

# Another characterization of quasisupermodularity

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## Abstract

An ordering on a lattice is quasisupermodular if and only if inserting it into any parametric optimization problem with the single crossing property cannot destroy the monotonicity of the set of optima. More detailed conditions for the monotonicity of the set of optima in a parameter influencing the preference ordering, as well as in the sublattice of feasible choices, are also obtained. *JEL* Classification Number: C 72.

*Key words:* Best response correspondence; increasing correspondence; single crossing; quasisupermodular ordering

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# 1 Introduction

The concept of strategic complementarity was first developed in a cardinal form, around the notion of a supermodular function (Topkis, 1978, 1979; Veinott, 1989; Vives, 1990; Milgrom and Roberts, 1990). Milgrom and Shannon (1994) gave the idea an ordinal expression and obtained a neat characterization result, Theorem 4.

For our purposes here, that result is better perceived as two independent statements, related to “type B,” respectively, “type A” problems in the terminology of Quah (2007). First (Milgrom and Shannon, 1994, Corollary 1), the set of optimal choices depends on the sublattice of available choices in a monotone way if and only if the utility function is quasisupermodular. Second, if a quasisupermodular utility function is perturbed, the set of optimal choices from every sublattice ascends if and only if the single crossing conditions hold.

In the standard model of a strategic game, where each player’s utility depends on the choices of others but the strategy set does not, Milgrom and Shannon’s necessity result does not mean that we cannot have increasing best responses without the quasisupermodularity of preferences. Indeed, there are weaker sufficient conditions in the literature (Kukushkin et al., 2005, Lemma 3.1).

The main objective of this paper is to develop an analog of Milgrom and Shannon’s Theorem 4 for situations where no comparison between optimal choices from different sets is ever made. It turns out that necessity has to be interpreted in a much broader sense: the immersion of a given optimization problem into various parametric settings has to be considered. We do not restrict ourselves to the Veinott order on sublattices, but add four more ways to extend an order relation from points to subsets; altogether, we keep in mind five definitions of an increasing correspondence. Quasisupermodularity is partitioned into four independent constituent parts, each of which is necessary and sufficient (in that sophisticated sense) for a kind (actually, two kinds) of the monotonicity of optima.

We also develop four analogs of Corollary 1 of Milgrom and Shannon (1994) for situations where the Veinott order is used to compare feasible sublattices with one another, while comparisons between the sets of optima are made with the four extensions of the basic order from points to subsets. The conditions on the preference ordering are: strict quasisupermodularity, two “halves” of quasisupermodularity proper, and “weak quasisupermodularity.” Quasisupermodularity itself, as well as its two “halves,” are distinguished by playing the same roles in both “type A” and “type B” problems.

Section 2 reproduces the standard notion of a choice function generated by the maximization of a binary relation; in Section 3, we define a number of extensions of an order from points to subsets. In Section 4, a range of single crossing conditions is formulated and their (rather straightforward) connections with the monotonicity of optima on chains are described. Section 5 contains the definitions of four partial versions of quasisupermodularity and the proofs of their basic properties; the central results of the paper are collected in Section 6. Section 7 is about “type B” problems. A few concluding remarks

are in Section 8.

## 2 Preferences and choice

Throughout the paper, we assume a set  $A$  of alternatives given. There is an agent whose preferences over the alternatives are expressed by a binary relation  $\succ$  on  $A$ , which is assumed to be an *ordering*, i.e., *irreflexive*, *transitive*, and *negatively transitive* ( $z \not\succeq y \not\succeq x \Rightarrow z \not\succeq x$ ). Then the “non-strict preference” relation  $\succeq$  defined by  $y \succeq x \Leftrightarrow x \not\succeq y$  is reflexive, transitive, and total.

Orderings can also be defined in terms of representations in chains:  $\succ$  is an ordering if and only if there is a chain  $\mathcal{C}$  and a mapping  $u: A \rightarrow \mathcal{C}$  such that  $y \succ x \Leftrightarrow u(y) > u(x)$  for all  $x, y \in A$  (then  $y \succeq x \Leftrightarrow u(y) \geq u(x)$ ). The most usual assumption in game theory is that the preferences of a player are described by a *utility function*  $u: A \rightarrow \mathbb{R}$ . Here we work in a purely ordinal framework, so it is natural to replace  $\mathbb{R}$  with an arbitrary chain.

As is usual in decision theory, we allow for the possibility that only a subset  $X \subseteq A$  may be available for choice. The set of all subsets of  $A$  is denoted  $\mathfrak{B}_A$ . Given  $X \in \mathfrak{B}_A$ , we define

$$M(X, \succ) := \{x \in X \mid \nexists y \in X [y \succ x]\} = \{x \in X \mid \forall y \in X [x \succeq y]\}, \quad (1)$$

the set of *maximizers* of  $\succ$  on  $X$ . A very helpful observation is that  $y \succ x$  whenever  $\succ$  is an ordering,  $x, y \in X$ , and  $x \notin M(X, \succ) \ni y$  (“revealed preference”).

Clearly,  $M(X, \succ) \neq \emptyset$  if  $X$  is a finite nonempty subset of  $A$ . We do not restrict ourselves to finite subsets here; nor do we study more general conditions for the existence of maximizers. In a sense, we brush aside the distinction between an empty set  $M(X, \succ)$  and a nonempty one. A rationalization for this attitude is given in Section 3 below.

With game-theoretic applications in mind, we consider *parametric families*  $\langle \succ^t \rangle_{t \in T}$  of orderings on  $A$ ; the parameter  $t$  may be interpreted as (an aggregate of) the choice(s) of other agent(s). Given a parametric family and  $X \in \mathfrak{B}_A$ , the *best response correspondence*  $\mathcal{R}^X: T \rightarrow \mathfrak{B}_X \subseteq \mathfrak{B}_A$  is defined in the usual way:

$$\mathcal{R}^X(t) := M(X, \succ^t). \quad (2)$$

Admittedly, the preferences of a player in the standard strategic game model are described by an ordering on  $X \times T$  rather than by a parametric family of orderings on  $X$ . It is impossible to study, say, strong equilibria or the (in)efficiency of Nash equilibria in the latter framework. On the other hand, a parametric family of orderings is adequate when the subject is the existence of a Nash equilibrium or the behavior of individual adaptive dynamics. Some twenty five years ago, Olga Bondareva argued that the proper definition of a non-cooperative game (in an ordinal context) must stipulate that each player is only able to compare strategy profiles differing in her own choice. Although one does not have to accept this, rather extreme, view, there is something to it.

### 3 Monotonicity

We always assume  $A$  to be a *partially ordered set* (a *poset*). Most often, it is a *lattice*, in which case  $\mathfrak{L}_A$  denotes the set of all sublattices of  $A$ . The exact definitions are assumed commonly known. Given a lattice  $A$  and  $x, y \in A$ , we denote  $L(x, y) := \{x, y, x \vee y, x \wedge y\}$ , the minimal sublattice of  $A$  containing both  $x$  and  $y$ ; clearly,  $\#L(x, y) \in \{1, 2, 4\}$ .

The reversal of an order ( $y < x \Leftrightarrow x > y$ ) produces an order again; moreover, a lattice remains a lattice. Having proved a theorem, we can replace, in all assumptions and the statement itself, the relations and operations  $>$ ,  $\geq$ ,  $\vee$ , etc. with  $<$ ,  $\leq$ ,  $\wedge$ , etc., and obtain another valid theorem. The use of this simple observation (referred to as “duality”) leads to considerable economy in the total length of proofs.

When considering a parametric family of preference relations, we assume that  $T$  is also a poset. A mapping  $r: T \rightarrow A$  is *increasing* if  $r(t') \geq r(t)$  whenever  $t' > t$ ; when it comes to correspondences  $R: T \rightarrow \mathfrak{B}_A$ , we have to extend the order from  $A$  to  $\mathfrak{B}_A$ . Following Veinott (1989), we consider four ways to do so for a lattice  $A$ :

$$Y \geq^\wedge X \Leftrightarrow \forall y \in Y \forall x \in X [y \wedge x \in X]; \quad (3a)$$

$$Y \geq^\vee X \Leftrightarrow \forall y \in Y \forall x \in X [y \vee x \in Y]; \quad (3b)$$

$$Y \geq^{\vee t} X \Leftrightarrow [Y \geq^\vee X \ \& \ Y \geq^\wedge X]; \quad (3c)$$

$$Y \geq^{\vee v} X \Leftrightarrow \forall y \in Y \forall x \in X [y \vee x \in Y \ \text{or} \ y \wedge x \in X]. \quad (3d)$$

Another relation can be defined for any poset  $A$ :

$$Y \gg X \Leftrightarrow \forall y \in Y \forall x \in X [y \geq x]. \quad (3e)$$

Clearly,  $Y \gg X$  implies every other relation (3). None of the relations is an order, even on nonempty subsets:  $\gg$  and  $\geq^{\vee t}$  are antisymmetric and transitive, but generally not reflexive; neither  $\geq^{\vee v}$ , nor  $\geq^\wedge$  or  $\geq^\vee$  need even be transitive.

