Better response dynamics and Nash equilibrium in discontinuous games

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Abstract

Philip Reny's approach to games with discontinuous utility functions can work outside its original context. The existence of Nash equilibrium and the possibility to approach the equilibrium set with a finite number of individual improvements are established, under conditions weaker than the better reply security, for three classes of strategic games: potential games, games with strategic complements, and aggregative games with appropriate monotonicity conditions.
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1 Introduction

Reny (1999) made a significant step in the development of sufficient conditions for Nash equilibrium existence in games with discontinuous utility functions. A feature common to games considered by Reny and most of his followers, see, e.g., McLennan et al. (2011) or Prokopovych (2013), is that the strategy sets are convex and each utility function is quasiconcave in own argument. Bich (2009) relaxes the quasiconcavity, but not at all radically.

In this paper, we extend Reny's approach to three different classes of strategic games: potential games; games with strategic complements; aggregative games with appropriate monotonicity conditions. Besides, our attention is switched from the mere existence of a Nash equilibrium to the possibility to approach the equilibrium set with a finite "individual improvement path." What unites the three classes is that the existence of a Nash equilibrium in none of them has anything to do with convexity. Moreover, it is much easier to prove and understand in the case of a *finite* game; in an infinite game, there may be no equilibrium at all, to say nothing of its approachability, without some topological assumptions. And for each class of games, we obtain a set of such assumptions that could not be derived from the previous literature.

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Following Reny (2016), we consider games with purely ordinal preferences, i.e., where utility functions take values from arbitrary chains rather than the real line. Inevitably, we only consider pure strategies. Our (i.e., essentially, Reny's) topological assumptions do not ensure the existence of the best responses; therefore, the standard fixed point theorems cannot be applied directly. Instead, we consider finite subgames, where Nash equilibria not only exist, but can be reached, starting from an arbitrary strategy profile, with a finite number of individual improvements. The "finite deviation" assumptions ensure the possibility to find a finite subgame every Nash equilibrium of which is arbitrarily close to the set of Nash equilibria of the original game. Thus, we obtain the "very weak finite improvement" property of the original game: the set of Nash equilibria is nonempty and can be approached with a finite number of individual improvements starting anywhere in the set of strategy profiles.

We understand potential games in a much broader sense than Monderer and Shapley (1996), viz. we consider games where individual improvements are acyclic. Thus, our Theorem 1 generalizes the main result of Kukushkin (2011), which in its turn generalized the good old "acyclicity plus open lower contour sets" theorem (Bergstrom, 1975; Walker, 1977). As an application to economics, we show that the assumptions of Theorem 1 hold in a rather general class of Bertrand competition games (Propositions 4.1 and 4.2).

Strategic complements are also understood in a more general, ordinal sense, as in Milgrom and Shannon (1994), rather than in the cardinal one, as in Vives (1990). Moreover, we do not fix a list of requirements a game must satisfy to deserve the badge of "Strategic Complements." The point is that there are various versions of the single crossing and quasisupermodularity conditions in the literature (Milgrom and Shannon, 1994; LiCalzi and Veinott, 1992; Shannon, 1995; Quah, 2007; Quah and Strulovici, 2009; Kukushkin, 2013b) and "trade-offs" between them are possible, i.e., a stronger interpretation of one property coupled with a weaker interpretation of the other may have the same implications as a weaker interpretation of the first property together with a stronger interpretation of the second. Our Theorems 2 and $2'$ extend the main result of Kukushkin et al. (2005) to infinite games, even with some strengthening.

While the only known way to establish the existence of an equilibrium in a potential game of Section 4 or an aggregative game of Section 6 consists in following improvement paths, in the case of strategic complements there is also an option of invoking Tarski's fixed point theorem, which ensures equilibrium existence without giving much information on better, or even best, response dynamics (e.g., Theorem 5.1 of Vives (1990) establishes the convergence of Cournot tâtonnement to equilibrium only if the starting point belongs to a rather specific area in the set of strategy profiles). The fact that the mere existence of an equilibrium can be obtained under weaker assumptions than in our Theorem 2 may be of interest to some readers. (An anonymous referee even refused to see any value in studying improvement dynamics when the existence of an equilibrium can be established by other means.) Accordingly, a list of such assumptions is given in Propositions 5.1 and 5.2. A comparison with an earlier result on the existence of Nash equilibrium in a discontinuous game with a version of strategic complements, Theorem 2 of Prokopovych and Yannelis (2017), is in Section 7.5.

