The single crossing conditions for incomplete preferences

Nikolai S. Kukushkin*

January 18, 2014

Abstract

We study the implications of the single crossing conditions for preferences described by binary relations. All restrictions imposed on the preferences are satisfied in the case of approximate optimization of a bounded-above utility function. In the context of the choice of a single agent, the transitivity of strict preferences ensures that the best response correspondence is increasing in the sense of a natural preorder; if the preferences are represented by an interval order, there is an increasing selection from the best response correspondence. In a strategic game, a Nash equilibrium exists and can be reached from any strategy profile after a finite number of best response improvements if all strategy sets are chains, the single crossing conditions hold w.r.t. pairs [one player's strategy, a profile of other players' strategies], and the strict preference relations are transitive. If, additionally, there are just two players, every best response improvement path reaches a Nash equilibrium after a finite number of steps. If each player is only affected by a linear combination of the strategies of others, the single crossing conditions hold w.r.t. pairs [one player's strategy, an aggregate of the strategies of others], and the preference relations are interval orders, then a Nash equilibrium exists and can be reached from any strategy profile with a finite best response path.

Key words: strong acyclicity; single crossing; Cournot tatonnement; Nash equilibrium; aggregative game
MSC2010 Classifications: 91A10, 91A44
JEL Classification: C 72.

1 Introduction

Although the description of preferences of the players with utility functions remains the most popular in game theory, an interest is growing in alternative approaches, in particular, in various forms of bounded rationality. Not surprisingly, familiar techniques often prove inapplicable in a broader context, or, at least, have to be modified substantially.

This paper explores how the replacement of utility functions with more general preference relations will affect classical results on strategic complementarity and monotone comparative statics. Naturally, real-valued utilities are indispensable in a cardinal framework, "supermodular games" (Topkis 1979; Veinott, 1989; Vives 1990; Milgrom and Roberts 1990). A purely ordinal version, started by Milgrom and Shannon (1994) and developed further in later papers (Athey, 2001; Quah, 2007; Quah and Strulovici,

*Russian Academy of Sciences, Dorodnicyn Computing Center, 40, Vavilova, Moscow 119333 Russia
E-mail: ququns@inbox.ru HomePage

2009; Reny, 2011), *can* be expressed in the language of binary relations, provided that the non-strict preference relation is transitive and complete.

However, there is no obvious way to extend those results to a wider class of preference relations, as, say, ε-optimization of a numeric utility function, or a Pareto combination of several utility functions ("multi-criteria optimization"). For instance, the previous literature, to the best of my knowledge, contains no general existence result for ε-Nash equilibria in games of strategic complementarity where the best responses may fail to exist.

Here we formulate the single crossing conditions in the language of binary relations, and then study their implications in three different contexts: the choice of an alternative by a single agent, whose preferences are influenced by an exogenous parameter; games of strategic complements; aggregative games (which may exhibit strategic complements, strategic substitutes, or a combination of both). In each case, some familiar implications hold for preferences of a more general kind, while others do not. To be more precise, very much depends on the "degree of rationality" of the preferences.

Two general restrictions are imposed throughout. First, the feasible sets (or strategy sets) in the main theorems are linearly ordered (chains). This restriction is virtually inessential in the case of aggregative games, but quite biting in two others. Unfortunately, the straightforward extension of the notion of quasisupermodularity to incomplete preferences does not have the desirable implications and the prospects for more sophisticated extensions remain unclear.

Another universal assumption is the strict acyclicity of preferences: no agent can carry out an infinite sequence of unilateral improvements. The existence of the best responses is thereby ensured without imposing any topological restrictions. Due to that assumption, the results here resemble what happens in *finite* games (to the extent that there is any similarity with complete preferences). Also importantly, it is satisfied in the case of ε-optimization of a bounded-above utility function.

Section 2 introduces basic notions concerning preferences and choice. In particular, we describe the three "levels of rationality" of preferences, which play an important role in all the results. In Section 3, the single crossing conditions are translated into the language of binary relations, and their implications for monotone comparative statics in the context of the choice of a single agent are obtained. Propositions 3.3 and 3.5 show the monotonicity of the best response correspondence in terms of natural extensions of the basic order from points to subsets. Theorem 3.8 establishes the existence of an increasing selection from the best response correspondence when both available choices and parameters form chains, and the preferences are described by interval orders.

In Section 4, we define strategic games where the preferences of each player are described by a family of binary relations on own strategies, with the choices of others as parameters; we also define adaptive ("best response") dynamics. Actually, we consider two different scenarios that coincide for complete preferences. In the case of ε-optimization, the difference is whether to demand that the new, ε-optimal, strategy should be a noticeable (more than ε) improvement over the current strategy or not.

Section 5 contains the theorems about games with strategic complementarity, i.e., where the single crossing conditions hold with respect to pairs <one player's strategy, a profile of other players' strategies>. Theorem 5.1 shows that a Nash equilibrium exists and can be reached from any strategy profile after a finite number of best response improvements if the strategy sets are chains and the preference relations are transitive. If there are just two players, *every* best response improvement path reaches a Nash equilibrium after a finite number of steps (Theorem 5.2). There is a certain similarity with Theorems 2 and 3, respectively, from Kukushkin et al. (2005). Theorem 5.4 is about monotone comparative statics of Nash equilibrium: if the preferences of the players are perturbed in such a way that the single crossing conditions hold, then the whole set of Nash equilibria goes "upwards" (in the same sense as in Proposition 3.3).

Section 6 is about *aggregative games* (Novshek 1985; Kukushkin 1994, 2004, 2005; Dubey et al. 2006; Jensen 2010). In Theorem 6.1, each player is only affected by a linear combination of the strategies of others with a symmetrical matrix of coefficients, the single crossing conditions hold with respect to pairs <one player's strategy, an aggregate of the strategies of others>, and the preference relations are interval orders. Then a Nash equilibrium exists and can be reached from any strategy profile with a finite best response path (possibly, with insignificant improvements along the way). The proof is based on Theorem 3.8 and on a trick invented by Huang (2002) for studying fictitious play and then adapted to aggregative games by Dubey et al. (2006). A similar approach works for the maximum/minimum aggregation (Theorems 6.2 and 6.3).

A few concluding remarks are in Section 7. More complicated proofs and more sophisticated examples are deferred to Appendices.

2 Basic notions

Let the (strict) *preferences* of an agent over alternatives from a set X be described by a binary relation $\succ$, $y \succ x$ meaning that the availability of the alternative y makes x unacceptable. It is natural to pay special attention to the set of maximal elements (*maximizers*)

$$M(X, \succ) := \{x \in X \mid \nexists\, y \in X\, [y \succ x]\}. \tag{1}$$

Until appropriate assumptions are imposed, it is quite possible that $M(X, \succ) = \emptyset$.

The usual description of preferences with a utility function $u\colon X \to \mathbb{R}$ fits into this scheme if we define

$$y \succ x \rightleftharpoons u(y) > u(x) \tag{2}$$

for all $x, y \in X$. Then $M(X, \succ) = \operatorname{Argmax}_{x \in X} u(x)$.

The main emphasis in this paper is on preferences that do not admit a representation (2), even if $\mathbb{R}$ is replaced with an arbitrary chain. It should be stressed at the start that *arbitrary* binary relations are not good for anything (Example A.1 below). We consider two "rationality" conditions weaker than (2).

The milder condition is that $\succ$ is irreflexive and transitive, i.e., a (strict) partial order. The stronger one is that $\succ$ is an *interval order* (see, e.g., Fishburn, 1985), i.e., there are a chain $\mathcal{C}$ and two mappings $u^+, u^-\colon X \to \mathcal{C}$ such that, for all $x, y \in X$:

$$u^+(x) \geq u^-(x); \tag{3a}$$

$$y \succ x \iff u^-(y) > u^+(x). \tag{3b}$$

An example is given by preferences of this kind:

$$y \succ x \rightleftharpoons u(y) > u(x) + \varepsilon, \tag{4}$$

where u is real-valued and $\varepsilon \geq 0$. Obviously, $M(X, \succ) = \{x \in X \mid u(x) \geq \sup_{x' \in X} u(x') - \varepsilon\}$ in this case (with $M(X, \succ) = \emptyset$ if the supremum is $+\infty$).

Our final condition imposed throughout is *strong acyclicity* of preferences: there exists no infinite sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ such that $x^{k+1} \succ x^k$ for each k. This assumption ensures that $M(X, \succ) \neq \emptyset$. It obviously implies irreflexivity and the impossibility of finite improvement cycles (acyclicity proper). For preferences described by (4), it holds if $\varepsilon > 0$ and u is bounded above.

The use of those conditions in this paper is based on the following simple statements.

Proposition 2.1. *Let $\succ$ be strongly acyclic and transitive on X, and let $x \in X \setminus M(X, \succ)$. Then there is $y \in M(X, \succ)$ such that $y \succ x$.*

Proof. Since $x \notin M(X, \succ)$, there is $x^1 \in X$ such that $x^1 \succ x$. If $x^1 \in M(X, \succ)$, we are home. Otherwise, there is $x^2 \in X$ such that $x^2 \succ x^1$, and so on. Since $\succ$ is strongly acyclic, we must reach $x^m \in M(X, \succ)$ at some stage; by transitivity, $x^m \succ x$. □

Proposition 2.2. *Let $\succ$ be a strongly acyclic interval order on X and let $\{x^0, \ldots, x^m\} \subseteq X \setminus M(X, \succ)$. Then there is $y \in M(X, \succ)$ such that $y \succ x^k$ for all $k \in \{0, \ldots, m\}$.*

Proof. Without restricting generality, we may assume $u^+(x^0) = \max_k u^+(x^k)$. By Proposition 2.1, there is $y \in M(X, \succ)$ such that $y \succ x^0$. Then $u^-(y) > u^+(x^0) \geq u^+(x^k)$ for each k, hence $y \succ x^k$. □

3 Single crossing and monotone comparative statics

Henceforth, the preferences are described by a family $\langle \succ^s \rangle_{s \in S}$ of binary relations, rather than a single relation, parameter s reflecting external influences, e.g., the choices of other agents. We define the *best response correspondence*:

$$\mathcal{R}(s) := M(X, \succ^s). \tag{5}$$

We always assume alternatives and parameters to form *partially ordered sets* (*posets*). A parametric family $\langle \succ^s \rangle_{s \in S}$ on X has the *single crossing* property if these conditions hold:

$$\forall x, y \in X\, \forall s, s' \in S\, \big[[s' > s \;\&\; y \succ^s x \;\&\; y > x] \Rightarrow y \succ^{s'} x\big]; \tag{6a}$$

$$\forall x, y \in X\, \forall s, s' \in S\, \big[[s' > s \;\&\; y \succ^{s'} x \;\&\; y < x] \Rightarrow y \succ^s x\big]. \tag{6b}$$

This definition is equivalent to Milgrom and Shannon's (1994) if every $\succ^s$ can be represented, in the sense of (2), by a numeric function $u(x, s)$.

For a family of preference relations defined by ε-optimization (4), both conditions (6) hold if $u(x, s)$ satisfies Topkis's (1979) *increasing differences* condition:

$$\forall x, y \in X\, \forall s, s' \in S\, \big[[s' \geq s \;\&\; y \geq x] \Rightarrow u(y, s') - u(x, s') \geq u(y, s) - u(x, s)\big]. \tag{7}$$

Under rather mild rationality assumptions, the single crossing conditions make possible monotone comparative statics results.

Proposition 3.1. *Let a parametric family $\langle \succcurlyeq^s \rangle_{s \in S}$ of strongly acyclic and transitive binary relations on a chain X satisfy condition (6b). Let $s, s' \in S$, $s' > s$, and $x \in \mathcal{R}(s)$. Then $x' > x$ whenever $x' \succcurlyeq^{s'} x$. Moreover, if $x \notin \mathcal{R}(s')$, then there exists $x' \in \mathcal{R}(s')$ such that $x' \succcurlyeq^{s'} x$ and $x' > x$.*

Proof. If $x' \succcurlyeq^{s'} x$ and $x' < x$, we have $x' \succcurlyeq^{s} x$ by (6b), which contradicts the assumption $x \in \mathcal{R}(s)$. If $x \notin \mathcal{R}(s')$, then $x' \in \mathcal{R}(s')$ such that $x' \succcurlyeq^{s'} x$ exists by Proposition 2.1; $x' > x$ holds by the first claim. $\square$

Proposition 3.2. *Let a parametric family $\langle \succcurlyeq^s \rangle_{s \in S}$ of strongly acyclic and transitive binary relations on a chain X satisfy condition (6a). Let $s, s' \in S$, $s' > s$, and $x' \in \mathcal{R}(s')$. Then $x' > x$ whenever $x \succcurlyeq^{s} x'$. Moreover, if $x' \notin \mathcal{R}(s)$, then there exists $x \in \mathcal{R}(s)$ such that $x \succcurlyeq^{s} x'$ and $x' > x$.*

The proof is dual to that of Proposition 3.1.