**Remark.** When the order on  $A$  is reversed,  $Y \geq^\wedge X$  transforms into  $X \geq^\vee Y$ ,  $Y \geq^\vee X$  into  $X \geq^\wedge Y$ , and  $Y \geq^* X$  into  $X \geq^* Y$  for  $\geq^*$  defined by (3c), (3d), or (3e).

Let  $\geq^*$  denote one of the relations (3) and  $T$  be a poset. A correspondences  $R: T \rightarrow \mathfrak{B}_A$  is *increasing w.r.t.  $\geq^*$*  if  $R(t') \geq^* R(t)$  whenever  $t' > t$ . Veinott (1989) called correspondences increasing w.r.t.  $\geq^{\vee t}$  ( $\geq^{\vee v}$ ) in this sense (weakly) ascending. Correspondences increasing w.r.t.  $\gg$  may be called strongly ascending; those increasing w.r.t.  $\geq^\wedge$  ( $\geq^\vee$ ), “meet” (“join”) ascending.

Such monotonicity is closely related to the existence of *monotone selections* from  $R$  [i.e., increasing mappings  $r: T \rightarrow A$  such that  $r(t) \in R(t)$  for every  $t \in T$ ], provided  $R(t) \neq \emptyset$  for all  $t$ . If a correspondence  $R: T \rightarrow \mathfrak{B}_A$  is increasing w.r.t.  $\gg$ , then every selection from  $R$  is increasing. If  $R$  is increasing w.r.t.  $\geq^\wedge$ ,  $\geq^\vee$ , or  $\geq^{\vee v}$ , then a monotone selection exists under a completeness assumption about every value  $R(t)$  (Veinott, 1989, Theorem 3.2; Kukushkin, 2009, Proposition 3.1 and Theorem 1); naturally, a stronger

assumption is needed in the last case. If  $R$  is increasing w.r.t.  $\succeq^{\vee t}$ , then no completeness assumption at all is needed provided  $A$  is a sublattice of the Cartesian product of a finite number of chains (Kukushkin, 2009, Theorem 2).

We are interested in conditions on the preferences ensuring monotonicity, w.r.t. one or another of the relations (3), of correspondences  $\mathcal{R}^X$  defined by (2). The monotonicity of a single correspondence  $\mathcal{R}^X$  may happen just “by accident”; however, when a wide enough class of admissible subsets  $X$  is taken into account, necessity results become obtainable. Separation between existence and monotonicity is possible because each of the relations (3) holds trivially if either  $Y$  or  $X$  is empty (non-existence cannot destroy monotonicity).

An ordering  $\succ$  on a poset  $A$  is (*strictly*) *increasing* if  $y \succeq x$  ( $y \succ x$ ) whenever  $y > x$ ; dually,  $\succ$  is (*strictly*) *decreasing* if  $y \succeq x$  ( $y \succ x$ ) whenever  $x > y$ . The well-known Szpilrajn theorem asserts the existence of a strictly increasing total order on every poset. When the preferences are increasing or decreasing, most of the following becomes trivial. Naturally, we are interested in less straightforward connections between preferences and order.

## 4 Single crossing

It is well known that the “single crossing” conditions of various kinds (Milgrom and Shannon, 1994) are important for the monotonicity of best responses. Those conditions are most conveniently presented with the help of a ternary relation on the set of binary relations on a given set: “ $\triangleright_1$  is closer to  $\triangleright_0$  than  $\triangleright_2$  is”; similar observations were made by Quah and Strulovici (2009) and Alexei Savvateev (a seminar presentation, 2007). In the following, the role of  $\triangleright_0$  is always played by the basic order on  $A$ , while  $\triangleright_1$  and  $\triangleright_2$  are (strict or non-strict) preference relations.

Let  $\succ$  and  $\preceq$  be orderings on a poset  $A$ . We consider four conditions:

$$\forall x, y \in A [y > x \ \& \ y \succ x \Rightarrow y \preceq x]; \quad (4a)$$

$$\forall x, y \in A [y > x \ \& \ y \preceq x \Rightarrow y \preceq x]; \quad (4b)$$

$$\forall x, y \in A [y > x \ \& \ y \succeq x \Rightarrow y \preceq x]; \quad (4c)$$

$$\forall x, y \in A [y > x \ \& \ y \succ x \Rightarrow y \preceq x]. \quad (4d)$$

Each condition defines a binary relation on the set of orderings on  $A$ . The first two are preorders. The third is transitive, but generally not reflexive. The last relation is reflexive, but need not be transitive.

Given a poset  $A$  and a parametric family  $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$  of orderings on  $A$ , we say that  $\mathcal{U}$  satisfies the *meet single crossing* condition if (4a) holds for  $\preceq$  as  $\succ$  and  $\preceq^t$  as  $\preceq$  whenever  $t, t' \in T$  and  $t' > t$ . Similarly,  $\mathcal{U}$  satisfies the *join, strict, or weak single crossing* condition if (4b), (4c), or (4d) holds under the same circumstances. We say that  $\mathcal{U}$  satisfies the *single crossing* condition if it satisfies both join and meet single crossing conditions. Our terminology coincides with that of Milgrom and Shannon (1994) when  $\mathcal{U}$  is represented by

a utility function (they did not explicitly define the meet, join, and weak single crossing conditions, though).

For more convenience in further referencing, we consider four “reversed” versions of conditions (4):

$$\forall x, y \in A [y < x \ \& \ y \succcurlyeq x \Rightarrow y \preccurlyeq x]; \quad (5a)$$

$$\forall x, y \in A [y < x \ \& \ y \succcurlyeq x \Rightarrow y \preccurlyeq x]; \quad (5b)$$

$$\forall x, y \in A [y < x \ \& \ y \succcurlyeq x \Rightarrow y \preccurlyeq x]; \quad (5c)$$

$$\forall x, y \in A [y < x \ \& \ y \succcurlyeq x \Rightarrow y \preccurlyeq x]. \quad (5d)$$

It is easily checked that each condition (5) is equivalent to the corresponding condition (4) after the exchange of the roles of  $\succcurlyeq$  and  $\preccurlyeq$ . Therefore, the (meet, join, strict, or weak) single crossing conditions could be defined with references to (5) as well.

**Remark.** If the order on  $A$  is reversed, conditions (4a), (4b), (4c), and (4d) transform into (5b), (5a), (5c), and (5d), respectively. If both orders on  $A$  and  $T$  are reversed, meet and join single crossing conditions transform into each other; their conjunction, as well as strict and weak single crossing conditions, are “self-dual” in this sense.

**Proposition 4.1.** *Let  $A$  and  $T$  be posets, and  $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$  be a parametric family of orderings on  $A$ . Then the following statements are equivalent.*

1.  $\mathcal{U}$  satisfies the meet single crossing condition.
2. There holds  $\mathcal{R}^X(t') \geq^\wedge \mathcal{R}^X(t)$  whenever  $t', t \in T$ ,  $t' > t$ , and  $X \in \mathfrak{B}_A$  is a chain.
3. There holds  $\mathcal{R}^X(t') \geq^\wedge \mathcal{R}^X(t)$  whenever  $t', t \in T$ ,  $t' > t$ , and  $X \in \mathfrak{B}_A$  is a chain with  $\#X = 2$ .

*Proof.* Let Statement 1 hold,  $t', t \in T$ ,  $t' > t$ , and  $X \in \mathfrak{B}_A$  be a chain. We have to show  $\mathcal{R}^X(t') \geq^\wedge \mathcal{R}^X(t)$ ; let  $y \in \mathcal{R}^X(t')$  and  $x \in \mathcal{R}^X(t)$ . If  $y \geq x$ , we are home immediately; let  $x > y$ . If  $y \in \mathcal{R}^X(t)$ , we are home again. If  $y \notin \mathcal{R}^X(t)$ , then  $x \not\preccurlyeq y$ , hence  $x \preccurlyeq^{t'} y$  by (4a), contradicting the assumption  $y \in \mathcal{R}^X(t')$ .

