Strategic complementarity and substitutability without transitive indifference

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Abstract

We study what useful implications strategic complementarity or substitutability may have when the indifference relation(s) need not be transitive. Two results are obtained about the existence of a monotone selection from the best response correspondence when both strategies and parameters form chains. The first of them implies the existence of an epsilon equilibrium in a game with strategic substitutes and appropriate aggregation; the second is also essential for games with strategic substitutes, but for those where the preferences are more complicated. Two more results are obtained about the existence of a Nash equilibrium in games with strategic complementarities where strategy sets are chains, but monotone selections from the best response correspondences need not exist. Again, the first implies the existence of an epsilon-equilibrium; the second is applicable to more complicated preferences.

Key words strict acyclicity; interval order; single crossing; monotone selection; Nash equilibrium

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1 Introduction

The standard way to describe preferences of the players in game theory – with utility functions – looks severely restrictive when compared with what is available in choice theory (Fishburn, 1973; Sen, 1984; Aizerman and Aleskerov, 1995). The desirability of bridging the gap has been recognized since, at the latest, Aumann (1962, 1964). However, familiar approaches quite often do not work in a broader context, or have to be modified substantially.

This paper strives to find out what equilibrium existence results could be derived from strategic complementarity or substitutability when the preferences are defined by binary relations such that incomparability need not be transitive. The study of games with strategic complementarities was started in a cardinal framework, "supermodular games" (Topkis, 1979; Veinott, 1989; Vives, 1990; Milgrom and Roberts, 1990). Milgrom and Shannon (1994) developed a purely ordinal version, but their approach only works when the preferences of each player are described with an ordering (i.e., indifference *is* transitive). The same is valid with respect to later papers (Athey, 2001; Quah, 2007;

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Quah and Strulovici, 2009; Reny, 2011). If a broader class of preference relations is allowed, the whole edifice collapses.

Suppose the utility function is bounded above, but need not attain a maximum; then ε-optimization suggests itself strongly, which means allowing intransitive indifference. The (ε-)best response correspondence need not be ascending even if the utility function is supermodular and the increasing differences condition holds; it is only *weakly* ascending under these strong assumptions. The set of (ε-)best responses to a particular profile of strategies of other players need not be a lattice and need not be complete, hence the existence of a monotone selection cannot be derived even from Veinott (1989, Theorem 3.2), so there is nothing to apply Tarski's fixed point theorem to. To the best of my knowledge, the previous literature contains no existence result for ε-Nash equilibria in games with strategic complementarity (to say nothing of strategic substitutability) where the existence of the best responses is not guaranteed. "Multi-criterial optimization" may be mentioned as another source of similar (in some aspects, even worse) problems.

The main point of this paper is that something can be obtained even in such situations. One "only" has to apply roundabout techniques and reconcile oneself to less impressive results. We do not specifically address ε-optimization or Pareto dominance. Instead, we start with preferences described by an arbitrary binary relation and then consequently impose restrictions under which tangible results are possible.

Theorems 1 and 2 establish the existence of a monotone selection from the best response correspondence when both available choices and parameters form chains. Proposition 5 about the existence of an ε-Nash equilibrium in games with strategic complements or substitutes and an appropriate aggregation easily follows. It should be noted that every equilibrium existence result in the literature on games with *decreasing* best responses hinges on the presence of scalar aggregation in the utilities and the availability of monotone selections (Novshek, 1985; Kukushkin, 1994, 2003, 2004, 2005; Dubey et al., 2006; Jensen, 2010), hence the restriction to chains is natural.

No aggregation in the utilities is needed for the existence of a Nash equilibrium in the standard theory of games with strategic complementarities. Theorems 3 and 4 show the fact to hold in a more general setting; only transitivity of strict preference is required. In particular, both theorems may work in the absence of monotone selections. Unfortunately, we still have to assume that every strategy set is a chain although there is no counterexample with multi-dimensional strategies.

Section 2 introduces conditions on preferences that ensure the existence of optimal choices and some weak analogs of the "revealed preference" property. Section 3 contains two theorems on the existence of monotone selections; Section 4, two theorems on the existence of a Nash equilibrium in the absence of monotone selections. More complicated proofs are deferred to Section 5; concluding remarks are collected in Section 6.

2 Preferences and choice

A *strict order* is an irreflexive and transitive binary relation. A set with a given strict order is called a *partially ordered set* (*poset*); when the order is total, i.e., every two different points are comparable, the poset is called a *chain.*

Let the preferences of an agent over alternatives from a set X be described by a binary relation

$\succ$. For every $Y \subseteq X$, we denote

$$M(Y, \succ) := \{x \in Y \mid \nexists\, y \in Y\, [y \succ x]\}, \tag{1}$$

the set of "optimal," or rather acceptable, choices from Y. Most often in game theory in general, and in complementarity studies in particular, the preferences are represented with a *utility function* $u\colon X \to \mathbb{R}$, i.e.,

$$y \succ x \iff u(y) > u(x) \tag{2}$$

for all $x, y \in X$. Actually, utility functions as such are not necessary to derive the existence of a Nash equilibrium from Tarski's fixed point theorem and to obtain the usual monotone comparative statics results. What *is* needed for the standard techniques to work is this, "revealed preference," property:

$$\forall x, y \in X\, [x \notin M(X, \succ) \ni y \Rightarrow y \succ x], \tag{3}$$

supplemented with $M(X, \succ) \neq \emptyset$, naturally.

Condition (3) is ensured whenever $\succ$ is an *ordering*, i.e., a *negatively transitive* strict order: $z \not\succ y \not\succ x \Rightarrow z \not\succ x$. Equivalently, $\succ$ is an ordering if and only if there are a chain L and a mapping $u\colon X \to L$ such that (2) holds for all $x, y \in X$.

Here we rely on properties weaker than (3). A binary relation $\succ$ has the *NM-property* on a subset $Y \subseteq X$ if

$$\forall x \in Y \setminus M(Y, \succ)\, \exists y \in M(Y, \succ)\, [y \succ x]. \tag{4}$$

$\succ$ has the *strong NM-property* on a subset $Y \subseteq X$ if

$$\forall \{x^0, \ldots, x^m\} \subseteq Y \setminus M(Y, \succ)\, \exists y \in M(Y, \succ)\, \forall k \in \{0, \ldots, m\}\, [y \succ x^k]. \tag{5}$$

It turns out that, roughly speaking, the strong NM-property (plus single crossing) is conducive to the existence of a monotone selection from the best response correspondence, while the NM-property is conducive to the existence of a Nash equilibrium under strategic complementarity.

A strict order $\succ$ is called an *interval order* if it satisfies the condition

$$\forall x, y, a, b \in X\, \big[\, [y \succ x \;\&\; a \succ b] \Rightarrow [y \succ b \;\text{ or }\; a \succ x]\, \big]. \tag{6}$$

Equivalently, $\succ$ is an interval order if and only if there are a chain L and two mappings $u^+, u^-\colon X \to L$ such that, for all $x, y \in X$,

$$u^+(x) \geq u^-(x); \quad y \succ x \iff u^-(y) > u^+(x).$$

A relation $\succ$ is *strictly acyclic* if there exists no *infinite improvement path*, i.e., no sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ such that $x^{k+1} \succ x^k$ for all k. As an example, let $u\colon X \to \mathbb{R}$ be bounded above and $\varepsilon > 0$; let the preference relation be

$$y \succ x \rightleftharpoons u(y) > u(x) + \varepsilon. \tag{7}$$

It is easily seen that $\succ$ is a strictly acyclic interval order (even a *semiorder*). $M(Y, \succ)$ consists of all ε-maxima of u on Y.

Routine proofs of the two following statements are given for completeness.

Proposition 1. *Let $\succ$ be a binary relation on a set X. Then $\succ$ has the NM-property on every nonempty subset $Y \subseteq X$ if and only if it is strictly acyclic and transitive.*

Proof. To prove the sufficiency, we assume $x^* \in Y \setminus M(Y, \succ)$. There is $y^1 \in Y$ such that $y^1 \succ x^*$. If $y^1 \in M(Y, \succ)$, we are home; otherwise, there is $y^2 \in Y$ such that $y^2 \succ y^1 \succ x^*$. Iterating this argument, we obtain an improvement path $x^*, y^1, y^2, \ldots$ Since $\succ$ is strictly acyclic, the path ends, at some stage, with $y^m \in M(Y, \succ)$. Since $\succ$ is transitive, we have $y^m \succ x^*$.

Conversely, if $\succ$ admits an infinite improvement path $\langle x^k \rangle_{k \in \mathbb{N}}$, then $M(\{x^k\}_{k \in \mathbb{N}}, \succ) = \emptyset$. If $z \succ y \succ x$, but $z \not\succ x$, then $M(\{x, y, z\}, \succ) = \{z\}$, hence (4) does not hold for $Y = \{x, y, z\}$ and $x^* = x$. $\square$

Remark. Strict acyclicity alone is necessary and sufficient for the property that $M(Y, \succ) \neq \emptyset$ whenever $X \supseteq Y \neq \emptyset$.

