement paths without cycling, as in the normal forms of games with perfect information (Kukushkin, 2002), in finite games with additive aggregation and some monotonicity conditions (Kukushkin, 2001), or in finite pseudosupermodular games (Theorem 5.2 below).

Theorem 3.2 applies to certain congestion games (Holzman and Law-Yone, 1997), to games with partial rivalry (Milchtaich, 1996; Konishi et al., 1997), and to voting by veto (Kukushkin, 1999b); Theorem 3.3, to finite "Germeier–Vatel" models (Kukushkin, 1999b). Theorem 3.4 works for infinite potential games as defined in Kukushkin (1999a), examples of which may be found in Monderer and Shapley (1996) and Kukushkin (1994); it is also referred to in the proofs of Theorems 5.2 and 5.3 below. Theorem 3.5 applies to compact-continuous "Germeier–Vatel" models (Kukushkin, 1999b).

4 Convex-Concave Games

Let $u(\cdot)$ be a real-valued function defined on a convex subset of a topological vector space; we call u *strictly quasiconcave* if

$$u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\} \tag{4.1}$$

with a strict inequality whenever $\lambda \in]0, 1[$ and $u(x) \neq u(y)$.

Lemma 4.1. *Let u be a strictly quasiconcave function, $\lambda_k \geq 0$ for $k = 1, \ldots, m$, $\sum_k \lambda_k = 1$, and $x^1, \ldots, x^m$ belong to the domain of u; then*

$$u(\sum_k \lambda_k x^k) \geq \min_k u(x^k)$$

with a strict inequality whenever $\lambda_k > 0$ for all k and $\max_k u(x^k) > \min_k u(x^k)$.

The proof goes by a straightforward recursion using (4.1).

Theorem 4.2. *A strategic game possesses almost enough Nash equilibria if each X_i is a compact and convex subset of a topological vector space, and each u_i is upper semicontinuous in x, continuous in x_{-i}, and strictly quasiconcave in x_i.*

Proof. Let X' be a generalized Nash equilibrium. For every $i \in N$ and $x'_{-i} \in X'_{-i}$, $R_i(x'_{-i}) \subseteq X_i$ is closed and convex. We denote $Y_{-i} = \{x'_{-i} \in X'_{-i} | \quad R_i(x'_{-i}) \subseteq X'_i\}$, $M = \{i \in N | \quad Y_{-i} \neq \emptyset\}$. Let $i \in N$ and $x'_{-i} \in X'_{-i}$; if there exists $x_i \in R_i(x'_{-i}) \setminus X'_i$, then $u_i(x_i, x'_{-i}) >$

$u_i(x'_i, x'_{-i})$ for all $x'_i \notin R_i(x'_{-i})$, hence (2.1) implies $X'_i \subseteq R_i(x'_{-i})$. Therefore, if $Y_{-i} = \emptyset$, the Nash condition for player i, $u_i(x') \geq u_i(x_i, x'_{-i})$ for all $x_i \in X_i$, holds at all $x' \in X'$, so we may just forget about such i (more formally, we may fix $x'_i \in X'_i$ for each $i \notin M$ arbitrarily). Denote $Y_i = \operatorname{conv} \bigcup_{x'_{-i} \in Y_{-i}} R_i(x'_{-i})$, $Z_i = \operatorname{cl} Y_i$, and $Z = \prod_{i \in M} Z_i$.

Let us show $Y_i \subseteq X'_i$ for each $i \in M$; suppose not: $x_i = \sum_k \lambda_k x_i^k \notin X'_i$ with all $\lambda_k > 0$, while $x_i^k \in R_i(x_{-i}^k)$ and $x_{-i}^k \in Y_{-i}$, hence $x_i^k \in X'_i$, for all k. Now if $x_i^k \in R_i(x_{-i}^1)$ for all k, then $x_i \in R_i(x_{-i}^1) \subseteq X'_i$; thus we have to assume $x_i^k \notin R_i(x_{-i}^1)$ for some k, then $u_i(x_i, x_{-i}^1) > u_i(x_i^k, x_{-i}^1)$ by Lemma 4.1, contradicting (2.1) (for $x_i \notin X'_i$, $x_i^k \in X'_i$, and $x_{-i}^1 \in X'_{-i}$).

