Monotone comparative statics: Changes in preferences vs changes in the feasible set

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Abstract

Let a preference ordering on a lattice be perturbed. As is well known, single crossing conditions are necessary and sufficient for a monotone reaction of the set of optimal choices from every chain. Actually, there are several interpretations of monotonicity and several corresponding single crossing conditions. We describe restrictions on the preferences that ensure a monotone reaction of the set of optimal choices from every sublattice whenever a perturbation of preferences satisfies the corresponding single crossing condition. Quasisupermodularity is necessary if we want monotonicity in every conceivable sense; otherwise, weaker conditions will do.

Key words strategic complementarity; monotone comparative statics; best response correspondence; single crossing; quasisupermodularity

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1 Introduction

The concept of strategic complementarity was first developed in a cardinal form, around the notion of a supermodular function (Topkis, 1978, 1979; Veinott, 1989; Vives, 1990; Milgrom and Roberts, 1990). Milgrom and Shannon (1994) gave the idea an ordinal expression and obtained a neat characterization result, Theorem 4 in their paper. For our purposes here, that result is better perceived as two independent statements on comparative statics, the first relating to changes in the feasible set and the second to changes in the utility function; in Quah's (2007) terminology, these are, respectively, "type B" and "type A" problems.

Concerning type B problems, Milgrom and Shannon (1994, Corollary 1) showed that the set of optimal choices depends on the sublattice of available choices in a monotone way if and only if the utility function is quasisupermodular. For type A problems, they showed that, given a parametric family of quasisupermodular utility functions, the set of optimal choices from every sublattice is monotone in the parameter if and only if the single crossing condition holds.

It should be stressed that Milgrom and Shannon established the necessity of quasisupermodularity only in type B problems; actually, there are sufficient conditions for increasing best responses in strategic games that *do not* imply this property of utility functions (Kukushkin et al., 2005, Lemma 3.1). Also restricted to type B problems are similar necessity results concerning weak (Shannon, 1995) or strong (Milgrom and Shannon, 1994; Shannon, 1995) monotonicity.

This paper presents characterization results for type A problems. Our approach has these three distinguishing features. First, we describe preferences with binary relations (orderings) rather than utility functions. Second, we simultaneously keep in mind several interpretations of monotonicity, following in this respect LiCalzi and Veinott (1992).

Finally, we concentrate on a single perturbation of preferences. As is well known, single crossing conditions (of various kinds) are necessary and sufficient for a monotone reaction of the set of optimal choices from every *chain*. We describe the exact restrictions on the preferences that ensure a monotone reaction of the set of optimal choices from every *sublattice* whenever a perturbation of preferences satisfies the corresponding single crossing condition. Roughly speaking, we partition each condition that has already emerged in type B problems into two "halves," the first ensuring a monotone response when the preference ordering becomes more like the basic order, the second, in the opposite case.

The understanding of consequences of a single perturbation allows us to obtain new sufficient conditions for increasing best responses that do not imply quasisupermodularity, as well as to characterize preference orderings that can be inserted into parametric settings without destroying the monotonicity of the set of optima. In particular, quasisupermodularity *is* necessary in type A problems if we want monotonicity in every conceivable sense; otherwise, weaker conditions will do.

Our principal motivation is technical, even aesthetic: Since monotonicity considerations play a significant role in economic theory, in particular, in game theory, a deeper understanding of interrelationships between various properties cannot be useless. A specific promising area is the existence of pure strategy monotone equilibria in Bayesian games (like, e.g., first-price auctions). As Athey (2001) was first to show, ordinal notions like single crossing or quasisupermodularity can be surprisingly helpful in solving the equilibrium existence problem in such games. Generalizing Athey, Reny (2011) found that rather weak versions of the conditions (not considered in Milgrom and Shannon, 1994, or Shannon, 1995) are actually sufficient to derive quite strong conclusions. The new, weaker, versions developed here may help obtain even stronger results.

Section 2 reproduces the standard notion of a choice function generated by the maximization of a binary relation; in Section 3, we define a number of extensions of an order from points to subsets. In Sections 4 and 5, we reproduce conditions related, respectively, to type A problems on chains (single crossing) and type B problems on sublattices (various versions of quasisupermodularity).

The central results of the paper are in Sections 6 and 7. The former contains the conditions related to type A problems on sublattices and their characterizations; the latter considers parametric optimization. A few concluding remarks are in Section 8.

2 Preferences and choice

Throughout the paper, we assume a set A of alternatives given. There is an agent whose preferences over the alternatives are expressed by a binary relation $\succ$ on A, which is assumed to be an *ordering*, i.e., *irreflexive*, *transitive*, and *negatively transitive* ($z \not\succ y \not\succ x \Rightarrow z \not\succ x$). Then the "non-strict preference" relation $\succeq$ defined by $y \succeq x \rightleftharpoons x \not\succ y$ is reflexive, transitive, and total; the indifference $y \sim x \rightleftharpoons [x \not\succ y \;\&\; y \not\succ x]$ is an equivalence relation.

Orderings can also be defined in terms of representations in chains: $\succ$ is an ordering if and only if there is a chain $\mathcal{C}$ and a mapping $u\colon A \to \mathcal{C}$ such that $y \succ x \iff u(y) > u(x)$ for all $x, y \in A$ (then $y \succeq x \iff u(y) \geq u(x)$). The most usual assumption in game theory is that the preferences of a player are described by a *utility function* $u\colon A \to \mathbb{R}$. Here we work in a purely ordinal framework, so it is natural to replace $\mathbb{R}$ with an arbitrary chain (e.g., $\mathbb{R}^m$ with a lexicographic order).

As is usual in decision theory, we allow for the possibility that only a subset $X \subseteq A$ may be available for choice. The set of all subsets of A is denoted $\mathfrak{B}_A$. Given $X \in \mathfrak{B}_A$, we define

$$M(X, \succ) := \{x \in X \mid \nexists y \in X\,[y \succ x]\} = \{x \in X \mid \forall y \in X\,[x \succeq y]\}, \tag{1}$$

the set of *maximizers* of $\succ$ on X. A very helpful observation is that $y \succ x$ whenever $\succ$ is an ordering, $x \in X$, and $x \notin M(X, \succ) \ni y$ ("revealed preference").

Clearly, $M(X, \succ) \neq \emptyset$ if X is a finite nonempty subset of A. We do not restrict ourselves to finite subsets here; nor do we study more general conditions for the existence of maximizers. In a sense, we brush aside the distinction between an empty set $M(X, \succ)$ and a nonempty one. A rationalization for this attitude, following Shannon (1995, p. 213), is given in Section 3.

We also consider *parametric families* $\langle \succ^t \rangle_{t \in T}$ of orderings on A, the parameter t reflecting the impact of actions of other players or impersonal external forces (e.g., prices in outside markets).

Given a parametric family and $X \in \mathfrak{B}_A$, the *best response correspondence* $\mathcal{R}^X : T \to \mathfrak{B}_X \subseteq \mathfrak{B}_A$ is defined in the usual way:

$$\mathcal{R}^X(t) := M(X, \succ^t). \tag{2}$$

Describing the preferences of a player in a strategic game by a parametric family of orderings on the player's strategy set X, we lose some information available in the standard model (an ordering on $X \times T$ would be an adequate ordinal description). It is impossible to discuss, say, strong equilibria or the (in)efficiency of Nash equilibria in our framework. On the other hand, a parametric family of orderings is adequate when the subject is the existence or comparative statics of Nash equilibria, or individual adaptive dynamics.

Remark. Some twenty five years ago, Olga Bondareva argued that the proper definition of a non-cooperative game (in an ordinal context) must stipulate that each player is only able to compare strategy profiles differing in her own choice. Although one does not have to accept this, rather extreme, view, there is something to it.

3 Monotonicity

We always assume A to be a *partially ordered set* (a *poset*). Most often, it is a *lattice*, in which case $\mathfrak{L}_A$ denotes the set of all sublattices of A. The exact definitions are assumed commonly known. Given a lattice A and $x, y \in A$, we denote $L(x, y) := \{x, y, x \vee y, x \wedge y\}$, the minimal sublattice of A containing both x and y; clearly, $\#L(x, y) \in \{1, 2, 4\}$.

The reversal of an order ($y < x \rightleftharpoons x > y$) produces an order again; moreover, a lattice remains a lattice. Having proved a theorem, we can replace, in all assumptions and the statement itself, the relations and operations $>$, $\geq$, $\vee$, etc. with $<$, $\leq$, $\wedge$, etc., and obtain another valid theorem. The use of this simple observation (referred to as "duality") leads to considerable economy in the total length of proofs.

To discuss monotonicity, we extend the order from A to $\mathfrak{B}_A$. Following Veinott (1989), we consider four ways to do so for a lattice A:

$$Y \geq^{\wedge} X \rightleftharpoons \forall y \in Y \, \forall x \in X \, [y \wedge x \in X]; \tag{3a}$$

$$Y \geq^{\vee} X \rightleftharpoons \forall y \in Y \, \forall x \in X \, [y \vee x \in Y]; \tag{3b}$$

$$Y \geq^{\mathrm{Vt}} X \rightleftharpoons [Y \geq^{\vee} X \; \& \; Y \geq^{\wedge} X]; \tag{3c}$$

$$Y \geq^{\mathrm{wV}} X \rightleftharpoons \forall y \in Y \, \forall x \in X \, [y \vee x \in Y \;\text{ or }\; y \wedge x \in X]. \tag{3d}$$

One more relation can be defined for any poset A:

$$Y \ggg X \rightleftharpoons \forall y \in Y \, \forall x \in X \, [y \geq x]. \tag{4}$$

Sometimes, we employ a few more relations on $\mathfrak{B}_A$ that also make sense for any poset A:

$$Y \geqslant^{\mathrm{Up}} X \rightleftharpoons \forall y \in Y \setminus X \, \forall x \in X \, [y > x]; \tag{5a}$$

$$Y \geqslant^{\mathrm{Dn}} X \rightleftharpoons \forall y \in Y \, \forall x \in X \setminus Y \, [y > x]; \tag{5b}$$

$$Y \geqslant X \rightleftharpoons [Y \geqslant^{\mathrm{Up}} X \;\&\; Y \geqslant^{\mathrm{Dn}} X]; \tag{5c}$$

$$Y \geqslant^{\mathrm{w}} X \rightleftharpoons \forall y \in Y \setminus X \, \forall x \in X \setminus Y \, [y > x]. \tag{5d}$$

Each of the relations (3), (4), or (5) holds trivially if either Y or X is empty; it is this fact that makes separation between existence and monotonicity possible (nonexistence cannot destroy monotonicity).