Proposition 2. *Let $\succ$ be a binary relation on a set X. Then $\succ$ has the strong NM-property on every nonempty subset $Y \subseteq X$ if and only if it is a strictly acyclic interval order.*

Proof. To prove the sufficiency, we assume $\{x^0, \ldots, x^m\} \subseteq Y \setminus M(Y, \succ)$. When $m = 0$, we just invoke Proposition 1. Then we argue by induction. For $m > 0$, the induction hypothesis implies the existence of $y' \in M(Y, \succ)$ such that $y' \succ x^k$ for each $k = 0, \ldots, m-1$; we also have $y'' \in M(Y, \succ)$ such that $y'' \succ x^m$. For each $k = 0, \ldots, m-1$, we apply (6) to x^k, y', y'', x^m, obtaining that either $y' \succ x^m$ or $y'' \succ x^k$ for each $k = 0, \ldots, m-1$. In either case, we are home.

Conversely, if (6) does not hold, we have $M(\{x, y, a, b\}, \succ) = \{y, a\}$, hence (5) does not hold for $Y = \{x, y, a, b\}$ and $\{x, b\} \subseteq Y \setminus M(Y, \succ)$. $\square$

Various versions of compactness-continuity may be substituted for strict acyclicity. We consider just one of them, expressed in terms of order rather than topology. A chain X is *complete* if the least upper bound $\sup Y$ and the greatest lower bound $\inf Y$ exist in X for every subset $Y \subseteq X$. A subset Y of a complete chain X is *subcomplete* if $\sup Z \in Y$ and $\inf Z \in Y$ for every nonempty subset $Z \subseteq Y$. We denote $\mathfrak{C}_X$ the set of nonempty subcomplete subsets of X.

Assuming X a complete chain and given $Y \subseteq X$, we denote $Y^{\leftarrow} := Y \setminus \{\inf Y\}$ and $Y^{\rightarrow} := Y \setminus \{\sup Y\}$. Then we define a very weak version of upper semicontinuity:

$$\forall Y \in \mathfrak{C}_X \left[\left(\sup Y^{\rightarrow} = \sup Y \;\&\; \forall x, y \in Y^{\rightarrow} \, [y > x \Rightarrow y \succ x] \right) \Rightarrow \forall x \in Y^{\rightarrow} \, [\sup Y \succ x] \right]; \tag{8a}$$

$$\forall Y \in \mathfrak{C}_X \left[\left(\inf Y^{\leftarrow} = \inf Y \;\&\; \forall x, y \in Y^{\leftarrow} \, [x > y \Rightarrow y \succ x] \right) \Rightarrow \forall x \in Y^{\leftarrow} \, [\inf Y \succ x] \right]. \tag{8b}$$

Remark. Every strictly acyclic relation satisfies (8) vacuously: both pairs of "left hand side" conditions are incompatible in this case.

Proposition 3. *Let $\succ$ be a binary relation on a complete chain X. Then $\succ$ has the NM-property on every $Y \in \mathfrak{C}_X$ if and only if it is a strict order satisfying both conditions* (8).

The proof is deferred to Section 5.1.

Remark. There is an obvious, if vague, analogy to Theorem 1 of Kukushkin (2008); conditions (8) are similar to "ω-transitivity" there.

Proposition 4. *Let $\succ$ be a binary relation on a complete chain X. Then $\succ$ has the strong NM-property on every $Y \in \mathfrak{C}_X$ if and only if it is an interval order satisfying both conditions* (8)*.*

The sufficiency is proven with a reference to Proposition 3 combined with the same argument as in the proof of Proposition 2. The necessity for $\succ$ to be an interval order is proven in the same way as in Proposition 2; the necessity of conditions (8) immediately follows from Proposition 3.

3 Monotone selections

We consider a *parametric family* $\langle \stackrel{s}{\succ} \rangle_{s\in S}$ of binary relations on X; the parameter s reflects outside influences (e.g., the choices of other agents). To simplify notations, we define the *best response correspondence*:

$$\mathcal{R}(s) := M(X, \stackrel{s}{\succ}). \tag{9}$$

Henceforth, we always assume X and S to be posets (most often, just chains). A mapping $r\colon S \to X$ is *increasing* if $r(s'') \geq r(s')$ whenever $s', s'' \in S$ and $s'' \geq s'$. A *monotone selection* from $\mathcal{R}$ is an increasing mapping $r\colon S \to X$ such that $r(s) \in \mathcal{R}(s)$ for every $s \in S$.

A parametric family $\langle \stackrel{s}{\succ} \rangle_{s\in S}$ has the *single crossing* property if these conditions hold:

$$\forall x, y \in X\, \forall s, s' \in S\, \big[[s' > s \;\&\; y \stackrel{s}{\succ} x \;\&\; y > x] \Rightarrow y \stackrel{s'}{\succ} x\big]; \tag{10a}$$

$$\forall x, y \in X\, \forall s, s' \in S\, \big[[s' > s \;\&\; y \stackrel{s'}{\succ} x \;\&\; y < x] \Rightarrow y \stackrel{s}{\succ} x\big]. \tag{10b}$$

This definition is equivalent to Milgrom and Shannon's (1994) if every $\stackrel{s}{\succ}$ is an ordering represented by a numeric function.

For a family of preference relations defined by ε-optimization (7) with a parameter s in the function, both conditions (10) hold if $u(x, s)$ satisfies Topkis's (1979) *increasing differences* condition:

$$\forall x, y \in X\, \forall s, s' \in S\, \big[[s' \geq s \;\&\; y \geq x] \Rightarrow u(y, s') - u(x, s') \geq u(y, s) - u(x, s)\big]. \tag{11}$$

When X and S are chains, the condition is equivalent to the *supermodularity* of u (as a function on the lattice $X \times S$).

Theorem 1. *Let X and S be chains. Let a parametric family $\langle \stackrel{s}{\succ} \rangle_{s\in S}$ of strictly acyclic relations on X satisfy single crossing conditions* (10)*. Let every $\stackrel{s}{\succ}$ $(s \in S)$ have the strong NM-property on X. Then there exists a monotone selection from $\mathcal{R}$ on S.*

The proof is deferred to Section 5.2.

Corollary. *Let X and S be chains. Let a parametric family $\langle \stackrel{s}{\succ} \rangle_{s\in S}$ of strictly acyclic interval orders on X satisfy single crossing conditions* (10)*. Then there exists a monotone selection from $\mathcal{R}$ on S.*

An application of Theorem 2 from Kukushkin (2005) to monotone selections from ε-best response correspondences existing by our Theorem 1 immediately gives us, e.g., this result.

Proposition 5. *Let Γ be a strategic game with a compact strategy set $X_i \subset \mathbb{R}$ for each $i \in N$. Let each utility function be $u_i(x_N) = U_i(x_i, \sum_{j\neq i} a_{ij}x_j)$, where $a_{ij} = a_{ji} \in \mathbb{R}$ whenever $j \neq i$. Let each $U_i(\cdot, s)$ be bounded above and let the increasing differences condition* (11) *be satisfied by each $U_i(x_i, s)$. Then Γ possesses an ε-Nash equilibrium for every $\varepsilon > 0$.*

Remark. When $a_{ij} \geq 0$ for all $j \neq i$, we have a game with strategic complementarity; when $a_{ij} \leq 0$ for all $j \neq i$, a game with strategic substitutability. A more general situation with coefficients of both signs is also possible. The linear aggregate of the choices of other players can be replaced with a polylinear combination, or the (minus) minimum/maximum of them (Kukushkin, 2003, Theorems 7 and 8).

If the strong NM-property assumption is just dropped, Theorem 1 becomes wrong even for finite sets X and S, see Example 3 below. Under the assumption that every $\succ^s$ has the NM-property on X, Theorem 1 fails in full generality (Example 1 below), but is valid for finite X or S.

Proposition 6. *Let X and S be chains. Let a parametric family $\langle \succ^s \rangle_{s\in S}$ of binary relations on X satisfy* (10a)*. Let every $\succ^s$ ($s \in S$) have the NM-property on X. If either X or S is finite, then there exists a monotone selection from $\mathcal{R}$ on S.*

Proof. Let S be finite. We start with $s^+ := \max S$ and pick $r(s^+) \in \mathcal{R}(s^+)$ arbitrarily. Then we move along S downwards, denoting $s + 1$ the point in S immediately above s. If $r(s + 1) \in \mathcal{R}(s)$, we set $r(s) := r(s + 1)$; otherwise, we invoke (4) and pick $r(s) \in \mathcal{R}(s)$ such that $r(s) \succ^s r(s + 1)$. The inequality $r(s) > r(s+1)$ would, by (10a), imply $r(s) \succ^{s+1} r(s+1)$, contradicting the induction hypothesis; therefore, $r(s) \leq r(s + 1)$ for all $s \in S$.

Now let X be finite. For every $s \in S$, we set $r(s) := \min \mathcal{R}(s)$. The inequalities $s' > s$ and $r(s) > r(s')$ would imply $r(s') \notin \mathcal{R}(s)$, hence $\mathcal{R}(s) \ni y \succ^s r(s')$ by (4). By the definition of $r(s)$, we have $y \geq r(s) > r(s')$, hence $y \succ^{s'} r(s')$ by (10a), contradicting the definition of $r(s')$. ☐

Remark. By duality, (10a) can be replaced with (10b).