Let Statement 1 be violated: there are  $t', t \in T$  and  $x, y \in A$  such that  $t' > t$ ,  $y > x$ ,  $y \not\preccurlyeq x$ , but  $x \succcurlyeq^{t'} y$ . Then we define  $X := \{x, y\}$  and immediately obtain  $x \in \mathcal{R}^X(t') \setminus \mathcal{R}^X(t)$  while  $\mathcal{R}^X(t) = \{y\}$ , hence  $\mathcal{R}^X(t') \geq^\wedge \mathcal{R}^X(t)$  does not hold, i.e., Statement 3 is invalid.  $\square$

**Proposition 4.2.** *Let  $A$  and  $T$  be posets, and  $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$  be a parametric family of orderings on  $A$ . Then the following statements are equivalent.*

1.  $\mathcal{U}$  satisfies the join single crossing condition.
2. There holds  $\mathcal{R}^X(t') \geq^\vee \mathcal{R}^X(t)$  whenever  $t', t \in T$ ,  $t' > t$ , and  $X \in \mathfrak{B}_A$  is a chain.
3. There holds  $\mathcal{R}^X(t') \geq^\vee \mathcal{R}^X(t)$  whenever  $t', t \in T$ ,  $t' > t$ , and  $X \in \mathfrak{B}_A$  is a chain with  $\#X = 2$ .

The proof is dual to that of Proposition 4.1.

**Proposition 4.3 (Milgrom and Shannon, 1994).** *Let  $A$  and  $T$  be posets, and  $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$  be a parametric family of orderings on  $A$ . Then the following statements are equivalent.*

1.  $\mathcal{U}$  satisfies the single crossing condition.
2. There holds  $\mathcal{R}^X(t') \geq^{Vt} \mathcal{R}^X(t)$  whenever  $t', t \in T$ ,  $t' > t$ , and  $X \in \mathfrak{B}_A$  is a chain.
3. There holds  $\mathcal{R}^X(t') \geq^{Vt} \mathcal{R}^X(t)$  whenever  $t', t \in T$ ,  $t' > t$ , and  $X \in \mathfrak{B}_A$  is a chain with  $\#X = 2$ .

The equivalence immediately follows from Propositions 4.1 and 4.2.

**Proposition 4.4.** *Let  $A$  and  $T$  be posets, and  $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$  be a parametric family of orderings on  $A$ . Then the following statements are equivalent.*

1.  $\mathcal{U}$  satisfies the strict single crossing condition.
2. There holds  $\mathcal{R}^X(t') \gg \mathcal{R}^X(t)$  whenever  $t', t \in T$ ,  $t' > t$ , and  $X \in \mathfrak{B}_A$  is a chain.
3. There holds  $\mathcal{R}^X(t') \gg \mathcal{R}^X(t)$  whenever  $t', t \in T$ ,  $t' > t$ , and  $X \in \mathfrak{B}_A$  is a chain with  $\#X = 2$ .

*Proof.* Let the strict single crossing condition hold,  $t', t \in T$ , and  $t' > t$ . We have to show  $\mathcal{R}^X(t') \gg \mathcal{R}^X(t)$ ; let  $y \in \mathcal{R}^X(t')$  and  $x \in \mathcal{R}^X(t)$ . If  $y \geq x$ , we are home; let  $x > y$ . We have  $x \succeq^t y$  since  $x \in \mathcal{R}^X(t)$ ; applying (4c), we obtain  $x \succ^{t'} y$ , which contradicts  $y \in \mathcal{R}^X(t')$ .

Let the strict single crossing condition be violated: there are  $t', t \in T$  and  $x, y \in A$  such that  $t' > t$ ,  $y > x$ ,  $y \succeq^t x$ , but  $x \succeq^{t'} y$ . Then we define  $X := \{x, y\}$  and immediately obtain  $y \in \mathcal{R}^X(t)$  while  $x \in \mathcal{R}^X(t')$ , hence  $\mathcal{R}^X(t') \gg \mathcal{R}^X(t)$  does not hold.  $\square$

**Proposition 4.5.** *Let  $A$  and  $T$  be posets, and  $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$  be a parametric family of orderings on  $A$ . Then the following statements are equivalent.*

1.  $\mathcal{U}$  satisfies the weak single crossing condition.
2. There holds  $\mathcal{R}^X(t') \geq^{wV} \mathcal{R}^X(t)$  whenever  $t', t \in T$ ,  $t' > t$ , and  $X \in \mathfrak{B}_A$  is a chain.
3. There holds  $\mathcal{R}^X(t') \geq^{wV} \mathcal{R}^X(t)$  whenever  $t', t \in T$ ,  $t' > t$ , and  $X \in \mathfrak{B}_A$  is a chain with  $\#X = 2$ .

*Proof.* Let the weak single crossing condition hold,  $t', t \in T$ , and  $t' > t$ . We have to show  $\mathcal{R}^X(t') \geq^{wV} \mathcal{R}^X(t)$ ; let  $y \in \mathcal{R}^X(t')$  and  $x \in \mathcal{R}^X(t)$ . If  $y \geq x$ , we are home; let  $x > y$ . We have to show that either  $y \in \mathcal{R}^X(t)$  or  $x \in \mathcal{R}^X(t')$ . Since  $x \in \mathcal{R}^X(t)$ , we have  $x \succeq^t y$ . If  $y \succeq^t x$  as well, we have  $y \in \mathcal{R}^X(t)$ ; otherwise, we apply (4d), obtaining  $x \succeq^{t'} y$ , which implies  $x \in \mathcal{R}^X(t')$ .

Let the weak single crossing condition be violated: there are  $t', t \in T$  and  $x, y \in A$  such that  $t' > t$ ,  $y > x$ ,  $y \succ^t x$ , but  $x \succ^{t'} y$ . Then we define  $X := \{x, y\}$  and immediately obtain  $\mathcal{R}^X(t) = \{y\}$  while  $\mathcal{R}^X(t') = \{x\}$ , hence  $\mathcal{R}^X(t') \not\geq^{wv} \mathcal{R}^X(t)$  does not hold.  $\square$

## 5 Quasisupermodularity

Naturally, one does not have to be satisfied with maximization on chains, although scalar strategies are met in economics models most often. The *necessity* of single crossing conditions, obviously, holds on any class of admissible subsets that contains all finite chains (but not otherwise, see Quah and Strulovici, 2009). The sufficiency is less robust.

**Example 5.1.** Let  $A := \{(0, 0), (0, 1), (1, 0), (1, 1)\} \subset \mathbb{R}^2$ ,  $T := \{0, 1\}$ , and a function  $u: A \times T \rightarrow \mathbb{R}$  be defined by the following matrices (the axes are directed upwards and rightwards):

$$\begin{array}{cc} t = 0 & t = 1 \\ \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} & \begin{bmatrix} 5 & 4 \\ 0 & 6 \end{bmatrix} \end{array}$$

Clearly,  $u$  satisfies the strict single crossing condition, even the strictly increasing differences condition. However,  $\mathcal{R}^A(0) = \{(0, 1)\}$ , while  $\mathcal{R}^A(1) = \{(1, 0)\}$ . Therefore,  $\mathcal{R}^A$  is not increasing w.r.t. any relation (3); there is no monotone selection either.

Milgrom and Shannon (1994) called a function  $u$  on a lattice  $A$  *quasisupermodular* if

$$\forall x, y \in A \left[ \text{sign}(u(x \vee y) - u(y)) \geq \text{sign}(u(x) - u(x \wedge y)) \right]. \quad (6)$$

The condition is purely ordinal and can easily be reformulated in terms of a preference ordering (Alexei Savvateev, a seminar presentation, 2007):

$$\forall x, y \in A \left[ x \succ y \wedge x \Rightarrow y \vee x \succ y \right]; \quad (7a)$$

$$\forall x, y \in A \left[ x \succeq y \wedge x \Rightarrow y \vee x \succeq y \right]. \quad (7b)$$

In the light of the following results, it seems reasonable to call (7a) *meet quasisupermodularity* and (7b) *join quasisupermodularity*. Each “half” of quasisupermodularity can be meaningfully partitioned into two halves of its own:

$$\forall x, y \in A \left[ x \succ y \wedge x \Rightarrow [(y \vee x \succ x) \text{ or } (y \vee x \succ y)] \right]; \quad (8a)$$

$$\forall x, y \in A \left[ y \succeq y \vee x \Rightarrow [(y \wedge x \succeq x) \text{ or } (y \wedge x \succeq y)] \right]; \quad (8b)$$

$$\forall x, y \in A \left[ x \succeq y \wedge x \Rightarrow [(y \vee x \succeq x) \text{ or } (y \vee x \succeq y)] \right]; \quad (8c)$$

$$\forall x, y \in A \left[ y \succ y \vee x \Rightarrow [(y \wedge x \succ x) \text{ or } (y \wedge x \succ y)] \right]. \quad (8d)$$

**Remark.** Each condition (8) holds trivially when  $x$  and  $y$  are comparable in the basic order.

**Proposition 5.2.** *An ordering  $\succ$  on a lattice  $A$  satisfies condition (7a) if and only if it satisfies (8a) and (8b);  $\succ$  satisfies (7b) if and only if it satisfies (8c) and (8d).*

*Proof.* The necessity is obvious. To prove the sufficiency, we suppose the contrary. Let  $x \succ y \wedge x$ , but  $y \not\geq y \vee x$ ; then  $y \vee x \succ x$  by (8a), hence  $y \succ y \wedge x$  by transitivity, which contradicts (8b). The proof of the equivalence (7b)  $\equiv$  [(8c) & (8d)] is dual.  $\square$

**Example 5.3.** Let  $A := \{(0, 0), (0, 1), (1, 0), (1, 1)\} \subset \mathbb{R}^2$ ; we consider four orderings on  $A$  represented by these matrices (the axes are directed upwards and rightwards):

$$\mathbf{a.} \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \quad \mathbf{b.} \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} \quad \mathbf{c.} \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \quad \mathbf{d.} \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}.$$

The ordering represented by the matrix ‘‘a’’ satisfies all conditions (8) except (8a), and similarly with other matrices.