If attention is restricted to *nonempty* subsets, then $\geqslant$ is a partial order, i.e., reflexive, antisymmetric and transitive relation. $\ggg$ and $\geq^{\mathrm{Vt}}$ are antisymmetric and transitive, but generally not reflexive; none of the other relations (3) or (5) need even be transitive. If the order on A is reversed, $Y \geq^{\wedge} X$ transforms into $X \geq^{\vee} Y$, $Y \geq^{\vee} X$ into $X \geq^{\wedge} Y$, $Y \geqslant^{\mathrm{Dn}} X$ transforms into $X \geqslant^{\mathrm{Up}} Y$, $Y \geqslant^{\mathrm{Up}} X$ into $X \geqslant^{\mathrm{Dn}} Y$, and $Y \geq^{*} X$ into $X \geq^{*} Y$ for $\geq^{*}$ defined by any other relation (3), (4), or (5).

Remark. The relation $\geq^{\mathrm{Vt}}$ is often called "strong set order." It was introduced into game-theoretic literature by Topkis (1978, p. 308), who gave the relation no particular name, but unambiguously ascribed it to "Veinott (personal communication)." Actually, $\ggg$ is stronger than any relation (3) or (5).

To facilitate comparisons with LiCalzi and Veinott (1992), we also define an auxiliary relation on $\mathfrak{B}_A$:

$$Y \bowtie X \rightleftharpoons \forall y \in Y \, \forall x \in X \, [y \geq x \;\text{ or }\; x \geq y]. \tag{6}$$

The conjunction of any one relation (3) with $\bowtie$ is equivalent to the conjunction of the corresponding relation (5) with $\bowtie$. In particular, relations (3) are equivalent to relations (5) when A is a chain.

When considering a parametric family of preference relations, we assume that T is also a poset. Let $\geq^{*}$ denote one of the relations (3), (4), or (5). A correspondence $R\colon T \to \mathfrak{B}_A$ is *increasing w.r.t.* $\geq^{*}$ if $R(t') \geq^{*} R(t)$ whenever $t' > t$. Veinott (1989) called correspondences increasing w.r.t. $\geq^{\mathrm{Vt}}$ ($\geq^{\mathrm{wV}}$) in this sense (*weakly*) *ascending*. Topkis (1978) called correspondences increasing w.r.t. $\ggg$ *strongly ascending*; those increasing w.r.t. $\geq^{\wedge}$ ($\geq^{\vee}$), may be called "meet" ("join") ascending; those increasing w.r.t. $\geqslant$, "semi-strongly" ascending.

Such monotonicity is closely related to the existence of *monotone selections* from R [i.e., increasing mappings $r\colon T \to A$ such that $r(t) \in R(t)$ for every $t \in T$], provided $R(t) \neq \emptyset$ for all t. If a correspondence $R\colon T \to \mathfrak{B}_A$ is increasing w.r.t. $\ggg$, then every selection from R is increasing. If R is increasing w.r.t. $\geq^{\wedge}$, $\geq^{\vee}$, or $\geq^{\mathrm{wV}}$, then a monotone selection exists under a completeness assumption about every value $R(t)$ (Veinott, 1989, Theorem 3.2; Kukushkin, 2009, Proposition 3.1 and Theorem 1); naturally, a stronger assumption is needed in the last case. If R is increasing w.r.t. $\geq^{\mathrm{Vt}}$, then no completeness assumption at all is needed provided A is a sublattice of the Cartesian product of a finite number of chains (Kukushkin, 2009, Theorem 2). If R is increasing w.r.t. $\geqslant$, then no further restrictions are needed for the existence of a monotone selection.

We are interested in conditions on the preferences ensuring monotonicity, w.r.t. one or another of the relations (3), (4), or (5), of correspondences $\mathcal{R}^X$ defined by (2). The monotonicity of a single correspondence $\mathcal{R}^X$ may happen just "by accident"; however, when a wide enough class of admissible subsets X is taken into account, necessity results become obtainable. In Sections 4–6, we concentrate on a single perturbation of $\succ$ or X, returning to the correspondence $\mathcal{R}$ as a whole in Section 7.

A general theorem about ordered sets, see, e.g., Dushnik and Miller (1941), plays an important technical role here.

Szpilrajn Theorem. *On every poset A, there exists a linear extension of the basic order, i.e., an irreflexive and transitive binary relation $\gg$ such that $\big[y \neq x \Rightarrow [y \gg x \quad or \quad x \gg y]\big]$ and $\big[y > x \Rightarrow y \gg x\big]$ for all $x, y \in A$.*

4 Type A problems on chains: Single crossing

It is well known that the "single crossing" conditions of various kinds (Milgrom and Shannon, 1994; Shannon, 1995) are important for the monotonicity of best responses. Those conditions are most conveniently presented with the help of a ternary relation on the set of binary relations on a given set: "$\rhd_1$ is closer to $\rhd_0$ than $\rhd_2$ is"; similar observations were made by Quah and Strulovici (2009) and Alexei Savvateev (a seminar presentation, 2007). In the following, the role of $\rhd_0$ is always played by the basic order on A, while $\rhd_1$ and $\rhd_2$ are (strict or non-strict) preference relations.

Let $\succ$ and $\succ'$ be orderings on a poset A. We consider four conditions:

$$\forall x, y \in A \left[y > x \;\&\; y \succ x \Rightarrow y \succ' x\right]; \tag{7a}$$

$$\forall x, y \in A \left[y > x \;\&\; y \succeq x \Rightarrow y \succeq' x\right]; \tag{7b}$$

$$\forall x, y \in A \left[y > x \;\&\; y \succeq x \Rightarrow y \succ' x\right]; \tag{7c}$$

$$\forall x, y \in A \left[y > x \;\&\; y \succ x \Rightarrow y \succeq' x\right]. \tag{7d}$$

Each condition defines a binary relation on the set of orderings on A. The first two are preorders. The third is transitive, but generally not reflexive. The last relation is reflexive, but need not be transitive.

For more convenience in further referencing, we consider four "reversed" versions of conditions (7):

$$\forall x, y \in A \left[y < x \;\&\; y \succeq x \Rightarrow y \succeq' x\right]; \tag{8a}$$

$$\forall x, y \in A \left[y < x \;\&\; y \succ x \Rightarrow y \succ' x\right]; \tag{8b}$$

$$\forall x, y \in A \left[y < x \;\&\; y \succeq x \Rightarrow y \succ' x\right]; \tag{8c}$$

$$\forall x, y \in A \left[y < x \;\&\; y \succ x \Rightarrow y \succeq' x\right]. \tag{8d}$$

It is easily checked that each condition (8) is equivalent to the corresponding condition (7) after the exchange of the roles of $\succ$ and $\succ'$. If the order on A is reversed, conditions (7a), (7b), (7c), and (7d) transform into (8b), (8a), (8c), and (8d), respectively.

An ordering $\succ$ on a poset A is (*strictly*) *increasing* if $y \succeq x$ ($y \succ x$) whenever $y > x$. The Szpilrajn Theorem shows the existence of a strictly increasing ordering on every poset. If $\succ$ is [strictly] increasing, then (8b) [(8c)] holds for every $\succ'$. (*Strictly*) *decreasing* orderings are defined dually and have dual properties.

Proposition 1. *Let $\succ$ and $\succ'$ be orderings on a poset A. Then the following statements are equivalent.*

1. *Condition* (7a) *holds.*
2. *There holds $M(X, \succ') \geq^{\wedge} M(X, \succ)$ whenever $X \in \mathfrak{B}_A$ is a chain.*
3. *There holds $M(X, \succ') \geq^{\wedge} M(X, \succ)$ whenever $X \in \mathfrak{B}_A$ is a chain with $\#X = 2$.*

Proof. Let Statement 1 hold, $X \in \mathfrak{B}_A$ be a chain, $y \in M(X, \succ')$ and $x \in M(X, \succ)$. If $y \geq x$, we are home immediately; let $x > y$. If $y \in M(X, \succ)$, we are home again. If $y \notin M(X, \succ)$, then $x \succ y$, hence $x \succ' y$ by (7a), contradicting the assumption $y \in M(X, \succ')$.

Let Statement 1 be violated: there are $x, y \in A$ such that $y > x$, $y \succ x$, but $x \succeq' y$. Then we define $X := \{x, y\}$ and immediately obtain $x \in M(X, \succ') \setminus M(X, \succ)$ while $M(X, \succ) = \{y\}$, hence $M(X, \succ') \geq^{\wedge} M(X, \succ)$ does not hold, i.e., Statement 3 is invalid. $\square$

Proposition 2. *Let $\succ$ and $\succ'$ be orderings on a poset A. Then the following statements are equivalent.*

1. *Condition* (7b) *holds.*
2. *There holds $M(X, \succ') \geq^{\vee} M(X, \succ)$ whenever $X \in \mathfrak{B}_A$ is a chain.*
3. *There holds $M(X, \succ') \geq^{\vee} M(X, \succ)$ whenever $X \in \mathfrak{B}_A$ is a chain with $\#X = 2$.*

The proof is dual to that of Proposition 1.