In contrast to strategic complements, strategic substitutes, by themselves, are not conducive to the existence of Nash equilibrium. In a game with additive aggregation, however, they do ensure the existence of an equilibrium as was shown by Novshek (1985), see also Kukushkin (1994). Dubey et al.

(2006), having modified a construction invented by Huang (2002) for different purposes, created a tool applicable to some non-additive aggregation rules as well. Kukushkin (2005) used the tool to show the convergence of Cournot tâtonnement to equilibrium in aggregative games exhibiting strategic complements, strategic substitutes, or a combination of both. The most general description of aggregation rules for which that trick can still work was given by Jensen (2010). Our Theorem 3 establishes the existence and approachability of Nash equilibrium in games with Jensen aggregation rules where the best responses may fail to exist.

Section 2 contains basic definitions and notations associated with a strategic game. In Section 3, we reproduce Reny's original notions and more general topological conditions, which, via a technical Proposition 3.4, play the key role in the rest of the paper. In Sections 4, 5, and 6, we consecutively apply Proposition 3.4 to potential games, games with strategic complements, and aggregative games. Several related questions of secondary importance are discussed in Section 7. More complicated (or just tedious) proofs (of Proposition 3.2, Propositions 4.1 and 4.2, Theorem 2 and Theorem 3) are deferred to Appendix.

2 Basic definitions

A *strategic game* Γ is defined by a finite set of *players* N and, for each $i \in N$, a *strategy set* X_i, a chain $\mathcal{C}_i$ (a *utility scale*), and a "generalized" *utility function* $u_i \colon X_N \to \mathcal{C}_i$, where $X_N := \prod_{i \in N} X_i$ is the set of *strategy profiles*. For each $i \in N$, we denote $X_{-i} := \prod_{j \in N \setminus \{i\}} X_j$, and often use notation like $(x_i, x_{-i}) \in X_N$.

With every strategic game, we associate this *individual improvement relation* $\rhd^{\mathrm{Ind}}$ on X_N ($i \in N$, $y_N, x_N \in X_N$):

$$y_N \rhd_i^{\mathrm{Ind}} x_N \rightleftharpoons [y_{-i} = x_{-i} \ \& \ u_i(y_N) > u_i(x_N)];$$
$$y_N \rhd^{\mathrm{Ind}} x_N \rightleftharpoons \exists i \in N \, [y_N \rhd_i^{\mathrm{Ind}} x_N].$$

By definition, a Nash equilibrium is a *maximizer* of the relation $\rhd^{\mathrm{Ind}}$ on X_N, i.e., a strategy profile $x_N \in X_N$ such that $y_N \rhd^{\mathrm{Ind}} x_N$ holds for no $y_N \in X_N$. The set of Nash equilibria is denoted $\mathrm{E}(\Gamma) \subseteq X_N$.

An (*individual*) *improvement path* is a (finite or infinite) sequence $\langle x_N^k \rangle_{k=0,1,\ldots}$ such that $x_N^{k+1} \rhd^{\mathrm{Ind}} x_N^k$ whenever $k \geq 0$ and x_N^{k+1} is defined. A strategic game Γ has the *finite improvement property* (*FIP*, Monderer and Shapley, 1996) iff there is no infinite improvement path. Γ has the *weak finite improvement property* (*weak FIP*) iff, for every strategy profile $x_N^0 \in X_N$, there is a finite improvement path $x_N^0, \ldots, x_N^m$ such that $x_N^m \in \mathrm{E}(\Gamma)$. Obviously, FIP implies weak FIP: *every* improvement path in a game with FIP ends at a Nash equilibrium after a finite number of steps. Both properties look more natural for a finite game although they may be observed in an infinite game now and then.

Henceforth, the strategy sets X_i are assumed to be topological spaces; each chain $\mathcal{C}_i$ is endowed with its order interval topology; the sets X_N, $\mathcal{C}_N := \prod_{i \in N} \mathcal{C}_i$, X_{-i}, and $X_N \times \mathcal{C}_N$ are endowed with their product topologies. The topological closure of a subset Y of any one of those spaces is denoted $\operatorname{cl} Y$. We say that Γ has the *very weak FIP* (Kukushkin, 2011) iff, for every $x_N^0 \in X_N$, there is $y_N \in \mathrm{E}(\Gamma)$ such that for every open neighborhood O of y_N there is a finite improvement path $x_N^0, \ldots, x_N^m$ with $x_N^m \in O$.