To obtain characterization results for preferences ensuring monotonicity of  $\mathcal{R}^X$  for sublattices  $X \in \mathfrak{L}_A$ , we have to modify the problem itself. The sufficiency parts of Propositions 4.1–4.5 can be interpreted as the monotonicity of the correspondence  $M(X, \cdot)$  w.r.t. relations (4) on the set of orderings when  $X$  is a chain. Here, each condition (8) is shown to be necessary and sufficient for a kind (actually, two kinds) of such monotonicity on sublattices.

**Proposition 5.4.** *Let  $A$  be a lattice and  $\succ$  be an ordering on  $A$ . Then the following statements are equivalent.*

1.  $\succ$  satisfies (8a).
2. There holds  $M(X, \succ) \geq^\wedge M(X, \succ')$  whenever  $X \in \mathfrak{L}_A$ ,  $\succ'$  is an ordering on  $X$ , and (4a) holds on  $X$ .
3. There holds  $M(X, \succ) \geq^{\vee} M(X, \succ')$  whenever  $X \in \mathfrak{L}_A$ ,  $\succ'$  is an ordering on  $X$ , and (4d) holds on  $X$ .
4. There holds  $M(X, \succ) \geq^\wedge M(X, \succ')$  whenever  $X \in \mathfrak{L}_A$ ,  $\#X \leq 4$ , and  $\succ'$  is an ordering on  $A$  such that (4a) and (4b) hold on  $A$ .
5. There holds  $M(X, \succ) \geq^{\vee} M(X, \succ')$  whenever  $X \in \mathfrak{L}_A$ ,  $\#X \leq 4$ , and  $\succ'$  is an ordering on  $A$  such that (4a) holds on  $A$ .

*Proof.* The implications Statement 2  $\Rightarrow$  Statement 4 and Statement 3  $\Rightarrow$  Statement 5 are obvious.

Statement 1  $\Rightarrow$  Statement 2. Let (8a) and (4a) hold. We have to show that  $y \wedge x \in M(X, \succ)$  whenever  $y \in M(X, \succ')$  and  $x \in M(X, \succ)$ . Supposing the contrary, we have  $x \succ y \wedge x$ , hence  $y \vee x \succ y$  by (8a) and the optimality of  $x$ . Therefore,  $y \vee x \not\prec y$  by (4a), contradicting the optimality of  $y$ .

Statement 1  $\Rightarrow$  Statement 3. Let (8a) and (4d) hold. We have to show  $M(X, \preceq) \succeq^{wV} M(X, \succ)$ ; let  $y \in M(X, \preceq)$  and  $x \in M(X, \succ)$ . If  $y \wedge x \in M(X, \succ)$ , we are home; otherwise,  $x \succ y \wedge x$ , hence  $y \vee x \succ y$  by (8a) and the optimality of  $x$ . Therefore,  $y \vee x \preceq y$  by (4d), hence  $y \vee x \in M(X, \preceq)$ .

Statement 4  $\Rightarrow$  Statement 1. Let (8a) be violated: there are  $x, y \in A$  such that  $x \succ y \wedge x$ , but  $y \succeq y \vee x$  and  $x \succeq y \vee x$ . Without restricting generality,  $x \succeq y$ . We define  $X := L(x, y)$ , so  $y \wedge x \notin M(X, \succ) \ni x$ , and  $Y := \{z \in A \mid z \geq y\}$ ; our assumptions imply  $x \notin Y$ . Then we define an ordering  $\preceq$  on  $A$ : it coincides with  $\succ$  on  $A \setminus Y$  and on  $Y$ , whereas  $z' \preceq z$  whenever  $z \notin Y \ni z'$ . Both (4a) and (4b) are obvious: whenever  $z \in Y$  and  $z' > z$  we have  $z' \in Y$  as well. Meanwhile,  $y \in M(X, \preceq)$ , hence  $M(X, \preceq) \succeq^\wedge M(X, \succ)$  does not hold, i.e., Statement 4 is invalid.

Statement 5  $\Rightarrow$  Statement 1. Let (8a) be violated. We pick  $x$  and  $y$  as in the previous paragraph, define  $X := L(x, y)$ , so  $y \wedge x \notin M(X, \succ) \ni x$  again, and then define  $\preceq$  in the same manner, but with  $Y := \{z \in A \mid z \succ y \ \& \ z > y\} \cup \{y\}$ . Clearly,  $M(X, \preceq) = \{y\}$ , hence  $M(X, \preceq) \succeq^{wV} M(X, \succ)$  does not hold. Since  $\preceq$  and  $\succ$  satisfy (4a), Statement 5 is invalid.  $\square$

**Proposition 5.5.** *Let  $A$  be a lattice and  $\succ$  be an ordering on  $A$ . Then the following statements are equivalent.*

1.  $\succ$  satisfies (8b).
2. There holds  $M(X, \succ) \succeq^\wedge M(X, \preceq)$  whenever  $X \in \mathfrak{L}_A$ ,  $\preceq$  is an ordering on  $X$ , and (5a) holds on  $X$ .
3. There holds  $M(X, \succ) \ggg M(X, \preceq)$  whenever  $X \in \mathfrak{L}_A$ ,  $\preceq$  is an ordering on  $X$ , and (5c) holds on  $X$ .
4. There holds  $M(X, \succ) \succeq^\wedge M(X, \preceq)$  whenever  $X \in \mathfrak{L}_A$ ,  $\#X \leq 4$ , and  $\preceq$  is an ordering on  $A$  such that (5c) holds on  $A$ .

*Proof.* Let (8b) hold,  $X \in \mathfrak{L}_A$ ,  $x \in M(X, \preceq)$  and  $y \in M(X, \succ)$ . Let us show that  $y \wedge x \in M(X, \preceq)$  if (5a) holds. Supposing the contrary, we have  $x \preceq y \wedge x$ , hence  $x \succ y \wedge x$  by (5a). Since  $y \succeq x$ , (8b) implies that  $y \vee x \succ y$ , which contradicts the optimality of  $y$ . Let us show that  $y \geq x$  if (5c) holds. Supposing the contrary, we have  $x > y \wedge x$ ; since  $y \succeq y \vee x$  and  $y \succeq x$ , we have  $y \wedge x \succeq x$  by (8b). Therefore,  $y \wedge x \preceq x$  by (5c), which contradicts the optimality of  $x$ .

Let (8b) be violated: there are  $x, y \in A$  such that  $x \succ y \wedge x$  and  $y \succ y \wedge x$ , but  $y \succeq y \vee x$ . Without restricting generality,  $y \succeq x$ ; we define  $X := L(x, y)$ , so  $y \in M(X, \succ)$ . Then we define an ordering  $\preceq$  on  $A$  in the same manner as in the proof of Proposition 5.4, but with  $Y := \{z \in A \mid z \leq x\}$ . On every equivalence class  $E$  of  $\preceq$ , we pick a strictly increasing total order  $\ggg_E$ , existing by the Szpilrajn theorem. Then we define  $\preceq'$  as a lexicography:  $z' \preceq' z$  if  $z' \preceq z$ , or if they belong to the same equivalence class  $E$  and  $z \ggg_E z'$ . Clearly,  $\preceq'$  is a total order. Let  $z' > z$  and  $z' \preceq z$ , hence  $z' \preceq' z$ . By the definition of  $\preceq'$ , we have  $z' \preceq z$ . If  $z' \in Y$ , then  $z \in Y$  as well, hence  $z' \succ z$ ; if  $z' \notin Y$ , then

$z \notin Y$  and  $z' \succ z$  again. Therefore, (5c) holds for  $\succ'$  and  $\succ$ . Meanwhile,  $M(X, \succ') = \{x\}$ , hence  $M(X, \succ) \not\geq^{\wedge} M(X, \succ')$  does not hold.  $\square$

**Proposition 5.6.** *Let  $A$  be a lattice and  $\succ$  be an ordering on  $A$ . Then the following statements are equivalent.*

1.  $\succ$  satisfies (8c).
2. There holds  $M(X, \succ) \geq^{\vee} M(X, \succ')$  whenever  $X \in \mathfrak{L}_A$ ,  $\succ'$  is an ordering on  $X$ , and (4b) holds on  $X$ .
3. There holds  $M(X, \succ) \gg M(X, \succ')$  whenever  $X \in \mathfrak{L}_A$ ,  $\succ'$  is an ordering on  $X$ , and (4c) holds on  $X$ .
4. There holds  $M(X, \succ) \geq^{\vee} M(X, \succ')$  whenever  $X \in \mathfrak{L}_A$ ,  $\#X \leq 4$ , and  $\succ'$  is an ordering on  $A$  such that (4c) holds on  $A$ .