Proposition 3 (Milgrom and Shannon, 1994). *Let $\succ$ and $\succ'$ be orderings on a poset A. Then the following statements are equivalent.*

1. *Conditions* (7a) *and* (7b) *hold.*
2. *There holds $M(X, \succ') \geq^{\mathrm{Vt}} M(X, \succ)$ whenever $X \in \mathfrak{B}_A$ is a chain.*
3. *There holds $M(X, \succ') \geq^{\mathrm{Vt}} M(X, \succ)$ whenever $X \in \mathfrak{B}_A$ is a chain with $\#X = 2$.*

Proposition 4 (Shannon, 1995). *Let $\succ$ and $\succ'$ be orderings on a poset A. Then the following statements are equivalent.*

1. *Condition* (7c) *holds.*
2. *There holds* $M(X, \succ') \ggg M(X, \succ)$ *whenever* $X \in \mathfrak{B}_A$ *is a chain.*
3. *There holds* $M(X, \succ') \ggg M(X, \succ)$ *whenever* $X \in \mathfrak{B}_A$ *is a chain with* $\#X = 2$.

Proposition 5 (Shannon, 1995). *Let* $\succ$ *and* $\succ'$ *be orderings on a poset* A. *Then the following statements are equivalent.*

1. *Condition* (7d) *holds.*
2. *There holds* $M(X, \succ') \geq^{\mathrm{wV}} M(X, \succ)$ *whenever* $X \in \mathfrak{B}_A$ *is a chain.*
3. *There holds* $M(X, \succ') \geq^{\mathrm{wV}} M(X, \succ)$ *whenever* $X \in \mathfrak{B}_A$ *is a chain with* $\#X = 2$.

Type B problems when A is a chain present no difficulty.

Proposition 6. *Let* $\succ$ *be an ordering on a chain* A *and let* $\geq^*$ *be one of relations* (3c), (4), *or* (5c) *on* $\mathfrak{B}_A$. *Then* $M(Y, \succ) \geq^* M(X, \succ)$ *whenever* $Y, X \in \mathfrak{B}_A$ *and* $Y \geq^* X$.

A straightforward proof is omitted.

5 Type B problems on sublattices: Quasisupermodularity

Naturally, one does not have to be satisfied with maximization on chains, although scalar strategies are met in economics models most often. When the attention is switched to *lattices*, both type A and type B problems require more complicated answers than those given by Propositions 1–5 or Proposition 6, respectively. We start with a reproduction of basic conditions and results related to type B problems. First of all, when the relation $\geq^{\mathrm{Vt}}$ is restricted to $\mathfrak{L}_A$, it becomes a partial order; accordingly, sublattices of A are usually compared w.r.t. this relation.

Milgrom and Shannon (1994) introduced the notion of a *quasisupermodular* (*QSM*) function on a lattice. Their definition can easily be reformulated in terms of a preference ordering (Alexei Savvateev, a seminar presentation, 2007):

$$\forall x, y \in A \left[x \succ y \wedge x \Rightarrow y \vee x \succ y \right]; \tag{9a}$$

$$\forall x, y \in A \left[x \succeq y \wedge x \Rightarrow y \vee x \succeq y \right]. \tag{9b}$$

In light of the following results, it seems reasonable to call (9a) *meet quasisupermodularity* ($\wedge$-*QSM*) and (9b) *join quasisupermodularity* ($\vee$-*QSM*).

An ordering $\succ$ on a lattice A is *strictly quasisupermodular* (*SQSM*) if

$$\forall x, y \in A \left[[y \vee x > x > y \wedge x \;\&\; x \succeq y \wedge x] \Rightarrow y \vee x \succ y \right]; \tag{9c}$$

$\succ$ is *weakly quasisupermodular* (*wQSM*) if

$$\forall x, y \in A \left[x \succ y \wedge x \Rightarrow y \vee x \succeq y\right]. \tag{9d}$$

It is easy to see that strict quasisupermodularity (9c) implies every other condition (9), hence quasisupermodularity as well. Weak quasisupermodularity (9d) is implied by every other condition (9).

Remark. Reny (2011, Section 4.1) applies the term "weak quasisupermodularity" to what is called "join quasisupermodularity" (9b) here; our usage follows Shannon (1995) and, in a broader sense, Veinott (1989).

Proposition 7 (LiCalzi and Veinott, 1992, Theorem 1). *Let $\succ$ be an ordering on a lattice A. Then the following statements are equivalent.*

1. *Condition* (9d) *holds, i.e., $\succ$ is wQSM.*
2. *There holds $M(Y, \succ) \geq^{\mathrm{wV}} M(X, \succ)$ whenever $Y, X \in \mathfrak{L}_A$ and $Y \geq^{\mathrm{Vt}} X$.*
3. *There holds $M(Y, \succ) \geq^{\mathrm{wV}} M(X, \succ)$ whenever $Y, X \in \mathfrak{L}_A$, $Y \geq^{\mathrm{Vt}} X$, and $\#Y = \#X = 2$.*

Proposition 8 (LiCalzi and Veinott, 1992, Theorem 3). *Let $\succ$ be an ordering on a lattice A. Then the following statements are equivalent.*

1. *Condition* (9a) *holds, i.e., $\succ$ is $\wedge$-QSM.*
2. *There holds $M(Y, \succ) \geq^{\wedge} M(X, \succ)$ whenever $Y, X \in \mathfrak{L}_A$ and $Y \geq^{\mathrm{Vt}} X$.*
3. *There holds $M(Y, \succ) \geq^{\wedge} M(X, \succ)$ whenever $Y, X \in \mathfrak{L}_A$, $Y \geq^{\mathrm{Vt}} X$, and $\#Y = \#X = 2$.*

Proposition 9 (LiCalzi and Veinott, 1992, Theorem 2). *Let $\succ$ be an ordering on a lattice A. Then the following statements are equivalent.*

1. *Condition* (9b) *holds, i.e., $\succ$ is $\vee$-QSM.*
2. *There holds $M(Y, \succ) \geq^{\vee} M(X, \succ)$ whenever $Y, X \in \mathfrak{L}_A$ and $Y \geq^{\mathrm{Vt}} X$.*
3. *There holds $M(Y, \succ) \geq^{\vee} M(X, \succ)$ whenever $Y, X \in \mathfrak{L}_A$, $Y \geq^{\mathrm{Vt}} X$, and $\#Y = \#X = 2$.*

Remark. Propositions 8 and 9 together imply Corollary 1 of Milgrom and Shannon (1994).

Proposition 10 (LiCalzi and Veinott, 1992, Theorem 5). *Let $\succ$ be an ordering on a lattice A. Then the following statements are equivalent.*

1. *Condition* (9c) *holds, i.e., $\succ$ is SQSM.*
2. *There hold $M(Y, \succ) \geqslant M(X, \succ)$ and $M(Y, \succ) \bowtie M(X, \succ)$ whenever $Y, X \in \mathfrak{L}_A$ and $Y \geq^{\mathrm{Vt}} X$.*

3. There holds $M(Y, \succ) \geqslant^{\mathrm{w}} M(X, \succ)$ *whenever* $Y, X \in \mathfrak{L}_A$, $Y \geq^{\mathrm{Vt}} X$, *and* $\#Y = \#X = 2$.

Remark. Actually, LiCalzi and Veinott (1992) did not consider relations (5), only the conjunction of $\geq^{\mathrm{Vt}}$ and $\bowtie$, which is equivalent to the conjunction of $\geqslant$ and $\bowtie$.

Since every lattice of two points is a chain, Propositions 7–10 show that monotone comparative statics in type B problems on chains or on sublattices require the same list of conditions.

In principle, we might study the monotonicity of $M(\cdot, \succ) \colon \mathfrak{L}_A \to \mathfrak{B}_A$, or $\mathfrak{B}_A \to \mathfrak{B}_A$, when one of "order" relations (3), (4), or (5) is imposed on both the source and target. However, the answers would be straightforward and not especially interesting: the monotonicity of $M(X, \succ)$ in $X \in \mathfrak{B}_A$ w.r.t. $\ggg$ or $\geqslant$ holds for any $\succ$, cf. Proposition 6; the monotonicity of $M(X, \succ)$ in $X \in \mathfrak{B}_A$ or $X \in \mathfrak{L}_A$ w.r.t. $\geq^{\wedge}$ or $\geqslant^{\mathrm{Up}}$ holds if and only if $\succ$ is decreasing; the monotonicity of $M(X, \succ)$ in $X \in \mathfrak{B}_A$ or $X \in \mathfrak{L}_A$ w.r.t. $\geq^{\vee}$ or $\geqslant^{\mathrm{Dn}}$ holds if and only if $\succ$ is increasing; the monotonicity of $M(X, \succ)$ in $X \in \mathfrak{L}_A$ or $X \in \mathfrak{B}_A$ w.r.t. $\geq^{\mathrm{wV}}$ or $\geqslant^{\mathrm{w}}$ holds if and only if the agent is indifferent between all outcomes. There is nothing surprising in that. The relations $\ggg$ and $\geqslant$ are so strong that it does not matter exactly what is optimized; the other relations, on the contrary, are so weak that severe restrictions on $\succ$ are necessary to obtain monotonicity of optima even in a very weak sense.

6 Type A problems on sublattices

Returning to type A problems, we start by noticing that the *necessity* of conditions (7) for a monotone response to a perturbation of preferences holds on any class of admissible subsets that contains all finite chains (but not otherwise, see Quah and Strulovici, 2009). The sufficiency is less robust.

Example 1. Let $A := \{(0,0), (0,1), (1,0), (1,1)\} \subset \mathbb{R}^2$, and two orderings on A be defined by the following matrices (the axes are directed upwards and rightwards):

$$\begin{array}{cc} \succ & \succ' \\ \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} & \begin{bmatrix} 5 & 4 \\ 0 & 6 \end{bmatrix} \end{array}$$

Clearly, (7c) holds; moreover, even the strictly increasing differences condition holds. However, $M(A, \succ) = \{(0,1)\}$, while $M(A, \succ') = \{(1,0)\}$. Therefore, $M(A, \succ') \geq^{*} M(A, \succ)$ does not hold for any relation (3), (4), or (5) as $\geq^{*}$.