Slightly relaxing the requirement, we say that Γ has the *very-very weak FIP* iff, for every $x_N^0 \in X_N$, there is $y_N \in \operatorname{cl} \mathrm{E}(\Gamma)$ such that for every open neighborhood O of y_N there is a finite improvement path $x_N^0, \ldots, x_N^m$ with $x_N^m \in O$.

Remark. If X_N is a *metric space* with a metric d, then the very-very weak FIP can be reformulated as follows: for every $x_N^0 \in X_N$ and every $\varepsilon > 0$, there are $y_N \in \mathrm{E}(\Gamma)$ and a finite improvement path $x_N^0, \ldots, x_N^m$ such that $d(y_N, x_N^m) < \varepsilon$. In this case, the difference between the very weak FIP and the very-very weak FIP is whether the same $y_N \in \mathrm{E}(\Gamma)$ can be chosen for all $\varepsilon > 0$ or not.

Proposition 2.1. *A strategic game Γ has the very weak FIP if and only if, for every $x_N^0 \in X_N$ and every open neighborhood O of $\mathrm{E}(\Gamma)$, there is a finite improvement path $x_N^0, \ldots, x_N^m$ such that $x_N^m \in O$.*

Proof. The necessity is obvious: every open neighborhood of $\mathrm{E}(\Gamma)$ is simultaneously an open neighborhood of $y_N \in \mathrm{E}(\Gamma)$ from the definition of the very weak FIP. To prove the sufficiency, we suppose the contrary: for every $y_N \in \mathrm{E}(\Gamma)$, there is an open neighborhood $O(y_N) \ni y_N$ such that no finite improvement path started at x_N^0 ever reaches $O(y_N)$. Then we set $O := \bigcup_{y_N \in \mathrm{E}(\Gamma)} O(y_N)$; in the case of $\mathrm{E}(\Gamma) = \emptyset$, $O := \emptyset$. Now O is an open neighborhood of $\mathrm{E}(\Gamma)$; therefore, there must be a finite improvement path $x_N^0, \ldots, x_N^m$ such that $x_N^m \in O$. If $\mathrm{E}(\Gamma) = \emptyset$, we have $x_N^m \in \emptyset$; otherwise, there holds $x_N^m \in O(y_N)$ for some $y_N \in \mathrm{E}(\Gamma)$. In either case, we have a contradiction. $\square$

Proposition 2.2. *A strategic game Γ has the very-very weak FIP if and only if, for every $x_N^0 \in X_N$ and every open neighborhood O of $\operatorname{cl} \mathrm{E}(\Gamma)$, there is a finite improvement path $x_N^0, \ldots, x_N^m$ such that $x_N^m \in O$.*

The proof is essentially the same as that of Proposition 2.1; only $\mathrm{E}(\Gamma)$ should be replaced with $\operatorname{cl} \mathrm{E}(\Gamma)$.

3 Better-reply security and finite deviation

We start with auxiliary notations. Considering functions u_i as components of a mapping $u_N \colon X_N \to \mathcal{C}_N$, we denote G the graph of the mapping, i.e., the set of pairs $\langle x_N, u_N(x_N) \rangle \in X_N \times \mathcal{C}_N$ for all $x_N \in X_N$. For every $x_N \in X_N$, we denote $\bar{G}(x_N) := \{v_N \in \mathcal{C}_N \mid (x_N, v_N) \in \operatorname{cl} G\}$ and perceive $\bar{G}$ as a correspondence from X_N to $\mathcal{C}_N$.

Then, we reproduce Reny's (1999) definitions. Player $i \in N$ *can secure a payoff of $\alpha \in \mathcal{C}_i$ at $x_N^* \in X_N$* iff there exists $y_i \in X_i$ such that $u_i(y_i, x_{-i}) \geq \alpha$ for all x_{-i} in some open neighborhood of x_{-i}^*. A game Γ is *better-reply secure* iff, whenever x_N is not a Nash equilibrium and $v_N \in \bar{G}(x_N)$, some player i can secure a payoff strictly above v_i at x_N.

Let $Y \subseteq X_N$ be a set of strategy profiles and Z be a set of pairs $\langle i, y_i \rangle$ ($i \in N$, $y_i \in X_i$). We say that Z *dominates* Y iff for every $x_N \in Y$ there holds $u_i(y_i, x_{-i}) > u_i(x_N)$, i.e., $(y_i, x_{-i}) \rhd^{\mathrm{Ind}} x_N$, for (at least one) $\langle i, y_i \rangle \in Z$. When Z is finite, we say that Y is *finitely dominated* (with Z).