Remark. The first claim in either Proposition 3.1 or Proposition 3.2 does not need strong acyclicity, nor transitivity.

Given a poset X, we, following Smithson (1971), define a preorder (reflexive and transitive binary relation) on the set of subsets of X by:

$$Y \geq Z \rightleftharpoons \forall y \in Y\, \exists z \in Z\, [y \geq z]\ \&\ \forall z \in Z\, \exists y \in Y\, [y \geq z]. \tag{8}$$

Proposition 3.3. *Let a parametric family $\langle \succcurlyeq^s \rangle_{s \in S}$ of strongly acyclic and transitive binary relations on a chain X satisfy both conditions (6). Then $\mathcal{R}(s') \geq \mathcal{R}(s)$, in the sense of (8), whenever $s, s' \in S$ and $s' \geq s$.*

Immediately follows from Propositions 3.1 and 3.2.

It is worthwhile to compare this comparative statics result with what has been obtained for preferences described by utility functions (2). Assuming X a *lattice*, we consider two binary relations on its subsets, both invented by the late A.F. Veinott, Jr.:

$$Y \geq^{\mathrm{Vt}} Z \rightleftharpoons \forall y \in Y\, \forall z \in Z\, [y \wedge z \in Z\ \&\ y \vee z \in Y]; \tag{9}$$

$$Y \geq^{\mathrm{wV}} Z \rightleftharpoons \forall y \in Y\, \forall z \in Z\, [y \wedge z \in Z \ \text{ or } \ y \vee z \in Y]. \tag{10}$$

When $Y \neq \emptyset \neq Z$, (9) implies both (8) and (10), which, generally, do not imply each other. $\geq^{\mathrm{Vt}}$, dubbed "strong set order" in Topkis (1979), is transitive and anti-symmetric on all nonempty subsets and reflexive on sublattices; $\geq^{\mathrm{wV}}$, generally, is not even transitive. Veinott (1989) called correspondences increasing in the sense of (9) *ascending*, and correspondences increasing in the sense of (10) *weakly ascending*.

Milgrom and Shannon (1994) introduced the notion of a *quasisupermodular* function on a lattice:

$$\forall x, y \in X\, \big[u(x) > u(y \wedge x) \Rightarrow u(y \vee x) > u(y)\big]; \tag{11a}$$

$$\forall x, y \in X\, \big[u(y) > u(y \vee x) \Rightarrow u(y \wedge x) > u(x)\big]. \tag{11b}$$

If X is a chain, conditions (11) hold trivially.

Theorem 4 of Milgrom and Shannon (1994) states, among other things, that $\mathcal{R}$ is ascending, i.e., increasing in the sense of (9), if X is a lattice, every $\succcurlyeq^s$ admits a representation (2) with a quasisupermodular function u, and the single crossing conditions (6) hold. This claim is much stronger than that of Proposition 3.3; however, it does not hold for "less rational" preferences.

Example 3.4. Let $X := \{0, 1, 2\}$ and $S := \{0, 1\}$; let the preferences of an agent be defined by ε-optimization (4) of the utility function $u(x, s)$ depicted in the following matrix, with $\varepsilon := 2$:

$$\begin{bmatrix} 0 & \underline{1} & \underline{3} \\ \underline{3} & 0 & \underline{1} \end{bmatrix},$$

where own choice, x, is on the abscissae axis (directed to the right), and parameter s, on the ordinates axis (directed upwards). Conditions (6), even (7), are easy to check. However, $\mathcal{R}$ is not ascending: $2 \in \mathcal{R}(0)$ and $1 \in \mathcal{R}(1)$, but $1 \notin \mathcal{R}(0)$; therefore, $\mathcal{R}(1) \geq^{\mathrm{Vt}} \mathcal{R}(0)$ does not hold.

It is no accident that $\mathcal{R}$ in Example 3.4 is *weakly* ascending.

Proposition 3.5. *Let a parametric family $\langle \succ^s \rangle_{s \in S}$ of interval orders on a lattice X satisfy both conditions (6). Let every $\succ^s$ admit a representation (3) with a quasisupermodular function u^+. Then $\mathcal{R}(s') \geq^{\mathrm{wV}} \mathcal{R}(s)$ whenever $s, s' \in S$ and $s' \geq s$.*

Proof. Let $s' \geq s$, $x \in \mathcal{R}(s)$, and $y \in \mathcal{R}(s')$. We have to show that either $x \wedge y \in \mathcal{R}(s)$ or $x \vee y \in \mathcal{R}(s')$. Let $x \wedge y \notin \mathcal{R}(s)$, i.e., there is $z \in X$ such that $z \succ^s x \wedge y$; then $u^+(x, s) > u^+(x \wedge y, s)$ since we would have $z \succ^s x$ otherwise. Applying (11a), we obtain $u^+(x \vee y, s) > u^+(y, s)$; applying (6a) if $s' > s$, we obtain $u^+(x \vee y, s') > u^+(y, s')$. Therefore, $x \vee y \in \mathcal{R}(s')$. □

Remark. Conditions (6b) and (11b) were not used in the proof. Moreover, the strict inequality in the right hand side of (11a) can be replaced with a non-strict one (weak quasisupermodularity). A dual argument shows that (6b) and (11b), even with a non-strict inequality in the right hand side of the latter, would also be sufficient.

In the case of a parametric family of interval orders on a chain, conditions (11) become vacuous, so the single crossing conditions (6) ensure that $\mathcal{R}$ is increasing in the sense of both (8) and (10), which holds in Example 3.4. However, the assumptions of Proposition 3.3 are insufficient for that.

Example 3.6. Let $X := \{0, 1, 2, 3\}$ and $S := \{0, 1\}$. Let the preferences of an agent be defined by the strong Pareto combination of two utility functions $u_i(x, s)$, i.e.,

$$y \succ^s x \rightleftharpoons \forall i \in \{1, 2\}\, [u_i(y, s) > u_i(x, s)],$$

with the functions depicted in the following matrix:

$$\begin{bmatrix} (0,0) & \underline{(3,0)} & (1,5) & \underline{(2,6)} \\ \underline{(5,2)} & (4,1) & \underline{(1,5)} & (0,0) \end{bmatrix},$$

where own choice, x, is on the abscissae axis (directed to the right), and parameter s, on the ordinates axis (directed upwards). Either function u_i satisfies conditions (6), even (7). However, $\mathcal{R}$ is not even weakly ascending: $2 \in \mathcal{R}(0)$ and $1 \in \mathcal{R}(1)$, but $2 \notin \mathcal{R}(1)$ and $1 \notin \mathcal{R}(0)$; therefore, $\mathcal{R}(1) \geq^{\mathrm{wV}} \mathcal{R}(0)$ does not hold.

Proposition 3.7. *Let a parametric family $\langle \succ^s \rangle_{s \in S}$ of strongly acyclic and transitive binary relations on a chain X satisfy both conditions (6). Let $s, s', s'' \in S$, $s < s' < s''$ and $x \in \mathcal{R}(s) \cap \mathcal{R}(s'')$. Then $x \in \mathcal{R}(s')$.*

Proof. Supposing the contrary, we invoke Proposition 3.1 and obtain $x' \in \mathcal{R}(s')$ such that $x' > x$ and $x' \succ^{s'} x$, which contradicts Proposition 3.2 (the first statement) applied to $s'' > s'$, x and x'. $\square$

Given a parametric family $\langle \succ^s \rangle_{s \in S}$ on X, an *increasing selection* from $\mathcal{R}$ is a mapping $r \colon S \to X$ such that $r(s) \in \mathcal{R}(s)$ for every $s \in S$ and $r(s'') \geq r(s')$ whenever $s', s'' \in S$ and $s'' \geq s'$. The existence of an increasing selection from $\mathcal{R}$ can be viewed as a reasonable alternative definition of increasing best responses; besides, it is indispensable when deriving the existence of a Nash equilibrium from Tarski's fixed point theorem. Assuming that $\mathcal{R}$ is increasing in the sense of (8) and every $\mathcal{R}(s)$ contains the greatest, $\bigvee \mathcal{R}(s)$, [or the least, $\bigwedge \mathcal{R}(s)$] element, it is easy to see that either $r^{\max}(s) := \bigvee \mathcal{R}(s)$ or, respectively, $r^{\min}(s) := \bigwedge \mathcal{R}(s)$ is increasing. The usual practice in the study of games with strategic complementarities is to make continuity assumptions that ensure the existence of both the greatest and the least best responses (quasisupermodularity alone is insufficient for that). Obviously, this approach is unworkable when one resorts to ε-optimization just because of a lack of continuity.

Generally, a correspondence increasing in the sense of (8) need not admit an increasing selection; moreover, it need not admit a fixed point when X and S coincide (Roddy and Schröder, 2005, Example 2.3). Example A.2 below shows that an increasing selection may fail to exist under the assumptions of Proposition 3.3; such selections exist in Example 3.6 because everything is finite there (Kukushkin, 2013b). The situation changes when the preferences are "more rational."

Theorem 3.8. *Let X and S be chains such that both $\min S$ and $\max S$ exist. Let a parametric family $\langle \succ^s \rangle_{s \in S}$ of strongly acyclic interval orders on X satisfy single crossing conditions (6). Then there exists an increasing selection from $\mathcal{R}$ on S such that $r(S)$ is finite, i.e., r takes a finite number of values.*

Sketch of a proof. The proof is based on Propositions 2.2, 3.1, 3.2, and 3.7: We start with an arbitrary pair $x^0 \in \mathcal{R}(s^0)$ and define $r(s) := x^0$ whenever $x^0 \in \mathcal{R}(s)$. If, by chance, r is thus defined on the whole S, we are home. Otherwise, we pick $s^1 \in S$ such that $x^0 \notin \mathcal{R}(s^1)$ and, applying Proposition 3.1 if $s^1 > s^0$, or Proposition 3.2 if $s^1 < s^0$, obtain $x^1 \in \mathcal{R}(s^1)$ such that $x^1 \succ^{s^1} x^0$. Now we set $r(s) := x^1$ whenever $x^1 \in \mathcal{R}(s)$ and that assignment is compatible with monotonicity. Then we continue in the same way; however, we may have to invoke Proposition 2.2 before Propositions 3.1 or 3.2. Strong acyclicity (at end points of S) ensures that the process stops at some stage, producing r defined on the whole S.

A detailed argument is given in Section A.1. $\square$

4 Strategic games and Cournot tâtonnement

We define a *strategic game* in a way that is not quite standard. Instead of utility functions, each player's preferences are described by binary relations on the set of own strategies, parameterized by the choices of others. Thus, we ignore each player's preferences between outcomes which differ in other players' choices; such preferences play no part in the definition of a Nash equilibrium or in myopic adaptive dynamics.

Remark. Olga Bondareva (1979) championed the idea that the preferences of players in a noncooperative game *should* be described in this way. We do not follow any ideology here, just notational convenience.

There is a finite set N of players and a set X_i of strategies for each $i \in N$. We denote $X_N := \prod_{i\in N} X_i$ and $X_{-i} := \prod_{j\neq i} X_j$. Each player i's (strict) *preferences* are described by a parametric family of binary relations $\succ_i^{x_{-i}}$ ($x_{-i} \in X_{-i}$) on X_i. Then we have the *best response correspondence* $\mathcal{R}_i$ for each player $i \in N$, defined by (5) with $S = X_{-i}$, $X = X_i$, and $\succ_i^{x_{-i}}$ as $\succ^s$.