Thus, $Y_i \subseteq X'_i$, hence $Z_i \subseteq X'_i$ because X'_i is closed, hence $Z \subseteq X'_M$. Now the existence of a Nash equilibrium in Z (among the players $i \in M$) follows from the Kakutani theorem in a standard way. Finally, (2.1) implies that an equilibrium in Z is an equilibrium in X_M; as to the players $i \notin M$, they have already been accounted for. □

Remark. A shorter proof is possible, but the current proof can be used in the following theorem as well.

Remark. Example 2.1 shows that "almost" cannot be dropped in the theorem.

Theorem 4.3. *The mixed extension of a finite strategic game possesses enough Nash equilibria.*

Proof. Obviously, the previous theorem applies. The only stage in its proof where the closedness of X' was used at all is deriving $Z_i \subseteq X'_i$ from $Y_i \subseteq X'_i$. Now each $R_i(x'_{-i})$ is the convex hull of a finite number of pure strategies; therefore, Y_i is also the convex hull of a finite number of pure strategies, hence is closed, hence $Z_i = Y_i \subseteq X'_i$. □

It remains unclear so far whether Theorem 4.2 could be proved with a reference to Theorem 3.4. The conditions of Theorem 4.3 do not imply those of Theorem 3.1.

Example 4.1. Let us consider the mixed extension of the following (bi)matrix game:

$$\begin{matrix} (-1,1) & (1,-1) \\ (1,-1) & (-1,1) \end{matrix}$$

There is a unique Nash equilibrium. Imagining a finite improvement path leading to the equilibrium from any other strategy profile, we easily see that there must be a stage when one of the players has ultimately chosen his equilibrium strategy, $\langle 1/2, 1/2 \rangle$, while the partner still chooses another strategy. Now the first player does not change his strategy because his choice is supposed to be final, while the second player cannot improve because his both pure strategies bring the same result.

For mixed extensions of infinite (compact-continuous) games, we can only apply Theorem 4.2, deriving the presence of almost enough Nash equilibria. Again, Example 2.1 (with a cardinal interpretation of the utilities) shows that "almost" cannot be dropped: mixed strategies assigning a positive probability to $]0, 1]$ form a generalized Nash equilibrium. Restricting the choice of each player to $x_i > 0$, we easily see that $x_i = 1/2$ strictly dominates any $x_i > 1/2$ (for either $i = 1, 2$); iterating this argument, we see that no strategy

$x_i > 0$ survives iterated deletion of strictly dominated strategies, hence this generalized Nash equilibrium contains no singleton (mixed) Nash equilibrium.

It is important to stress that the mere fact that the Kakutani (or even Brouwer) theorem can be applied to the best responses by no means ensures that the game possesses almost enough Nash equilibria.

Example 4.2. Let $N = \{1, 2\}$, $X_1 = X_2 = \{(\rho, \varphi)| \quad 0 \le \rho \le 1\}$ (regarded as a subset of the plane with polar coordinates), $u_1((\rho_1, \varphi_1), (\rho_2, \varphi_2)) = V(\rho_1, \rho_2) + \chi(\rho_1, \rho_2) \cdot \chi(\varphi_1, \varphi_2)$ and $u_2((\rho_1, \varphi_1), (\rho_2, \varphi_2)) = u_1((\rho_2, \varphi_2), (\rho_1, \varphi_1 \oplus \varphi^0))$, where $V(\rho_1, \rho_2) = \min\{2\rho_1, 3\rho_2 - \rho_1\}$, $\chi(s_1, s_2) = 1$ if $s_1 = s_2$ and $\chi(s_1, s_2) = 0$ otherwise, $\oplus$ denotes addition modulo 2π, and $0 < \varphi^0 < 2\pi$. The best responses are easy to find: $R_1(\rho_2, \varphi_2) = \{(\rho_2, \varphi_2)\}$ and $R_2(\rho_1, \varphi_1) = \{(\rho_1, \varphi_1 \oplus \varphi^0)\}$; both are continuous. Therefore, the existence of a Nash equilibrium is ensured by the Brouwer theorem; indeed, the origin is a unique equilibrium, providing the players with utility levels $\langle 1, 1\rangle$. If the players agree to choose $\rho_1 = 1$ and $\rho_2 = 1$, they receive the utility levels $\langle 2, 2\rangle$, at least, and the agreement is self-policing.