A game Γ has the *R-finite deviation property* iff, for every $\bar{x}_N \in X_N \setminus \mathrm{E}(\Gamma)$, there is an open neighborhood of $\bar{x}_N$ which is finitely dominated. Γ has the *P-finite deviation property* iff, for every $\bar{x}_N \in X_N \setminus \mathrm{E}(\Gamma)$, there is an open neighborhood O of $\bar{x}_N$ such that $O \setminus \mathrm{E}(\Gamma)$ is finitely dominated. Γ

has the *Q-finite deviation property* iff, for every $\bar{x}_N \in X_N \setminus \operatorname{cl} \mathrm{E}(\Gamma)$, there is an open neighborhood of $\bar{x}_N$ which is finitely dominated.

$Y \subseteq X_N$ is *singly dominated* iff it is dominated with a set Z containing at most one pair $\langle i, y_i \rangle$ for each $i \in N$. A game Γ has the *R-single deviation property* [*Q-single deviation property*] iff, for every $\bar{x}_N \in X_N \setminus \mathrm{E}(\Gamma)$ [$\bar{x}_N \in X_N \setminus \operatorname{cl} \mathrm{E}(\Gamma)$], there is an open neighborhood of $\bar{x}_N$ which is singly dominated. Γ has the *P-single deviations property* iff, for every $\bar{x}_N \in X_N \setminus \mathrm{E}(\Gamma)$, there is an open neighborhood O of $\bar{x}_N$ such that $O \setminus \mathrm{E}(\Gamma)$ is singly dominated.

R-single/finite deviation properties were introduced by Reny (2011), under the names of just *single/ finite deviation properties*. P-single deviation property was introduced by Prokopovych (2013), under the name of *weak single deviation property*. The other definitions are given here by analogy, for the completeness of the picture. This set of implications is obvious:

$$\begin{array}{ccccc} \text{R-single deviation} & \Rightarrow & \text{P-single deviation} & \Rightarrow & \text{Q-single deviation} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{R-finite deviation} & \Rightarrow & \text{P-finite deviation} & \Rightarrow & \text{Q-finite deviation.} \end{array}$$

Example 3.1. Let us consider a strategic game Γ where $N := \{1, 2\}$; $X_1 := \{0, 1, 2\}$; $X_2 := \{0\} \cup \{1/h\}_{h=1,2,\ldots} \subset \mathbb{R}$; $u_1(0, x_2) := 0$ for all x_2;

$$u_1(1, x_2) := \begin{cases} 3, & x_2 = 0 \ \text{ or } \ x_2 = 1/(2h+1); \\ -3, & x_2 = 1/(2h); \end{cases}$$

$$u_1(2, x_2) := \begin{cases} 3, & x_2 = 0 \ \text{ or } \ x_2 = 1/(2h); \\ -3, & x_2 = 1/(2h+1); \end{cases}$$

$u_2(x_1, x_2) := -x_1 \cdot x_2$ for all x_1, x_2.

Clearly, $\mathrm{E}(\Gamma) = \{(1, 0), (2, 0)\}$. Γ has the weak FIP since a Nash equilibrium can be reached from any strategy profile after, at most, two individual improvements. The game has the R-finite deviation property, but not even Q-single deviation property: both $x_1 = 1$ and $x_1 = 2$ are necessary to dominate any open neighborhood of $(0, 0)$.

A one-person game with the P-single deviation property, but without the R-finite deviation property is easy to produce. Example 4.2 presents a game with the Q-single deviation property, but without the P-finite deviation property.

Proposition 3.1. *If a game Γ has the R-finite deviation property, then* $\mathrm{E}(\Gamma)$ *is closed.*

A straightforward proof is omitted. Note that $\mathrm{E}(\Gamma)$ may be empty. Obviously, R-finite deviation cannot be replaced with P-single deviation.

Proposition 3.2. *If a game Γ is better-reply secure and* $\operatorname{cl} u_N(X_N)$ *is compact, then Γ has the R-single deviation property.*

The statement is rather close to Reny (2011, Theorem 1) and Prokopovych (2013, Lemma 2), and implies both. Since our assumptions are broader, a complete proof is given in Appendix, Section A.