Example 1. Let $X := [-2, 2]$, $S := [-1, 1]$ (both with natural orders), and relations $\succ^s$ be defined by

$$y \succ^s x \rightleftharpoons [u_1(y, s) > u_1(x, s) \ \&\ u_2(y, s) > u_2(x, s)], \tag{12}$$

where $u\colon X \times S \to \mathbb{R}^2$ is this: $u(1, s) := \langle 5, 2\rangle$ and $u(-1, s) := \langle 2, 5\rangle$ for all $s \in S$; $u(2, s) := u(-2, s) := u(x, s) := \langle 0, 0\rangle$ for all $x \in]-1, 1[$ and $s \in S$; whenever $x \in]1, 2[$ and $s \geq 0$,

$$u_1(x, s) := \begin{cases} x + s - 1, & \text{if } x + s \leq 2, \\ x + s + 4, & \text{if } x + s > 2, \end{cases}$$

while $u_2(x, s) := 6 - x - s$; whenever $x \in]-2, -1[$ and $s \geq 0$, $u(x, s) := \langle x + 6, -1 - x\rangle$; finally, $u_i(x, s)$ for all $s < 0$, $i = 1, 2$, and $x \in]-2, -1[\cup]1, 2[$ is such that the equality

$$u_i(x, s) = u_{3-i}(-x, -s) \tag{13}$$

holds for all $s \in S$, $i = 1, 2$, and $x \in X$.

The very form of (12) ensures that every $\succcurlyeq^s$ is irreflexive and transitive. Whenever $x \in \{-2\} \cup]-1,1[\cup \{2\}$ and $y \in]-2,-1] \cup [1,2[$, $y \succcurlyeq^s x$ for every $s \in S$. Whenever $x, y \in]-2,-1[$ or $x, y \in]1,2[$, $y \succcurlyeq^s x$ does not hold for any $s \in S$. Let $s \geq 0$; if $-2 < x < -1$, then $u_1(x) < 5$ and $u_2(x) \leq 1$, hence $1 \succcurlyeq^s x$; if $1 < x \leq 2 - s < 2$, then $u_1(x) \leq 1$ and $u_2(x) < 5$, hence $-1 \succcurlyeq^s x$; if $2 - s < y < 2$, then $u_1(y) > 6$ and $u_2(y) > 3$, hence $y \succcurlyeq^s 1$. "Dually," by (13), $y \succcurlyeq^s -1 \succcurlyeq^s x$ whenever $s < 0$, $-2 < y < -2 - s$, and $1 < x < 2$; $1 \succcurlyeq^s x$ whenever $s < 0$ and $-2 - s \leq x < -1$. Thus, we see that every relation $\succcurlyeq^s$ is strictly acyclic: no more than three consecutive improvements can be made from any starting point (e.g., $2 - s/2 \succcurlyeq^s 1 \succcurlyeq^s -1.5 \succcurlyeq^s -2$ when $s > 0$). Single crossing conditions (10) are also easy to check.

Suppose there is a monotone selection r from R. If $r(s) > -1$ for some $s > 0$, then $2 > r(s) > 2-s$; defining $s' := 2-r(s) > 0$, we have $s' < s$, hence $r(s') \leq r(s)$, hence $r(s') < 2-s'$, hence $r(s') \in R(s')$ is only possible if $r(s') = -1$. Therefore, $r(s) = -1$ for some $s > 0$; dually, $r(s) = 1$ for some $s < 0$. We have a contradiction, i.e., there is no monotone selection: Theorem 1 cannot be extended to strictly acyclic and transitive preference relations.

Remark. It is easy to see that the orderings defined by functions u_1 and u_2 in Example 1 are *not* strictly acyclic. Moreover, if, given arbitrary chains X and S, we considered a preference relation

$$y \succcurlyeq^s x \rightleftharpoons [u_1(y,s) > u_1(x,s) + \varepsilon \;\&\; u_2(y,s) > u_2(x,s) + \varepsilon]$$

with $\varepsilon > 0$ and both u_i bounded above in x for every $s \in S$, then $\mathcal{R}(s)$ would contain the set of ε-optima of $u_1(x,s) + u_2(x,s)$, which admits a monotone selection by Theorem 1. Thus, the example cannot claim to present typical problems with parametric multi-criterial ε-optimization.

Theorem 2. *Let X and S be chains, and X be complete. Let a parametric family $\langle \succcurlyeq^s \rangle_{s \in S}$ of transitive binary relations on X satisfy single crossing conditions* (10)*. Let every $\succcurlyeq^s$ satisfy both conditions* (8) *and have the strong NM-property on X. Then there exists a monotone selection from $\mathcal{R}$ on S.*

The proof is deferred to Section 5.3.

4 Nash equilibrium without monotone selections

Let us consider a modification of the standard notion of a strategic game. There is a finite set N of players and a poset X_i of strategies for each $i \in N$. We denote $X_N := \prod_{i \in N} X_i$ and $X_{-i} := \prod_{j \neq i} X_j$; both are posets with the Cartesian product of the orders on components. Each player i's *preferences* are described by a parametric family of binary relations $\succ_i^{x_{-i}}$ ($x_{-i} \in X_{-i}$) on X_i; the player's *best response correspondence* $\mathcal{R}_i$ is defined by (9) with $S := X_{-i}$. A *Nash equilibrium* is $x_N \in X_N$ such that $x_i \in \mathcal{R}_i(x_{-i})$ for each $i \in N$.

When each player's preferences are defined with a utility function $u_i(x_N)$, our definition of a Nash equilibrium is equivalent to the standard one. It may be worthwhile to note that the question of, say, (in)efficiency of equilibria makes no sense in our framework. Assuming that the preferences are defined in the style of (7), our definition transforms into that of an ε-Nash equilibrium.

Theorem 3. *Let Γ be a strategic game where each X_i is a chain such that both* $\min X_i$ *and* $\max X_i$ *exist. Let the parametric family of preference relations of each player satisfy both conditions* (10)*.*

Let every relation $\succ_i^{x_{-i}}$ be strictly acyclic and have the NM-property on X_i. Then Γ possesses a Nash equilibrium.

The proof is deferred to Section 5.4.

Example 2. Let $N := \{1,2\}$, $X_1 := X_2 :=]0,1]$ (with the natural order); let preferences of the players be defined by (7) with utility functions $u_1(x_1,x_2) := -|2x_1 - x_2|/x_2$ and $u_2(x_1,x_2) := -|x_1 - x_2|/x_1$, and $\varepsilon \in]0,(3-\sqrt{5})/2[$. All assumptions of Theorem 3 (or Theorem 4 for that matter) are satisfied except for the existence of $\min X_i$; single crossing conditions (10) hold because both utility functions are supermodular. There is no (ε-)Nash equilibrium: $x_2 > (1-\varepsilon)x_1$ whenever $x_2 \in \mathcal{R}_2(x_1)$, while $x_1 > (2-\varepsilon)x_2$ whenever $x_1 \in \mathcal{R}_1(x_2)$; therefore, there should hold $x_2 > (1-\varepsilon)(2-\varepsilon)x_2 > x_2$ at any equilibrium.

Theorem 4. *Let Γ be a strategic game where each X_i is a complete chain. Let the parametric family of preference relations of each player satisfy both conditions* (10)*. Let every relation $\succ_i^{x_{-i}}$ be a strict order satisfying both conditions* (8)*. Then Γ possesses a Nash equilibrium.*

The proof is deferred to Section 5.5.

Example 1 shows that the assumptions of Theorems 3 or 4 do not ensure the existence of monotone selections from the best response correspondences. Both theorems become just wrong without NM-property, even for finite sets X_i.

Example 3. Let $N := \{1,2\}$, $X_1 := \{0,1,2,3,4\}$ and $X_2 := \{5,6\}$ (both with natural orders); let preference relations $\succ_i^{x_{-i}}$ be defined by: $2 \succ_1^5 4 \succ_1^5 0 \succ_1^5 1 \succ_1^5 3$; $1 \succ_1^6 3 \succ_1^6 2 \succ_1^6 4 \succ_1^6 0$; $5 \succ_2^{x_1} 6$ whenever $x_1 \leq 1$; $6 \succ_2^{x_1} 5$ whenever $x_1 \geq 2$. The preferences of player 1 are intransitive, but single crossing conditions (10) are easy to check: (10a) is nontrivial only for $4 \succ_1^5 0$; (10b), only for $1 \succ_1^6 3$ and $2 \succ_1^6 4$. Player 2's preferences are described by a family of total orders; (10) are obvious. There is no Nash equilibrium: $\mathcal{R}_1(5) = \{2\}$ and $\mathcal{R}_1(6) = \{1\}$, whereas $\mathcal{R}_2(2) = \{6\}$ and $\mathcal{R}_2(1) = \{5\}$. It may be noted that $\mathcal{R}_1$ admits no monotone selection.

5 Proofs

The proofs of theorems where conditions (8) are involved are based on *transfinite recursion*; it seems worthwhile to start with an outline of the method.

A poset is *well ordered* if every subset contains its minimum (hence the poset itself is a chain). When dealing with a well ordered poset Δ, we always denote $0 := \min \Delta$ and $[0,\alpha] := \{\beta \in \Delta \mid \beta \leq \alpha\}$. The *successor* of $\alpha \in \Delta$, denoted $\alpha + 1$, is uniquely defined as $\min\{\beta \in \Delta \mid \beta > \alpha\}$ (unless $\alpha = \max \Delta$, but it does not matter).

The principle of transfinite recursion allows us to consider a mapping $\lambda \colon \Delta \to \mathcal{X}$ well defined if we have defined $\lambda(0) \in \mathcal{X}$ and described how $\lambda(\alpha) \in \mathcal{X}$ should be constructed given $\lambda(\beta) \in \mathcal{X}$ for all $\beta < \alpha$. Quite often, the definition of $\lambda(\alpha+1)$ is based on $\lambda(\alpha)$ alone, so *all* $\beta < \alpha$ are only involved when α is a *limit*, i.e., not the successor to any $\beta \in \Delta$.