Although the best responses in the example are well defined, the conditions of Theorem 2.2 are not met (so the upper hemi-continuity of the best responses is "accidental"). It seems worthwhile to produce a more complicated example with continuous utilities.

Example 4.3. Again, there are two players with the same discs as strategy sets, $X_1 = X_2 = \{(\rho, \varphi)| \quad 0 \le \rho \le 1\}$, and continuous utilities: $u_1((\rho_1, \varphi_1), (\rho_2, \varphi_2)) = V(\rho_1, \rho_2) + \rho_2 \cdot \eta'(\rho_1, \rho_2) \cdot \eta''(\varphi_1, \varphi_2)$ and $u_2((\rho_1, \varphi_1), (\rho_2, \varphi_2)) = u_1((\rho_2, \varphi_2), (\rho_1, \varphi_1 \oplus \varphi^0))$, where $V(\rho_1, \rho_2) = \min\{\rho_1, 4\rho_2 - \rho_1\}$, $r(\rho) = \min\{2\rho, 1\}$, both η' and η'' are continuous, $\eta''(\varphi_1, \varphi_2) = 1$ if $\varphi_1 = \varphi_2$ and $0 \le \eta''(\varphi_1, \varphi_2) < 1$ otherwise, $\eta'(\rho_1, \rho_2) = 1$ if $\rho_1 = r(\rho_2)$, $0 < \eta'(\rho_1, \rho_2) < 1$ whenever $0 < |\rho_1 - r(\rho_2)| < 1/3$ and $\eta'(\rho_1, \rho_2) = 0$ otherwise; finally, $0 < \varphi^0 < 2\pi$ again.

The best responses are $R_1(\rho_2, \varphi_2) = \{(r(\rho_2), \varphi_2)\}$ and $R_2(\rho_1, \varphi_1) = \{(r(\rho_1), \varphi_1 \oplus \varphi^0)\}$; again, the origin is a unique Nash equilibrium, providing the players with utility levels $\langle 0, 0\rangle$. Meanwhile, if $\rho_1 \ge 1/3$, then $u_2((\rho_1, \varphi_1), (\rho_2, \varphi_2)) < 1/3$ when $\rho_2 < 1/3$, and $u_2((\rho_1, \varphi_1), (\rho_2, \varphi_2)) \ge 1/3$ when $\rho_2 \ge 1/3$; a similar statement holds if the roles of the players are reversed. Thus, the conditions $\rho_i \ge 1/3$ for both i define a generalized Nash equilibrium Pareto dominating the unique singleton equilibrium.

5 Games with Strategic Complementarities

In this section, we assume two structures on each X_i, topology and order, such that the order is continuous in the topology, each X_i is a complete lattice and a compact space, and each utility function u_i is upper semicontinuous in own variable x_i. Under the assumptions, $R_i(x_{-i}) \ne \emptyset$ for all $i \in N$ and $x_{-i} \in X_{-i}$. The orders on each X_i induce an order (their product) on each X_{-i} and on X.