The proof is dual to that of Proposition 5.5.

**Proposition 5.7.** *Let  $A$  be a lattice and  $\succ$  be an ordering on  $A$ . Then the following statements are equivalent.*

1.  $\succ$  satisfies (8d).
2. There holds  $M(X, \succ) \geq^{\vee} M(X, \succ')$  whenever  $X \in \mathfrak{L}_A$ ,  $\succ'$  is an ordering on  $X$ , and (5b) holds on  $X$ .
3. There holds  $M(X, \succ) \geq^{\text{wV}} M(X, \succ')$  whenever  $X \in \mathfrak{L}_A$ ,  $\succ'$  is an ordering on  $X$ , and (5d) holds on  $X$ .
4. There holds  $M(X, \succ) \geq^{\vee} M(X, \succ')$  whenever  $X \in \mathfrak{L}_A$ ,  $\#X \leq 4$ , and  $\succ'$  is an ordering on  $A$  such that (5a) and (5b) hold on  $A$ .
5. There holds  $M(X, \succ) \geq^{\text{wV}} M(X, \succ')$  whenever  $X \in \mathfrak{L}_A$ ,  $\#X \leq 4$ , and  $\succ'$  is an ordering on  $A$  such that (5b) holds on  $A$ .

The proof is dual to that of Proposition 5.4.

**Proposition 5.8.** *Let  $A$  be a lattice and  $\succ$  be an ordering on  $A$ . Then the following statements are equivalent.*

1. Both conditions (8a) and (8c) hold.
2. There holds  $M(X, \succ) \geq^{\text{Vt}} M(X, \succ')$  whenever  $X \in \mathfrak{L}_A$ ,  $\succ'$  is an ordering on  $X$ , and (4a) and (4b) hold on  $X$ .
3. There holds  $M(X, \succ) \geq^{\text{Vt}} M(X, \succ')$  whenever  $X \in \mathfrak{L}_A$ ,  $\#X \leq 4$ , and  $\succ'$  is an ordering on  $A$  such that (4a) and (4b) hold on  $A$ .

The equivalence immediately follows from Propositions 5.4 and 5.6.

**Proposition 5.9.** *Let  $A$  be a lattice and  $\succ$  be an ordering on  $A$ . Then the following statements are equivalent.*

1. Both conditions (8b) and (8d) hold.
2. There holds  $M(X, \succ) \geq^{\vee^t} M(X, \preceq)$  whenever  $X \in \mathfrak{L}_A$ ,  $\preceq$  is an ordering on  $X$ , and (5a) and (5b) hold on  $X$ .
3. There holds  $M(X, \succ) \geq^{\vee^t} M(X, \preceq)$  whenever  $X \in \mathfrak{L}_A$ ,  $\#X \leq 4$ , and  $\preceq$  is an ordering on  $A$  such that (5a) and (5b) hold on  $A$ .

The equivalence immediately follows from Propositions 5.5 and 5.7.

**Remark.** Agliardi (2000) called a function  $u$  on a lattice  $A$  *pseudosupermodular* if the ordering represented by  $u$  satisfies (8a) and (8c). In the light of Propositions 5.8 and 5.9, it might be appropriate to call an ordering *pseudosupermodular upwards* [*downwards*] if it satisfies (8a) and (8c) [(8b) and (8d)].

Sufficient conditions for monotonicity of the best responses (in one sense or another) are obtained as easy corollaries. Given a poset  $T$ , a *monotone pseudopartition* of  $T$  consists of two subsets  $T^\uparrow, T^\downarrow \subseteq S$  such that  $\forall t', t \in T [t' > t \Rightarrow [t \in T^\uparrow \text{ or } t' \in T^\downarrow]]$ . Clearly, any two points outside  $T^\uparrow \cup T^\downarrow$  must be incomparable.

**Proposition 5.10.** *Let  $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$  be a parametric family of orderings on a lattice  $A$ ; let  $\mathcal{U}$  satisfy the strict single crossing condition. Let there be a monotone pseudopartition  $\langle T^\uparrow, T^\downarrow \rangle$  of  $T$  such that  $\preceq^t$  satisfies (8c) for  $t \in T^\uparrow$  and (8b) for  $t \in T^\downarrow$ . Then every  $\mathcal{R}^X$  ( $X \in \mathfrak{L}_A$ ) is increasing w.r.t.  $\ggg$ .*

*Proof.* Let  $t' > t$ . If  $t \in T^\uparrow$ , then (8c) holds with  $\preceq^t$  as  $\succ$  while (4c) holds with  $\preceq^{t'}$  as  $\preceq$  and  $\preceq^t$  as  $\succ$ . Therefore,  $\mathcal{R}^X(t') \ggg \mathcal{R}^X(t)$  for every  $X \in \mathfrak{L}_A$  by Statement 3 of Proposition 5.6. If  $t' \in T^\downarrow$ , then (8b) holds with  $\preceq^{t'}$  as  $\succ$  while (5c) holds with  $\preceq^t$  as  $\preceq$  and  $\preceq^{t'}$  as  $\succ$ . Therefore,  $\mathcal{R}^X(t') \ggg \mathcal{R}^X(t)$  for every  $X \in \mathfrak{L}_A$  by Statement 3 of Proposition 5.5.  $\square$

**Proposition 5.11.** *Let  $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$  be a parametric family of orderings on a lattice  $A$ ; let  $\mathcal{U}$  satisfy the meet single crossing condition. Let there be a monotone pseudopartition  $\langle T^\uparrow, T^\downarrow \rangle$  of  $T$  such that  $\preceq^t$  satisfies (8a) for  $t \in T^\uparrow$  and (8b) for  $t \in T^\downarrow$ . Then every  $\mathcal{R}^X$  ( $X \in \mathfrak{L}_A$ ) is increasing w.r.t.  $\geq^\wedge$ .*

**Proposition 5.12.** *Let  $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$  be a parametric family of orderings on a lattice  $A$ ; let  $\mathcal{U}$  satisfy the join single crossing condition. Let there be a monotone pseudopartition  $\langle T^\uparrow, T^\downarrow \rangle$  of  $T$  such that  $\preceq^t$  satisfies (8c) for  $t \in T^\uparrow$  and (8d) for  $t \in T^\downarrow$ . Then every  $\mathcal{R}^X$  ( $X \in \mathfrak{L}_A$ ) is increasing w.r.t.  $\geq^\vee$ .*

**Proposition 5.13.** *Let  $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$  be a parametric family of orderings on a lattice  $A$ ; let  $\mathcal{U}$  satisfy the single crossing condition. Let there be a monotone pseudopartition  $\langle T^\uparrow, T^\downarrow \rangle$  of  $T$  such that  $\preceq^t$  satisfies (8a) and (8c) for  $t \in T^\uparrow$ , and (8b) and (8d) for  $t \in T^\downarrow$ . Then every  $\mathcal{R}^X$  ( $X \in \mathfrak{L}_A$ ) is increasing w.r.t.  $\geq^{\vee^t}$ .*

**Proposition 5.14.** *Let  $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$  be a parametric family of orderings on a lattice  $A$ ; let  $\mathcal{U}$  satisfy the weak single crossing condition. Let there be a monotone pseudopartition  $\langle T^\uparrow, T^\downarrow \rangle$  of  $T$  such that  $\succ^t$  satisfies (8a) for  $t \in T^\uparrow$  and (8d) for  $t \in T^\downarrow$ . Then every  $\mathcal{R}^X$  ( $X \in \mathfrak{L}_A$ ) is increasing w.r.t.  $\succeq^{wV}$ .*

Each proof is quite similar to that of Proposition 5.10.

**Remark.** Lemma 3.1 of Kukushkin et al. (2005) immediately follows from Proposition 5.13 ( $T^\uparrow = T$ ).

## 6 Main characterization results

**Theorem 1.** *An ordering  $\succ$  on a lattice  $A$  is quasisupermodular if and only if it has both following properties.*

1. *There holds  $M(X, \succ) \succeq^{Vt} M(X, \succ)$  whenever  $X \in \mathfrak{L}_A$ ,  $\succ$  is an ordering on  $X$ , and (4a) and (4b) hold on  $X$ .*
2. *There holds  $M(X, \succ) \succeq^{Vt} M(X, \succ)$  whenever  $X \in \mathfrak{L}_A$ ,  $\succ$  is an ordering on  $X$ , and (5a) and (5b) hold on  $X$ .*

Moreover, the “if” part survives the restriction of both requirements to  $X \in \mathfrak{L}_A$  with  $\#X \leq 4$ .