The application of this technique to the proof of a theorem starts with an appropriate poset $\mathcal{X}$ such that the statement of the theorem is equivalent to the existence of a point in $\mathcal{X}$ satisfying a certain condition. In the case of Proposition 3, $\mathcal{X}$ is the set of feasible choices; in Theorem 4, it is the set of strategy profiles; in Theorem 2, it is, roughly speaking, the set of "partial monotone selections." The Zermelo Theorem – Every set can be well ordered – implies the existence of an infinite well ordered set Δ with a cardinality greater than that of $\mathcal{X}$. Then we construct an increasing mapping $\lambda\colon \Delta \to \mathcal{X}$ with the property that an equality $\lambda(\alpha') = \lambda(\alpha)$ with $\alpha' > \alpha$ is only possible when $\lambda(\alpha)$ is a point we need. Since the cardinality of Δ is greater than that of $\mathcal{X}$, the equality must occur at some stage. The definition of $\lambda(\alpha+1)$ given $\lambda(\alpha)$ is specific in each case, while $\lambda(\alpha)$ for a limit α is always $\lambda(\alpha) := \sup\{\lambda(\beta)\}_{\beta<\alpha}$. The latter fact ensures that $\lambda([0,\alpha]) \in \mathfrak{C}_X$ and $\lambda(\alpha) = \sup\lambda([0,\alpha]) = \sup\big(\lambda([0,\alpha]) \setminus \{\lambda(\alpha)\}\big)$, hence condition (8a) can be applied.

The dual version of the scheme, where $\lambda\colon \Delta \to \mathcal{X}$ is decreasing and $\lambda(\alpha) := \inf\{\lambda(\beta)\}_{\beta<\alpha}$ for every limit α, so (8b) can be applied, is valid too, but remains behind the scene.

5.1 Proof of Proposition 3

The necessity part is rather straightforward. First, $\succ$ must be a strict order for the same reason as in Proposition 1. Let the "left hand side" condition in (8a) be satisfied for $Y \subseteq X$. Since $\sup Y^{\to} = \sup Y$, for every $y \in Y^{\to}$ there is $y' \in Y^{\to}$ such that $y' > y$, hence $y' \succ y$. Therefore, $M(Y,\succ) \cap Y^{\to} = \emptyset$, hence (4) implies that $M(Y,\succ) = \{\sup Y\}$ and $\sup Y \succ y$ for every $y \in Y^{\to}$, i.e., (8a) holds. The necessity of (8b) is proven dually.

We start the sufficiency proof with the definition of two auxiliary strict orders:

$$y \underset{>}{\succ} x \rightleftharpoons [y \succ x \;\&\; y > x];$$
$$y \underset{<}{\succ} x \rightleftharpoons [y \succ x \;\&\; y < x].$$

Lemma 5.1.1. *Let X be a complete chain and $\succ$ be a strict order on X satisfying* (8a). *Then $\underset{>}{\succ}$ has the NM-property on X.*

Proof. Let $x^0 \in X \setminus M(X, \underset{>}{\succ})$ and let Δ be a well ordered set of a cardinality greater than that of X. We define a mapping $\lambda\colon \Delta \to X$ by transfinite recursion. First, $\lambda(0) := x^0$. Whenever $\lambda(\alpha) \notin M(X, \underset{>}{\succ})$, we pick $\lambda(\alpha+1) \in X$ such that $\lambda(\alpha+1) \underset{>}{\succ} \lambda(\alpha)$. If $\lambda(\alpha) \in M(X, \underset{>}{\succ})$, we set $\lambda(\alpha+1) := \lambda(\alpha)$. Whenever α is a limit, we set $\lambda(\alpha) := \sup\{\lambda(\beta)\}_{\beta<\alpha}$. Since $\lambda(\alpha+1) > \lambda(\alpha)$ whenever $\lambda(\alpha) \notin M(X, \underset{>}{\succ})$, there must be $\bar\alpha \in \Delta$ such that $\lambda(\bar\alpha) \in M(X, \underset{>}{\succ})$, hence $\lambda(\alpha) = \lambda(\bar\alpha) \in M(X, \underset{>}{\succ})$ for all $\alpha \geq \bar\alpha$. Without restricting generality, we assume that $\bar\alpha$ is minimal with the property.

Straightforward transfinite induction in $\alpha \in \Delta$ shows that

$$\forall \beta \in \Delta\left[[\alpha > \beta \;\&\; \bar\alpha > \beta] \Rightarrow \lambda(\alpha) \underset{>}{\succ} \lambda(\beta)\right] \tag{14}$$

for all $\alpha \in \Delta$: $\lambda(1) \underset{>}{\succ} \lambda(0)$ by the definition of λ; if (14) holds for α, then $\lambda(\alpha+1) \underset{>}{\succ} \lambda(\alpha)$ by the definition of λ and $\lambda(\alpha+1) \underset{>}{\succ} \lambda(\beta)$ for all $\beta < \alpha$ by transitivity; if α is a limit and (14) holds for all $\alpha' < \alpha$, then (14) for α follows from (8a) because the monotonicity condition holds by the induction hypothesis.

Thus, $M(X, \underset{>}{\succ}) \ni \lambda(\bar\alpha) \underset{>}{\succ} \lambda(0) = x^0$ and we are home. $\square$

Lemma 5.1.2. *Let X be a complete chain and $\succ$ be a strict order on X satisfying* (8b)*. Then $\underset{<}{\succ}$ has the NM-property on X.*

The proof is dual to that of Lemma 5.1.1.

The final argument is similar to the proof of Lemma 5.1.1. Let $Y \in \mathfrak{C}_X$ and $x^0 \in Y \setminus M(Y, \succ)$. Without restricting generality, $x^0 \notin M(Y, \underset{<}{\succ})$; by Lemma 5.1.2, there is $x^* \in M(Y, \underset{<}{\succ})$ such that $x^* \succ x^0$. By transfinite recursion, we define a mapping $\lambda \colon \Delta \to Y$, where Δ is a well ordered set of a cardinality greater than that of Y, such that $\lambda(\alpha) \in M(Y, \underset{<}{\succ})$ for all $\alpha \in \Delta$ and $\lambda(\alpha) \underset{>}{\succ} \lambda(\beta)$ whenever $\alpha > \beta$ unless $\lambda(\beta) \in M(Y, \succ)$, in which case $\lambda(\alpha) = \lambda(\beta)$. First, we set $\lambda(0) := x^*$.

Whenever $\lambda(\alpha) \in M(Y, \underset{>}{\succ})$, we have $\lambda(\alpha) \in M(Y, \succ)$ as well; we set $\lambda(\alpha + 1) := \lambda(\alpha)$ in this case, hence $\lambda(\alpha') = \lambda(\alpha)$ for all $\alpha' > \alpha$. If $\lambda(\alpha) \notin M(Y, \underset{>}{\succ})$, we pick $y^* \in Y$ such that $y^* \underset{>}{\succ} \lambda(\alpha)$. If $y^* \in M(Y, \underset{<}{\succ})$, we set $\lambda(\alpha+1) := y^*$; otherwise, we apply Lemma 5.1.2, obtaining $\lambda(\alpha+1) \in M(Y, \underset{<}{\succ})$ such that $\lambda(\alpha + 1) \underset{<}{\succ} y^*$. Since $\lambda(\alpha + 1) \succ \lambda(\alpha)$ and $\lambda(\alpha) \in M(Y, \underset{<}{\succ})$, we have $\lambda(\alpha + 1) > \lambda(\alpha)$, hence $\lambda(\alpha+1) \underset{>}{\succ} \lambda(\alpha)$ in this case, hence $\lambda(\alpha+1) \underset{>}{\succ} \lambda(\beta)$ for all $\beta < \alpha$ by the induction hypothesis.

Whenever α is a limit, we set $\lambda(\alpha) := \sup\{\lambda(\beta)\}_{\beta<\alpha}$; $\lambda(\alpha) \in Y$ because $Y \in \mathfrak{C}_X$. If there is $\beta < \alpha$ such that $\lambda(\beta) \in M(Y, \succ)$, then $\lambda(\alpha) = \lambda(\beta)$. Otherwise, the monotonicity condition in the left hand side of (8a) holds by the induction hypothesis; therefore, $\lambda(\alpha) \underset{>}{\succ} \lambda(\beta)$ for every $\beta < \alpha$ by (8a). Besides, $\lambda(\alpha) \in M(Y, \underset{<}{\succ})$: the existence of $y \in Y$ such that $y \underset{<}{\succ} \lambda(\alpha)$ would immediately imply $y < \lambda(\beta)$ for some $\beta < \alpha$, hence $y \underset{<}{\succ} \lambda(\beta)$, contradicting the induction hypothesis $\lambda(\beta) \in M(Y, \underset{<}{\succ})$.

Since $\lambda(\alpha + 1) > \lambda(\alpha)$ whenever $\lambda(\alpha) \notin M(Y, \succ)$, while the cardinality of Δ is greater than that of Y, there must be $\bar{\alpha} \in \Delta$ such that $\lambda(\bar{\alpha}) \in M(X, \succ)$, hence $\lambda(\alpha) = \lambda(\bar{\alpha}) \in M(X, \succ)$ for all $\alpha \geq \bar{\alpha}$. Then we have $\lambda(\bar{\alpha}) \succ \lambda(0) \succ x^0$.