A *Nash equilibrium* is a strategy profile $x_N^0 \in X_N$ such that $x_i^0 \in \mathcal{R}_i(x_{-i}^0)$ for all $i \in N$.

If the players' preferences are described with (ordinal) utility functions,

$$y_i \succ_i^{x_{-i}} x_i \iff u_i(y_i, x_{-i}) > u_i(x_i, x_{-i}), \tag{12}$$

then $\mathcal{R}_i(x_{-i}) = \operatorname{Argmax}_{x_i\in X_i} u_i(x_i, x_{-i})$ and our definition of a Nash equilibrium is equivalent to the standard one. However, we are interested in more general preferences here.

A conceptual subtlety should be noted. When (as is usually the case) the definition of a strategic game includes utility functions, rather than preference relations of a more general kind, "ε-Nash equilibrium" is an alternative notion of equilibrium, similar to "Nash equilibrium," but obviously different. Here, "ε-Nash equilibrium" is exactly the same thing as "Nash equilibrium," only applied to a game with modified preferences: ε-optimization (4) instead of (12).

When describing best response dynamics in a strategic game, the following binary relations on the set of strategy profiles prove convenient ($i \in N$, y_N, $x_N \in X_N$):

$$y_N \triangleright_i^{\mathrm{BR}} x_N \rightleftharpoons [y_{-i} = x_{-i} \;\&\; y_i \succ_i^{x_{-i}} x_i \;\&\; y_i \in \mathcal{R}_i(x_{-i})]; \tag{13a}$$

$$y_N \triangleright^{\mathrm{BR}} x_N \rightleftharpoons \exists i \in N\, [y_N \triangleright_i^{\mathrm{BR}} x_N] \tag{13b}$$

(*Best Response improvement* relation);

$$y_N \triangleright_i^{[\mathrm{BR}]} x_N \rightleftharpoons [y_{-i} = x_{-i} \;\&\; x_i \notin \mathcal{R}_i(x_{-i}) \ni y_i]; \tag{14a}$$

$$y_N \triangleright^{[\mathrm{BR}]} x_N \rightleftharpoons \exists i \in N\, [y_N \triangleright_i^{[\mathrm{BR}]} x_N] \tag{14b}$$

(*Best Response "quasi-improvement"* relation).

If $y_N \triangleright_i^{\mathrm{BR}} x_N$, then $y_N \triangleright_i^{[\mathrm{BR}]} x_N$. The converse holds for preferences representable by a utility function (12), but generally not otherwise. In the case of ε-optimization (4), the difference between $\triangleright_i^{\mathrm{BR}}$ and $\triangleright_i^{[\mathrm{BR}]}$ is whether to demand that the "superior" strategy should, besides being ε-optimal, be a significant (more than ε) improvement over the "inferior" strategy, or not. If the preferences are even less similar to those described by a utility function (e.g., a Pareto combination of several utility functions), a quasi-improvement may be no improvement at all.

By definition, every Nash equilibrium is a maximizer of $\triangleright^{\mathrm{BR}}$ and $\triangleright^{[\mathrm{BR}]}$; the converse need not be true. However, if $x_N^0 \in X_N$ is a maximizer of $\triangleright^{[\mathrm{BR}]}$ and $\mathcal{R}_i(x_{-i}^0) \neq \emptyset$ for each $i \in N$, then x_N^0 is a Nash equilibrium. Similarly, $x_N^0 \in X_N$ is a Nash equilibrium if it is a maximizer of $\triangleright^{\mathrm{BR}}$ and each $\succ_i^{x_{-i}^0}$ is strongly acyclic and transitive (immediately follows from Proposition 2.1).

A (finite or infinite) sequence of strategy profiles $\langle x_N^k \rangle_{k\in\mathbb{N}}$ is a *best response* (*quasi*)-*improvement path* if $x_N^{k+1} \triangleright^{\mathrm{BR}} x_N^k$ ($x_N^{k+1} \triangleright^{[\mathrm{BR}]} x_N^k$) whenever $k \geq 0$ and x_N^{k+1} is defined. A game Γ has the *finite best response improvement property* (*FBRP*) if it admits no infinite best response improvement path, and every maximal (i.e., allowing no extension further) best response improvement path ends at a Nash equilibrium. Γ has the *weak FBRP* if, for every $x_N^0 \in X_N$, there is a finite best response improvement path $x_N^0, \ldots, x_N^m$ such that x_N^m is a Nash equilibrium. The (*weak*) *finite best response quasi-improvement property*, (weak) F[BR]P, is defined similarly.

Remark. The second condition in the definition of the FBRP is superfluous when the preferences are defined by utility functions and the best responses exist everywhere. In the general case, however, we could not say that the weak FBRP is weaker than the FBRP without the requirement (see Example A.1 below).

The notion of a *restricted FBRP*, a property intermediate between FBRP and weak FBRP, was defined in Kukushkin (2004, Section 6, p. 103). Here we employ a similar version of F[BR]P. Let, for every $i \in N$ and $x_{-i} \in X_{-i}$, a nonempty subset $\mathcal{R}_i^*(x_{-i}) \subseteq \mathcal{R}_i(x_{-i})$ be given ("admissible best responses"). We define the corresponding *admissible best response quasi-improvement* relation by:

$$y_N \trianglerighteq_i^{[\mathrm{BR}]^*} x_N \rightleftharpoons [y_{-i} = x_{-i} \;\&\; x_i \notin \mathcal{R}_i(x_{-i}) \;\&\; y_i \in \mathcal{R}_i^*(x_{-i})]; \tag{15a}$$

$$y_N \triangleright^{[\mathrm{BR}]^*} x_N \rightleftharpoons \exists i \in N\, [y_N \trianglerighteq_i^{[\mathrm{BR}]^*} x_N]. \tag{15b}$$

An *admissible best response quasi-improvement path* is a (finite or infinite) sequence of strategy profiles $\langle x_N^k \rangle_{k \in \mathbb{N}}$ such that $x_N^{k+1} \triangleright^{[\mathrm{BR}]^*} x_N^k$ whenever $k \geq 0$ and x_N^{k+1} is defined.

A game Γ has a *restricted finite best response quasi-improvement property* (restricted F[BR]P) if there is a collection of admissible best response correspondences $\mathcal{R}_i^*$ such that Γ admits no infinite admissible best response quasi-improvement path. (Since the condition $\mathcal{R}_i^*(x_{-i}) \neq \emptyset$ was included in the definition, a restricted F[BR]P implies the weak F[BR]P.) As noted above, whether the "quasi-improvement-related" dynamic properties deserve much interest by themselves depends on the degree of rationality of the preferences. Nonetheless, even the weakest of those properties, the weak F[BR]P, implies the existence of a Nash equilibrium in any case.

5 Strategic complements

In all the theorems to follow, we consider games where strategy sets X_i are posets, actually, chains. Then $X_N = \prod_{i \in N} X_i$ and all $X_{-i} = \prod_{j \neq i} X_j$ are posets too with the Cartesian product of the orders on components. When applied to the preferences of the players in a strategic game, the single crossing conditions (6) look as follows:

$$\forall i \in N\, \forall x_i, y_i \in X_i\, \forall x_{-i}, y_{-i} \in X_{-i}\, \big[[y_{-i} \geq x_{-i} \;\&\; y_i > x_i \;\&\; y_i \succ_i^{x_{-i}} x_i] \Rightarrow y_i \succ_i^{y_{-i}} x_i\big]; \tag{16a}$$

$$\forall i \in N\, \forall x_i, y_i \in X_i\, \forall x_{-i}, y_{-i} \in X_{-i}\, \big[[y_{-i} \geq x_{-i} \;\&\; y_i < x_i \;\&\; y_i \succ_i^{y_{-i}} x_i] \Rightarrow y_i \succ_i^{x_{-i}} x_i\big]. \tag{16b}$$

Theorem 5.1. *Let each X_i in a game Γ be a chain containing its maximum and minimum. Let the parametric family of preference relations of each player satisfy both conditions (16) and every $\succ_i^{x_{-i}}$ be strongly acyclic and transitive. Then Γ has the weak FBRP. In other words, Nash equilibria exist and can be reached from any strategy profile after a finite number of best response improvements (13).*

Proof. We define

$$X^{\uparrow} := \{x_N \in X_N \mid \forall i \in N\, \forall y_N \in X_N\, [y_N \triangleright_i^{\mathrm{BR}} x_N \Rightarrow y_i > x_i]\}. \tag{17}$$

Lemma 5.1.1. *If $x_N \in X^{\uparrow}$ and $y_N \triangleright^{\mathrm{BR}} x_N$, then $y_N \in X^{\uparrow}$ too.*

Proof of Lemma 5.1.1. Let $y_N \triangleright_i^{\mathrm{BR}} x_N$; then $y_i > x_i$ since $x_N \in X^{\uparrow}$. Suppose, to the contrary, that there are $z_N \in X_N$ and $j \in N$ such that $z_N \triangleright_j^{\mathrm{BR}} y_N$ and $y_j > z_j$. Since $y_i \in \mathcal{R}_i(y_{-i})$, we have $j \neq i$, hence $x_j = y_j > z_j$ and $z_{-j} = y_{-j} > x_{-j}$. Therefore, $z_j \succ_j^{x_{-j}} x_j$ by (6b). By Proposition 3.1, applied to $y_{-j} > x_{-j}$ and $z_j \in \mathcal{R}_j(y_{-j})$, we obtain that either $z_j \in \mathcal{R}_j(x_{-j})$ or there is $z'_j \in \mathcal{R}_j(x_{-j})$ such that $z'_j \succ_j^{x_{-j}} z_j$ and $z'_j < z_j$. In the first case, we immediately have a contradiction with the assumption $x_N \in X^{\uparrow}$; in the second case, we invoke the transitivity of $\succ_j^{x_{-j}}$ first. □

If $x_N^0 \in X^{\uparrow}$, but is not an equilibrium, we pick an arbitrary $x_N^1 \in X_N$ such that $x_N^1 \triangleright^{\mathrm{BR}} x_N^0$; then $x_N^1 \in X^{\uparrow}$ by Lemma 5.1.1. Iterating this operation, we obtain a best response improvement path $\langle x_N^k \rangle_k$ such that $x_N^k \in X^{\uparrow}$ whenever x_N^k is defined. Besides, $x_i^{k+1} > x_i^k$ whenever $x_N^{k+1} \triangleright_i^{\mathrm{BR}} x_N^k$; by (6a), we have $x_i^{k+1} \succ_i^{\max X_{-i}} x_i^k$ for all such k. If the path is infinite, then we will have an infinite number of improvements for, at least, one i (actually, two), contradicting the assumed strong acyclicity. Therefore, it must stop at some stage, and that is only possible at an equilibrium.

If $x_N^0 \notin X^{\uparrow}$, we pick $i \in N$ and $x_N^1 \in X_N$ such that $x_N^1 \triangleright_i^{\mathrm{BR}} x_N^0$ and $x_i^1 < x_i^0$; if $x_N^1 \notin X^{\uparrow}$, we behave similarly. Iterating this operation as long as $x_N^k \notin X^{\uparrow}$, we obtain a best response improvement path $\langle x_N^k \rangle_k$ such that $x_i^{k+1} < x_i^k$ whenever $x_N^{k+1} \triangleright_i^{\mathrm{BR}} x_N^k$. The path cannot be infinite for the same (or rather dual) reason as in the previous paragraph. Once $x_N^k \in X^{\uparrow}$, we already know that an infinite best response improvement path is impossible. □

Theorem 5.2. *If a two-person game Γ satisfies all assumptions of Theorem 5.1, then it has the FBRP. In other words,* every *best response improvement path in Γ reaches a Nash equilibrium after a finite number of steps.*

Proof. Without restricting generality, we may assume $N = \{1, 2\}$. Suppose to the contrary that $\langle x_N^k \rangle_{k \in \mathbb{N}}$ is an infinite best response improvement path. Since we could start the path anyplace, we may assume that, for all $k \in \mathbb{N}$,

$$x_1^{2k} \notin \mathcal{R}_1(x_2^{2k}) \ni x_1^{2k+1} = x_1^{2k+2}; \quad \mathcal{R}_2(x_1^{2k}) \ni x_2^{2k} = x_2^{2k+1} \notin \mathcal{R}_2(x_1^{2k+1}).$$

Again without restricting generality, we may assume $x_1^1 > x_1^0$. By Proposition 3.1 (second statement), applied to x_1^0 as s, x_1^1 as s', x_2^0 as x, and x_2^2 as x', we obtain that $x_2^2 > x_2^0$. A straightforward inductive argument shows that $x_2^{2k+2} > x_2^{2k+1}$ and $x_1^{2k+1} > x_1^{2k}$ for all $k \in \mathbb{N}$. Now the relation $x_2^{2k+2} \succ_2^{x_1^{2k+1}} x_2^{2k}$ and condition (6a) imply $x_2^{2k+2} \succ_2^{\max X_1} x_2^{2k}$ for all $k \in \mathbb{N}$, which fact contradicts the strong acyclicity of $\succ_2^{\max X_1}$.