*Proof.* The equivalence immediately follows from Propositions 5.8 and 5.9. □

**Theorem 2.** *An ordering  $\succ$  on a lattice  $A$  is quasisupermodular if and only if it has all the following properties.*

1. *There holds  $M(X, \succ) \gg M(X, \succ)$  whenever  $X \in \mathfrak{L}_A$ ,  $\succ$  is an ordering on  $X$ , and (4c) holds on  $X$ .*
2. *There holds  $M(X, \succ) \gg M(X, \succ)$  whenever  $X \in \mathfrak{L}_A$ ,  $\succ$  is an ordering on  $X$ , and (5c) holds on  $X$ .*
3. *There holds  $M(X, \succ) \succeq^{wV} M(X, \succ)$  whenever  $X \in \mathfrak{L}_A$ ,  $\succ$  is an ordering on  $X$ , and (4d) holds on  $X$ .*
4. *There holds  $M(X, \succ) \succeq^{wV} M(X, \succ)$  whenever  $X \in \mathfrak{L}_A$ ,  $\succ$  is an ordering on  $X$ , and (5d) holds on  $X$ .*

Moreover, the “if” part survives the restriction of both requirements to  $X \in \mathfrak{L}_A$  with  $\#X \leq 4$ .

*Proof.* The equivalence immediately follows from Propositions 5.4–5.7. □

**Proposition 6.1.** *An ordering  $\succ$  on a lattice  $A$  is meet quasisupermodular if and only if it has both following properties.*

1. *There holds  $M(X, \succ) \geq^{\wedge} M(X, \succ)$  whenever  $X \in \mathfrak{L}_A$ ,  $\succ$  is an ordering on  $X$ , and (4a) holds on  $X$ .*
2. *There holds  $M(X, \succ) \geq^{\wedge} M(X, \succ)$  whenever  $X \in \mathfrak{L}_A$ ,  $\succ$  is an ordering on  $X$ , and (5a) holds on  $X$ .*

Moreover, the “if” part survives the restriction of both requirements to  $X \in \mathfrak{L}_A$  with  $\#X \leq 4$ .

*Proof.* The equivalence immediately follows from Propositions 5.4 and 5.5. □

**Proposition 6.2.** *An ordering  $\succ$  on a lattice  $A$  is join quasisupermodular if and only if it has both following properties.*

1. *There holds  $M(X, \succ) \geq^{\vee} M(X, \succ)$  whenever  $X \in \mathfrak{L}_A$ ,  $\succ$  is an ordering on  $X$ , and (4b) holds on  $X$ .*
2. *There holds  $M(X, \succ) \geq^{\vee} M(X, \succ)$  whenever  $X \in \mathfrak{L}_A$ ,  $\succ$  is an ordering on  $X$ , and (5b) holds on  $X$ .*

Moreover, the “if” part survives the restriction of both requirements to  $X \in \mathfrak{L}_A$  with  $\#X \leq 4$ .

*Proof.* The equivalence immediately follows from Propositions 5.6 and 5.7. □

To give our characterization results a final form, somewhat cumbersome terminology is needed. Let  $T$  and  $\bar{T}$  be two posets such that  $T = \bar{T} \setminus \{\bar{t}\}$  ( $\bar{t} \in \bar{T}$ ); let  $\bar{\mathcal{U}} = \langle \succ^t \rangle_{t \in \bar{T}}$  be a parametric family of orderings on a lattice  $A$ , and  $\mathcal{U} := \langle \succ^t \rangle_{t \in T}$ . We say that  $\bar{\mathcal{U}}$  is an *extension of  $\mathcal{U}$  with the single crossing property* if (4a) and (4b) hold for  $\succ^{\bar{t}}$  as  $\succ$  and  $\succ^t$  as  $\succ$  whenever  $T \ni t > \bar{t}$ , whereas (4a) and (4b) hold for  $\succ^t$  as  $\succ$  and  $\succ^{\bar{t}}$  as  $\succ$  whenever  $\bar{T} \ni \bar{t} > t \in T$ . Similarly,  $\bar{\mathcal{U}}$  is an *extension of  $\mathcal{U}$  with the strict [weak, meet, or join] single crossing property* if (4c) [(4d), (4a), or (4b)] holds for  $\succ^{\bar{t}}$  as  $\succ$  and  $\succ^t$  as  $\succ$  whenever  $T \ni t > \bar{t}$ , whereas (4c) [(4d), (4a), or (4b)] holds for  $\succ^t$  as  $\succ$  and  $\succ^{\bar{t}}$  as  $\succ$  whenever  $\bar{T} \ni \bar{t} > t \in T$ .

We say that an ordering  $\succ$  on a lattice  $A$  *preserves ascendance* if, whenever  $\mathcal{U}$  is a parametric family of orderings on  $A$  such that  $\mathcal{R}^X$  defined by (2) is increasing w.r.t.  $\geq^{\vee t}$  for every  $X \in \mathfrak{L}_A$ , and  $\bar{\mathcal{U}}$  is an extension of  $\mathcal{U}$  with the single crossing property such that  $\succ^{\bar{t}}$  coincides with  $\succ$ , every correspondence  $\bar{\mathcal{R}}^X$  ( $X \in \mathfrak{L}_A$ ) defined by (2) for  $\bar{\mathcal{U}}$  is increasing w.r.t.  $\geq^{\vee t}$ . Similarly, an ordering  $\succ$  on a lattice  $A$  *preserves strong [weak, meet, or join] ascendance* if, whenever  $\mathcal{U}$  is a parametric family of orderings on  $A$  such that  $\mathcal{R}^X$  defined by (2) is increasing w.r.t.  $\ggg [\geq^{\vee V}, \geq^{\wedge}, \geq^{\vee}]$  for every  $X \in \mathfrak{L}_A$ , and  $\bar{\mathcal{U}}$  is an extension of  $\mathcal{U}$  with the strict [weak, meet, or join] single crossing property such that  $\succ^{\bar{t}}$  coincides with  $\succ$ , every correspondence  $\bar{\mathcal{R}}^X$  ( $X \in \mathfrak{L}_A$ ) defined by (2) for  $\bar{\mathcal{U}}$  is increasing w.r.t.  $\ggg [\geq^{\vee V}, \geq^{\wedge}, \geq^{\vee}]$ .

**Theorem 3.** *Let  $\succ$  be an ordering on a lattice  $A$ . Then the following statements are equivalent.*

1.  $\succ$  is quasisupermodular.
2.  $\succ$  preserves ascendance.
3.  $\succ$  preserves both strong ascendance and weak ascendance.

*Proof.* The equivalence immediately follows from Theorems 1 and 2. □

**Proposition 6.3.** *An ordering  $\succ$  on a lattice  $A$  is meet (join) quasisupermodular if and only if it preserves meet (join) ascendance.*

*Proof.* The equivalence immediately follows from Proposition 6.1 (6.2). □

## 7 “Type B” problems

Given an ordering  $\succ$  on a lattice  $A$ , we may consider  $M(\cdot, \succ)$  as a mapping  $\mathfrak{B}_A \rightarrow \mathfrak{B}_A$  or  $\mathfrak{L}_A \rightarrow \mathfrak{B}_A$ . Milgrom and Shannon’s Corollary to Theorem 4 states that the quasisupermodularity of  $\succ$  is necessary and sufficient for the correspondence to be increasing w.r.t. the Veinott order  $\succeq^{\text{vt}}$  on both the source and target. It seems natural to ask what happens to this result if other relations (3) are referred to.

If we retain the idea that monotonicity should be defined w.r.t. the same relation on both the source and target, the answers will be straightforward and not especially interesting: the monotonicity of  $M(X, \succ)$  in  $X \in \mathfrak{B}_A$  w.r.t.  $\ggg$  holds for any  $\succ$ ; the monotonicity of  $M(X, \succ)$  in  $X \in \mathfrak{B}_A$  or  $X \in \mathfrak{L}_A$  w.r.t.  $\succeq^{\vee}$  ( $\succeq^{\wedge}$ ) holds if and only if  $\succ$  is in(de)creasing; the monotonicity of  $M(X, \succ)$  in  $X \in \mathfrak{L}_A$  w.r.t.  $\succeq^{\text{wV}}$  is only possible if the agent is indifferent between all outcomes. There is nothing surprising in that. The relation  $\ggg$  is so strong that it does not matter exactly what is optimized; the relations  $\succeq^{\vee}$ ,  $\succeq^{\wedge}$ , and  $\succeq^{\text{wV}}$ , on the contrary, are so weak that severe restrictions on  $\succ$  are necessary to obtain monotonicity of optima even in a very weak sense.