In the main theorem, we also assume that each utility function u_i has the properties of single crossing in (x_i, x_{-i}) (Milgrom and Shannon, 1994) and of pseudosupermodularity in x_i (Agliardi, 2000):

$$[y_i \ge x_i \,\&\, y_{-i} \ge x_{-i}] \Rightarrow [\mathrm{sign}(u_i(y) - u_i(x_i, y_{-i})) \ge \mathrm{sign}(u_i(y_i, x_{-i}) - u_i(x))] \tag{5.1}$$

and

$$\operatorname{sign}(\max\{u_i(x_i \vee y_i, z_{-i}) - u_i(x_i, z_{-i}), u_i(x_i \vee y_i, z_{-i}) - u_i(y_i, z_{-i})\}) \geq \operatorname{sign}(\max\{u_i(x_i, z_{-i}) - u_i(x_i \wedge y_i, z_{-i}), u_i(y_i, z_{-i}) - u_i(x_i \wedge y_i, z_{-i})\}), \quad (5.2)$$

where $i \in N$, $x_i, y_i \in X_i$, $x_{-i}, y_{-i}, z_{-i} \in X_i$, and $\operatorname{sign}(t)$ is -1 if $t < 0$, 0 if $t = 0$, and 1 if $t > 0$ (although subtraction is used in both definitions, the properties themselves are purely ordinal).

Lemma 5.1. *If a game satisfies (5.1) and (5.2), then, for each $i \in N$, $x_i, y_i \in X_i$, and $x_{-i}, y_{-i} \in X_i$,*

$$[y_{-i} \geq x_{-i} \,\&\, y_i \in R_i(y_{-i}) \,\&\, x_i \in R_i(x_{-i})] \Rightarrow [y_i \vee x_i \in R_i(y_{-i}) \,\&\, y_i \wedge x_i \in R_i(x_{-i})].$$

The statement means that $R_i(x_{-i})$ is a sublattice of X_i (pick $y_{-i} = x_{-i}$) and $R_i(\cdot)$ is increasing w.r.t. the strong set order defined by Veinott (see Topkis, 1979).

Proof. Indeed, $x_i \in R_i(x_{-i})$ implies $u_i(x) \geq u_i(x_i \wedge y_i, x_{-i})$ and $u_i(x) \geq u_i(y_i, x_{-i})$, hence, by (5.2), $u_i(x_i \vee y_i, x_{-i}) \geq u_i(y_i, x_{-i})$, hence, by (5.1), $u_i(x_i \vee y_i, y_{-i}) \geq u_i(y)$, hence $x_i \vee y_i \in R_i(y_{-i})$. On the other hand, $y_i \in R_i(y_{-i})$ implies that $u_i(y) \geq u_i(x_i \vee y_i, y_{-i})$, hence, by (5.1), $u_i(y_i, x_{-i}) \geq u_i(x_i \vee y_i, x_{-i})$, hence, by (5.2), $u_i(x_i \wedge y_i, x_{-i}) \geq u_i(x)$, hence $x_i \wedge y_i \in R_i(x_{-i})$. □

Remark. The lemma is obviously inspired by Proposition 3 of Agliardi (2000), but is formally independent of it.

Theorem 5.2. *If a game satisfies (5.1) and (5.2), then it possesses almost enough Nash equilibria.*

Proof. To produce an improvement path from an arbitrary strategy profile to an equilibrium, we impose the following rules: (1) If, at a current profile, there exist profitable deviations upwards, one of them must be chosen. (2) Otherwise, a most profitable (for the deviating player) deviation downwards must be chosen. It turns out that a path abiding by these rules cannot cycle and only stops at a Nash equilibrium.