By Proposition 2.1, if a best response improvement path cannot be extended beyond x_N^m, that profile must be a Nash equilibrium. □

Remark. An anonymous referee raised a question of how the difference between $n > 2$ and $n = 2$ could be explained. As was shown in the proof of Theorem 5.1, no infinite best response improvement path could be started from $X^{\uparrow}$, or, dually, from $X^{\downarrow}$. When $n > 2$, a best response improvement path can avoid both $X^{\uparrow}$ and $X^{\downarrow}$ indefinitely (Kukushkin et al. 2005, Example 4); when $n = 2$, it must hit one or the other after, at most, two steps.

Interestingly, there are no restrictions on the chains X_i apart from the existence of their maxima and minima; those assumptions, however, are essential.

Example 5.3. Let $N := \{1, 2\}$, $X_1 := X_2 :=]0, 1]$ (with the natural order); let preferences of the players be defined by (4) with utility functions $u_1(x_1, x_2) := \min\{2x_1 - x_2, (x_2 - 2x_1)/x_2\}$ and $u_2(x_1, x_2) := \min\{2x_2 - x_1, (x_1 - 2x_2)/x_1\}$, and $0 < \varepsilon < 1$. All assumptions of Theorem 5.2 are satisfied except for the existence of $\min X_i$; single crossing conditions (16) hold because both utility functions are supermodular. There is no (ε-)Nash equilibrium: $x_2 \le (1 + \varepsilon)x_1/2$ whenever $x_2 \in \mathcal{R}_2(x_1)$, while $x_1 \le (1 + \varepsilon)x_2/2$ whenever $x_1 \in \mathcal{R}_1(x_2)$; therefore, there should hold $x_1 \le (1 + \varepsilon)^2 x_1/4 < x_1$ at any equilibrium.

To obtain a comparative statics result for Nash equilibria, we adapt conditions (6) to perturbations of each player's preferences in a strategic game (perturbed preferences are denoted by $\ggcurly_i^{x_{-i}}$):

$$\forall i \in N \, \forall x_i, y_i \in X_i \, \forall x_{-i} \in X_{-i} \, \big[[y_i \succ_i^{x_{-i}} x_i \;\&\; y_i > x_i] \Rightarrow y_i \ggcurly_i^{x_{-i}} x_i\big]; \tag{18a}$$

$$\forall i \in N \, \forall x_i, y_i \in X_i \, \forall x_{-i} \in X_{-i} \, \big[[y_i \ggcurly_i^{x_{-i}} x_i \;\&\; y_i < x_i] \Rightarrow y_i \succ_i^{x_{-i}} x_i\big]. \tag{18b}$$

Theorem 5.4. *Let Γ' and Γ be two games with identical sets of players and strategies, and let both satisfy all assumptions of Theorem 5.1. Let the preference relations $\succ_i^{x_{-i}}$ in Γ and $\ggcurly_i^{x_{-i}}$ in Γ' satisfy conditions (18). Let $\mathrm{E}(\Gamma)$ and $\mathrm{E}(\Gamma')$ denote the sets of Nash equilibria in Γ and Γ', respectively. Then $\mathrm{E}(\Gamma') \ge \mathrm{E}(\Gamma)$ in the sense of (8). In other words, for every Nash equilibrium x_N in Γ there is a Nash equilibrium x'_N in Γ' such that $x'_N \ge x_N$, and for every Nash equilibrium x'_N in Γ' there is a Nash equilibrium x_N in Γ such that $x'_N \ge x_N$.*

Proof. Let $x_N \in \mathrm{E}(\Gamma)$. If, simultaneously, $x_N \in \mathrm{E}(\Gamma')$, then we are home immediately. Otherwise, $x_N \in X^{\uparrow}$ in Γ': Supposing that $y_i \ggcurly_i^{x_{-i}} x_i$ and $y_i < x_i$, we immediately obtain $y_i \succ_i^{x_{-i}} x_i$ by (18b). Exactly as in the proof of Theorem 5.1, we obtain an increasing best response improvement path reaching a Nash equilibrium x'_N after a finite number of steps. Clearly, $x'_N \ge x_N$.

If $x'_N \in \mathrm{E}(\Gamma') \setminus \mathrm{E}(\Gamma)$, then $x' \in X^{\downarrow} := \{x_N \in X_N \mid \forall i \in N \, \forall y_N \in X_N \, [y_N \overset{\mathrm{BR}}{\triangleright_i} x_N \Rightarrow y_i < x_i]\}$ in Γ dually to the preceding paragraph. Now we invoke the dual to Lemma 5.1.1 and obtain $x_N \in \mathrm{E}(\Gamma)$ such that $x'_N \ge x_N$ in the same way. □

A similar result for games with complete preferences was obtained in Lippman et al. (1987) by similar argument. Since strong acyclicity was not assumed there, those authors had to employ transfinite recursion.

6 Aggregative games

An *aggregative game* is a strategic game where each X_i is a subset of $\mathbb{R}$ and there are mappings $\sigma_i \colon X_{-i} \to \mathbb{R}$ (*aggregation rules*) such that every preference relation $\succ_i^{x_{-i}}$ only depends on $\sigma_i(x_{-i})$. For each $i \in N$, we denote $S_i := \sigma_i(X_{-i}) \subset \mathbb{R}$ in this case, and use notations $\succ_i^{s_i}$ instead of $\succ_i^{x_{-i}}$ and $\mathcal{R}_i(s_i) := M(X_i, \succ^{s_i})$.

Throughout this section, we assume that the single crossing conditions (6) hold for each $\succ_i^{s_i}$. If the preferences are defined by ε-optimization (4) with parameter s_i in the utility function, these conditions follow from the same increasing differences condition (7). Since both X_i and S_i are chains, that condition is equivalent to the *supermodularity* of u_i (as a function on the lattice $X_i \times S_i$).

If each σ_i is increasing in each x_j, then conditions (6) imply (16), hence Theorem 5.1 is applicable. Strictly speaking, aggregation adds something even in this case: Theorem 5.1 does not imply Theorem 6.1 or Theorem 6.2 if $n > 2$. It is much more interesting, however, that aggregation and single crossing conditions (6) can ensure the existence of a Nash equilibrium and nice dynamic properties when not all σ_i are increasing in all x_j, i.e., in the absence of strategic complementarity.

Novshek (1985) was the first to notice that decreasing best responses are conducive to the existence of a Nash equilibrium in a game with additive aggregation. Kukushkin (2004) showed the acyclicity of the best responses under the same conditions. Similar results have been obtained for a wider scope of aggregation rules in Dubey et al. (2006), Kukushkin (2005), and Jensen (2010). (Naturally, the preferences were complete in all those papers.) In each case, the existence of a monotone selection from each best response correspondence was essential.

Theorem 3.8 allows us to apply the same technique to aggregative games where preferences are described by interval orders. Without aiming at the highest generality, we just present a couple of appropriate collections of aggregation rules. The first case, obviously important from the viewpoint of possible applications, is linear aggregation of a rather general kind.

Theorem 6.1. *Let Γ be an aggregative game where each strategy set contains its maximum and minimum, $\sigma_i(x_{-i}) = \sum_{j \neq i} a_{ij} x_j$ with $a_{ij} = a_{ji} \in \mathbb{R}$ whenever $j \neq i$, and every $\succ_i^{s_i}$ is a strongly acyclic interval order. Let the parametric family of preference relations of each player satisfy both conditions (6). Then Γ has a restricted F[BR]P.*

Remark. When $a_{ij} \geq 0$ for all $j \neq i$, we have a game with strategic complements; when $a_{ij} \leq 0$ for all $j \neq i$, a game with strategic substitutes. A more general situation with coefficients of both signs is also allowed.

Sketch of a proof. Applying Theorem 3.8, we obtain an increasing selection $r_i \colon S_i \to X_i$ from the best responses and define an *admissible best response quasi-improvement* relation by (15) with

$$\mathcal{R}_i^*(x_{-i}) := \{r_i(\sigma_i(x_{-i}))\}.$$

To show the impossibility of an infinite admissible best response quasi-improvement path, some auxiliary constructions are needed, which generally follow Dubey et al. (2006), who, in their turn, used a trick developed by Huang (2002) for different purposes. The finiteness of each $r_i(S_i)$ simplifies something; the absence of upper hemicontinuity demands more subtlety, and here we follow Kukushkin (2005).

A detailed argument is given in Section A.2. □

If the preferences of the players are described just by utility functions, then $\triangleright^{\mathrm{BR}}$ and $\triangleright^{[\mathrm{BR}]}$ are equivalent, hence F[BR]P and FBRP become the same thing. However, the FBRP cannot be asserted in Theorem 6.1, even for a finite game with such nice preferences, see Example A.4.

Corollary. *Let Γ be a strategic game with a strategy set $X_i \subset \mathbb{R}$ for each $i \in N$ and utility functions of the form $u_i(x_N) = U_i(x_i, \sum_{j \neq i} a_{ij} x_j)$, where $a_{ij} = a_{ji} \in \mathbb{R}$ whenever $j \neq i$. Let each X_i contain its* *maximum and minimum, each $U_i(\cdot, s_i)$ be bounded above, and the increasing differences condition (7) be satisfied by each U_i. Then Γ possesses an ε-Nash equilibrium for every $\varepsilon > 0$.*

Another example is the maximum/minimum aggregation. For economics applications, it may look exotic although it allows reasonable interpretations (Hirshleifer 1983; Boncinelli and Pin 2012). It is also quite interesting from a purely technical viewpoint: Strong acyclicity, assumed here, allows us virtually to derive Theorems 6.2 and 6.3 from Theorem 6.1. Similar results in a more general setting are also valid, but require absolutely different techniques (Kukushkin, 2003, Theorems 7 and 8). There is no direct analog of coefficients a_{ij} in the theorems to follow; we only allow each player to be affected by the maximal/minimal choice of her "neighbors," $j \in I(i)$.

Theorem 6.2. *Let Γ be an aggregative game where each strategy set contains its maximum and minimum, $\sigma_i(x_{-i}) = \max_{j\in I(i)} x_j$ with $j \in I(i) \iff i \in I(j)$, and every $\succ_i^{s_i}$ is a strongly acyclic interval order. Let the parametric family of preference relations of each player satisfy both conditions (6). Then Γ has a restricted F[BR]P.*

Theorem 6.3. *Let Γ be an aggregative game where each strategy set contains its maximum and minimum, $\sigma_i(x_{-i}) = -\max_{j\in I(i)} x_j$ with $j \in I(i) \iff i \in I(j)$, and every $\succ_i^{s_i}$ is a strongly acyclic interval order. Let the parametric family of preference relations of each player satisfy both conditions (6). Then Γ has a restricted F[BR]P.*

Both proofs, relying heavily on that of Theorem 6.1, are deferred to Section A.3.

Remark. Theorem 6.2 is about a game with strategic complements; Theorem 6.3, about a game with strategic substitutes. An intermediate case where the signs of some x_j are reversed before taking the maximum remains uninvestigated (there is no ground to expect a nice result here, but no explicit counterexample either).