Let us restrict attention to  $X \in \mathfrak{L}_A \setminus \{\emptyset\}$ . Then  $\succeq^{\text{vt}}$  is an order while the other relations (3) need not be. On the other hand,  $M(X, \succ)$  need not belong to  $\mathfrak{L}_A$  if we impose no *a priori* restrictions on  $\succ$ , hence all relations (3) are in the same position. It turns out that the monotonicity of  $M(\cdot, \succ)$  as a mapping  $\mathfrak{L}_A \rightarrow \mathfrak{B}_A$  with  $\succeq^{\text{vt}}$  on the source and various relations (3) on the target is equivalent to “QSM-style” properties of  $\succ$ .

We call an ordering  $\succ$  on a lattice  $A$  *weakly* (respectively, *strictly*) *quasisupermodular* if

$$\forall x, y \in A [x \succ y \wedge x \Rightarrow y \vee x \succeq y], \tag{9a}$$

or respectively,

$$\forall x, y \in A [[x \succeq y \wedge x \ \& \ y \vee x > x > y \wedge x] \Rightarrow y \vee x \succ y]. \tag{9b}$$

It is easy to see that strict quasisupermodularity (9b) implies quasisupermodularity, i.e., both conditions (7) or all conditions (8). Either condition (7), in turn, implies weak quasisupermodularity. Every ordering in Example 5.3 is weakly quasisupermodular, hence (9a) does not imply any condition (8). Similarly, none of the conditions (8) implies (9a) by itself.

**Example 7.1.** Let  $A := \{(0, 0), (0, 1), (1, 0), (1, 1)\} \subset \mathbb{R}^2$ ; we consider two orderings on  $A$  represented by these matrices (the axes are directed upwards and rightwards):

$$\mathbf{a.} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \mathbf{b.} \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}.$$

Neither ordering is weakly quasisupermodular; that represented by the matrix “a” satisfies (8a) and (8c); that represented by the matrix “b,” (8b) and (8d).

**Proposition 7.2.** *An ordering  $\succ$  on a lattice  $A$  satisfies (9a) if it satisfies (8a) or (8c), and satisfies (8b) or (8d).*

*Proof.* Suppose the contrary:  $x \succ y \wedge x$ , but  $y \succ y \vee x$ . Then  $y \vee x \succeq x$  by (8a) or (8c), hence  $y \succ y \wedge x$  by transitivity, which is incompatible with either (8b) or (8d).  $\square$

**Remark.** In the light of Propositions 5.4–5.7, conditions (8a) and (8c) may be called “upward-looking,” while (8b) and (8d), “downward-looking.” Thus, Proposition 7.2 and Example 7.1 show that a pair of conditions (8) looking in different directions implies (9a), while a pair looking in the same direction does not. There may be a simple explanation for this mnemonic rule, but it has not been found yet.

Conditions (9), together with (7), characterize orderings ensuring “Type B” monotone comparative statics.

**Proposition 7.3.** *Let  $\succ$  be an ordering on a lattice  $A$ . Then the mapping  $F: \mathfrak{L}_A \rightarrow \mathfrak{B}_A$  defined by  $F(X) := M(X, \succ)$  is increasing w.r.t.  $\succeq^{\vee t}$  on  $\mathfrak{L}_A$  and  $\succeq^{w\vee}$  on  $\mathfrak{B}_A$  if and only if  $\succ$  is weakly quasisupermodular.*

*Proof.* Let (9a) hold,  $X, X' \in \mathfrak{L}_A$ , and  $X' \succeq^{\vee t} X$ . Whenever  $x \in M(X, \succ)$  and  $y \in M(X', \succ)$ , we have  $y \wedge x \in X$  and  $y \vee x \in X'$ . If  $y \wedge x \notin M(X, \succ)$ , then  $x \succ y \wedge x$ , hence  $y \vee x \in M(X', \succ)$  by (9a). Therefore,  $F(X') \succeq^{w\vee} F(X)$ .

Let (9a) be violated: there are  $x, y \in A$  such that  $x \succ y \wedge x$  and  $y \succ y \vee x$ . It follows immediately that  $x$  and  $y$  are incomparable in the basic order. We define  $X := \{x, y \wedge x\}$  and  $X' := \{y, y \vee x\}$ . Now  $F(X) = M(X, \succ) = \{x\}$  while  $F(X') = \{y\}$ , hence  $F(X') \succeq^{w\vee} F(X)$  does not hold.  $\square$

**Proposition 7.4.** *Let  $\succ$  be an ordering on a lattice  $A$ . Then the mapping  $F: \mathfrak{L}_A \rightarrow \mathfrak{B}_A$  defined by  $F(X) := M(X, \succ)$  is increasing w.r.t.  $\succeq^{\vee t}$  on  $\mathfrak{L}_A$  and  $\ggg$  on  $\mathfrak{B}_A$  if and only if  $\succ$  is strictly quasisupermodular.*

*Proof.* Let (9b) hold,  $X, X' \in \mathfrak{L}_A$ , and  $X' \succeq^{\vee t} X$ . Whenever  $x \in M(X, \succ)$  and  $y \in M(X', \succ)$ , we have  $y \wedge x \in X$  and  $y \vee x \in X'$ . If  $y \not\geq x$ , then we have  $x \succeq y \wedge x$  and  $y \vee x > x > y \wedge x$ , hence  $y \vee x \succ y$  by (9a), contradicting  $y \in M(X', \succ)$ .

Let (9b) be violated: there are  $x, y \in A$  such that  $x \succeq y \wedge x$  and  $y \succeq y \vee x$  while  $x$  and  $y$  are incomparable in the basic order. We define  $X := \{x, y \wedge x\}$  and  $X' := \{y, y \vee x\}$ . Now  $F(X) = M(X, \succ) \ni x$  while  $F(X') \ni y$ , hence  $F(X') \not\gg F(X)$  does not hold.  $\square$

**Proposition 7.5.** *Let  $\succ$  be an ordering on a lattice  $A$ . Then the mapping  $F: \mathfrak{L}_A \rightarrow \mathfrak{B}_A$  defined by  $F(X) := M(X, \succ)$  is increasing w.r.t.  $\succeq^{\vee t}$  on  $\mathfrak{L}_A$  and  $\succeq^\wedge$  on  $\mathfrak{B}_A$  if and only if  $\succ$  is meet quasisupermodular.*

*Proof.* Let (7a) hold,  $X, X' \in \mathfrak{L}_A$ , and  $X' \succeq^{\vee t} X$ . Whenever  $x \in M(X, \succ)$  and  $y \in M(X', \succ)$ , we have  $y \wedge x \in X$  and  $y \vee x \in X'$ . If  $y \wedge x \notin M(X, \succ)$ , then  $x \succ y \wedge x$ , hence  $y \vee x \succ y$  by (7a), contradicting  $y \in M(X', \succ)$ .

Let (7a) be violated: there are  $x, y \in A$  such that  $x \succ y \wedge x$  and  $y \succeq y \vee x$ . It follows immediately that  $x$  and  $y$  are incomparable in the basic order. We define  $X := \{x, y \wedge x\}$  and  $X' := \{y, y \vee x\}$ . Now  $F(X) = M(X, \succ) = \{x\}$  while  $F(X') \ni y$ , hence  $F(X') \not\geq F(X)$  does not hold.  $\square$

**Proposition 7.6.** *Let  $\succ$  be an ordering on a lattice  $A$ . Then the mapping  $F: \mathfrak{L}_A \rightarrow \mathfrak{B}_A$  defined by  $F(X) := M(X, \succ)$  is increasing w.r.t.  $\succeq^{\vee t}$  on  $\mathfrak{L}_A$  and  $\succeq^\vee$  on  $\mathfrak{B}_A$  if and only if  $\succ$  is join quasisupermodular.*

The proof is dual to that of Proposition 7.5.

**Remark.** Propositions 7.5 and 7.6 together imply Corollary 1 of Milgrom and Shannon (1994).

An alternative approach to monotone comparative statics can be based on monotone selections from  $M(\cdot, \succ)$ . Then, however, separation between the existence and monotonicity of optima becomes impossible, hence the prospects for comprehensive characterization results are rather dim. A neat result is obtainable when attention is restricted to the poset  $\mathfrak{F}_A$  of all finite nonempty sublattices of a lattice  $A$  with the Veinott order  $\succeq^{\vee t}$ .

**Proposition 7.7.** *Let  $\succ$  be an ordering on a lattice  $A$ . Then:*

1. *The mapping  $F: \mathfrak{F}_A \rightarrow \mathfrak{B}_A$  defined by  $F(X) := M(X, \succ)$  admits a monotone selection if and only if  $\succ$  is weakly quasisupermodular.*
2. *Every selection from the same  $F$  is increasing if and only if  $\succ$  is strictly quasisupermodular.*

*Proof.* If (9a) holds, then  $F$  is increasing by Proposition 7.3, hence a monotone selection exists by Theorem 3.2 from Veinott (1989) or Theorem 1 from Kukushkin (2009). If (9a) is violated, we consider the same  $X$  and  $X'$  as in the necessity proof in Proposition 7.3. Clearly, there is no monotone selection there.