Monotone response of every set $M(X, \succ)$ ($X \in \mathfrak{L}_A$) to every perturbation of preferences satisfying an appropriate single crossing condition (7), or (8), is restored if we impose restrictions on $\succ$. Roughly speaking, we partition each condition (9) into two "halves" – "upward-looking" and "downward-looking" ones. The following list of conditions is obtained by applying a uniform procedure to each condition (9), viz. we retain the same left hand side and replace the right hand side with the disjunction of two alternatives:

$$\forall x, y \in A \left[x \succ y \wedge x \Rightarrow [(y \vee x \succ x) \text{ or } (y \vee x \succ y)]\right]; \tag{10a}$$
$$\forall x, y \in A \left[x \succeq y \wedge x \Rightarrow [(y \vee x \succeq x) \text{ or } (y \vee x \succeq y)]\right]; \tag{10b}$$
$$\forall x, y \in A \left[[y \vee x > x > y \wedge x \;\&\; x \succeq y \wedge x] \Rightarrow [(y \vee x \succ x) \text{ or } (y \vee x \succ y)]\right]; \tag{10c}$$
$$\forall x, y \in A \left[x \succ y \wedge x \Rightarrow [(y \vee x \succeq x) \text{ or } (y \vee x \succeq y)]\right]. \tag{10d}$$

To obtain the next list, we re-write each condition (9) using the tautology $(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)$, and then apply the same procedure. In other words, we have the negation of the right hand side of the appropriate condition (9) in the left hand side and, again, the disjunction of two alternatives in the right hand side:

$$\forall x, y \in A \left[y \succeq y \vee x \Rightarrow [(y \wedge x \succeq x) \text{ or } (y \wedge x \succeq y)]\right]; \tag{11a}$$
$$\forall x, y \in A \left[y \succ y \vee x \Rightarrow [(y \wedge x \succ x) \text{ or } (y \wedge x \succ y)]\right]; \tag{11b}$$
$$\forall x, y \in A \left[[y \vee x > x > y \wedge x \;\&\; y \succeq y \vee x] \Rightarrow [(y \wedge x \succ x) \text{ or } (y \wedge x \succ y)]\right]; \tag{11c}$$
$$\forall x, y \in A \left[y \succ y \vee x \Rightarrow [(y \wedge x \succeq x) \text{ or } (y \wedge x \succeq y)]\right]. \tag{11d}$$

Remark. Each of conditions (10) and (11) holds trivially when x and y are comparable in the basic order.

Proposition 11. *An ordering $\succ$ on a lattice A satisfies any one condition* (9) *if and only if it satisfies both corresponding conditions* (10) *and* (11).

Proof. The necessity is obvious in each case. To prove the sufficiency for (9a), we suppose the contrary. Let $x \succ y \wedge x$, but $y \succeq y \vee x$; then $y \vee x \succ x$ by (10a), hence $y \succ y \wedge x$ by transitivity, which contradicts (11a). The proof of the implication [(10b) & (11b)] $\Rightarrow$ (9b) is dual. The proofs of the other implications are similar. □

If the order on A is reversed, conditions (10) and (11) transform into each other; moreover, "meet-related" conditions become "join-related" and vice versa.

Conditions (10), as well as (11), are ordered between themselves in the same way as (9): (c) implies all others; (d) is implied by all others. There is no other implication between the conditions.

Example 2. Let $A := \{(0,0), (0,1), (1,0), (1,1)\} \subset \mathbb{R}^2$; we consider four orderings on A represented by these matrices (the axes are directed upwards and rightwards):

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

The ordering represented by the first matrix satisfies (10c), hence all conditions (10), but not (11d), hence none of (11). The second matrix satisfies (10c) and (11b), but not (11a). The third, (10c) and (11a), but not (11b). The last one satisfies (10c), (11a), and (11b), but not (11c). The impossibility to derive conditions (10) from (11) is shown dually.

None of conditions (10) or (11) seems to play any role in type B problems even though they have been derived from conditions (9). In type A problems, however, each of the eight is necessary and sufficient for a kind (actually, two kinds) of the monotonicity of optima.

Proposition 12. *Let A be a lattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ satisfies* (10a)*.*
2. *There holds $M(X, \succ') \geq^{\wedge} M(X, \succ)$ whenever $X \in \mathfrak{L}_A$, $\succ'$ is an ordering on X, and* (7a) *holds on X.*
3. *There holds $M(X, \succ') \geq^{\mathrm{w}\vee} M(X, \succ)$ whenever $X \in \mathfrak{L}_A$, $\succ'$ is an ordering on X, and* (7d) *holds on X.*
4. *There holds $M(X, \succ') \geq^{\wedge} M(X, \succ)$ whenever $X \in \mathfrak{L}_A$, $\#X \leq 4$, and $\succ'$ is an ordering on A such that* (7a) *and* (7b) *hold on A.*
5. *There holds $M(X, \succ') \geq^{\mathrm{w}\vee} M(X, \succ)$ whenever $X \in \mathfrak{L}_A$, $\#X \leq 4$, and $\succ'$ is an ordering on A such that* (7a) *holds on A.*

Proof. The implications [Statement 2 $\Rightarrow$ Statement 4] and [Statement 3 $\Rightarrow$ Statement 5] are obvious. Therefore, it is enough to show that (10a) implies both Statements 2 and 3, while the negation of (10a) implies the negation of both Statements 4 and 5.

Let (10a) hold, $X \in \mathfrak{L}_A$, $x \in M(X, \succ)$, and $y \in M(X, \succ')$. Then $x \succeq y \vee x$; assuming that $y \wedge x \notin M(X, \succ)$, we have $x \succ y \wedge x$, hence $y \vee x \succ y$ by (10a). If (7a) holds, we immediately obtain $y \vee x \succ' y$, which contradicts the optimality of y. If (7d) holds, we obtain $y \vee x \succeq' y$, hence $y \vee x \in M(X, \succ')$. Since x and y were arbitrary, we obtain $M(X, \succ') \geq^{\wedge} M(X, \succ)$ in the first case and $M(X, \succ') \geq^{\mathrm{w}\vee} M(X, \succ)$ in the second.

Let (10a) be violated: there are $x, y \in A$ such that $x \succ y \wedge x$, but $y \succeq y \vee x$ and $x \succeq y \vee x$. Exchanging the labels "x" and "y" if needed, we assume $x \succeq y$. Then we define $X := L(x, y)$, so $y \wedge x \notin M(X, \succ) \ni x$.

To show that Statement 4 does not hold, we denote $Y := \{z \in A \mid z \geq y\}$. Our assumptions imply $x \notin Y$: otherwise, we would have $y \vee x = x$ and $y \wedge x = y$, hence $x \succ y$ and $y \succeq x$ simultaneously. Now we define an ordering $\succ'$ on A by setting $z' \succ' z$ if and only if one of these conditions holds: (i) $z' \succ z$ and $z, z' \in Y$; (ii) $z' \succ z$ and $z, z' \in A \setminus Y$; (iii) $z \notin Y \ni z'$. Both (7a) and (7b) are obvious: whenever $z \in Y$ and $z' > z$, we have $z' \in Y$ as well. Meanwhile, $y \in M(X, \succ')$, hence $M(X, \succ') \geq^{\wedge} M(X, \succ)$ does not hold, i.e., Statement 4 is invalid.

To show that Statement 5 does not hold, we define $\succ'$ in the same manner as in the previous paragraph, but with $Y := \{z \in A \mid z \succ y \ \&\ z > y\} \cup \{y\}$. Clearly, $M(X, \succ') = \{y\}$, hence $M(X, \succ') \geq^{\mathrm{w}\vee} M(X, \succ)$ does not hold. Meanwhile, (7a) is obvious: whenever $z' > z$, $z' \succ z$, and $z \in Y$, we have $z' \in Y$ as well. Thus, Statement 5 is invalid. $\square$

Proposition 13. *Let A be a lattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ satisfies* (10b)*.*

2. *There holds $M(X, \succ') \geq^{\vee} M(X, \succ)$ whenever $X \in \mathfrak{L}_A$, $\succ'$ is an ordering on X, and* (7b) *holds on X.*

3. *There holds $M(X, \succ') \ggg M(X, \succ)$ whenever $X \in \mathfrak{L}_A$, $\succ'$ is an ordering on X, and* (7c) *holds on X.*

4. *There holds $M(X, \succ') \geq^{\vee} M(X, \succ)$ whenever $X \in \mathfrak{L}_A$, $\#X \leq 4$, and $\succ'$ is an ordering on A such that* (7c) *holds on A.*

Proof. The implications [Statement 2 $\Rightarrow$ Statement 4 $\Leftarrow$ Statement 3] are obvious. Therefore, it is enough to show that (10b) implies both Statements 2 and 3, while the negation of (10b) implies the negation of Statement 4.

Let (10b) hold, $X \in \mathfrak{L}_A$, $x \in M(X, \succ)$ and $y \in M(X, \succ')$. Then $x \succeq y$ and $x \succeq y \wedge x$; therefore, $y \vee x \succeq y$ by (10b). If (7b) holds, then $y \vee x \succeq' y$, hence $y \vee x \in M(X, \succ')$. Let us show that $y \geq x$ if (7c) holds. Supposing the contrary, we have $y \vee x > y$; therefore, relation $y \vee x \succeq y$ implies $y \vee x \succ' y$ by (7c), which contradicts the optimality of y. Since x and y were arbitrary, we obtain $M(X, \succ') \geq^{\vee} M(X, \succ)$ in the first case and $M(X, \succ') \ggg M(X, \succ)$ in the second.