Corollary. *Let Γ be a strategic game with a strategy set $X_i \subset \mathbb{R}$ for each $i \in N$ and utility functions of the form $u_i(x_N) = U_i(x_i, -\max_{j\in I(i)} x_j)$, where $j \in I(i) \iff i \in I(j)$. Let each X_i contain its maximum and minimum, each $U_i(\cdot, s_i)$ be bounded above, and the increasing differences condition (7) be satisfied by each U_i. Then Γ possesses an ε-Nash equilibrium for every $\varepsilon > 0$.*

Proposition 6.4. *Let Γ be an aggregative game where each strategy set contains its maximum and minimum, $\sigma_i(x_{-i}) = \min_{j\in I(i)} x_j$ with $j \in I(i) \iff i \in I(j)$, and every $\succ_i^{s_i}$ is a strongly acyclic interval order. Let the parametric family of preference relations of each player satisfy both conditions (6). Then Γ has a restricted F[BR]P.*

Proposition 6.5. *Let Γ be an aggregative game where each strategy set contains its maximum and minimum, $\sigma_i(x_{-i}) = -\min_{j\in I(i)} x_j$ with $j \in I(i) \iff i \in I(j)$, and every $\succ_i^{s_i}$ is a strongly acyclic interval order. Let the parametric family of preference relations of each player satisfy both conditions (6). Then Γ has a restricted F[BR]P.*

If a game satisfies the assumptions of Proposition 6.4 or 6.5, then it satisfies the assumptions of Theorem 6.2 or 6.3 after the order on each strategy set is reversed (each X_i is replaced with $-X_i$ and min with max).

Naturally, Proposition 6.5 admits a corollary virtually identical to that of Theorem 6.3. Similar statements related to Theorem 6.2 and Proposition 6.4 immediately follow from Theorem 5.1.

The broadest class of aggregation rules for which the Huang-Dubey-Haimanko-Zapechelnyuk trick is known to work is in Jensen (2010). Most likely, an analog of Theorem 6.1 for those rules is valid as well, but the very formulation would require plenty of additional notations.

7 Concluding remarks

7.1. The preference relation described by (4) is a *semiorder* rather than just an interval order. In the general theory of ordered sets, semiorders play a role not less important than interval orders. Here, however, we obtained no result valid for semiorders, but not for interval orders in general.

7.2. The property asserted in Proposition 2.1 was called "the NM-property" in Kukushkin (2008). A more exact term would be "the external stability, in the sense of von Neumann and Morgenstern, of the set of maximizers." In a similar way, the property asserted in Proposition 2.2 could be called "the strong NM-property." It is easy to see that Propositions 3.1–3.3 and Theorem 5.2 remain valid if transitivity is replaced with the NM-property on X (or X_i). Similarly, Theorems 3.8 and 6.1–6.3 remain valid if, instead of being an interval order, every preference relation is assumed to have the strong NM-property on X (or X_i). (Naturally, strong acyclicity is still needed in all cases.) As to Theorem 5.1, transitivity was used in the proof there, and it remains unclear whether the NM-property would be sufficient. (The strong NM-property *is* enough, but that property is not weaker than transitivity.)

7.3. Strictly speaking, the usual monotone comparative statics results under strategic complementarity in "type A" problems (in terms of Quah, 2007) do not need a representation (2), even if $\mathbb{R}$ is replaced with an arbitrary chain. Let us say that a binary relation $\succ$ has the *revealed preference property* on a set X if

$$\forall x, y \in X \, [x \notin M(X, \succ) \ni y \Rightarrow y \succ x]. \tag{19}$$

It is quite straightforward to check that the part of Theorem 4 of Milgrom and Shannon (1994) concerning "type A" problems remains valid for preferences described by a binary relation with the revealed preference property on X (provided the single crossing and quasisupermodularity conditions are translated into the language of binary relations). The same statement is valid with respect to the sufficiency parts of the main results of Kukushkin (2013a) (where various versions of the single crossing and quasisupermodularity conditions *are* expressed in the language of binary relations).

On the other hand, this generalization may not be of more than some technical interest: there seems to be no natural restriction on preferences that is weaker than (2), but ensures (19). Moreover, if $\succ$ has the "revealed preference" property on every subset $Y \subset X$ and $M(Y, \succ) \neq \emptyset$ for every finite subset $Y \subset X$, then $\succ$ must admit a representation (2). (For that reason, (19) cannot play any role in "type B" problems.) Similarly, if we demand the (strong) NM-property to hold on every nonempty subset, then the converse to Proposition 2.1 (Proposition 2.2) becomes valid.

7.4. If S in Theorem 3.8 does not contain either minimum or maximum, then an increasing selection still exists, but $r(S)$ need not be finite; hence that selection cannot be used in the proof of Theorem 6.1. Actually, Example 5.3 shows that an equilibrium may fail to exist in this case. (Every two-person game with scalar strategies is aggregative by our definition.)

7.5. It remains unclear whether the assumption that both X and S are chains can be dropped or weakened in Theorem 3.8. From the game-theoretic viewpoint, however, the question does not seem pressing. The existence of an equilibrium in a game with strategic complementarities does not require increasing selections (Theorem 5.1). If the best responses are, say, decreasing, then, indeed, all existence results in the literature need increasing selections, but they also need the assumption that each player is only affected by a scalar aggregate of the partners/rivals' choices. In principle, Theorem 1 of Jensen (2010) *could* be applicable to a game with non-scalar strategies, so an extension of our Theorem 3.8 to non-scalar X *could* be useful; however, no interesting example of such a game has emerged so far.

On the other hand, an extension of Theorem 3.8 to non-scalar S would allow us to add monotone comparative statics statements to Theorems 6.1–6.3. Such an extension does not look implausible (as long as X remains scalar), but there is no clear-cut theorem as yet. On a still other hand, though, monotone comparative statics of equilibria in games with strategic complementarities is established in Theorem 5.4 without any need for monotone selections, whereas a similar study of perturbation of the set of Nash equilibria in more general aggregative games is a much more tricky business, which requires even stronger restrictions, see Acemoglu and Jensen (2013).

7.6. If every $\mathcal{R}_i(x_{-i})$ is a singleton, then a restricted FBRP (F[BR]P) is equivalent to the FBRP (F[BR]P). If, additionally, every $\succ_i^{x_{-i}}$ has the NM-property, then the FBRP and F[BR]P are equivalent. If, additionally, $\#N = 2$, then the FBRP is equivalent to the weak FBRP.

7.7. It is instructive to compare our Theorems 5.1 and 5.2 with Theorems 2 and 3, respectively, from Kukushkin et al. (2005). The assertive parts are the same, whereas the assumptions are incomparable: we do not require X_i's to be finite, nor a representation (2) of the preferences, here; on the other hand, non-scalar strategies were allowed there (to a certain extent).

7.8. A closer look at the proofs of Theorems 5.1 and 5.2 shows that both remain valid if the maxima and minima exist in all X_i but one.

7.9. If the objective and the findings of this paper are interpreted very narrowly as establishing the existence of an ε-Nash equilibrium in discontinuous supermodular games, then one might plausibly argue that an alternative approach may be more convenient. Let there be a supermodular game where strategy sets are complete sublattices of $\mathbb{R}^m$, but utility functions are not even upper semicontinuous in own strategy, so there is no Nash equilibrium. Then we can replace each utility function with its "upper semicontinuous closure," i.e., define

$$\bar{u}_i(x_i, x_{-i}) := \sup_{x_i^k \to x_i} \limsup_k u_i(x_i^k, x_{-i}),$$

take a Nash equilibrium of the modified game, and look for an ε-Nash equilibrium of the original game somewhere in its vicinity. Aggregative games as defined in Section 6 could be treated in the same way.

That approach can, indeed, work smoothly when all discontinuities are rather simple, as is often the case in typical economics models. Suppose we consider a Cournot oligopoly where the inverse demand function satisfies all assumptions of Theorem 3 from Novshek (1985), but the cost functions are not lower semicontinuous: when the output reaches certain threshold level(s), the cost jumps up. Then a Cournot equilibrium may fail to exist, but an ε-equilibrium can be found in exactly the way described above. Consider modified cost functions which jump up when the output *exceeds* the same threshold level(s); the modified model possesses an equilibrium. If the output of every firm at an equilibrium in the modified model is not at a threshold level, then it is an equilibrium in the original model as well. Otherwise, the relevant firm(s) should produce a little bit less, and it will be an ε-equilibrium.

On the other hand, to obtain a general existence result in this way may not be easy. As Example A.3 below shows, there may be no ε-Nash equilibrium of a discontinuous game close to an arbitrary equilibrium in the modified game. One may also be interested in more than the mere existence of an ε-Nash equilibrium. It seems impossible to obtain Theorems 5.1 or 5.2 without considering ε-improvements explicitly; and those theorems hold for a wider class of preferences.

7.10. As was noted after Theorem 5.4, a similar result for games with complete preferences was established quite some time ago. Moreover, under the usual topological assumptions, there exist both the greatest and the least Nash equilibrium for every parameter, which provide increasing selections from the set of Nash equilibria. To the best of my knowledge, however, nothing is known about the monotonicity of the set of Nash equilibria in a parameter in the sense of (9), or even (10), as well as about the existence of increasing selections from the same set without those continuity assumptions, even for preferences admitting a representation (2).

7.11. Theorems 6.2 and 6.3 can be extended, with only minor changes in the proofs, to "lexicographic aggregation" such as the Leximax or Leximin orderings. Aggregation rules σ_i should then be mappings from X_{-i} to chains "longer" than $\mathbb{R}$.

7.12. A nice coincidence is worth mentioning. As noted in Section 4, "quasi-improvement-related" dynamics admit a reasonable interpretation when preference relations are interval orders, but, generally, not otherwise. And indeed, all results about such dynamics assume preferences described by interval orders.

7.13. The fact that we had to assume each strategy set in each theorem to be a chain is extremely irritating. Unfortunately, I have no idea at the moment whether and how the assumption could be dispensed with. On the other hand, a conjecture that complete preferences, or at least, (19), are indispensable when dealing with multi-dimensional strategies seems premature: there is no counterexample, nor even a hint wherefrom such an example could emerge.

Acknowledgments

Financial support from the Russian Foundation for Basic Research (projects 11-07-00162 and 14-07-00075) is acknowledged. I have benefitted from fruitful contacts with Vladimir Danilov, Francisco Marhuenda, Hervé Moulin, John Quah, Kevin Reffett, Alexei Savvateev, and Satoru Takahashi. Helpful comment and suggestions from an anonymous associate editor and three referees are also appreciated.

A Appendices

A.1 Proof of Theorem 3.8

We call a subset $S' \subseteq S$ an *interval* if $s \in S'$ whenever $s' < s < s''$ and $s', s'' \in S'$. The intersection of any number of intervals is an interval too. By Proposition 3.7, the set $\{s \in S \mid x \in \mathcal{R}(s)\}$, for every

$x \in X$, is an interval (perhaps, empty one).

The key role is played by the following recursive definition of sequences $x^k \in X$, $s^k \in S$, and $S^k \subseteq S$ $(k \in \mathbb{N})$ such that:

$$s^k \in S^k; \tag{20a}$$

$$S^k \text{ is an interval;} \tag{20b}$$

$$\forall s \in S^k \left[x^k \in \mathcal{R}(s)\right]; \tag{20c}$$

$$\forall m < k \left[S^k \cap S^m = \emptyset\right]; \tag{20d}$$

$$\forall m < k \left[[s^k < s^m \Rightarrow x^k < x^m] \;\&\; [s^k > s^m \Rightarrow x^k > x^m]\right]; \tag{20e}$$

$$\forall m < k \left[x^k \succcurlyeq^{s^k} x^m \;\text{ or }\; x^m \in \mathcal{R}(s^k)\right]; \tag{20f}$$

$$\forall s \in S \left[[x^k \in \mathcal{R}(s) \;\&\; s \notin S^k] \Rightarrow \exists m < k \left(s \in S^m \;\text{ or }\; s < s^m < s^k \;\text{ or }\; s^k < s^m < s\right)\right]. \tag{20g}$$

We start with an arbitrary $s^0 \in S$, pick $x^0 \in \mathcal{R}(s^0)$, and set $S^0 := \{s \in S \mid x^0 \in \mathcal{R}(s)\}$. Now (20a), (20c), and (20g) for $k = 0$ immediately follow from the definitions; (20b), from Proposition 3.7; (20d), (20e), and (20f) hold vacuously.