Let us introduce some notation:

$$X^+ = \{x \in X | \quad \exists y \in X [y > x \,\&\, y \triangleright x]\}, \; X^- = X \setminus X^+,$$

$$R_i^-(x) = \{y_i \in X_i | \quad y_i \leq x_i \,\&\, \forall z_i \in X_i \; [z_i \leq x_i \Rightarrow u_i(y_i, x_{-i}) \geq u_i(z_i, x_{-i})]\},$$

$$y \mathrel{\triangleright\!\!\triangleright}_i x \iff y \triangleright_i x \,\&\, [y_i > x_i \vee (x \in X^- \,\&\, y_i \in R_i^-(x))],$$

$$y \mathrel{\triangleright\!\!\triangleright} x \iff \exists i \in N \; [y \mathrel{\triangleright\!\!\triangleright}_i x],$$

$$y \succ x \iff [y \in X^- \,\&\, x \in X^+] \vee [x, y \in X^+ \,\&\, y > x] \vee [x, y \in X^- \,\&\, y < x].$$

By definition, $y \mathrel{\triangleright\!\!\triangleright} x \Rightarrow y \triangleright x$; it is easy to check that $\succ$ is irreflexive and transitive.

Lemma 5.2.1. *If $x \in X$ is a maximizer for $\mathrel{\triangleright\!\!\triangleright}$, then x is a Nash equilibrium (i.e., a maximizer for $\triangleright$).*

Proof. Suppose the contrary: $x \in X$ is a maximizer for $\triangleright\!\!\triangleright$ (i.e., there is no profitable deviation either upwards or downwards), but not a Nash equilibrium, i.e., there exist $i \in N$ and $y_i \in X_i$ such that $u_i(y_i, x_{-i}) > u_i(x)$ (then y_i must be incomparable with x_i). We have $x_i \wedge y_i < x_i$, hence $u_i(x_i \wedge y_i, x_{-i}) \le u_i(x) < u_i(y_i, x_{-i})$; similarly, $x_i \vee y_i > x_i$, hence $u_i(x_i \vee y_i, x_{-i}) \le u_i(x) < u_i(y_i, x_{-i})$. Now we have a contradiction with (5.2). □

Lemma 5.2.2. *If $y \triangleright\!\!\triangleright x$, then $y \succ x$.*

Proof. The only point worth discussing is the incompatibility of $y \triangleright\!\!\triangleright_j x$, $y < x$, and $x \in X^-$ with $y \in X^+$. Suppose the contrary: there are $i \in N$ and $z_i > y_i$ such that

$$u_i(z_i, y_{-i}) > u_i(y). \tag{5.3}$$

Let us consider two alternatives.

If $i = j$ (hence $y_{-i} = x_{-i}$), $z_i > x_i$ contradicts $x \in X^-$ while $z_i < x_i$ contradicts $y_i \in R_i^-(x)$; therefore, we have to assume that z_i and x_i are incomparable, hence $z_i \vee x_i > x_i$. Now $y_i \in R_i^-(x)$ implies $u_i(y) \ge u_i(z_i \wedge x_i, x_{-i})$, hence, by (5.3) and (5.2), $u_i(z_i \vee x_i, x_{-i}) > u_i(x)$, contradicting $x \in X^-$.

Thus, we are led to $i \ne j$, hence $y_i = x_i$ and $y_{-i} < x_{-i}$. Now (5.3) and (5.1) imply $u_i(z_i, x_{-i}) > u_i(x)$, again contradicting $x \in X^-$. □

Lemma 5.2.3. *If $x^k \to x^\omega$ and $x^{k+1} \succ x^k$ for all $k = 0, 1, \ldots$, then $x^\omega \succ x^0$.*

Proof. As in Lemma 5.2.2, we only have to show that $x^k \in X^-$ and $x^{k+1} < x^k$ for all k imply $x^\omega = \lim_{k\to\infty} x^k \in X^-$. Suppose the contrary: there exist $i \in N$ and $y_i \in X_i$ such that $y_i > x_i^\omega$ (hence $y_i > x_i^k$ for all k large enough) and $u_i(y_i, x_{-i}^\omega) > u_i(x^\omega)$, hence

$$u_i(y_i, x_{-i}^\omega) > u_i(x_i^k, x_{-i}^\omega) \tag{5.4}$$

for all k large enough. Obviously, $x_{-i}^\omega \le x_{-i}^k$ for any k; therefore, (5.4) and (5.1) imply $u_i(y_i, x_{-i}^k) > u_i(x^k)$, hence $x^k \in X^+$ again. □