5.2 Proof of Theorem 1

A subset $S' \subseteq S$ is an *interval* if $s \in S'$ whenever $s' < s < s''$ and $s', s'' \in S'$. The intersection of any number of intervals is an interval too. Let S be a chain, $S' \subseteq S$ be an interval, and $s \in S \setminus S'$; then either $s > s'$ for all $s' \in S'$, or $s' > s$ for all $s' \in S'$. We write $s > S'$ in the first case, and $s < S'$ in the second.

Lemma 5.2.1. *Let a parametric family $\langle \overset{s}{\succ} \rangle_{s \in S}$ of binary relations on a chain X satisfy both conditions* (10)*. Let every $\overset{s}{\succ}$ have the NM-property on X. Then the set $\{s \in S \mid x \in \mathcal{R}(s)\}$, for every $x \in X$, is an interval.*

Proof. Suppose the contrary: $s' < s < s''$ and $x \in \mathcal{R}(s') \cap \mathcal{R}(s'')$, but $x \notin \mathcal{R}(s)$. By (4), we can pick $x^* \in \mathcal{R}(s)$ such that $x^* \overset{s}{\succ} x$. If $x^* > x$, we have $x^* \overset{s''}{\succ} x$ by (10a), contradicting the assumed $x \in \mathcal{R}(s'')$. If $x^* < x$, we have $x^* \overset{s'}{\succ} x$ by (10b) with the same contradiction. $\square$

The key role is played by the following recursive definition of sequences $x^k \in X$, $s^k \in S$, $S^k \subseteq S$, and $\vartheta^k \in \{-1, 1\}$ ($k \in \mathbb{N}$) such that, in particular,

$$s^k \in S^k; \tag{15a}$$

$$S^k \text{ is an interval;} \tag{15b}$$

$$\forall s \in S^k \left[x^k \in \mathcal{R}(s)\right]; \tag{15c}$$

$$\forall m < k \left[S^k \cap S^m = \emptyset\right]; \tag{15d}$$

$$\forall s \in S \left[[x^k \in \mathcal{R}(s) \;\&\; s < S^k] \Rightarrow \exists m < k \,[s \in S^m \;\text{ or }\; s < s^m < s^k]\right]; \tag{15e}$$

$$\forall s \in S \left[[x^k \in \mathcal{R}(s) \;\&\; s > S^k] \Rightarrow \exists m < k \,[s \in S^m \;\text{ or }\; s > s^m > s^k]\right]; \tag{15f}$$

$$\forall m < k \left[[s^k < s^m \Rightarrow x^k < x^m] \;\&\; [s^k > s^m \Rightarrow x^k > x^m]\right]; \tag{15g}$$

$$\forall m < k \left[x^k \succ^{s^k} x^m \;\text{ or }\; x^m \in \mathcal{R}(s^k)\right]. \tag{15h}$$

We start with an arbitrary $s^0 \in S$, pick $x^0 \in \mathcal{R}(s^0)$, and set $S^0 := \{s \in S \mid x^0 \in \mathcal{R}(s)\}$ and $\vartheta^0 := 1$. Now (15a), (15c), (15e) and (15f) for $k = 0$ immediately follow from the definitions; (15b), from Lemma 5.2.1; (15d), (15g), and (15h) hold vacuously.

Let $k \in \mathbb{N} \setminus \{0\}$, and let x^m, s^m, S^m satisfying (15) have been defined for all $m < k$. We define $\Sigma^k := \bigcup_{m<k} S^m$. For every $s \in \Sigma^k$, there is a unique, by (15d), $\mu(s) < k$ such that $s \in S^{\mu(s)}$. By (15c), $r(s) := x^{\mu(s)}$ is a selection from $\mathcal{R}$ on Σ^k. The conditions (15b) and (15g) imply that r is increasing. If $\Sigma^k = S$, then we already have a monotone selection, so we stop the process.

Otherwise, we proceed in accordance with the following rules. First, we look for $s \in S \setminus \Sigma^k$ such that both $K_k^-(s) := \{m < k \mid s^m < s\}$ and $K_k^+(s) := \{m < k \mid s^m > s\}$ are not empty; if successful, we pick one of them as s^k and set $\vartheta^k := \vartheta^{k-1}$. Otherwise, i.e., if Σ^k is an interval, we set $\vartheta^k := -\vartheta^{k-1}$. Then, if $\vartheta^{k-1} = -1$, we first look for $s^k \in S \setminus \Sigma^k$ such that $K_k^-(s^k) = \emptyset$; if $\vartheta^{k-1} = 1$, we first look for $s^k \in S \setminus \Sigma^k$ such that $K_k^+(s^k) = \emptyset$. If the search is unsuccessful in either case, we pick $s^k \in S \setminus \Sigma^k$ arbitrarily.

We denote $K^* := \{m < k \mid x^m \notin \mathcal{R}(s^k)\}$, $m^- := \operatorname{argmax}_{m \in K_k^-(s^k)} s^m$, $m^+ := \operatorname{argmin}_{m \in K_k^+(s^k)} s^m$, and $I := [s^{m^-}, s^{m^+}]$. If one of $K_k^\pm(s^k)$ is empty, the respective $m^\pm$ is left undefined, in which case $I := \{s \in S \mid s^{m^-} < s\}$ or $I := \{s \in S \mid s < s^{m^+}\}$. By (5), we can pick $x^k \in \mathcal{R}(s^k)$ such that $x^k \succ^{s^k} x^m$ for each $m \in K^*$, hence (15h) holds. Finally, we define $S^k := \{s \in S \setminus \Sigma^k \mid x^k \in \mathcal{R}(s)\} \cap I$. Now the conditions (15a), (15c), and (15d) immediately follow from the definitions; (15b), (15e) and (15f), from Lemma 5.2.1.

Checking (15g) needs a bit more effort. If we assume that $x^{m^-} \in \mathcal{R}(s^k)$, then the condition (15e) for m^- and s^k implies the existence of $m < m^-$ such that $s^{m^-} < s^m < s^k$, contradicting the definition of m^-; therefore, $m^- \in K^*$, hence $x^k \succ^{s^k} x^{m^-}$ by the choice of x^k. If $x^k < x^{m^-}$ then $x^k \succ^{s^{m^-}} x^{m^-}$ by (10b), contradicting (15c) for m^-. Therefore, $x^k > x^{m^-} \geq x^m$ for all $m \in K_k^-(s^k)$. A dual argument shows that $x^k < x^{m^+} \leq x^m$ for all $m \in K_k^+(s^k)$. Thus, (15g) holds.

To summarize, either we obtain a monotone selection on some step, or our sequences are defined [and satisfy (15)] for all $k \in \mathbb{N}$.

Lemma 5.2.2. *Let there be a sequence $\langle k_n \rangle_{n \in \mathbb{N}}$ such that $k_{n+1} > k_n$ and $s^{k_{n+1}} > s^{k_n}$ for all $n \in \mathbb{N}$; then there is no $s \in S$ such that $s \geq s^{k_n}$ for all $n \in \mathbb{N}$.*

Proof. We denote $H := \{h \in \mathbb{N} \mid \exists n \in \mathbb{N} \,[s^h < s^{k_n}]\} \supseteq \{k_n\}_{n \in \mathbb{N}}$ and recursively define a sequence $\langle \varkappa_n \rangle_{n \in \mathbb{N}}$ in this way: $\varkappa_0 := k_0$; given $\varkappa_n$, $\varkappa_{n+1} := \min\{h \in H \mid s^h > s^{\varkappa_n}\}\,[\neq \emptyset]$. Obviously, the sequence $\langle \varkappa_n \rangle_{n \in \mathbb{N}}$ satisfies the same monotonicity conditions as $\langle k_n \rangle_{n \in \mathbb{N}}$.

For every $n \in \mathbb{N}$, we have $x^{\varkappa_{n+1}} > x^{\varkappa_h}$ by (15g) and $x^{\varkappa_{n+1}} \not\succ^{s^{\varkappa_{h+1}}} x^{\varkappa_n}$ by (15e) and the minimality of $\varkappa_{n+1}$. If an upper bound s for s^{k_n} existed, it would be an upper bound for $s^{\varkappa_n}$ as well because of the definition of H. Therefore, we would have $x^{\varkappa_{n+1}} \succ^{s} x^{\varkappa_n}$ by (10a) for all $n \in \mathbb{N}$, contradicting the strict acyclicity of $\succ^{s}$. □

Lemma 5.2.3. *Let there be a sequence $\langle k_n \rangle_{n\in\mathbb{N}}$ such that $k_{n+1} > k_n$ and $s^{k_{n+1}} < s^{k_n}$ for all $n \in \mathbb{N}$; then there is no $s \in S$ such that $s \le s^{k_n}$ for all $n \in \mathbb{N}$.*

The proof is dual to that of Lemma 5.2.2.

Let us assume our sequences defined for all $k \in \mathbb{N}$, and define $\Sigma^\infty := \bigcup_{k\in\mathbb{N}} S^k$. The same $r(s) := x^{\mu(s)}$ is a monotone selection from $\mathcal{R}$ on Σ^∞. The final step of the proof consists in showing that $\Sigma^\infty = S$.