Let $k \in \mathbb{N} \setminus \{0\}$, and let x^m, s^m, S^m satisfying (20) have been defined for all $m < k$. We define $\bar{S}^k := \bigcup_{m<k} S^m$. For every $s \in \bar{S}^k$, there is a unique, by (20d), $\mu(s) < k$ such that $s \in S^{\mu(s)}$. By (20c), $r(s) := x^{\mu(s)}$ is a selection from $\mathcal{R}$ on $\bar{S}^k$. The conditions (20b) and (20e) imply that r is increasing. If $\bar{S}^k = S$, then we already have an increasing selection, so we stop the process.

Otherwise, we pick $s^k \in S \setminus \bar{S}^k$ arbitrarily and denote $K^- := \{m < k \mid s^m < s^k\}$, $K^+ := \{m < k \mid s^m > s^k\}$ $K^* := \{m < k \mid x^m \notin \mathcal{R}(s^k)\}$, $m^- := \operatorname{argmax}_{m \in K^-} s^m$, $m^+ := \operatorname{argmin}_{m \in K^+} s^m$, and $I := \{s \in S \mid s^{m^-} < s < s^{m^+}\}$. If one of $K^\pm$ is empty (both cannot be), the respective $m^\pm$ is left undefined, in which case $I := \{s \in S \mid s^{m^-} < s\}$ or $I := \{s \in S \mid s < s^{m^+}\}$.

By Proposition 2.2 applied to $\succcurlyeq^{s^k}$, we can pick $x^k \in \mathcal{R}(s^k)$ such that $x^k \succcurlyeq^{s^k} x^m$ for each $m \in K^*$, hence (20f) holds. Finally, we define $S^k := \{s \in S \setminus \bar{S}^k \mid x^k \in \mathcal{R}(s)\} \cap I$. Now the conditions (20a), (20c), and (20d) immediately follow from the definitions; (20b) and (20g), from Proposition 3.7.

Checking (20e) needs a bit more effort. If we assume that $x^{m^-} \in \mathcal{R}(s^k)$, then the condition (20g) for m^- and s^k implies the existence of $m < m^-$ such that $s^{m^-} < s^m < s^k$, contradicting the definition of m^-; therefore, $x^k \succcurlyeq^{s^k} x^{m^-}$ by (20f), hence $x^k > x^{m^-}$ by Proposition 3.1. Therefore, $x^k > x^m$ for all $m \in K^-$. A dual argument shows that $x^k < x^{m^+} \leq x^m$ for all $m \in K^+$. Thus, (20e) holds.

To summarize, either we obtain an increasing selection on some step, or our sequences are defined [and satisfy (20)] for all $k \in \mathbb{N}$.

Lemma A.1.1. *If conditions (20) hold for all $k \in \mathbb{N}$, then there exists an increasing sequence $\langle k_h \rangle_{h \in \mathbb{N}}$ such that s^{k_h} is either monotone increasing or monotone decreasing in h, and $x^{k_{h+1}} \succcurlyeq^{s^{k_{h+1}}} x^{k_h}$ for each $h \in \mathbb{N}$.*

Proof. We denote $\mathbb{N}^\downarrow$, respectively, $\mathbb{N}^\uparrow$, the set of $k \in \mathbb{N}$ such that $s^m < s^k$, or $s^m > s^k$, holds for an infinite number of $m \in \mathbb{N}$. Clearly, $\mathbb{N} = \mathbb{N}^\downarrow \cup \mathbb{N}^\uparrow$; without restricting generality, $\mathbb{N}^\downarrow \neq \emptyset$. We consider two alternatives.

Let there exist $\min\{s^k \mid k \in \mathbb{N}^\downarrow\} =: s^*$; then the set $\{m \in \mathbb{N} \mid s^m < s^k\}$ is finite for every $s^k < s^*$, hence the set $\{m \in \mathbb{N} \mid s^k < s^m < s^*\}$ is infinite. We define $k_0 := \min\{k \in \mathbb{N} \mid s^k < s^*\}$, and then recursively define k_{h+1} as the least $k \in \mathbb{N}$ for which $s^{k_h} < s^k < s^*$. The minimality of k_h ensures that $k_{h+1} > k_h$. Whenever $s^{k_h} < s^m < s^{k_{h+1}}$, we have $m > k_{h+1}$ by the same minimality; therefore, $x^{k_h} \notin \mathcal{R}(s^{k_{h+1}})$ by (20g), hence $x^{k_{h+1}} \succ^{s^{k_{h+1}}} x^{k_h}$ by (20f).

Let $\min\{s^k \mid k \in \mathbb{N}^{\downarrow}\}$ not exist; then the set $\{m \in \mathbb{N}^{\downarrow} \mid s^m < s^k\}$ is nonempty (actually, infinite) for every $k \in \mathbb{N}^{\downarrow}$. We set $k_0 := \min \mathbb{N}^{\downarrow}$, and then recursively define k_{h+1} as the least $k \in \mathbb{N}^{\downarrow}$ for which $s^k < s^{k_h}$. The minimality of k_h again ensures that $k_{h+1} > k_h$. Whenever $s^{k_{h+1}} < s^m < s^{k_h}$, we have $m \in \mathbb{N}^{\downarrow}$, hence $m > k_{h+1}$; therefore, $x^{k_h} \notin \mathcal{R}(s^{k_{h+1}})$ by (20g), hence $x^{k_{h+1}} \succ^{s^{k_{h+1}}} x^{k_h}$ by (20f). □

The final step of the proof consists in showing that the existence of a sequence described in Lemma A.1.1 contradicts the strong acyclicity assumption. If s^{k_h} is increasing, the relations $x^{k_{h+1}} \succ^{s^{k_{h+1}}} x^{k_h}$ "translate," by (6a), to $x^{k_{h+1}} \succ^{\max S} x^{k_h}$ for each $h \in \mathbb{N}$. If s^{k_h} is decreasing, we obtain $x^{k_{h+1}} \succ^{\min S} x^{k_h}$ for each $h \in \mathbb{N}$ by (6b).

A.2 Proof of Theorem 6.1

Since each X_i contains its maximum and minimum, the same holds for each S_i. Applying Theorem 3.8, we obtain an increasing selection $r_i \colon S_i \to X_i$ from the best responses such that $r_i(S_i)$ is finite for each $i \in N$. Now we define an *admissible best response quasi-improvement* relation by (15) with

$$\mathcal{R}_i^*(x_{-i}) := \{r_i(\sigma_i(x_{-i}))\}.$$

Before showing the impossibility of an infinite admissible best response quasi-improvement path, we develop quite a number of auxiliary constructions.

For each $i \in N$, we denote $r_i(S_i) =: \{x_i^1, \ldots, x_i^{m_i}\}$, assuming $x_i^{h+1} > x_i^h$ for all relevant h. For every $x_i \in X_i$, we define $\eta_i(x_i) \in \{0, 1, \ldots, m_i\}$ as the minimal h such that $x_i^{h+1} > x_i$; if no such h exists, we set $\eta_i(x_i) := m_i$. Then we define $s_i^0 := \min S_i$ and $s_i^h := \sup\{s_i \in S_i \mid r_i(s_i) = x_i^h\}$ for $h \in \{1, \ldots, m_i\}$; clearly, $s_i^{m_i} = \max S_i$. Then we denote $\Delta_i^h := s_i^h - s_i^{h-1}\,[\geq 0]$ for $h = 1, \ldots, m_i$. Note that $\Delta_i^h = 0$ is possible for some h (if $r_i(s_i) = x_i^h$ for a unique $s_i = s_i^h = s_i^{h-1}$), in which case $\Delta_i^{h'} > 0$ for (typically, both) adjacent h'.

For every $x_N \in X_N$, we define a set $N^0(x_N) := \{i \in N \mid x_i \in r_i(S_i)\}$ and a function

$$P(x_N) := \sum_{i \in N} \Big[-x_i \cdot s_i^{\eta_i(x_i)} + \sum_{j \in N\colon j \neq i} \frac{1}{2}\, a_{ij} \cdot x_i \cdot x_j + \sum_{h=1}^{\eta_i(x_i)} x_i^h \cdot \Delta_i^h \Big]. \tag{21}$$

For each $i \in N$, we define a binary relation $\triangleright_i$ on $r_i(S_i)$ by setting (for each $h \in \{1, \ldots, m_i - 1\}$ such that $s_i^h \in S_i$) $x_i^{h+1} \triangleright_i x_i^h$ if $r_i(s_i^h) = x_i^{h+1}$, and $x_i^h \triangleright_i x_i^{h+1}$ if $r_i(s_i^h) = x_i^h$. Clearly, $x_i^h \triangleright_i x_i^{h+1}$ and $x_i^h \triangleright_i x_i^{h-1}$ whenever $\Delta_i^h = 0$ (provided $h < m_i$ and $h > 0$, respectively).

Lemma A.2.1. *Let $i \in N$, $z_i, y_i, x_i \in r_i(S_i)$, and $z_i \triangleright_i y_i \triangleright_i x_i$. Then either $z_i > y_i > x_i$ or $z_i < y_i < x_i$.*

Immediately follows from the definitions.

Then we extend $\triangleright_i$ to the whole X_i, setting $y_i \triangleright_i x_i$ whenever $x_i \notin r_i(S_i) \ni y_i$, and define $\mathrel{\triangleright\!\!\triangleright}_i$ as the transitive closure of $\triangleright_i$ on X_i.

Lemma A.2.2. *Each relation $\mathrel{\triangleright\!\!\triangleright}_i$ is irreflexive and transitive.*

Immediately follows from Lemma A.2.1.

Finally, we define a potential:

$$y_N \ggcurly x_N \rightleftharpoons \big[N^0(y_N) \supset N^0(x_N) \text{ or } [N^0(y_N) = N^0(x_N) \,\&\, P(y_N) > P(x_N)] \text{ or } \big(N^0(y_N) = N^0(x_N) \,\&\, P(y_N) = P(x_N) \,\&\, \forall i \in N\, [y_i = x_i \text{ or } y_i \rhd\!\rhd_i x_i] \,\&\, \exists i \in N\, [y_i \rhd\!\rhd_i x_i]\big)\big]. \quad (22)$$

Lemma A.2.3. *The relation $\ggcurly$ is irreflexive and transitive.*

Immediately follows from the definition.

Lemma A.2.4. *If $y_N \rhd_i^{[\mathrm{BR}]^*} x_N$, then $y_N \ggcurly x_N$.*

Proof. By definition, $y_i = r_i(\sigma_i(x_{-i}))$ and $y_{-i} = x_{-i}$, hence $i \in N^0(y_N)$. If $x_i \notin r_i(S_i)$, then we have $N^0(x_N) \subset N^0(y_N)$ since $y_j = x_j$ for all $j \neq i$; therefore, $y_N \ggcurly x_N$ by the first lexicographic component in (22). Otherwise, we have $N^0(x_N) = N^0(y_N)$; let us compare $P(y_N)$ and $P(x_N)$.

Let $y_i = x_i^{h''}$ and $x_i = x_i^{h'}$; we denote $\bar{s}_i := \sigma_i(x_{-i})$. Since $y_i = r_i(\bar{s}_i)$, we have $s_i^{h''-1} \leq \bar{s}_i \leq s_i^{h''}$. Since $\sum_{j \in N:\, j \neq i} a_{ij} \cdot x_j = \bar{s}_i$, we have

$$P(y_N) = y_i \cdot (\bar{s}_i - s_i^{h''}) + \sum_{h=1}^{h''} x_i^h \cdot \Delta_i^h + C(x_{-i}) = y_i \cdot (\bar{s}_i - s_i^{h''-1}) + \sum_{h=1}^{h''-1} x_i^h \cdot \Delta_i^h + C(x_{-i}). \quad (23)$$

Let us consider two alternatives.

1. Let $y_i > x_i$, i.e., $h'' > h'$. Similarly to (23), we have

$$P(x_N) = x_i \cdot (\bar{s}_i - s_i^{h'}) + \sum_{h=1}^{h'} x_i^h \cdot \Delta_i^h + C(x_{-i}) = x_i \cdot (\bar{s}_i - s_i^{h''-1}) + x_i \cdot (s_i^{h''-1} - s_i^{h'}) + \sum_{h=1}^{h'} x_i^h \cdot \Delta_i^h + C(x_{-i}). \quad (24)$$

Note that $C(x_{-i})$ is indeed the same.