Lemmas 5.2.2 and 5.2.3, and Theorem 2.2 of Kukushkin (2000) imply that *every* improvement path π satisfying $\pi(\alpha+1) \triangleright\!\!\triangleright \pi(\alpha)$ whenever $\pi(\alpha+1)$ can be defined ends at a maximizer for $\triangleright\!\!\triangleright$. Lemma 5.2.1 implies that the maximizer is a Nash equilibrium. A reference to Theorem 3.4 completes the proof. □

Remark. For a finite game, both Lemma 5.2.3 and the reference to Kukushkin (2000) are obviously redundant. Example 2.1 shows that "almost" in the formulation of Theorem 5.2 cannot be dropped: that game satisfies the conditions. If all strategies are scalar (i.e., each X_i is a chain), then (5.2) is satisfied automatically and may be dropped.

Similarly to Section 4, the mere fact that Tarski's theorem can be applied to the best responses does not ensure that the game possesses almost enough Nash equilibria. However, in this context it is possible to formulate conditions on the strategy sets and best responses ensuring that the game has almost enough Nash equilibria.

Example 5.1. Let us consider a two person game with $X_1 = X_2 = \{(0,0), (0,1), (1,0), (1,1)\} \subseteq I\!R^2$ (with the standard order) and utilities described by these "matrices" (where player 1 chooses a 2×2 "block," and player 2 a position in the block; the axes are directed upwards and to the right):

$$\begin{bmatrix} \underline{(9,7)} & (5,5) \\ (5,5) & \underline{(7,9)} \end{bmatrix} \quad \begin{bmatrix} (5,5) & (6,6) \\ (0,0) & (5,5) \end{bmatrix}$$

$$\begin{bmatrix} (5,5) & (0,0) \\ (6,6) & (5,5) \end{bmatrix} \quad \begin{bmatrix} \underline{(8,9)} & (5,5) \\ (5,5) & \underline{(9,8)} \end{bmatrix}$$

Although the statement of Lemma 5.1 holds for the game (there is even Topkis's (1979), cardinal increasing differences property), the condition (5.2) does not. Both best response functions are increasing and, in accordance with Tarski's theorem, there exist even two singleton equilibria: $\langle (0,0), (0,0) \rangle$ and $\langle (1,1), (1,1) \rangle$. However, there also exists a generalized Nash equilibrium, $\{(0,1), (1,0)\} \times \{(0,1), (1,0)\}$, ensuring a higher utility level (and less risky) for either player.

Theorem 5.3. *Let one of the strategy sets, say X_1, be a partially ordered set, and all other X_i's be chains, still assuming that there is a topology on each X_i such that the order is continuous in the topology, X_i is compact, and u_i is upper semicontinuous in x_i. Let each best response correspondence be increasing in the sense that*

$$[x''_{-i} \geq x'_{-i} \,\&\, x''_i \in R_i(x''_{-i}) \,\&\, x'_i \in R_i(x'_{-i})] \Rightarrow x''_i \geq x'_i.$$

Then the game possesses almost enough Nash equilibria.

Proof. We will reason similarly to the proof of Theorem 5.2, considering improvement paths abiding by the following rules: (1) If, at a current profile, every profitable best response deviation leads downwards, one of them must be chosen. (2) Otherwise, if there exist profitable best response deviations of player 1 not leading upwards, one of them must be chosen. (3) Finally, if neither condition is satisfied, a profitable best response deviation upwards must be chosen. Again, a path abiding by the rules cannot cycle and only stops at a Nash equilibrium.