Let us suppose that Σ^∞ is not an interval. Then there must be $s \in S \setminus \Sigma^\infty$ such that both $K_k^-(s)$ and $K_k^+(s)$, as defined in the recursive process, are nonempty for some $k \in \mathbb{N}$. We denote $s^- := \min\{s^m \mid m \le k\} < s$ and $s^+ := \max\{s^m \mid m \le k\} > s$. For every $h > k$, Σ^h is not an interval, hence we have $s^- < s^h < s^+$ for all $h > k$, hence $s^- \le s^h \le s^+$ for all $h \in \mathbb{N}$. Now we have a contradiction with Lemma 5.2.2 or Lemma 5.2.3: one can always find a strictly increasing or strictly decreasing subsequence in an infinite sequence without repetitions.

Let there be $s \in S \setminus \Sigma^\infty$ such that $s > s^k$ for all $k \in \mathbb{N}$. Then Lemma 5.2.2 immediately implies the existence of $\max\{s^k\}_{k\in\mathbb{N}} < s$; let $s^n \ge s^k$ for all $k \in \mathbb{N}$. We define $s^- := \min\{s^k \mid k \le n\} < s$; if $s^- \le s^k \le s^n$ for all $k \in \mathbb{N}$, we have the same contradiction as in the preceding paragraph. Otherwise, we define $h := \min\{k \in \mathbb{N} \mid s^k < s^-\}$; by definition, we have $s^h < s^k \le s^n$ for all $k < h$, hence $K_{h-1}^-(s^h) = \emptyset$. Now the description of the recursive process implies that Σ^h is an interval and $\vartheta^{h-1} = -1$ [because $K_{h-1}^+(s) = \emptyset$]. Therefore, $\vartheta^h = 1$, hence the inequality $s^k < s^n$ for $k > h$ is only possible if Σ^k is not an interval, hence we have $s^h < s^k < s^n$ for all $k > h$, hence $s^h \le s^k \le s^n$ for all k with the same contradiction again.

The case of $s \in S \setminus \Sigma^\infty$ such that $s < s^k$ for all k is treated dually. Thus, $\Sigma^\infty = S$ and the theorem is proven.

5.3 Proof of Theorem 2

We argue rather similarly to the proof of Theorem 1. The main difference is that the recursive process now is, generally, transfinite. This fact entails several complications; first of all, we cannot maintain (15h) any longer.

Let Δ be a well ordered set of a cardinality greater than that of S. By transfinite recursion, we construct a chain of subsets $\Sigma(\alpha) \subseteq S$ ($\alpha \in \Delta$) such that $\Sigma(\beta) \subseteq \Sigma(\alpha)$ whenever $\beta < \alpha$, with an equality only possible when $\Sigma(\beta) = S$; we also construct increasing mappings ("partial monotone selections") $r_\alpha \colon \Sigma(\alpha + 1) \to X$ such that $r_\alpha(s) \in \mathcal{R}(s)$ for every $s \in \Sigma(\alpha + 1)$ and $r_\alpha|_{\Sigma(\beta+1)} = r_\beta$ whenever $\beta < \alpha$. Since the cardinality of Δ is greater than that of S, there must be $\bar\alpha \in \Delta$ such that $\Sigma(\bar\alpha) = \Sigma(\bar\alpha + 1) = S$ hence $r_{\bar\alpha} \colon S \to X$ is a monotone selection from $\mathcal{R}$.

We start with $\Sigma(0) := \emptyset$. The recursive definition of $\Sigma(\alpha) \subseteq S$ for $\alpha > 0$ uses a number of auxiliary constructions recursively defined whenever $\Sigma(\alpha) \subset S$, namely $\sigma(\alpha) \in S$, $S(\alpha) \subseteq S$, $\xi(\alpha) \in X$, and

$\vartheta(\alpha) \in \{-1, 0, 1\}$ such that:

$$\sigma(\alpha) \in S(\alpha); \tag{16a}$$

$$S(\alpha) \text{ is an interval}; \tag{16b}$$

$$\forall s \in S(\alpha)\, \big[\xi(\alpha) \in \mathcal{R}(s)\big]; \tag{16c}$$

$$\forall \beta < \alpha\, \big[S(\alpha) \cap S(\beta) = \emptyset\big]; \tag{16d}$$

$$\forall s \in S\, \big[[\xi(\alpha) \in \mathcal{R}(s)\ \&\ s < S(\alpha)] \Rightarrow \exists \beta < \alpha\, [s \in S(\beta) \text{ or } s < \sigma(\beta) < \sigma(\alpha)]\big]; \tag{16e}$$

$$\forall s \in S\, \big[[\xi(\alpha) \in \mathcal{R}(s)\ \&\ s > S(\alpha)] \Rightarrow \exists \beta < \alpha\, [s \in S(\beta) \text{ or } s > \sigma(\beta) > \sigma(\alpha)]\big]; \tag{16f}$$

$$\forall \beta < \alpha\, \big[[\sigma(\alpha) < \sigma(\beta) \Rightarrow \xi(\alpha) < \xi(\beta)]\ \&\ [\sigma(\alpha) > \sigma(\beta) \Rightarrow \xi(\alpha) > \xi(\beta)]\big]; \tag{16g}$$

$$\vartheta(\alpha) \le 0 \Rightarrow \forall \beta < \alpha\, \big[\xi(\alpha) \succcurlyeq^{\sigma(\alpha)} \xi(\beta) \text{ or } \sigma(\beta) > \sigma(\alpha) \text{ or } \exists \gamma < \beta\, \big(\sigma(\gamma) \in [\sigma(\beta), \sigma(\alpha)]\,\big)\big]; \tag{16h}$$

$$\vartheta(\alpha) \ge 0 \Rightarrow \forall \beta < \alpha\, \big[\xi(\alpha) \succcurlyeq^{\sigma(\alpha)} \xi(\beta) \text{ or } \sigma(\beta) < \sigma(\alpha) \text{ or } \exists \gamma < \beta\, \big(\sigma(\gamma) \in [\sigma(\alpha), \sigma(\beta)]\,\big)\big]; \tag{16i}$$

$$\vartheta(\alpha) = -1 \Rightarrow \forall s < S(\alpha)\, \exists \beta < \alpha\, [\vartheta(\beta) \le 0\ \&\ s < \sigma(\beta) < \sigma(\alpha)]; \tag{16j}$$

$$\vartheta(\alpha) = 1 \Rightarrow \forall s > S(\alpha)\, \exists \beta < \alpha\, [\vartheta(\beta) \ge 0\ \&\ s > \sigma(\beta) > \sigma(\alpha)]. \tag{16k}$$

To start with, we pick $\sigma(0) \in S$ and $\xi(0) \in \mathcal{R}(\sigma(0))$ arbitrarily, and set $\vartheta(0) := 0$ and $S(0) := \{s \in S \mid \xi(0) \in \mathcal{R}(s)\}$. Now (16a), (16c), (16e) and (16f) for $\alpha = 0$ immediately follow from the definitions; (16b), from Lemma 5.2.1; (16d), (16g), (16h), (16i), (16j), and (16k) hold vacuously.

Let $\alpha \in \Delta \setminus \{0\}$, and let $\sigma(\beta) \in S$, $S(\beta) \subseteq S$, $\xi(\beta) \in X$, and $\vartheta(\beta)$ satisfying (16) have been defined for all $\beta < \alpha$. First of all, we define $\Sigma(\alpha) := \bigcup_{\beta<\alpha} S(\beta)$. For every $s \in \Sigma(\alpha)$, there is a unique, by (16d), $\varkappa(s) \in \Delta$ such that $\varkappa(s) < \alpha$ and $s \in S(\varkappa(s))$. By (16c), $r := \xi \circ \varkappa$ is a selection from $\mathcal{R}$ on $\Sigma(\alpha)$. The conditions (16b) and (16g) imply that r is increasing. If $\Sigma(\alpha) = S$, then we already have a monotone selection, so we effectively finish the process, setting $S(\alpha) := \emptyset$, hence $S(\beta) = \emptyset$ and $\Sigma(\beta) = S$ for all $\beta > \alpha$; there is no need to define $\sigma(\alpha)$, $\xi(\alpha)$, and $\vartheta(\alpha)$ in this case.

Otherwise, we pick $s^* \in S \setminus \Sigma(\alpha)$ arbitrarily and define $\Delta^- := \{\beta \in \Delta \mid \beta < \alpha\ \&\ \sigma(\beta) < s^*\}$ and $\Delta^+ := \{\beta \in \Delta \mid \beta < \alpha\ \&\ \sigma(\beta) > s^*\}$. Since $\alpha > 0$, both Δ^- and Δ^+ cannot be empty; if one of them is empty, everything related to it in the following should be just ignored. We also define $I := \{s \in S \setminus \Sigma(\alpha) \mid \forall \beta' \in \Delta^-\, \forall \beta'' \in \Delta^+\, [\sigma(\beta') < s < \sigma(\beta'')]\}\ [\ni s^*]$; (16a) and (16b) ensure that I is an interval.

Supposing $\Delta^- \neq \emptyset$, we define $x^- := \sup\{\xi(\beta)\}_{\beta\in\Delta^-}$ (its existence is ensured by the completeness of X), $\Delta^\uparrow := \{\beta \in \Delta^- \mid \forall \gamma \in \Delta^-\, [\gamma \ge \beta \text{ or } \sigma(\gamma) < \sigma(\beta)]\}$, and $X^\uparrow := \{\xi(\beta)\}_{\beta\in\Delta^\uparrow}$.