Subtracting (24) from (23), we obtain

$$P(y_N) - P(x_N) = (y_i - x_i) \cdot (\bar{s}_i - s_i^{h''-1}) + \sum_{h=h'+1}^{h''-1} (x_i^h - x_i) \cdot \Delta_i^h \geq 0. \quad (25)$$

If $P(y_N) > P(x_N)$, we are home by the second lexicographic component in (22). Let $P(y_N) = P(x_N)$; then both terms, the product and the sum, in (25) must equal zero. Since $y_i > x_i$, we have $\bar{s}_i = s_i^{h''-1}$, hence $y_i = x_i^{h''} \rhd_i x_i^{h''-1}$. If $h'' = h' + 1$ (so the sum is empty), then $y_i \rhd\!\rhd_i x_i$ and hence $y_N \ggcurly x_N$ by the third lexicographic component in (22).

Finally, let us show that the equality $P(y_N) = P(x_N)$ is incompatible with the inequality $h'' > h'+1$. If $h'' > h' + 2$, then the sum in (25) contains at least one strictly positive term. The only remaining possibility is $h'' - 1 = h' + 1$ and $\Delta_i^{h''-1} = 0$, i.e., $s_i^{h''-2} = s_i^{h''-1} = \bar{s}_i$. But then we must have $r_i(\bar{s}_i) = x_i^{h''-1}$, which contradicts $r_i(\bar{s}_i) = y_i = x_i^{h''}$.

2. Let $y_i < x_i$, i.e., $h'' < h'$. We ignore what follows the second equality sign in (23), and replace (24) with

$$P(x_N) = x_i \cdot (\bar{s}_i - s_i^{h'}) + \sum_{h=1}^{h'} x_i^h \cdot \Delta_i^h + C(x_{-i}) = x_i \cdot (\bar{s}_i - s_i^{h''}) + x_i \cdot (s_i^{h''} - s_i^{h'}) + \sum_{h=1}^{h'} x_i^h \cdot \Delta_i^h + C(x_{-i}). \quad (26)$$

Subtracting (26) from (23), we obtain

$$P(y_N) - P(x_N) = (x_i - y_i) \cdot (s_i^{h''} - \bar{s}_i) + \sum_{h=h''+1}^{h'} (x_i - x_i^h) \cdot \Delta_i^h. \quad (27)$$

Again, $P(y_N) \geq P(x_N)$, and an equality is only possible if $\bar{s}_i = s_i^{h''}$ and $h' = h'' + 1$, which means that $y_i \triangleright_i x_i$, hence $y_i \mathrel{\triangleright\!\!\triangleright}_i x_i$ as well. In other words, $y_N \ggcurly x_N$ again. □

Finally, let $\langle x_N^k \rangle_{k=0,1,\ldots}$ be an admissible best response quasi-improvement path, i.e., whenever x_N^{k+1} is defined, there holds $x_N^{k+1} \triangleright_i^{[\mathrm{BR}]^*} x_N^k$ for some (unique) $i \in N$. By Lemma A.2.4, we have $x_N^{k+1} \ggcurly x_N^k$. We set $N^* := \{i \in N \mid \exists k\, [x_i^{k+1} = r_i(x_{-i}^k)]\}$. If $i \in N \setminus N^*$, then x_i^k is the same for all k. Thus, our path moves upwards (in the sense of $\ggcurly$) in a finite set $\prod_{i \in N^*} r_i(S_i)$, hence it cannot be infinite.

A.3 Proof of Theorems 6.2 and 6.3

The two proofs are so similar that we do not have to distinguish almost to the very end.

Exactly as in the proof of Theorem 6.1, we apply Theorem 3.8, obtaining an increasing selection from the best responses $r_i\colon S_i \to X_i$ such that $r_i(S_i)$ is finite for each $i \in N$. Then we again define an *admissible best response quasi-improvement* relation by (15) with $\mathcal{R}_i^*(x_{-i}) := \{r_i(\sigma_i(x_{-i}))\}$.

Denoting $X_i^0 := r_i(S_i)$ and $\mathcal{X} := \bigcup_{i \in N} X_i^0 \subset \mathbb{R}$, we define strictly increasing mappings $\rho\colon \mathcal{X} \to \mathbb{N}$ by $\rho(x) := \#\{y \in \mathcal{X} \mid y < x\}$ (rank function) and $\varphi\colon \mathcal{X} \to \mathbb{R}$ by $\varphi(x) := n^{\rho(x)}$, where $n = \#N$.

Lemma A.3.1. *Let $I \subset N$, $y_I, x_I \in X_I^0$, and $\max_{i \in I} y_i > \max_{i \in I} x_i$. Then $\sum_{i \in I} \varphi(y_i) > \sum_{i \in I} \varphi(x_i)$.*

Proof. Let $\max_{i \in I} x_i = \mu$. Then $\sum_{i \in I} \varphi(y_i) \geq n^{\mu+1}$, while $\sum_{i \in I} \varphi(x_i) \leq \#I \cdot n^{\mu} < n^{\mu+1}$. □

Supposing, to the contrary, that $\langle x_N^k \rangle_{k \in \mathbb{N}}$ is an infinite admissible best response quasi-improvement path, we denote $N^* := \{i \in N \mid \exists k \in \mathbb{N}\, [x_i^{k+1} = r_i(x_i^k)]\}$ and consider two alternatives.

1. Let $N^* = N$. We pick $\bar{k} \in \mathbb{N}$ such that $x_i^k \in X_i^0$ whenever $k \geq \bar{k}$. Clearly, $\langle x_N^k \rangle_{k \geq \bar{k}}$ is an infinite admissible best response quasi-improvement path in a subgame where each player is restricted to strategies from X_i^0. On the other hand, Lemma A.3.1 implies that the subgame can be perceived as generated by the aggregation rules $\sigma_i^*(x_{-i}) = \sum_{j \in I(i)} x_j$ in the case of Theorem 6.2, or $\sigma_i^*(x_{-i}) = \sum_{j \in I(i)} (-x_j)$ in the case of Theorem 6.3. Therefore, it is covered by Theorem 6.1 in either case. The contradiction proves both theorems.

2. Let $N^* \subset N$. For each $i \in N \setminus N^*$, we have $x_i^k = x_i^0$ for all k. Therefore, we may consider a reduced game with the set of active players N^*, and each $i \in N \setminus N^*$ always choosing x_i^0. The game satisfies all assumptions of our theorem and $\langle x_N^k \rangle_{k \in \mathbb{N}}$ remains an infinite admissible best response quasi-improvement path; besides, Alternative 1 holds. Now the argument of the previous paragraph applies.

A.4 "Counterexamples"

Example A.1. Without the transitivity of preferences, even the existence of a Nash equilibrium does not follow from the single crossing conditions, even in a two-person game with finite chains X_i.

Let $N := \{1,2\}$, $X_1 := \{0,1,2,3,4\}$ and $X_2 := \{5,6\}$ (both with natural orders); let preference relations $\succ_i^{x_{-i}}$ be defined by: $2 \succ_1^5 4 \succ_1^5 0 \succ_1^5 1 \succ_1^5 3$; $1 \succ_1^6 3 \succ_1^6 2 \succ_1^6 4 \succ_1^6 0$; $5 \succ_2^{x_1} 6$ whenever $x_1 \le 1$; $6 \succ_2^{x_1} 5$ whenever $x_1 \ge 2$. The preferences of player 1 are intransitive, but single crossing conditions (16) are easy to check: (16a) is nontrivial only for $4 \succ_1^5 0$; (16b), only for $1 \succ_1^6 3$ and $2 \succ_1^6 4$. Player 2's preferences are described by a family of total orders; (16) are obvious. There is no Nash equilibrium: $\mathcal{R}_1(5) = \{2\}$ and $\mathcal{R}_1(6) = \{1\}$, whereas $\mathcal{R}_2(2) = \{6\}$ and $\mathcal{R}_2(1) = \{5\}$. It may be noted that $\mathcal{R}_1$ admits no increasing selection.

Remark. The game in Example A.1 admits no infinite best response improvement path, but does not have the FBRP, due to the second condition in the definition. However, it does admit infinite best response *quasi*-improvement paths.

Example A.2. Theorem 3.8 cannot be extended to strongly acyclic and transitive preference relations.

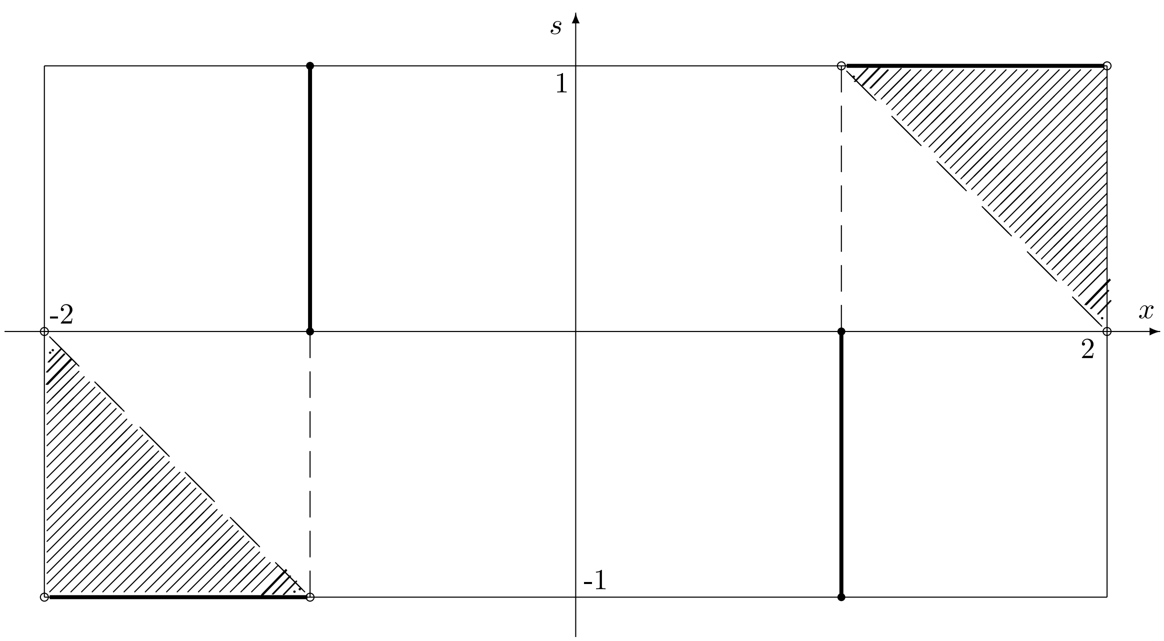

Figure 1: Best responses in Example A.2

Let $X := [-2,2]$, $S := [-1,1]$ (both with natural orders), and relations $\succ^s$ be defined by

$$y \succ^s x \rightleftharpoons [u_1(y,s) > u_1(x,s) \;\&\; u_2(y,s) > u_2(x,s)], \tag{28}$$

where $u\colon X \times S \to \mathbb{R}^2$ is this: $u(1,s) := \langle 5,2 \rangle$ and $u(-1,s) := \langle 2,5 \rangle$ for all $s \in S$; $u(2,s) := u(-2,s) :=$

$u(x, s) := \langle 0, 0 \rangle$ for all $x \in]-1, 1[$ and $s \in S$; whenever $x \in]1, 2[$ and $s \geq 0$,

$$u_1(x, s) := \begin{cases} x + s - 1, & \text{if } x + s \leq 2, \\ x + s + 4, & \text{if } x + s > 2, \end{cases}$$

while $u_2(x, s) := 6 - x - s$; whenever $x \in]-2, -1[$ and $s \geq 0$, $u(x, s) := \langle x + 6, -1 - x \rangle$; finally, $u_i(x, s)$ for all $s < 0$, $i = 1, 2$, and $x \in]-2, -1[\cup]1, 2[$ is such that the equality

$$u_i(x, s) = u_{3-i}(-x, -s) \tag{29}$$

holds for all $s \in S$, $i = 1, 2$, and $x \in X$.