Proposition 3.3. *If a game Γ has the Q-finite deviation property, and $Y \subseteq (X_N \setminus \mathrm{cl}\,\mathrm{E}(\Gamma))$ is compact, then Y is finitely dominated. If a game Γ has the P-finite deviation property, and $Y \subseteq (X_N \setminus \mathrm{E}(\Gamma))$ is compact, then Y is finitely dominated.*

Proof. Both statements are proven with essentially the same argument. By our assumption, there is an open neighborhood $O(x_N)$ of every $x_N \in Y$ which is finitely dominated, or such that $O(x_N) \setminus \mathrm{E}(\Gamma)$ is finitely dominated. Since Y is compact, it is covered by a finite number of those open neighborhoods. Taking the union of the appropriate sets of pairs $\langle i, y_i \rangle$, we see that Y is finitely dominated indeed. $\square$

To formulate our main technical result, we need a few definitions more.

A *subgame* Γ' of Γ is a strategic game defined by subsets $X'_i \subseteq X_i$ for all $i \in N$ and the restriction of the utility mapping u_N to $X'_N := \prod_{i \in N} X'_i$; we will use the notation $\Gamma' \leq \Gamma$. The individual improvement relation in a subgame is the restriction of $\rhd^{\mathrm{Ind}}$ to X'_N. If $x_N \in \mathrm{E}(\Gamma) \cap X'_N$, then $x_N \in \mathrm{E}(\Gamma')$; if $x_N \in \mathrm{E}(\Gamma')$, it need not belong to $\mathrm{E}(\Gamma)$. Γ has the *quasi weak FIP* iff, for every finite subgame Γ' of Γ, there is Γ'' such that $\Gamma' \leq \Gamma'' \leq \Gamma$ and Γ'' has the weak FIP.

Proposition 3.4. *Let a game Γ have the quasi weak FIP, and let X_N be compact. If Γ has the P-finite deviation property, then it has the very weak FIP. If Γ has the Q-finite deviation property, then it has the very-very weak FIP.*

Proof. As in the case of Propositions 2.1 and 2.2, the proofs of both assertions are essentially the same; only $\mathrm{E}(\Gamma)$ should be replaced with $\mathrm{cl}\,\mathrm{E}(\Gamma)$ in the second case. We consider the first statement and apply the criterion established in Proposition 2.1.

Let $O \supseteq \mathrm{E}(\Gamma)$ be open and let $x^0_N \in X_N \setminus O$. Since $X_N \setminus O$ is compact, it is finitely dominated by Proposition 3.3. Let Z be an appropriate finite set of pairs. For each $i \in N$, we define $X'_i := \{x^0_i\} \cup \{y_i \mid \langle i, y_i \rangle \in Z\} \subseteq X_i$. The sets X'_i define a finite subgame Γ' of Γ; by our assumption, there is Γ'' such that $\Gamma' \leq \Gamma'' \leq \Gamma$ and Γ'' has the weak FIP. Therefore, there is a finite improvement path $x^0_N, \ldots, x^m_N$ in Γ'' such that $x^m_N \in \mathrm{E}(\Gamma'')$. Now, we have either $x^m_N \in O$ or $x^m_N \notin O$. In the first case, we are home because $x^0_N, \ldots, x^m_N$ remains a finite improvement path in Γ. In the second case, we would have $x^m_N \in X_N \setminus O \subseteq X_N \setminus \mathrm{cl}\,\mathrm{E}(\Gamma)$ and hence there would be $\langle i, y_i \rangle \in Z$ such that $(y_i, x^m_{-i}) \rhd^{\mathrm{Ind}}_i x^m_N$, which is incompatible with $x^m_N \in \mathrm{E}(\Gamma'')$ since $y_i \in X'_i \subseteq X''_i$.

Since $O \supseteq \mathrm{E}(\Gamma)$ and $x^0_N \in X_N$ were arbitrary, Proposition 2.1 is applicable indeed and we are home. $\square$

4 Potential games

The relation $\rhd^{\mathrm{Ind}}$ is *acyclic* iff there is no *finite improvement cycle*, i.e., no improvement path for which $x^0_N = x^m_N$ with $m > 0$. A sufficient condition for that is the existence of a *generalized ordinal potential* (Monderer and Shapley, 1996), i.e., a function $P \colon X_N \to \mathbb{R}$ such that $P(y_N) > P(x_N)$ whenever $y_N \rhd^{\mathrm{Ind}} x_N$. (For a finite game, that condition is also necessary.)

Theorem 1. *Let Γ be a strategic game with compact strategy sets X_i. Let the individual improvement relation $\rhd^{\mathrm{Ind}}$ in Γ be acyclic. If Γ has the P-finite deviation property, then it has the very weak FIP.*

If Γ has the Q-finite deviation property, then it has the very-very weak FIP. In particular, Γ possesses a Nash equilibrium in either case.