Let us introduce some notation:

$$X^- = \{x \in X \mid \quad \forall i \in N\, [x_i \in R_i(x_{-i}) \vee \forall x'_i \in R_i(x_{-i})(x_i > x'_i)]\},$$

$$X^+ = \{x \in X \setminus X^- \mid \quad x_1 \in R_1(x_{-1}) \vee \forall x'_1 \in R_1(x_{-1})(x'_1 > x_1)\},$$

$$y \rhd\!\!\rhd x \iff \exists i \in N\ [y_{-i} = x_{-i} \,\&\, x_i \notin R_i(x_{-i}) \ni y_i \,\& \\ (x \notin X^- \cup X^+ \Rightarrow i = 1) \,\&\, (x \in X^+ \Rightarrow y_i > x_i)],$$

$$y \succ x \iff [y \in X^- \,\&\, x \notin X^-] \vee [y \in X^+ \,\&\, x \notin X^- \cup X^+] \vee \\ [x, y \in X^+ \,\&\, y > x] \vee [x, y \in X^- \,\&\, y < x].$$

Obviously, $y \rhd\!\!\rhd x$ implies $y \rhd x$; it is easy to check that $\succ$ is irreflexive and transitive.

Lemma 5.3.1. *If $x \in X$ is a maximizer for $\rhd\!\!\rhd$, then x is a Nash equilibrium.*

Proof. Suppose $x \in X^+$; then there exists a player $i \in N$ for which $x_i \notin R_i(x_{-i})$ and $x'_i > x_i$ for some $x'_i \in R_i(x_{-i})$. Denoting $y = \langle x'_i, x_{-i} \rangle$, we obtain $y \rhd\!\!\rhd x$, which contradicts the condition on x. Now let $x \notin X^+ \cup X^-$; then $x_1 \notin R_1(x_{-1})$ and, picking $x'_1 \in R_1(x_{-1})$ and denoting $y = \langle x'_1, x_{-1} \rangle$, we again obtain $y \rhd\!\!\rhd x$. Therefore, $x \in X^-$. Now if $x_i \notin R_i(x_{-i})$ for some $i \in N$, we may pick $x'_i \in R_i(x_{-i})$ and, denoting $y = \langle x'_i, x_{-i} \rangle$, obtain $y \rhd\!\!\rhd x$ once again. □

Lemma 5.3.2. *If $y \rhd\!\!\rhd x$, then $y \succ x$.*

Proof. If $x \notin X^+ \cup X^-$, then, by the definition of $\rhd\!\!\rhd$, $y_1 \in R_1(x_{-1}) = R_1(y_{-1})$, hence $y \in X^+ \cup X^-$, hence $y \succ x$. If $x \in X^-$, then, for some $i \in N$, $y_{-i} = x_{-i}$ and $y_i < x_i$, hence $y < x$; now it is sufficient to show $y \in X^-$. Suppose the contrary: there are $j \in N$ and $y'_j \in R_j(y_{-j})$ such that $y'_j > y_j \notin R_j(y_{-j})$; then $j \neq i$, hence $y_j = x_j$ and $y_{-j} < x_{-j}$. Now $y'_j > x_j$ implies $x_j \notin R_j(x_{-j})$, but then we have $x_j > x'_j \geq y'_j$ for any $x'_j \in R_j(x_{-j})$. Finally, if $x \in X^+$, then, similarly, $y_{-i} = x_{-i}$ and $y_i > x_i$ for some $i \in N$, hence $y > x$. Now if $y \in X^-$, then $y \succ x$ by definition; otherwise, the conditions $y_1 \notin R_1(y_{-1})$ and $y'_1 > y_1$ for some $y'_1 \in R_1(y_{-1})$ would imply a contradiction in the same (or rather dual) way as above, hence $y \in X^+$, hence $y \succ x$. □

Lemma 5.3.3. *If $x^k \to x^\omega$ and $x^{k+1} \succ x^k$ for all $k = 0, 1, \ldots,$ then $x^\omega \succ x^0$.*