The very form of (28) ensures that every $\succ^s$ is irreflexive and transitive. Whenever $x \in \{-2\} \cup]-1, 1[\cup \{2\}$ and $y \in]-2, -1] \cup [1, 2[$, $y \succ^s x$ for every $s \in S$. Whenever $x, y \in]-2, -1[$ or $x, y \in]1, 2[$, $y \succ^s x$ does not hold for any $s \in S$. Let $s \geq 0$; if $-2 < x < -1$, then $u_1(x) < 5$ and $u_2(x) \leq 1$, hence $1 \succ^s x$; if $1 < x \leq 2 - s < 2$, then $u_1(x) \leq 1$ and $u_2(x) < 5$, hence $-1 \succ^s x$; if $2 - s < y < 2$, then $u_1(y) > 6$ and $u_2(y) > 3$, hence $y \succ^s 1$. "Dually," by (29), $y \succ^s -1 \succ^s x$ whenever $s < 0$, $-2 < y < -2 - s$, and $1 < x < 2$; $1 \succ^s x$ whenever $s < 0$ and $-2 - s \leq x < -1$. Thus, $\mathcal{R}(s) = \{-1\} \cup]2 - s, 2[$ for $s > 0$, $\mathcal{R}(s) = \{1\} \cup]-2, -2 - s, 2[$ for $s < 0$, and $\mathcal{R}(0) = \{-1, 1\}$ (Figure 1). We see that every relation $\succ^s$ is strongly acyclic: no more than three consecutive improvements can be made from any starting point (e.g., $2 - s/2 \succ^s 1 \succ^s -1.5 \succ^s -2$ when $s > 0$). Single crossing conditions (6) are also easy to check.

Suppose there is an increasing selection r from $\mathcal{R}$. If $r(s) > -1$ for some $s > 0$, then $2 > r(s) > 2 - s$; defining $s' := 2 - r(s) > 0$, we have $s' < s$, hence $r(s') \leq r(s)$, hence $r(s') < 2 - s'$, hence $r(s') \in \mathcal{R}(s')$ is only possible if $r(s') = -1$. Therefore, $r(s) = -1$ for some $s > 0$; dually, $r(s) = 1$ for some $s < 0$. We have a contradiction, i.e., there is no increasing selection.

Example A.3. If discontinuous utility functions are replaced with their "upper semicontinuous closures," there may be no ε-Nash equilibrium of the original game close to a Nash equilibrium of the modified game.

Let $N := \{1, 2\}$, $X_1 := X_2 := [0, 1]$ (with the natural order), and the preferences of the players be defined by "isomorphic" utility functions

$$u_i(x_i, x_j) := \begin{cases} (1 + x_j)x_i, & \text{if } 0 \leq x_i < 1/2 \ \&\ 0 \leq x_j \leq 1, \\ (1 + x_j)x_i - 2, & \text{if } 1/2 \leq x_i < 3/4 \ \&\ 0 \leq x_j < 3/4, \\ (1 + x_j)x_i, & \text{if } 1/2 \leq x_i < 3/4 \ \&\ 3/4 \leq x_j \leq 1, \\ (x_j - 2)x_i, & \text{if } 3/4 \leq x_i \leq 1 \ \&\ 0 \leq x_j < 3/4, \\ (x_j - 2)x_i + 2, & \text{if } 3/4 \leq x_i \leq 1 \ \&\ 3/4 \leq x_j \leq 1. \end{cases}$$

It is easily checked that $u_i(x_i, x_j)$ thus defined is supermodular: whenever it increases in x_i, the rate of increase goes up as x_j increases; whenever it decreases in x_i, the rate of decrease goes down as x_j increases. For every x_j, the set $\mathcal{R}_i(x_j)$ is empty; therefore, there is no Nash equilibrium.

To obtain the upper semicontinuous closure of u_i, we have to modify it at $x_i = 1/2$ for $0 \leq x_j < 3/4$ and at $x_i = 3/4$ for all $0 \leq x_j \leq 1$. The resulting best responses are depicted as thick lines in Figure 2: $\bar{\mathcal{R}}_i(x_j) = \{1/2\}$ for $0 \leq x_j < 3/4$; $\bar{\mathcal{R}}_i(x_j) = \{3/4\}$ for $3/4 < x_j \leq 1$; $\bar{\mathcal{R}}_i(3/4) = \{1/2, 3/4\}$. That new game has two Nash equilibria: $(1/2, 1/2)$ and $(3/4, 3/4)$. Finally, if we switch from the original utilities

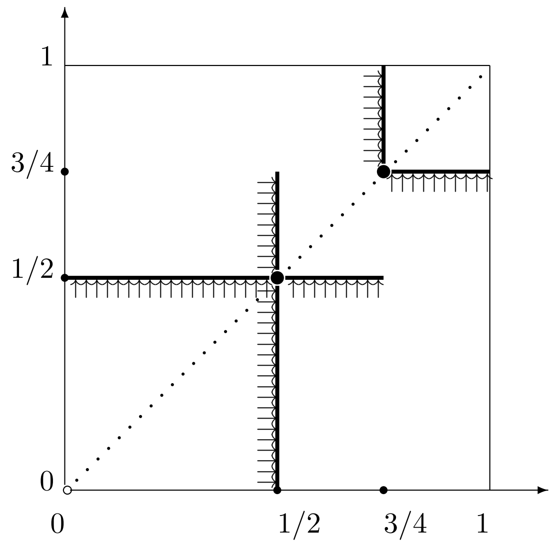

Figure 2: "Best" responses in Example A.3

u_i to preferences defined by (4) with a small enough $\varepsilon > 0$, then the "best" responses are depicted in the same figure by small arrows. We see that there is no ε-Nash equilibrium of the original game near the point $(3/4, 3/4)$, so any search in that vicinity would be futile.

Remark. In accordance with Theorem 5.2, it is impossible to make more than three consecutive ε-best response improvements in this game, whatever the starting point.

Example A.4. The FBRP cannot be asserted in Theorem 6.1, even for a finite game with the preferences described by utility functions.

Let $N := \{1, 2, 3\}$, $X_1 := \{0, 1, 2, 3, 4\}$, $X_2 := \{0, 1, 2, 3, 4, 5\}$, $X_3 := \{0, 1\}$; let the preferences of the players be defined by utility functions $u_i(x_N) = U_i(x_i, -\sum_{j \neq i} x_j)$. Clearly, we have an aggregative game as in Theorem 6.1 with $a_{ij} = -1$, hence $S_1 = \{-6, -5, \dots, 0\}$, $S_2 = \{-5, -4, \dots, 0\}$, $S_3 = \{-9, -8, \dots, 0\}$. Let the utilities be:

$$U_1: \begin{bmatrix} 2 & 3 & 3 & 3 & 4 \\ 2 & 3 & 3 & 3 & 2 \\ 2 & 3 & 3 & 3 & 2 \\ 2 & 3 & 3 & 3 & 2 \\ 2 & 3 & 3 & 3 & 2 \\ 2 & 3 & 3 & 3 & 2 \\ 2 & 1 & 1 & 1 & 0 \end{bmatrix} \quad U_2: \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad U_3: \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

where own choice, x_i, is on the abscissae axis, and s_i (= minus the sum of the partners' choices), on the ordinates axis. Conditions (6), even (7), are easy to check. By Theorem 6.1, the game has a restricted F[BR]P; actually, even a restricted FBRP. However, it does not have the FBRP since there is a best response improvement cycle:

$$\begin{array}{ccccccc} (3,0,0) & \xrightarrow{1} & (4,0,0) & \xrightarrow{3} & (4,0,1) & \xrightarrow{1} & (1,0,1) \\ \uparrow{\scriptstyle 2} & & & & & & \downarrow{\scriptstyle 2} \\ (3,5,0) & \xleftarrow{1} & (0,5,0) & \xleftarrow{3} & (0,5,1) & \xleftarrow{1} & (1,5,1) \end{array}.$$

Remark. The question of whether such an example is possible, was left open in Kukushkin (2004). If we retained the same sets X_i and utilities U_i, but redefined σ_i, setting $a_{ij} := 1$ for all $i, j \in N$, $j \neq i$, then there could be no best response improvement cycle (Kukushkin 2004, Theorem 1).

References

Acemoglu, D., and M.K. Jensen, 2013. Aggregate comparative statics. Games and Economic Behavior 80, 27–49.

Athey, S., 2001. Single crossing properties and the existence of pure strategy equilibria in games of incomplete information. Econometrica 69, 861–889.

Boncinelli, L., and P. Pin, 2012. Stochastic stability in best shot network games. Games and Economic Behavior 75, 538–554.

Bondareva, O.N., 1979. Development of game theoretical methods of optimization in cooperative games and their applications to multicriterial problems. In: Moiseev, N.N. (ed.) State of the Art in the Theory of Operations Research. Nauka, Moscow, 150–172. (In Russian)

Dubey, P., O. Haimanko, and A. Zapechelnyuk, 2006. Strategic complements and substitutes, and potential games. Games and Economic Behavior 54, 77–94.

Fishburn, P.C., 1985. Interval Orders and Interval Graphs: A Study of Partially Ordered Sets. Wiley, New York.

Hirshleifer, J., 1983. From weakest-link to best-shot: The voluntary provision of public goods. Public Choice, 41, 371–386.

Huang, Z., 2002. Fictitious play in games with a continuum of strategies. Ph.D. Thesis, State University of New York at Stony Brook, Department of Economics.

Jensen, M.K., 2010. Aggregative games and best-reply potentials. Economic Theory 43, 45–66.

Kukushkin, N.S., 1994. A fixed-point theorem for decreasing mappings. Economics Letters 46, 23–26.

Kukushkin, N.S., 2003. Acyclicity of Monotonic Endomorphisms. Russian Academy of Sciences, Dorodnicyn Computing Center, Moscow. A version available at http://www.ccas.ru/mmes/mmeda/ququ/MonoEndo.pdf

Kukushkin, N.S., 2004. Best response dynamics in finite games with additive aggregation. Games and Economic Behavior 48, 94–110.

Kukushkin, N.S., 2005. "Strategic supplements" in games with polylinear interactions. EconWPA Paper #0411008. Available at http://ideas.repec.org/p/wpa/wuwpga/0411008.html

Kukushkin, N.S., 2008. Maximizing an interval order on compact subsets of its domain. Mathematical Social Sciences 56, 195–206.

Kukushkin, N.S., 2013a. Monotone comparative statics: changes in preferences versus changes in the feasible set. Economic Theory 52, 1039–1060.

Kukushkin, N.S., 2013b. Increasing selections from increasing multifunctions. Order 30, 541–555.

Lippman, S.A., J.W. Mamer, and K.F. McCardle, 1987. Comparative statics in non-cooperative games via transfinitely iterated play. Journal of Economic Theory 41, 288–303.

Kukushkin, N.S., S. Takahashi, and T. Yamamori, 2005. Improvement dynamics in games with strategic complementarities. International Journal of Game Theory 33, 229–238.

Milgrom, P., and J. Roberts, 1990. Rationalizability, learning, and equilibrium in games with strategic complementarities. Econometrica 58, 1255–1277.

Milgrom, P., and C. Shannon, 1994. Monotone comparative statics. Econometrica 62, 157–180.

Novshek, W., 1985. On the existence of Cournot equilibrium. Review of Economic Studies 52, 85–98.

Quah, J., 2007. The comparative statics of constrained optimization problems. Econometrica 75, 401–431.

Quah, J., and B. Strulovici, 2009. Comparative statics, informativeness, and the interval dominance order. Econometrica 77, 1949–1992.

Reny, P.J., 2011. On the existence of monotone pure strategy equilibria in Bayesian games. Econometrica 79, 499–553.

Roddy, M.S., and B.S.W. Schröder, 2005. Isotone relations revisited. Discrete Mathematics 290, 229–248.

Smithson, R.E., 1971. Fixed points of order preserving multifunctions. Proceedings of the American Mathematical Society 28, 304–310.

Topkis, D.M., 1979. Equilibrium points in nonzero-sum n-person submodular games. SIAM Journal on Control and Optimization 17, 773–787.

Veinott, A.F., Jr., 1989. Lattice Programming. Unpublished lectures.

Vives, X., 1990. Nash equilibrium with strategic complementarities. Journal of Mathematical Economics 19, 305–321.