Proof. Let Γ' be a finite subgame of Γ. Since $\rhd^{\mathrm{Ind}}$ is acyclic in Γ, and hence in Γ' as well, Γ' even has the FIP. Therefore, Γ has the quasi weak FIP and hence Proposition 3.4 is applicable. $\square$

The assumptions of Theorem 1 are satisfied in Examples 4, 5, and 6 of Prokopovych and Yannelis (2017). The first two games even have the weak FIP: no more than two individual improvements are needed to reach a Nash equilibrium from every strategy profile. Example 6 actually belongs to a rather general class of games covered by Theorem 1.

We define a *simple Bertrand competition game* (with linear production costs and without biting capacity constraints) as follows. There is a finite set N of firms capable of producing a homogenous good. Each firm $i \in N$ is characterized by its constant marginal cost of production $c_i \geq 0$; its strategy is a price $x_i \in [c_i, K_i]$ (dumping is forbidden). When all prices are announced, consumers buy at the cheapest. The total demand is given by an upper semicontinuous and decreasing function $D \colon \mathbb{R}_+ \to \mathbb{R}_+$. To avoid pathologies, we assume that $\#N > 1$, $\max_i c_i < \min_i K_i$ and $D(p) > 0$ for some $p > \max_i c_i$. The firms which announced the lowest price share the total demand equally; the firms which announced higher prices produce nothing and sell nothing. Denoting $M(x_N) := \mathrm{Argmin}_i\, x_i \subseteq N$, the utility functions are

$$u_i(x_N) := \begin{cases} (x_i - c_i) \cdot D(x_i)/\#M(x_N), & i \in M(x_N); \\ 0, & i \notin M(x_N). \end{cases}$$

Proposition 4.1. *The individual improvement relation $\rhd^{\mathrm{Ind}}$ is acyclic in every simple Bertrand competition game Γ. If $\#\,\mathrm{Argmin}_{i\in N}\, c_i > 1$, then Γ has the R-single deviation property and hence satisfies the assumptions of Theorem 1.*

Theorem 1 is, thus, applicable whenever all firms have the same marginal cost of production. When a single firm is the most efficient producer, i.e., $\mathrm{Argmin}_{j\in N}\, c_j = \{i\}$, the exact shape of the demand function D (to be more precise, what price(s) would be optimal for player i as a monopolist) starts to matter. Denoting $\bar{c} := \min_{j\neq i} c_j\ [> c_i]$ in this case, we formulate five similar, but distinct, conditions:

$$\operatorname*{Argmax}_{x_i\in[c_i,\bar{c}]} \big((x_i - c_i)\cdot D(x_i)\big) \subseteq [c_i, \bar{c}[; \tag{1a}$$

$$\exists c > \bar{c}\, \Big[\operatorname*{Argmax}_{x_i\in[c_i,c]} \big((x_i - c_i)\cdot D(x_i)\big) \subseteq [c_i, \bar{c}] \Big]; \tag{1b}$$

$$\exists c > \bar{c}\, \Big[[c_i, \bar{c}] \cap \operatorname*{Argmax}_{x_i\in[c_i,c]} \big((x_i - c_i)\cdot D(x_i)\big) \neq \emptyset \Big]; \tag{1c}$$

$$\forall c < \bar{c}\, \Big[[c, \bar{c}[\cap \operatorname*{Argmax}_{x_i\in[c_i,\bar{c}]} \big((x_i - c_i)\cdot D(x_i)\big) \neq \emptyset \Big]; \tag{1d}$$

$$[c_i, \bar{c}[\cap \operatorname*{Argmax}_{x_i\in[c_i,\bar{c}]} \big((x_i - c_i)\cdot D(x_i)\big) \neq \emptyset. \tag{1e}$$

These implications hold: (1a) $\Rightarrow$ (1b) $\Rightarrow$ (1c) and (1a) $\Rightarrow$ (1e) $\Leftarrow$ (1d); the first follows from the upper semicontinuity of D; the others are obvious. All other implications between conditions (1) are, generally, wrong.