Let (10b) be violated: there are $x, y \in A$ such that $x \succeq y \wedge x$, but $x \succ y \vee x$ and $y \succ y \vee x$. Note that x and y must be incomparable in the basic order. Exchanging the labels "x" and "y" if needed, we assume $x \succeq y$. Then we define $X := L(x, y)$, so $x \in M(X, \succ)$. Now we define an ordering $\succ'$ on A in the same manner as in the proof of Proposition 12, with $Y := \{z \in A \mid z \geq y\}$. Invoking the Szpilrajn Theorem, we pick a linear order $\gg$ on A such that $z' \gg z$ whenever $z' > z$. Then we define $\succ''$ as a lexicography:

$$z' \succ'' z \rightleftharpoons \big[z' \succ' z \;\text{ or }\; [z' \sim' z \;\&\; z' \gg z]\big]. \tag{12}$$

To check (7c) for $\succ''$ and $\succ$, we suppose that $z' > z$ and $z' \succeq z$. Then $z' \gg z$ by the choice of $\gg$. If $z \in Y$, then $z' \in Y$ as well; therefore, $z' \succeq' z$. Applying (12), we see that $z' \succ'' z$ as it should be. Meanwhile, $M(X, \succ'') = \{y\}$, hence $M(X, \succ'') \geq^{\vee} M(X, \succ)$ does not hold. $\square$

Proposition 14. *Let A be a lattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *Both conditions* (10a) *and* (10b) *hold.*

2. *There holds $M(X, \succ') \geq^{\vee\mathrm{t}} M(X, \succ)$ whenever $X \in \mathfrak{L}_A$, $\succ'$ is an ordering on X, and* (7a) *and* (7b) *hold on X.*

3. *There holds $M(X, \succ') \geq^{\vee\mathrm{t}} M(X, \succ)$ whenever $X \in \mathfrak{L}_A$, $\#X \leq 4$, and $\succ'$ is an ordering on A such that* (7a) *and* (7b) *hold on A.*

Proof. The equivalence immediately follows from Propositions 12 and 13. $\square$

Remark. Agliardi (2000) called a function u on a lattice A *pseudosupermodular* if the ordering represented by u satisfies (10a) and (10b). However, she did not suggest anything like Proposition 14.

Proposition 15. *Let A be a lattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ satisfies* (11a)*.*
2. *There holds $M(X, \succ) \geq^{\wedge} M(X, \succ')$ whenever $X \in \mathfrak{L}_A$, $\succ'$ is an ordering on X, and* (8a) *holds on X.*
3. *There holds $M(X, \succ) \ggg M(X, \succ')$ whenever $X \in \mathfrak{L}_A$, $\succ'$ is an ordering on X, and* (8c) *holds on X.*
4. *There holds $M(X, \succ) \geq^{\wedge} M(X, \succ')$ whenever $X \in \mathfrak{L}_A$, $\#X \leq 4$, and $\succ'$ is an ordering on A such that* (8c) *holds on A.*

The proof is dual to that of Proposition 13.

Proposition 16. *Let A be a lattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ satisfies* (11b)*.*
2. *There holds $M(X, \succ) \geq^{\vee} M(X, \succ')$ whenever $X \in \mathfrak{L}_A$, $\succ'$ is an ordering on X, and* (8b) *holds on X.*
3. *There holds $M(X, \succ) \geq^{\mathrm{w}\vee} M(X, \succ')$ whenever $X \in \mathfrak{L}_A$, $\succ'$ is an ordering on X, and* (8d) *holds on X.*
4. *There holds $M(X, \succ) \geq^{\vee} M(X, \succ')$ whenever $X \in \mathfrak{L}_A$, $\#X \leq 4$, and $\succ'$ is an ordering on A such that* (8a) *and* (8b) *hold on A.*
5. *There holds $M(X, \succ) \geq^{\mathrm{w}\vee} M(X, \succ')$ whenever $X \in \mathfrak{L}_A$, $\#X \leq 4$, and $\succ'$ is an ordering on A such that* (8b) *holds on A.*

The proof is dual to that of Proposition 12.

Proposition 17. *Let A be a lattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *Both conditions* (11a) *and* (11b) *hold.*
2. *There holds $M(X, \succ) \geq^{\vee\mathrm{t}} M(X, \succ')$ whenever $X \in \mathfrak{L}_A$, $\succ'$ is an ordering on X, and* (8a) *and* (8b) *hold on X.*

3. *There holds* $M(X,\succ) \geq^{\mathrm{Vt}} M(X,\succ')$ *whenever* $X \in \mathfrak{L}_A$, $\#X \leq 4$, *and* $\succ'$ *is an ordering on* A *such that* (8a) *and* (8b) *hold on* A.

Proof. The equivalence immediately follows from Propositions 15 and 16. □

Theorem 1. *An ordering* $\succ$ *on a lattice* A *is quasisupermodular if and only if it has both following properties.*

1. *There holds* $M(X,\succ') \geq^{\mathrm{Vt}} M(X,\succ)$ *whenever* $X \in \mathfrak{L}_A$, $\succ'$ *is an ordering on* X, *and* (7a) *and* (7b) *hold on* X.
2. *There holds* $M(X,\succ) \geq^{\mathrm{Vt}} M(X,\succ')$ *whenever* $X \in \mathfrak{L}_A$, $\succ'$ *is an ordering on* X, *and* (8a) *and* (8b) *hold on* X.

Moreover, the "if" part survives the restriction of both requirements to $X \in \mathfrak{L}_A$ *with* $\#X \leq 4$ *and* $\succ'$ *defined on* A.

Proof. The equivalence immediately follows from Propositions 11, 14, and 17. □

Remark. The "only if" part adds rather little to Theorem 4 of Milgrom and Shannon (1994), but does not follow therefrom: $\succ'$ need not be QSM [and its quasisupermodularity would not follow even from (7c) or (8c)].

Theorem 2. *An ordering* $\succ$ *on a lattice* A *is quasisupermodular if and only if it has all the following properties.*

1. *There holds* $M(X,\succ') \ggg M(X,\succ)$ *whenever* $X \in \mathfrak{L}_A$, $\succ'$ *is an ordering on* X, *and* (7c) *holds on* X.
2. *There holds* $M(X,\succ) \ggg M(X,\succ')$ *whenever* $X \in \mathfrak{L}_A$, $\succ'$ *is an ordering on* X, *and* (8c) *holds on* X.
3. *There holds* $M(X,\succ') \geq^{\mathrm{wV}} M(X,\succ)$ *whenever* $X \in \mathfrak{L}_A$, $\succ'$ *is an ordering on* X, *and* (7d) *holds on* X.
4. *There holds* $M(X,\succ) \geq^{\mathrm{wV}} M(X,\succ')$ *whenever* $X \in \mathfrak{L}_A$, $\succ'$ *is an ordering on* X, *and* (8d) *holds on* X.

Moreover, the "if" part survives the restriction of all requirements to $X \in \mathfrak{L}_A$ *with* $\#X \leq 4$ *and* $\succ'$ *defined on* A.

Proof. The equivalence immediately follows from Propositions 11 and 12–16. □

Remark. Again, the "only if" part does not add very much to Theorems 2 and 3 of Shannon (1995), but is technically independent of them.

Proposition 18. *Let A be a lattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ satisfies* (10c)*.*
2. *There hold $M(X,\succ') \geqslant M(X,\succ)$ and $M(X,\succ') \bowtie M(X,\succ)$ whenever $X \in \mathfrak{L}_A$ and $\succ'$ is an ordering on X such that* (7a) *and* (7b) *hold on X.*
3. *There hold $M(X,\succ') \geqslant^{\mathrm{Up}} M(X,\succ)$ and $M(X,\succ') \bowtie M(X,\succ)$ whenever $X \in \mathfrak{L}_A$ and $\succ'$ is an ordering on X such that* (7a) *holds on X.*
4. *There holds $M(X,\succ') \geqslant^{\mathrm{Up}} M(X,\succ)$ whenever $X \in \mathfrak{L}_A$, $\#X \leq 4$, and $\succ'$ is an ordering on A such that* (7a) *and* (7b) *hold on A.*

Proof. The implications [Statement 2 $\Rightarrow$ Statement 4 $\Leftarrow$ Statement 3] are obvious. Therefore, it is enough to show that (10c) implies both Statements 2 and 3, while the negation of (10c) implies the negation of Statement 4.

Let (10c) hold, $X \in \mathfrak{L}_A$, (7a) hold on X, $x \in M(X,\succ)$, and $y \in M(X,\succ')$; then $x \succeq y$ and $x \succeq y \wedge x$. Assuming that y and x are incomparable in the basic order, we have $y \vee x \succ y$ by (10c), hence $y \vee x \succ' y$ by (7a), which contradicts the optimality of y. Therefore, $M(X,\succ') \bowtie M(X,\succ)$. If $y \geq x$, there can be no problem with (5a), nor (5c).

Let $x > y$. An assumption that $x \succ y$ would, by (7a), imply $x \succ' y$, again contradicting the optimality of y; therefore, $y \in M(X,\succ) \cap M(X,\succ')$, hence there is no problem with (5a). If (7b) holds as well, we have $x \succeq' y$ since $x \succeq y$, hence $x \in M(X,\succ) \cap M(X,\succ')$. Since x and y were arbitrary, we have $M(X,\succ') \geqslant M(X,\succ)$ in the first case and $M(X,\succ') \geqslant^{\mathrm{Up}} M(X,\succ)$ in the second.

Let (10c) be violated: there are $x, y \in A$ such that $y \vee x > x > y \wedge x$, $x \succeq y \wedge x$, $x \succeq y \vee x$, and $y \succeq y \vee x$. As usual, we may assume $x \succeq y$. Then we set $X := L(x,y)$, so $x \in M(X,\succ)$, and define an ordering $\succ'$ on A in the same manner as in the proof of Proposition 12, with $Y := \{z \in A \mid z \geq y\}$. Both (7a) and (7b) hold for the same reason as there. Meanwhile, $x \notin M(X,\succ') \ni y$, hence $M(X,\succ') \geqslant^{\mathrm{Up}} M(X,\succ)$ does not hold, i.e., Statement 4 is invalid. $\square$

Proposition 19. *Let A be a lattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ satisfies* (11c)*.*
2. *There hold $M(X,\succ) \geqslant M(X,\succ')$ and $M(X,\succ) \bowtie M(X,\succ')$ whenever $X \in \mathfrak{L}_A$ and $\succ'$ is an ordering on X such that* (8a) *and* (8b) *hold on X.*
3. *There hold $M(X,\succ) \geqslant^{\mathrm{Dn}} M(X,\succ')$ and $M(X,\succ) \bowtie M(X,\succ')$ whenever $X \in \mathfrak{L}_A$ and $\succ'$ is an ordering on X such that* (8b) *holds on X.*
4. *There holds $M(X,\succ) \geqslant^{\mathrm{Dn}} M(X,\succ')$ whenever $X \in \mathfrak{L}_A$, $\#X \leq 4$, and $\succ'$ is an ordering on A such that* (8a) *and* (8b) *hold on A.*

The proof is dual to that of Proposition 18.