The second statement immediately follows from Proposition 7.4.  $\square$

## 8 Concluding remarks

**8.1.** There seems to be no logical necessity for the same property of preference orderings to play prominent roles in both “Type A” and “Type B” problems. Nevertheless, quasisupermodularity does exactly that (Theorem 4 of Milgrom and Shannon, 1994, and Theorems 1–3 here). Actually, the same is true for each condition (7) separately: Propositions 5.4 and 5.5 (Propositions 5.6 and 5.7) establish connections with the monotonicity w.r.t.  $\succeq^\wedge$  ( $\succeq^\vee$ ) in “Type A” problems; Proposition 7.5 (Proposition 7.6), in “Type B” problems.

When it comes to the monotonicity w.r.t.  $\succeq^{wV}$  or  $\ggg$ , the situation changes. In the first case, the conjunction of (8a) and (8d), needed in “Type A” problems, is stronger than weak quasisupermodularity, needed in “Type B” problems. In the second case, strict quasisupermodularity, needed in “Type B” problems, is stronger than the conjunction of (8b) and (8c), needed in “Type A” problems. It is worth noting that if the main object of interest are monotone selections from  $M(\cdot, \succ)$ , the most relevant among relations (3) are just  $\succeq^{wV}$  and  $\ggg$ .

**8.2.** Five “order” relations (3) form a lattice (with the logical implication as order), which is *not* a sublattice of the lattice of all binary relations on  $\mathfrak{B}_A$ . Five single crossing conditions [four (4) and the conjunction of (4a) and (4b) – single crossing proper] form an isomorphic lattice; the same applies to five “QSM-style” conditions [both (7), both (9), and the conjunction of (7a) and (7b) – quasisupermodularity proper]. Neither is a sublattice of the lattice of all binary relations, respectively Boolean functions, on the set of orderings on  $A$ . Each of the three will become a sublattice if we add the disjunction of, respectively,  $\succeq^\wedge$  and  $\succeq^\vee$ , (4a) and (4b), or (7a) and (7b). However, there will be no analog of Propositions 4.1–4.5 or 7.3–7.6: the new conditions will be sufficient for the monotonicity w.r.t. the new “order,” but not necessary.

**Example 8.1.** Let  $A := \{0, 1, 2, 3\}$ ,  $T := \{0, 1\}$ , and a function  $u: A \times T \rightarrow \mathbb{R}$  be defined by the following matrix (the  $A$ -axis is directed rightwards; the  $T$ -axis, upwards):

$$\begin{array}{cccc} 3 & 3 & 5 & 4 \\ 0 & 1 & 2 & 2 \end{array}$$

Neither condition (4a) nor (4b) is satisfied for the orderings  $\succ'$  represented by  $u(\cdot, 1)$  and  $\succ$  represented by  $u(\cdot, 0)$ : (4a) is violated for  $x = 0$  and  $y = 1$ ; (4b), for  $x = 2$  and  $y = 3$ . On the other hand, every mapping  $\mathcal{R}^X$  ( $X \in \mathfrak{B}_A$ ) is increasing w.r.t. the disjunction of  $\succeq^\wedge$  and  $\succeq^\vee$ :  $\mathcal{R}^A(1) = \{2\}$  while  $\mathcal{R}^A(0) = \{2, 3\}$ , hence  $\mathcal{R}^A(1) \succeq^\wedge \mathcal{R}^A(0)$ ; on  $A \setminus \{3\}$  as well as on  $A \setminus \{2\}$ , (4a) holds; on  $A \setminus \{0\}$  as well as on  $A \setminus \{1\}$ , (4b) holds.

Similarly, the ordering  $\succ$  on the lattice  $A \times T$  represented by  $u$  satisfies neither condition (7): (7a) is violated in the leftmost  $2 \times 2$  cell; (7b), in the rightmost  $2 \times 2$  cell. On the other hand, the mapping  $M(\cdot, \succ): \mathfrak{L}_{A \times T} \rightarrow \mathfrak{B}_{A \times T}$  is increasing w.r.t.  $\succeq^{Vt}$  on the source and the disjunction of  $\succeq^\wedge$  and  $\succeq^\vee$  on the target. Let  $X, Y \in \mathfrak{L}_{A \times T}$  and  $Y \succeq^{Vt} X$ . By Theorem 2 of Milgrom and Shannon (1994), there are  $Z \in \mathfrak{L}_{A \times T}$  and  $a^+, a^- \in Z$

such that  $Y = \{a \in Z \mid a \geq a^+\}$  and  $X = \{a \in Z \mid a \leq a^-\}$ . If  $Z$  is a chain, we have  $M(Y, \succ) \geq^{\vee t} M(X, \succ)$  by Milgrom and Shannon's Corollary 1 because every ordering on a chain is quasisupermodular. If  $(2, 1) \notin Z \ni (3, 1)$ , then  $M(Y, \succ) = \{(3, 1)\}$ , hence  $M(Y, \succ) \gg M(X, \succ)$ . If  $(2, 1) \in Z$ , then  $M(Y, \succ) = \{(2, 1)\}$ , hence  $M(Y, \succ) \gg M(X, \succ)$  unless  $M(X, \succ) = \{(2, 0), (3, 0)\}$ , in which case  $M(Y, \succ) \geq^{\wedge} M(X, \succ)$ . Finally, if  $Z \cap \{(2, 1), (3, 1)\} = \emptyset$ , then  $Z$  is contained in the leftmost  $2 \times 2$  cell, where  $\succ$  is join quasisupermodular, hence  $M(Y, \succ) \geq^{\vee} M(X, \succ)$  by Proposition 7.6.

It may also be noted that the monotonicity w.r.t. the disjunction of  $\geq^{\wedge}$  and  $\geq^{\vee}$  seems not to lead to any new result on the existence of monotone selections.

**8.3.** In Section 4.1 of Reny (2009), the expressions “weak single crossing” and “weak quasisupermodularity” are applied to what is called here “join single crossing” and “join quasisupermodularity,” respectively. Our use of the epithet “weak” is consistent with Veinott's (1989) original usage; in particular, it always refers to self-dual conditions. (What is called “meet quasisupermodularity” here would have to be called something like “weak\* quasisupermodularity” if Reny considered meet semilattices.)

**8.4.** Assuming  $\succ$  represented with a mapping  $u$  from  $A$  to a chain, conditions (8) can be written in a “more algebraic” style, cf. Agliardi (2000) (or Veinott, 1989, for that matter).

$$\forall x, y \in A [u(y) \vee u(x) > u(y \wedge x) \Rightarrow u(y \vee x) > u(y) \wedge u(x)]; \quad (10a)$$

$$\forall x, y \in A [u(y) \vee u(x) \geq u(y \vee x) \Rightarrow u(y \wedge x) \geq u(y) \wedge u(x)]; \quad (10b)$$

$$\forall x, y \in A [u(y) \vee u(x) \geq u(y \wedge x) \Rightarrow u(y \vee x) \geq u(y) \wedge u(x)]; \quad (10c)$$

$$\forall x, y \in A [u(y) \vee u(x) > u(y \vee x) \Rightarrow u(y \wedge x) > u(y) \wedge u(x)]. \quad (10d)$$

Clearly, each condition (10) is equivalent to the corresponding condition (8).

**8.5.** The description of preferences with an ordering may seem very general, but it may also seem not general enough. Leaving aside the abstruse question of how much rationality in an agent's preferences it is right to assume, there is a mundane reason to go beyond orderings. Suppose the utility function  $u(x, t)$  is bounded above in  $x$  for every  $t$ , but need not attain a maximum; then  $\varepsilon$ -optimization suggests itself strongly, and this means considering a preference relation

$$y \succ^t x \iff u(y, t) > u(x, t) + \varepsilon$$

(with  $\varepsilon > 0$ ).  $\mathcal{R}^X(t)$  consists of all  $\varepsilon$ -maxima of  $u(\cdot, t)$ . The relation  $\succ^t$  is a *strongly acyclic semiorder*, but need not be an ordering. If  $u$  satisfies Topkis's (1978) *increasing differences* condition, then  $\{\succ^t\}_{t \in T}$  satisfies the single crossing conditions; if  $u$  is supermodular in the first argument, then  $\succ^t$  satisfies both (7). Nevertheless, none of the results of this paper is applicable even under so strong assumptions; actually,  $\mathcal{R}^X$  need not be ascending. The existence of a monotone selection can be proven when both  $X$  and  $T$  are chains; the existence of an  $\varepsilon$ -Nash equilibrium, when every strategy set is a chain (Kukushkin, 2009, Theorems 3 and 4). However, there is no similar result of any kind for non-scalar sets  $X$ .

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