Lemma 5.3.1. *$\vartheta(\beta) \le 0$ whenever $\beta \in \Delta^\uparrow$.*

Proof. Immediately follows from condition (16k) for β and s^*. □

Lemma 5.3.2. *For every $\gamma \in \Delta^-$, there is $\beta \in \Delta^\uparrow$ such that $\sigma(\beta) \ge \sigma(\gamma)$.*

Proof. We define $\mathrm{B} := \{\gamma' \in \Delta^- \mid \gamma' < \gamma\ \&\ \sigma(\gamma') > \sigma(\gamma)\}$. If $\mathrm{B} = \emptyset$, then $\gamma \in \Delta^\uparrow$. Otherwise, $\min \mathrm{B} \in \Delta^\uparrow$. □

Lemma 5.3.3. $x^- = \sup X^\uparrow$.

Proof. Immediately follows from Lemma 5.3.2 and (16g). □

Lemma 5.3.4. *For every $s \in I$, there holds $x^- \succcurlyeq^s \xi(\beta)$ for every $\beta \in \Delta^\uparrow$, except $\beta = \max \Delta^\uparrow$ if it exists (then $x^- = \xi(\max \Delta^\uparrow)$).*

Proof. Let $\beta, \beta' \in \Delta^\uparrow$ and $\beta' > \beta$; then $\sigma(\beta') > \sigma(\beta)$ by definition and $\xi(\beta') > \xi(\beta)$ by (16g). Lemma 5.3.1 and (16h) for β' imply $\xi(\beta') \succcurlyeq^{\sigma(\beta')} \xi(\beta)$ because the third disjunctive term in (16h) is incompatible with $\beta \in \Delta^\uparrow$. Therefore, $\xi(\beta') \succcurlyeq^s \xi(\beta)$ by (10a). We see that condition (8a) applies to $X^\uparrow$ and $\succcurlyeq^s$, hence $x^- \succcurlyeq^s \xi(\beta)$. □

Supposing $\Delta^+ \neq \emptyset$, we define $x^+ := \inf\{\xi(\beta)\}_{\beta\in\Delta^+}$, $\Delta^\downarrow := \{\beta \in \Delta^+ \mid \vartheta(\beta) \geq 0 \;\&\; \forall \gamma \in \Delta^+ [\gamma \geq \beta \text{ or } \sigma(\gamma) > \sigma(\beta)]\}$, and $X^\downarrow := \xi(\Delta^\downarrow)$.

Lemma 5.3.5. $\vartheta(\beta) \geq 0$ *whenever* $\beta \in \Delta^\downarrow$.

Lemma 5.3.6. *For every $\gamma \in \Delta^+$, there is $\beta \in \Delta^\downarrow$ such that $\sigma(\beta) \leq \sigma(\gamma)$.*

Lemma 5.3.7. $x^+ = \inf X^\downarrow$.

Lemma 5.3.8. *For every $s \in I$, there holds $x^+ \succcurlyeq^s \xi(\beta)$ for every $\beta \in \Delta^\downarrow$, except $\beta = \max \Delta^\downarrow$ if it exists (then $x^+ = \xi(\max \Delta^\downarrow)$).*

The proofs are dual to those of Lemmas 5.3.1, 5.3.2, 5.3.3, and 5.3.4.

Lemma 5.3.9. $x^- \leq x^+$ *(if both are defined).*

Proof. Whenever $\beta \in \Delta^+$ and $\gamma \in \Delta^-$, we have $\xi(\beta) \geq \xi(\gamma)$ by (16g) for $\max\{\beta, \gamma\} < \alpha$. Therefore, $x^- = \sup X^\uparrow \leq \inf X^\downarrow = x^+$. □

Lemma 5.3.10. *Let $s \in I$ and $y \in X$. If $y \succcurlyeq^s x^-$, then $y > x^-$. If $y \succcurlyeq^s x^+$, then $y < x^+$.*

Proof. Let $y < x^-$; by Lemma 5.3.3, there is $\beta \in \Delta^\uparrow$ such that $y < \xi(\beta)$. If $y \succcurlyeq^s x^-$, then, by Lemma 5.3.4, $y \succcurlyeq^s \xi(\beta)$, hence $y \succcurlyeq^{\sigma(\beta)} \xi(\beta)$ by (10b), which contradicts (16a) and (16c) for β. The case of $y > x^+$ is treated dually. □

Now we consider several alternatives.

A. Let there exist $s \in I$ such that neither x^-, nor x^+ belong to $\mathcal{R}(s)$. Then we pick one of them as $\sigma(\alpha)$, set $\vartheta(\alpha) := 0$, and, invoking (5), obtain $\xi(\alpha) \in \mathcal{R}(\sigma(\alpha))$ such that $\xi(\alpha) \succcurlyeq^{\sigma(\alpha)} x^-$ and $\xi(\alpha) \succcurlyeq^{\sigma(\alpha)} x^+$. Finally, we set $S(\alpha) := \{s \in I \mid \xi(\alpha) \in \mathcal{R}(s)\} \ni \sigma(\alpha)$.

B. Otherwise, we set $\sigma(\alpha) := s^*$ and consider two alternatives again. If $x^- \in \mathcal{R}(s^*)$, then we set $\vartheta(\alpha) := -1$, $\xi(\alpha) := x^-$, and $S(\alpha) := \{s \in I \mid x^- \in \mathcal{R}(s)\} \ni \sigma(\alpha)$. If $x^- \notin \mathcal{R}(s^*)$, then $x^+ \in \mathcal{R}(s^*)$ because the alternative A does not hold; we set $\vartheta(\alpha) := 1$, $\xi(\alpha) := x^+$, and $S(\alpha) := \{s \in I \mid x^+ \in \mathcal{R}(s)\} \ni \sigma(\alpha)$.

Let us check conditions (16). First, (16a), (16c), and (16d) immediately follow from the definitions; (16b), from Lemma 5.2.1.

If $s \in S$ satisfies the conditions in the left hand side of (16e), then $s \notin I$, hence there is $\beta \in \Delta^-$ such that $s < \sigma(\beta)$; obviously, the right hand side of (16e) holds with that β. Condition (16f) is checked dually.

Invoking Lemma 5.3.10 if the alternative A holds, we see that $x^- \le \xi(\alpha) \le x^+$; therefore, (16g) holds whenever $\xi(\beta) < x^-$ or $\xi(\beta) > x^+$. Let $\beta < \alpha$ and $x^- \le \xi(\beta) \le x^+$. If $\beta \in \Delta^-$, we have $\xi(\beta) = x^-$ and $\sigma(\beta) = \max\{\sigma(\gamma)\}_{\gamma \in \Delta^-}$, hence $x^- \notin \mathcal{R}(s^*)$ by (16f) for β and s^*, hence $\xi(\alpha) > x^-$. The case of $\beta \in \Delta^+$ is treated dually.

To check (16h), let us assume $\vartheta(\alpha) \le 0$, hence $\xi(\alpha) \not\succ^{\sigma(\alpha)} x^-$ or $\xi(\alpha) = x^-$. In the latter case, the existence of $\beta^* \in \Delta^\uparrow$ such that $\xi(\beta^*) = x^-$ would imply a contradiction with (16f) for β^* and $\sigma(\alpha)$ exactly as in the previous paragraph. Therefore, $\xi(\alpha) \not\succ^{\sigma(\alpha)} \xi(\beta)$ for every $\beta \in \Delta^\uparrow$ by Lemma 5.3.4 and (10a). Finally, the set $\Delta^- \setminus \Delta^\uparrow$ consists of $\beta \in \Delta^-$ for which there exists a $\gamma < \beta$ as in the last disjunctive term in (16h). Condition (16i) is checked dually.

Let us check (16j). If $\vartheta(\alpha) = -1$, then $\xi(\alpha) = x^- \in \mathcal{R}(\sigma(\alpha))$. If $s \in I \setminus S(\alpha)$, then $x^- \notin \mathcal{R}(s)$. By (4), there is $y \in \mathcal{R}(s)$ such that $y \not\succ^s x^-$; by Lemma 5.3.10, $y > x^-$. If $s < \sigma(\alpha)$ then $y \not\succ^{\sigma(\alpha)} x^-$ by (10a), which is incompatible with $x^- \in \mathcal{R}(\sigma(\alpha))$. Thus, $s < S(\alpha)$ is only possible if $s < I$. Then there is $\gamma \in \Delta^-$ such that $s < \sigma(\gamma)$; Lemma 5.3.2 implies the existence of $\beta \in \Delta^\uparrow$ such that $\sigma(\beta) \ge \sigma(\gamma)$; Lemma 5.3.1 implies that $\vartheta(\beta) \le 0$. Condition (16k) is checked dually.

The theorem is proven.

5.4 Proof of Theorem 3

The key role is played by the following recursive definition of a sequence $x_N^k \in X_N$ ($k \in \mathbb{N}$) such that $x_N^{k+1} \ge x_N^k$ and $x_i^{k+1} \in R_i(x_{-i}^k)$ for all $k \in \mathbb{N}$ and $i \in N$. By the latter condition, x_N^k is a Nash equilibrium if $x_N^{k+1} = x_N^k$. On the other hand, the sequence must stabilize at some stage because of the strict acyclicity assumption.