Proof. We only have to prove two implications: (i) if $x^k \in X^-$ for all k (hence $x^{k+1} < x^k$), then $x^\omega \in X^-$; (ii) if $x^k \in X^+$ for all k (hence $x^{k+1} > x^k$), then $x^\omega \in X^- \cup X^+$. Let us consider the first one. Pick $i \in N$ and consider two alternatives: either $x^\omega_{-i} < x^k_{-i}$ for all k, or $x^\omega_{-i} = x^k_{-i}$ for all k (large enough). Assuming the first alternative, we pick $x'_i \in R_i(x^\omega_{-i})$; by the monotonicity of R_i, we have $x'_i \leq x^k_i$ for every k, hence $x'_i \leq x^\omega_i$. Under the second alternative, we have either $x^k_i \in R_i(x^\omega_{-i})$ for all k, hence $x^\omega_i \in R_i(x^\omega_{-i})$, or $x^k_i \notin R_i(x^\omega_{-i})$ for arbitrarily large k, hence, for any $x'_i \in R_i(x^\omega_{-i})[= R_i(x^k_{-i})]$, $x'_i < x^k_i$, hence $x'_i \leq x^\omega_i$. Since $i \in N$ and $x'_i \in R_i(x^\omega_{-i})$ were arbitrary, $x^\omega \in X^-$. To prove the statement (ii), we apply a dual reasoning to $i = 1$. □

The end of the proof is exactly the same as in Theorem 5.2. □

6 References

Agliardi, E., 2000. A generalization of supermodularity. Economics Letters 68, 251-254.

Danilov, V.I., and A.I. Sotskov, 2002. Social Choice Mechanisms. Springer–Verlag, Berlin–Heidelberg–New York.

Gurvich, V.A., 1988. Equilibrium in pure strategies. Doklady Akademii Nauk SSSR 303, 789-793 (in Russian; English translation in Soviet Mathematics. Doklady 38, 597-602).

Holzman, R., and N. Law-Yone, 1997. Strong equilibrium in congestion games. Games and Economic Behavior 21, 85-101.

Konishi, H., M. Le Breton, and S. Weber, 1997. Equilibria in a model with partial rivalry. Journal of Economic Theory 72, 225-237.

Kreps, D.M., 1990. A Course in Microeconomic Theory. Princeton University Press, Princeton.

Kukushkin, N.S., 1994, A condition for the existence of a Nash equilibrium in games with public and private objectives. Games and Economic Behavior 7, 177-192.

Kukushkin, N.S., 1999a. Potential games: A purely ordinal approach. Economics Letters 64, 279-283.

Kukushkin, N.S., 1999b. Some classes of potential and semi-potential games. Universität Bielefeld, Institute of Mathematical Economics, Working Paper No.307.

Kukushkin, N.S., 2000. Potentials for binary relations and systems of reactions. Russian Academy of Sciences, Computing Center, Moscow.

Kukushkin, N.S., 2001. Best response dynamics in finite games with additive aggregation. (Manuscript)

Kukushkin, N.S., 2002. Perfect information and potential games. Games and Economic Behavior 38, 306-317.

Milchtaich, I., 1996. Congestion games with player-specific payoff functions. Games and Economic Behavior 13, 111-124.

Milgrom, P., and J. Roberts, 1990. Rationalizability, learning, and equilibrium in games with strategic complementarities. Econometrica 58, 1255-1277.

Milgrom, P., and C. Shannon 1994. Monotone comparative statics. Econometrica 62, 157-180.

Monderer, D., and L.S. Shapley, 1996. Potential games. Games and Economic Behavior 14, 124-143.

Natanson, I.P., 1974. The Theory of Functions of a Real Variable. Nauka, Moscow. (in Russian; there should be an English translation)

Rosenthal, R.W., 1973. A class of games possessing pure-strategy Nash equilibria. International Journal of Game Theory 2, 65-67.

Topkis, D.M., 1979. Equilibrium points in nonzero-sum n-person submodular games. SIAM Journal of Control and Optimization 17, 773-787.

Vives, X., 1990. Nash equilibrium with strategic complementarities. Journal of Mathematical Economics 19, 305-321.

Young, H.P., 1993. The evolution of conventions. Econometrica 61, 57-84.