Proposition 4.2. *Let Γ be a simple Bertrand competition game such that $\operatorname{Argmin}_{j\in N} c_j = \{i\}$. Then:*

4.2.1. *Condition (1a) is sufficient for Γ to have the R-single deviation property and necessary for Γ to have the R-finite deviation property.*

4.2.2. *The conjunction of conditions (1b) and (1e) is sufficient for Γ to have the P-single deviation property and necessary for Γ to have the P-finite deviation property.*

4.2.3. *The disjunction of conditions (1c) and (1d) is sufficient for Γ to have the Q-single deviation property and necessary for Γ to have the Q-finite deviation property.*

4.2.4. *Condition (1e) is necessary and sufficient for Γ to have the very weak FIP property.*

4.2.5. *The disjunction of all conditions (1) is sufficient for Γ to have the very-very weak FIP property and necessary for Γ to possess a Nash equilibrium.*

Intertwined proofs of both Propositions 4.1 and 4.2 are deferred to Appendix, Section B.

A simple Bertrand competition game may possess a Nash equilibrium without satisfying the assumptions of Theorem 1. However, this situation is inherently unstable: an arbitrarily small perturbation of coefficients c_j makes the theorem applicable.

Proposition 4.3. *Whenever a simple Bertrand competition game Γ possesses a Nash equilibrium, but does not satisfy the assumptions of Theorem 1, there is a player $j \in N$ such that for every $\varepsilon > 0$ there is $c'_j \in [c_j - \varepsilon, c_j]$ such that the simple Bertrand competition game Γ' where c_j is replaced with c'_j while everything else remains the same has the R-single deviation property and hence satisfies the assumptions of Theorem 1.*

Proof. Our assumption implies that $\operatorname{Argmin}_{j\in N} c_j = \{i\}$, (1e) holds, and all other conditions (1) do not. Thus, there is $x_i^+ \in [c_i, \bar{c}[$ such that $(x_i^+ - c_i)\cdot D(x_i^+) = (\bar{c} - c_i)\cdot D(\bar{c}) = \max_{x_i \in [c_i,\bar{c}]} \big((x_i - c_i)\cdot D(x_i)\big)$. The negation of (1d) implies the existence of $c^* \in [c_i, \bar{c}[$ such that $(x_i^+ - c_i)\cdot D(x_i^+) > (x_i - c_i)\cdot D(x_i)$ for all $x_i \in [c^*, \bar{c}[$; clearly, $x_i^+ < c^*$. We pick $j \in N$ for which $c_j = \bar{c}$. Given $\varepsilon > 0$, we set $c'_j := \max\{c^*, c_j - \varepsilon\}$. In the modified game Γ', we have $\bar{c}' = c'_j$. Thus, $(x_i^+ - c_i)\cdot D(x_i^+) > (\bar{c}' - c_i)\cdot D(\bar{c}')$, i.e., (1a) holds in Γ'. $\square$

Example 4.1. Let us consider a simple Bertrand competition game Γ where $N := \{1, 2\}$; $c_1 := 0$; $c_2 := 2$; $K_1 := K_2 := 10$;

$$D(p) := \begin{cases} 2, & 0 \le p \le 1; \\ 1, & 1 < p \le 3; \\ 0, & 3 < p. \end{cases}$$

The demand is formed by two big buyers; each is willing to buy just one unit of the good; the highest acceptable price is 1 for one buyer and 3 for the other. Let us denote $V(p) := pD(p)$ the profit of player 1 as a monopolist. Clearly, $\mathrm{Argmax}_{p\in[0,2]} V(p) = \{1, 2\}$, while $\mathrm{Argmax}_{p\in[0,c]} V(p) = \{c\}$ for every $c > 2 = \bar{c}$. Clearly, $\mathrm{E}(\Gamma) = \{(1, 2)\}$. Of all conditions (1), only (1e) holds, so Theorem 1 is not applicable. By Proposition 4.2, Γ has the very weak FIP property.

If, everything else remaining the same, c_2 is greater than 2, then condition (1e) no longer holds and the Nash equilibrium disappears. If c_2 is slightly less than 2, then, in accordance with Proposition 4.3, even (1a) holds and $\mathrm{E}(\Gamma)$ becomes $\{x_N \in X_N \mid x_1 = 1, x_2 \leq 2\}$.