Theorem 3. *An ordering $\succ$ on a lattice A is strictly quasisupermodular if and only if it has both following properties.*

1. *There hold $M(X,\succ') \geqslant M(X,\succ)$ and $M(X,\succ') \bowtie M(X,\succ)$ whenever $X \in \mathfrak{L}_A$ and $\succ'$ is an ordering on X such that* (7a) *and* (7b) *hold on X.*

2. *There hold $M(X,\succ) \geqslant M(X,\succ')$ and $M(X,\succ) \bowtie M(X,\succ')$ whenever $X \in \mathfrak{L}_A$ and $\succ'$ is an ordering on X such that* (8a) *and* (8b) *hold on X.*

Moreover, the "if" part survives the restriction of both requirements to $X \in \mathfrak{L}_A$ with $\#X \leq 4$ and $\succ'$ defined on A; it also survives dropping the component with $\bowtie$ in both requirements.

Proof. The equivalence immediately follows from Propositions 11, 18, and 19. □

Proposition 20. *Let A be a lattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ satisfies* (10d).

2. *There holds $M(X,\succ') \geq^{\wedge} M(X,\succ)$ whenever $X \in \mathfrak{L}_A$ and $\succ'$ is an ordering on X such that* (7c) *holds on X.*

3. *There holds $M(X,\succ') \geq^{\mathrm{w}\vee} M(X,\succ)$ whenever $X \in \mathfrak{L}_A$ and $\succ'$ is an ordering on X such that* (7b) *holds on X.*

4. *There holds $M(X,\succ') \geq^{\mathrm{w}\vee} M(X,\succ)$ whenever $X \in \mathfrak{L}_A$, $\#X \leq 4$, and $\succ'$ is an ordering on A such that* (7c) *holds on A.*

Proof. The implications [Statement 2 $\Rightarrow$ Statement 4 $\Leftarrow$ Statement 3] are obvious. Therefore, it is enough to show that (10d) implies Statements 2 and 3, while the negation of (10d) implies the negation of Statement 4.

Let (10d) hold, $X \in \mathfrak{L}_A$, $x \in M(X,\succ)$, and $y \in M(X,\succ')$; then $x \succeq y$ and $x \succeq y \wedge x$. Assuming that $x \succ y \wedge x$, we obtain $y \vee x \succeq y$ by (10d). If (7c) holds on X, then $y \vee x \succ' y$, which contradicts the optimality of y; therefore, $y \wedge x \in M(X,\succ)$. If (7b) holds, then $y \vee x \succeq' y$; therefore, $y \vee x \in M(X,\succ')$. Since x and y were arbitrary, we have $M(X,\succ') \geq^{\wedge} M(X,\succ)$ in the first case, and $M(X,\succ') \geq^{\mathrm{w}\vee} M(X,\succ)$ in the second.

Let (10d) be violated: there are $x, y \in A$ such that $x \succeq y$ and $x \succ y \wedge x$, but $y \succ y \vee x$. We define $X := L(x,y)$, so $y \wedge x \notin M(X,\succ) \ni x$, and orderings $\succ'$ and $\succ''$ on A in the same way as in the proof of Proposition 13. Condition (7c) for $\succ$ and $\succ''$ holds for the same reason as there. Meanwhile, $y \vee x \notin M(X,\succ'') \ni y$, hence $M(X,\succ'') \geq^{\mathrm{w}\vee} M(X,\succ)$ does not hold, i.e., Statement 4 is invalid. □

Proposition 21. *Let A be a lattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ satisfies* (11d).
2. *There holds $M(X, \succ) \geq^{\mathrm{V}} M(X, \succ')$ whenever $X \in \mathfrak{L}_A$ and $\succ'$ is an ordering on X such that* (8c) *holds on X.*
3. *There holds $M(X, \succ) \geq^{\mathrm{wV}} M(X, \succ')$ whenever $X \in \mathfrak{L}_A$ and $\succ'$ is an ordering on X such that* (8a) *holds on X.*
4. *There holds $M(X, \succ) \geq^{\mathrm{wV}} M(X, \succ')$ whenever $X \in \mathfrak{L}_A$, $\#X \leq 4$, and $\succ'$ is an ordering on A such that* (8c) *holds on A.*

The proof is dual to that of Proposition 20

Theorem 4. *An ordering $\succ$ on a lattice A is weakly quasisupermodular if and only if it has both following properties.*

1. *There holds $M(X, \succ') \geq^{\mathrm{wV}} M(X, \succ)$ whenever $X \in \mathfrak{L}_A$ and $\succ'$ is an ordering on X such that* (7a) *and* (7b) *hold on X.*
2. *There holds $M(X, \succ) \geq^{\mathrm{wV}} M(X, \succ')$ whenever $X \in \mathfrak{L}_A$ and $\succ'$ is an ordering on X such that* (8a) *and* (8b) *hold on X.*

Moreover, the "if" part survives the restriction of both requirements to $X \in \mathfrak{L}_A$ with $\#X \leq 4$ and $\succ'$ defined on A.

Proof. The equivalence immediately follows from Propositions 11, 20, and 21. □

7 Parametric optimization

In this section, we return to parametric families of orderings on A and the best response correspondences $\mathcal{R}^X : T \to \mathfrak{B}_X \subseteq \mathfrak{B}_A$ defined by (2).

Given posets A and T and a parametric family $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$ of orderings on A, we say that $\mathcal{U}$ satisfies the *meet single crossing* condition if (7a) holds for $\succ^t$ as $\succ$ and $\succ^{t'}$ as $\succ'$ whenever $t, t' \in T$ and $t' > t$. Similarly, $\mathcal{U}$ satisfies the *join*, *strict*, or *weak single crossing* condition if (7b), (7c), or (7d) holds under the same circumstances. We say that $\mathcal{U}$ satisfies the *single crossing* condition if it satisfies both join and meet single crossing conditions. Our terminology coincides with that of Milgrom and Shannon (1994) and Shannon (1995) when $\mathcal{U}$ is represented by a utility function (they did not explicitly define the meet and join single crossing conditions, though).

Remark. Reny (2011, Section 4.1) applies the term "weak single crossing" to what is called "join single crossing" here; our usage follows Shannon (1995) and, in a broader sense, Veinott (1989).

Clearly, the (meet, join, strict, or weak) single crossing conditions could be defined with references to (8) as well. If both orders on A and T are reversed, meet and join single crossing conditions transform into each other; their conjunction, as well as strict and weak single crossing conditions, are "self-dual" in this sense. Propositions 1–5 immediately imply that each of the single crossing conditions is equivalent to the monotonicity (in one sense or another) of correspondences $\mathcal{R}^X$ for all chains $X \in \mathfrak{B}_A$.

Propositions 12–21 give us sufficient conditions for the monotonicity of correspondences $\mathcal{R}^X$ for all sublattices $X \in \mathfrak{L}_A$. Given a poset T, a *monotone pseudopartition* of T consists of two subsets $T^{\uparrow}, T^{\downarrow} \subseteq T$ such that $\forall t', t \in T\, \big[t' > t \Rightarrow [t \in T^{\uparrow} \text{ or } t' \in T^{\downarrow}]\big]$. Clearly, any two points outside $T^{\uparrow} \cup T^{\downarrow}$ must be incomparable.

Proposition 22. *Let $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$ be a parametric family of orderings on a lattice A; let $\mathcal{U}$ satisfy the strict single crossing condition. Let there be a monotone pseudopartition $\langle T^{\uparrow}, T^{\downarrow} \rangle$ of T such that $\succ^t$ satisfies* (10b) *for $t \in T^{\uparrow}$ and* (11a) *for $t \in T^{\downarrow}$. Then every $\mathcal{R}^X$ $(X \in \mathfrak{L}_A)$ is increasing w.r.t. $\ggg$.*

Proof. Let $t' > t$. If $t \in T^{\uparrow}$, then (10b) holds with $\succ^t$ as $\succ$ while (7c) holds with $\succ^{t'}$ as $\succ'$ and $\succ^t$ as $\succ$. Therefore, $\mathcal{R}^X(t') \ggg \mathcal{R}^X(t)$ for every $X \in \mathfrak{L}_A$ by Statement 3 of Proposition 13. If $t' \in T^{\downarrow}$, then (11a) holds with $\succ^{t'}$ as $\succ$ while (8c) holds with with $\succ^t$ as $\succ'$ and $\succ^{t'}$ as $\succ$. Therefore, $\mathcal{R}^X(t') \ggg \mathcal{R}^X(t)$ for every $X \in \mathfrak{L}_A$ by Statement 3 of Proposition 15. ☐

Proposition 23. *Let $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$ be a parametric family of orderings on a lattice A; let $\mathcal{U}$ satisfy the meet single crossing condition. Let there be a monotone pseudopartition $\langle T^{\uparrow}, T^{\downarrow} \rangle$ of T such that $\succ^t$ satisfies* (10a) *for $t \in T^{\uparrow}$ and* (11a) *for $t \in T^{\downarrow}$. Then every $\mathcal{R}^X$ $(X \in \mathfrak{L}_A)$ is increasing w.r.t. $\geq^{\wedge}$.*