We define $x_i^0 := \min X_i$ for each $i \in N$. Given x_N^k, we, for each $i \in N$ independently, check whether $x_i^k \in R_i(x_{-i}^k)$ holds. If it does, we define $x_i^{k+1} := x_i^k$; otherwise, we invoke (4) and pick $x_i^{k+1} \in R_i(x_{-i}^k)$ such that $x_i^{k+1} \succ_i^{x_{-i}^k} x_i^k$. Supposing $x_i^{k+1} < x_i^k$ (hence $k > 0$), we obtain $x_i^{k+1} \succ_i^{x_{-i}^{k-1}} x_i^k$ by (10b), contradicting the induction hypothesis $x_i^k \in R_i(x_{-i}^{k-1})$. Therefore, $x_i^{k+1} > x_i^k$, hence $x_N^{k+1} \ge x_N^k$.

Supposing that $x_N^{k+1} > x_N^k$ for all $k \in \mathbb{N}$, we denote $x_{-i}^{\max} := (\max X_j)_{j \ne i} \in X_{-i}$ for each $i \in N$. Whenever $x_i^{k+1} \ne x_i^k$, we have $x_i^{k+1} \succ_i^{x_{-i}^k} x_i^k$ and $x_i^{k+1} > x_i^k$ as was shown in the previous paragraph; since $x_{-i}^{\max} \ge x_{-i}^k$, we have $x_i^{k+1} \succ_i^{x_{-i}^{\max}} x_i^k$ by (10a). Since N is finite, there must be $i \in N$ such that $x_i^{k+1} > x_i^k$ for an infinite number of k. Clearly, the elimination of repetitions in the sequence $\langle x_i^k \rangle_k$ makes it an infinite improvement path for the relation $\succ_i^{x_{-i}^{\max}}$, which contradicts the supposed strict acyclicity.

Remark. This proof obviously resembles Algorithm II of Topkis (1979). Note, however, that it collapses without the assumption that each X_i is a chain, while the original version of the algorithm, under the "revealed preference" property (3), has no use for the assumption.

5.5 Proof of Theorem 4

Let Δ be a well ordered set with a cardinality greater than that of X_N. By transfinite recursion, we construct a mapping $\xi_N \colon \Delta \to X_N$ such that, for all $\beta, \beta', \beta'' \in \Delta$, there hold:

$$\forall i \in N \left[\xi_i(\beta+1) \in \mathcal{R}_i(\xi_{-i}(\beta))\right]; \tag{17a}$$

$$\beta'' > \beta' \Rightarrow \xi_N(\beta'') \geq \xi_N(\beta'); \tag{17b}$$

$$\beta'' > \beta' \Rightarrow \forall i \in N\, [\xi_i(\beta'') = \xi_i(\beta') \text{ or } \xi_i(\beta'') \succ_i^{\xi_{-i}(\beta'')} \xi_i(\beta')]. \tag{17c}$$

First, we define $\xi_i(0) := \min X_i$ for each $i \in N$. Let $\alpha \in \Delta$ and $\xi_N(\beta)$ have been defined for all $\beta \leq \alpha$ so that (17a) holds for all $\beta < \alpha$ while (17b) and (17c) hold for all $\beta', \beta'' \leq \alpha$. For each $i \in N$, we define $\xi_i(\alpha+1) := \xi_i(\alpha)$ if $\xi_i(\alpha) \in \mathcal{R}_i(\xi_{-i}(\alpha))$, ensuring (17a) for $\beta = \alpha$ as well as the continuation of (17c). Otherwise, we pick $\xi_i(\alpha+1) \in \mathcal{R}_i(\xi_{-i}(\alpha))$ such that $\xi_i(\alpha+1) \succ_i^{\xi_{-i}(\alpha)} \xi_i(\alpha)$ (it exists by Proposition 3 and (4)), thus ensuring (17c) for $\beta'' = \alpha+1$ and $\beta' = \alpha$. Checking (17b) for $\beta'' = \alpha+1$, as well as (17c) for $\beta'' = \alpha+1$ and $\beta' < \alpha$, is postponed till after the definition of $\xi_i(\alpha)$ for limits.

Let α be a limit, and $\xi_N(\beta)$ satisfying (17) have been defined for all $\beta < \alpha$. Then we define $\xi_i(\alpha) := \sup_{\beta<\alpha} \xi_i(\beta)$ for each $i \in N$, ensuring (17b) for $\beta'' = \alpha$. By (10a), (17b) and (17c), we have $\xi_i(\beta') \succ_i^{\xi_{-i}(\alpha)} \xi_i(\beta)$ whenever $\beta', \beta < \alpha$ and $\xi_i(\beta') > \xi_i(\beta)$. If $\xi_i(\alpha) = \xi_i(\beta)$ for some $\beta < \alpha$, then (17c) for $\beta'' = \alpha$ is valid trivially; otherwise, the chain $\{\xi_i(\beta)\}_{0 \leq \beta < \alpha}$ satisfies the "left-hand-side" condition in (8a) for $\succ_i^{\xi_{-i}(\alpha)}$, hence $\xi_i(\alpha) \succ_i^{\xi_{-i}(\alpha)} \xi_i(\beta)$ for all $\beta < \alpha$, i.e., (17c) for $\beta'' = \alpha$ holds again.

Now let us return to a "successor step." If α itself is a successor, $\alpha = \alpha' + 1$, then the assumption that $\xi_i(\alpha+1) < \xi_i(\alpha)$ would imply $\xi_i(\alpha+1) \succ_i^{\xi_{-i}(\alpha')} \xi_i(\alpha)$ by (10b), contradicting (17a) for $\beta = \alpha'$; therefore, (17b) continues to hold. If α is a limit, the assumption $\xi_i(\alpha+1) < \xi_i(\alpha)$ would imply $\xi_i(\alpha+1) < \xi_i(\beta)$ for some $\beta < \alpha$, hence $\xi_i(\alpha+1) < \xi_i(\beta+1)$, and a contradiction with the condition $\xi_i(\beta+1) \in \mathcal{R}(\xi_{-i}(\beta))$ is obtained in exactly the same way. In either case, (17c) for $\beta'' = \alpha+1$ and $\beta' < \alpha$ holds by (10a).

The final argument is standard. We must have $\xi_N(\alpha) = \xi_N(\beta)$ for some $\beta < \alpha$. Then we have $\xi_N(\beta+1) = \xi_N(\beta)$ by (17b); therefore, $\xi_N(\beta)$ is a Nash equilibrium by (17a).

6 Concluding remarks

6.1. The analogy between our Proposition 3 and Theorem 1 of Kukushkin (2008) could be extended by noticing that conditions (8) are also *necessary* for just the non-emptiness of $M(Y, \succ)$ for every $Y \in \mathfrak{C}_X$ if $\succ$ is a semiorder, cf. Theorem 4.1 of Smith (1974) and Theorem 4 of Kukushkin (2008), but not otherwise; see Example 3 of Kukushkin (2008), where an interval order on a closed interval in the real line admits a maximizer on every compact (i.e., subcomplete) subset, but does not satisfy (8a).

6.2. The replacement of both conditions (10) with one of them in Proposition 6 comes at a cost. As is easily seen from the proof, the statement of Theorem 1 can be strengthened: whenever $s^0 \in S$ and $x^0 \in \mathcal{R}(s^0)$, there is a monotone selection r from $\mathcal{R}$ such that $r(s^0) = x^0$. Example 4 shows this statement wrong without both conditions (10), even when both S and X are finite and all $\succ^s$ are orderings. However, it becomes valid again if we add (10b) to the assumptions of Proposition 6; a routine modification of the proof is omitted.

Example 4. Let $X := \{0, 1\}$, $S := \{0, 1\}$ (both with natural orders), and relations $\succ^s$ be defined by $0 \succ^1 1$; condition (10a) holds vacuously while (10b) does not. We have $\mathcal{R}(0) = \{0, 1\}$ and $\mathcal{R}(1) = \{0\}$, so there is no monotone selection with $r(0) = 1$.

6.3. If both $\min S$ and $\max S$ exist in Theorem 1, then ϑ^k in the recursive construction becomes superfluous. Moreover, there exists a monotone selection r from $\mathcal{R}$ with a finite range $r(S)$: Lemmas 5.2.2 and 5.2.3 imply the impossibility of an infinite monotone subsequence of $\langle s^k \rangle_k$, hence the sequence must be finite.

6.4. It remains unclear whether the assumption that both X and S are chains can be dropped or weakened in Theorem 1. From the game-theoretic viewpoint, however, the question does not seem pressing. The existence of an ε-Nash equilibrium in a game with increasing best responses may hold in the absence of monotone selections as Theorem 3 and Example 1 demonstrate. If the best responses are, say, decreasing, then, indeed, all existence results in the literature need monotone selections, but they also need the assumption that each player is only affected by a scalar aggregate of the partners/rivals' choices. In principle, Theorem 1 of Jensen (2010) can be applied to a game with non-scalar strategies, so an extension of Theorem 1 to non-scalar X *could* be useful; however, no non-trivial example of such a game has emerged so far.

6.5. For games with strategic *substitutes* and preferences "less rational" than assumed in Theorems 1 and 2, e.g., where each player may keep in mind several objectives, there is neither equilibrium existence result, nor an example of nonexistence (in the presence of an appropriate aggregation as, say, in Proposition 5). On the other hand, the remark after Example 1 is relevant here too.

6.6. The assumption in Theorems 3 and 4 that each X_i is a chain is strong enough to be extremely irritating; however, I have no idea at the moment whether and how it could be dispensed with.

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