Proposition 24. *Let $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$ be a parametric family of orderings on a lattice A; let $\mathcal{U}$ satisfy the join single crossing condition. Let there be a monotone pseudopartition $\langle T^{\uparrow}, T^{\downarrow} \rangle$ of T such that $\succ^t$ satisfies* (10b) *for $t \in T^{\uparrow}$ and* (11b) *for $t \in T^{\downarrow}$. Then every $\mathcal{R}^X$ $(X \in \mathfrak{L}_A)$ is increasing w.r.t. $\geq^{\vee}$.*

Proposition 25. *Let $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$ be a parametric family of orderings on a lattice A; let $\mathcal{U}$ satisfy the single crossing condition. Let there be a monotone pseudopartition $\langle T^{\uparrow}, T^{\downarrow} \rangle$ of T such that $\succ^t$ satisfies* (10a) *and* (10b) *for $t \in T^{\uparrow}$, and* (11a) *and* (11b) *for $t \in T^{\downarrow}$. Then every $\mathcal{R}^X$ $(X \in \mathfrak{L}_A)$ is increasing w.r.t. $\geq^{\vee \mathrm{t}}$.*

Proposition 26. *Let $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$ be a parametric family of orderings on a lattice A; let $\mathcal{U}$ satisfy the weak single crossing condition. Let there be a monotone pseudopartition $\langle T^{\uparrow}, T^{\downarrow} \rangle$ of T such that $\succ^t$ satisfies* (10a) *for $t \in T^{\uparrow}$ and* (11b) *for $t \in T^{\downarrow}$. Then every $\mathcal{R}^X$ $(X \in \mathfrak{L}_A)$ is increasing w.r.t. $\geq^{\mathrm{w}\vee}$.*

Proposition 27. *Let $\mathcal{U} = \langle \succ^t \rangle_{t \in T}$ be a parametric family of orderings on a lattice A; let $\mathcal{U}$ satisfy the single crossing condition. Let there be a monotone pseudopartition $\langle T^{\uparrow}, T^{\downarrow} \rangle$ of T such that $\succ^t$ satisfies* (10c) *for $t \in T^{\uparrow}$, and* (11c) *for $t \in T^{\downarrow}$. Then every $\mathcal{R}^X$ $(X \in \mathfrak{L}_A)$ is increasing w.r.t. $\geqslant$.*

Proposition 28. *Let $\mathcal{U} = \langle \succcurlyeq^t \rangle_{t \in T}$ be a parametric family of orderings on a lattice A; let $\mathcal{U}$ satisfy the single crossing condition. Let there be a monotone pseudopartition $\langle T^\uparrow, T^\downarrow \rangle$ of T such that $\succcurlyeq^t$ satisfies* (10d) *for $t \in T^\uparrow$, and* (11d) *for $t \in T^\downarrow$. Then every $\mathcal{R}^X$ ($X \in \mathfrak{L}_A$) is increasing w.r.t. $\geq^{\mathrm{wV}}$.*

Each proof is quite similar to that of Proposition 22.

Remark. Lemma 3.1 of Kukushkin et al. (2005) immediately follows from Proposition 25 ($T^\uparrow = T$).

To obtain *characterization* results for parametric settings, somewhat cumbersome terminology is needed. Let T and $\bar{T}$ be two posets such that $T = \bar{T} \setminus \{\bar{t}\}$ ($\bar{t} \in \bar{T}$); let $\bar{\mathcal{U}} = \langle \succcurlyeq^t \rangle_{t \in \bar{T}}$ be a parametric family of orderings on a lattice A, and $\mathcal{U} := \langle \succcurlyeq^t \rangle_{t \in T}$. We say that $\bar{\mathcal{U}}$ is an *extension of $\mathcal{U}$ with the single crossing property* if (7a) and (7b) hold for $\succcurlyeq^{\bar{t}}$ as $\succ$ and $\succcurlyeq^t$ as $\succ'$ whenever $T \ni t > \bar{t}$, whereas (7a) and (7b) hold for $\succcurlyeq^t$ as $\succ$ and $\succcurlyeq^{\bar{t}}$ as $\succ'$ whenever $\bar{t} > t \in T$. Similarly, $\bar{\mathcal{U}}$ is an *extension of $\mathcal{U}$ with the strict* [*weak, meet,* or *join*] *single crossing property* if (7c) [(7d), (7a), or (7b)] holds for $\succcurlyeq^{\bar{t}}$ as $\succ$ and $\succcurlyeq^t$ as $\succ'$ whenever $T \ni t > \bar{t}$, whereas (7c) [(7d), (7a), or (7b)] holds for $\succcurlyeq^t$ as $\succ$ and $\succcurlyeq^{\bar{t}}$ as $\succ'$ whenever $\bar{t} > t \in T$.

We say that an ordering $\succ$ on a lattice A *preserves ascendance* if, whenever $\mathcal{U}$ is a parametric family of orderings on A such that $\mathcal{R}^X$ defined by (2) is increasing w.r.t. $\geq^{\mathrm{Vt}}$ for every $X \in \mathfrak{L}_A$, and $\bar{\mathcal{U}}$ is an extension of $\mathcal{U}$ with the single crossing property such that $\succcurlyeq^{\bar{t}}$ coincides with $\succ$, every correspondence $\bar{\mathcal{R}}^X$ ($X \in \mathfrak{L}_A$) defined by (2) for $\bar{\mathcal{U}}$ is increasing w.r.t. $\geq^{\mathrm{Vt}}$. Similarly, an ordering $\succ$ on a lattice A *preserves strong* [*weak, meet,* or *join*] *ascendance* if, whenever $\mathcal{U}$ is a parametric family of orderings on A such that $\mathcal{R}^X$ defined by (2) is increasing w.r.t. $\ggg$ [$\geq^{\mathrm{wV}}$, $\geq^\wedge$, $\geq^\vee$] for every $X \in \mathfrak{L}_A$, and $\bar{\mathcal{U}}$ is an extension of $\mathcal{U}$ with the strict [weak, meet, or join] single crossing property such that $\succcurlyeq^{\bar{t}}$ coincides with $\succ$, every correspondence $\bar{\mathcal{R}}^X$ ($X \in \mathfrak{L}_A$) defined by (2) for $\bar{\mathcal{U}}$ is increasing w.r.t. $\ggg$ [$\geq^{\mathrm{wV}}$, $\geq^\wedge$, $\geq^\vee$].

Proposition 29. *An ordering $\succ$ on a lattice A preserves meet (join) ascendance if and only if it is $\wedge$($\vee$)-QSM.*

Proof. The first equivalence immediately follows from Propositions 11, 12, and 15; the second, from Propositions 11, 13, and 16. □

Proposition 30. *An ordering $\succ$ on a lattice A preserves weak ascendance if and only if it satisfies* (10a) *and* (11b).

Proof. The equivalence immediately follows from Propositions 12 and 16. □

Proposition 31. *An ordering $\succ$ on a lattice A preserves strong ascendance if and only if it satisfies* (11a) *and* (10b).

Proof. The equivalence immediately follows from Propositions 13 and 15. □

Theorem 5. *Let $\succ$ be an ordering on a lattice A. Then the following statements are equivalent.*

1. $\succ$ *is quasisupermodular.*

2. $\succ$ *preserves ascendance.*

3. $\succ$ *preserves both strong ascendance and weak ascendance.*

Proof. The equivalence immediately follows from Theorems 1 and 2. $\square$

Remark. Once again, the implication [Statement 1 $\Rightarrow$ Statement 2] is rather close to being a corollary to Theorem 4 of Milgrom and Shannon (1994), but does not quite follow therefrom. The connection between the implication [Statement 1 $\Rightarrow$ Statement 3] and Theorems 2 and 3 of Shannon (1995) is even looser.

To "utilize" Propositions 18–21 in the same style, we say that an ordering $\succ$ on a lattice A *preserves semi-strong ascendance* [*weakly preserves weak ascendance*] if the monotonicity of every correspondence $\mathcal{R}^X$, $X \in \mathfrak{L}_A$, w.r.t. $\geqslant$ [$\geq^{\mathrm{wV}}$] is sustained after the insertion of $\succ$ with the single crossing property into any parametric family of orderings.

Proposition 32. *An ordering* $\succ$ *on a lattice A preserves semi-strong ascendance if and only if it is SQSM.*

Proof. The equivalence immediately follows from Propositions 18 and 19. $\square$

Proposition 33. *An ordering* $\succ$ *on a lattice A weakly preserves weak ascendance if and only if it is wQSM.*

Proof. The equivalence immediately follows from Propositions 20 and 21. $\square$

8 Concluding remarks

8.1. In most cases, the relations between different conditions relevant to type A (or type B for that matter) problems follow a simple rule: the stronger version of monotonicity we want, the stronger restriction should be imposed on preferences. In the case of Proposition 31 vs. Proposition 30, we have stronger monotonicity under stronger single crossing conditions, and indeed, the restrictions are incomparable. When it comes to Proposition 31 vs. Proposition 32, we again have stronger monotonicity under stronger single crossing conditions, but the first restriction on preferences is milder! I can suggest no *a priori* explanation why it should be so.

Another unexplained observation is that (strict) quasisupermodularity, as well as (9a) and (9b) separately, play the same roles in both type A and type B problems, whereas weak quasisupermodularity does not.

8.2. An alternative approach to monotone comparative statics can be based on monotone selections from $\mathcal{R}^X$ or $M(\cdot, \succ)$. Separation between the existence and monotonicity of optima becomes impossible in this case, hence the prospects for comprehensive characterization results are rather dim. Some partial results can be derived, nonetheless.

Proposition 34 below immediately implies that every selection from $M(\cdot, \succ) \colon \mathfrak{L}_A \to \mathfrak{B}_A$ is increasing if and only if $\succ$ is strictly$^+$ quasisupermodular. Theorem 1 of LiCalzi and Veinott (1992), plus Theorem 3.2 of Veinott (1989) or Theorem 1 of Kukushkin (2009), shows that the restriction of $M(\cdot, \succ)$ to the poset $\mathfrak{F}_A \subseteq \mathfrak{L}_A$ of all finite nonempty sublattices of A admits a monotone selection if and only if $\succ$ is weakly quasisupermodular. Invoking, additionally, Theorem 2 of Shannon (1995) and restricting attention to order upper semicontinuous orderings $\succ$, we see that weak quasisupermodularity is equivalent to the existence of a monotone selection from the restriction of $M(\cdot, \succ)$ to the poset $\mathfrak{M}_A \subseteq \mathfrak{L}_A$ of all subcomplete sublattices of A.

Similar approach to type A problems needs notions of the preservation of the existence, or universality, of monotone selections from $\mathcal{R}^X$, which can be formulated in a straightforward way. Proposition 30, respectively Proposition 31, then imply results similar to those of the preceding paragraph, where weak quasisupermodularity is replaced with the conjunction of (10a) and (11b), while strict$^+$ quasisupermodularity with the conjunction of (11a) and (10b).

Edlin and Shannon (1998), as well as Strulovici and Weber (2010), did study monotone comparative statics in terms of monotone selections; however, both additionally assumed smoothness and employed local considerations. Therefore, their results are not directly comparable to ours.

8.3. An anonymous referee has raised the question of what would happen to our results if the attention is restricted to preferences described by real-valued utility functions rather than orderings. All sufficiency statements obviously remain valid, so the question boils down to whether $\succ'$ and $\succ''$ in the proofs of necessity parts of Propositions 12, 13, 18, and 20 admit numeric representations if $\succ$ does. A positive answer is easy to obtain in the case of the first relation; in the second case, it is obvious for a countable A, but generally wrong otherwise – neither Szpilrajn's "construction," nor lexicography (12) preserve numerical representability.

Something could be done about that. First, we may separate $\geq^{\wedge}$ from $\ggg$ in Proposition 13: to prove the necessity of (10b) for the monotonicity in the former sense, only $\succ'$ satisfying (7a) and (7b) is needed, and there is no problem with numerical representability here. This is sufficient to save Proposition 14 and Theorem 1. We may also weaken the current Statement 3 of Proposition 13 only demanding (7c) to hold on X. Then Theorem 2 will be essentially saved; we only have to drop the last claim in the "moreover…" part. Finally, we may weaken the current Statement 4 in Proposition 20, only demanding (7a) and (7b) to hold; that statement is sufficient for Theorem 4. Whether Propositions 13 and 20 could be saved in their present form remains unclear.

8.4. Five "order" relations (3) and (4) form a lattice (with the logical implication as order), which is *not* a sublattice of the lattice of all binary relations on $\mathfrak{B}_A$. Five single crossing conditions [four (7) and the conjunction of (7a) and (7b) – single crossing proper] form an isomorphic lattice; the same applies to five "QSM-style" conditions [the four (9) and the conjunction of (9a) and (9b) –

quasisupermodularity proper], as well as their "halves," (10) and (11). Neither is a sublattice of the lattice of all binary relations, respectively Boolean functions, on the set of orderings on A. Each of the three will become a sublattice if we add the disjunction of, respectively, $\geq^{\wedge}$ and $\geq^{\vee}$, (7a) and (7b), or (9a) and (9b). However, there will be no analog of Propositions 1–5 or 7–9: the new conditions will be sufficient for the monotonicity w.r.t. the new "order," but not necessary.

Example 3. Let $A := \{0,1,2,3\}$, $T := \{0,1\}$, and a function $u\colon A \times T \to \mathbb{R}$ be defined by the following matrix (the A-axis is directed rightwards; the T-axis, upwards):

$$\begin{matrix} 3 & 3 & 5 & 4 \\ 0 & 1 & 2 & 2 \end{matrix}$$

Neither condition (7a) nor (7b) is satisfied for the orderings $\succ'$ represented by $u(\cdot,1)$ and $\succ$ represented by $u(\cdot,0)$: (7a) is violated for $x = 0$ and $y = 1$; (7b), for $x = 2$ and $y = 3$. On the other hand, every mapping $\mathcal{R}^X$ ($X \in \mathfrak{B}_A$) is increasing w.r.t. the disjunction of $\geq^{\wedge}$ and $\geq^{\vee}$: $\mathcal{R}^A(1) = \{2\}$ while $\mathcal{R}^A(0) = \{2,3\}$, hence $\mathcal{R}^A(1) \geq^{\wedge} \mathcal{R}^A(0)$; on $A \setminus \{3\}$ as well as on $A \setminus \{2\}$, (7a) holds; on $A \setminus \{0\}$ as well as on $A \setminus \{1\}$, (7b) holds.

Similarly, the ordering $\succ$ on the lattice $A \times T$ represented by u satisfies neither condition (9): (9a) is violated in the leftmost 2×2 cell; (9b), in the rightmost 2×2 cell. On the other hand, the mapping $M(\cdot,\succ)\colon \mathfrak{L}_{A\times T} \to \mathfrak{B}_{A\times T}$ is increasing w.r.t. $\geq^{\mathrm{Vt}}$ on the source and the disjunction of $\geq^{\wedge}$ and $\geq^{\vee}$ on the target. Let $X, Y \in \mathfrak{L}_{A\times T}$ and $Y \geq^{\mathrm{Vt}} X$. By Theorem 2 of Milgrom and Shannon (1994), there are $Z \in \mathfrak{L}_{A\times T}$ and $a^+, a^- \in Z$ such that $Y = \{a \in Z \mid a \geq a^+\}$ and $X = \{a \in Z \mid a \leq a^-\}$. If Z is a chain, we have $M(Y,\succ) \geq^{\mathrm{Vt}} M(X,\succ)$ by Milgrom and Shannon's Corollary 1 because every ordering on a chain is QSM. If $(2,1) \notin Z \ni (3,1)$, then $M(Y,\succ) = \{(3,1)\}$, hence $M(Y,\succ) \ggg M(X,\succ)$. If $(2,1) \in Z$, then $M(Y,\succ) = \{(2,1)\}$, hence $M(Y,\succ) \ggg M(X,\succ)$ unless $M(X,\succ) = \{(2,0),(3,0)\}$, in which case $M(Y,\succ) \geq^{\wedge} M(X,\succ)$. Finally, if $Z \cap \{(2,1),(3,1)\} = \emptyset$, then Z is contained in the leftmost 2×2 cell, where $\succ$ is $\vee$-QSM, hence $M(Y,\succ) \geq^{\vee} M(X,\succ)$ by Proposition 17.

It may also be noted that the monotonicity w.r.t. the disjunction of $\geq^{\wedge}$ and $\geq^{\vee}$ seems not to lead to any new result on the existence of monotone selections.

8.5. The description of preferences with an ordering may seem very general, but it may also seem not general enough. Leaving aside the abstruse question of how much rationality in an agent's preferences it is "right" to assume, there is a mundane reason to go beyond orderings. Suppose a utility function $u(x,t)$ is bounded above in x for every t, but need not attain a maximum; then ε-optimization suggests itself strongly, and this means considering a preference relation

$$y \succ^t x \rightleftharpoons u(y,t) > u(x,t) + \varepsilon$$

(with $\varepsilon > 0$). $\mathcal{R}^X(t)$ consists of all ε-maxima of $u(\cdot,t)$. The relation $\succ^t$ is a *strictly acyclic semiorder*, but need not be an ordering. If u satisfies Topkis's (1978) *increasing differences* condition, then $\{\succ^t\}_{t\in T}$ satisfies the single crossing conditions; if u is supermodular in the first argument, then $\succ^t$ is

QSM. Nevertheless, none of the results of this paper is applicable even under so strong assumptions; actually, $\mathcal{R}^X$ need not be ascending. The existence of a monotone selection can be proven when both X and T are chains; the existence of an ε-Nash equilibrium, when every strategy set is a chain (Kukushkin, 2009, Theorems 3 and 4). However, there is no similar result of any kind for non-scalar sets X.

8.6. The "order" relation $\ggg$ is never mentioned in Section 5; although Theorem 5 of LiCalzi and Veinott (1992) *does* contain a statement involving that relation, it can be easily derived from what is reproduced as Statement 2 of our Proposition 10 (if $Y \cap X = \emptyset$, then $Y \ggg X$ is equivalent to $Y \geqslant X$), hence should not be counted here. Actually, characterization results involving the relation $\ggg$ can be obtained in both type B and type A problems.

Let us call an ordering $\succ$ on a lattice A *strictly$^+$ quasisupermodular* (S^+QSM) if

$$\forall x, y \in A \left[[x > y \wedge x \;\&\; x \succeq y \wedge x] \Rightarrow y \vee x \succ y \right]. \tag{13}$$

Effectively, it means that $\succ$ is SQSM and no pair of points comparable in the basic order can be equivalent; in other words, the relation (7c) is reflexive on $\succ$. The restriction is quite exacting: if, loosely speaking, both the set of alternatives and the preferences are continuous, then (13) implies that $\succ$ is either strictly increasing or strictly decreasing.

Proposition 34. *Let $\succ$ be an ordering on a lattice A. Then the following statements are equivalent.*

1. *There holds* (13).
2. *There holds $M(Y, \succ) \ggg M(X, \succ)$ whenever $Y, X \in \mathfrak{L}_A$ and $Y \geq^{\mathrm{Vt}} X$.*
3. *There holds $M(X, \succ^{\perp}) \ggg M(X, \succ)$ whenever $X \in \mathfrak{L}_A$ and $\succ^{\perp}$ is an ordering on X such that* (7a) *holds on X, and there holds $M(X, \succ) \ggg M(X, \succ^{\perp})$ whenever $X \in \mathfrak{L}_A$ and $\succ^{\perp}$ is an ordering on X such that* (8b) *holds on X.*

Moreover, Statement 2 can be restricted to $Y, X \in \mathfrak{L}_A$ with $\#Y = \#X = 2$; Statement 3 can be restricted to $X \in \mathfrak{L}_A$ with $\#X \leq 4$ and $\succ^{\perp}$ defined (and satisfying the appropriate single crossing condition) on A.

A proof, similar to those above, is omitted; we also omit an exact formulation of the characterization of S$^+$QSM in terms of monotonicity preserved.

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