Example 4.2. Let us consider a simple Bertrand competition game Γ where $N := \{1, 2\}$; $c_1 := 0$; $c_2 := 1$; $K_1 := K_2 := 10$; $D(p) := \max\{(3 - p)/(1 + p), 0\}$. Again denoting, for $p < 3$, $V(p) := (3 - p)p/(1 + p) = 4p/(1 + p) - p$, profit of player 1 as a monopolist, we have $V'(p) = 4/(1 + p)^2 - 1$; therefore, $V'(p) > 0$ for $p < 1$ and $V'(p) < 0$ for $p > 1$. Of all conditions (1), only (1b) and (1c) hold. By Proposition 4.2, Γ has the Q-single deviation property, but not even the P-finite deviation property. Actually, the best response of player 1 is $x_1 = 1$ when $x_2 > 1$, and does not exist when $x_2 = 1$; therefore, $\mathrm{E}(\Gamma) = \{x_N \in X_N \mid x_1 = 1, x_2 > 1\}$.

It is instructive to ascertain that Γ does not have the very weak FIP property, only the very-very weak FIP. Let us consider a strategy profile $x_N = (x_1, 1)$ with $x_1 < 1$. Player 2 cannot improve at all; player 1 can improve choosing $y_1 \in \,]x_1, 1[$. Repeating such improvements, player 1 can, at most, realize an infinite improvement path converging to $x^\omega_N := (1, 1)$. Since x^ω_N belongs to $\mathrm{cl}\,\mathrm{E}(\Gamma)$, but not to $\mathrm{E}(\Gamma)$, we see that there is no very weak FIP indeed. An interesting point is that an “infinite improvement cycle” is possible in Γ (where all improvements are done by player 1). Choosing x^ω_N as a starting profile, we inevitably move to $(x_1, 1)$ with $x_1 < 1$ at the first step, and then can return back to x^ω_N in the limit.

If, everything else remaining the same, c_2 is greater than 1, then condition (1a) holds; hence the R-single deviation property obtains and $\mathrm{E}(\Gamma)$ becomes $\{x_N \in X_N \mid x_1 = 1\}$. If c_2 is slightly less than 1, then all conditions (1) are broken and all Nash equilibria disappear.

5 Strategic complements

We start with standard definitions useful for monotone comparative statics.

Let X and S be *partially ordered sets* (*posets*) and $\mathcal{C}$ be a chain. We say that a function $u\colon X \times S \to \mathcal{C}$ satisfies the *single crossing* conditions (Milgrom and Shannon, 1994) iff, for all $x, y \in X$ and $s, s' \in S$, there holds

$$[y > x \;\&\; s' > s \;\&\; u(y, s) > u(x, s)] \Rightarrow u(y, s') > u(x, s'); \tag{2a}$$

$$[y < x \;\&\; s' > s \;\&\; u(y, s') > u(x, s')] \Rightarrow u(y, s) > u(x, s). \tag{2b}$$

u satisfies the *weak single crossing* condition (Shannon, 1995) iff

$$[y > x \;\&\; s' > s \;\&\; u(y, s) > u(x, s)] \Rightarrow u(y, s') \geq u(x, s') \tag{3}$$

for all $x, y \in X$ and $s, s' \in S$. Either condition (2) implies (3).

Let X be a *lattice*. A function $u\colon X \to \mathcal{C}$ is *quasisupermodular* (Milgrom and Shannon, 1994; LiCalzi and Veinott, 1992) iff, whenever $y, x \in X$,

$$u(x) > u(y \wedge x) \Rightarrow u(y \vee x) > u(y); \tag{4a}$$
$$u(x) > u(y \vee x) \Rightarrow u(y \wedge x) > u(y). \tag{4b}$$

When X is a chain, both conditions (4) are satisfied in a trivial way for every function u. Kukushkin (2013b) partitioned conditions (4) into four independent conditions, two of which will be used here:

$$u(x) > u(y \wedge x) \Rightarrow u(y \vee x) > \min\{u(x), u(y)\}; \tag{5a}$$
$$u(x) > u(y \vee x) \Rightarrow u(y \wedge x) > \min\{u(x), u(y)\}. \tag{5b}$$

A function $u\colon X \to \mathcal{C}$ is *weakly quasisupermodular* (Shannon, 1995; LiCalzi and Veinott, 1992) iff, for all $x, y \in X$,

$$u(x) > u(y \wedge x) \Rightarrow u(y \vee x) \geq \min\{u(x), u(y)\}; \tag{6a}$$
$$u(x) > u(y \vee x) \Rightarrow u(y \wedge x) \geq \min\{u(x), u(y)\}. \tag{6b}$$

These implications are obvious: (4a) $\Rightarrow$ (5a) $\Rightarrow$ (6a); (4b) $\Rightarrow$ (5b) $\Rightarrow$ (6b). Meanwhile, (4a) does not imply (6b), and (4b) does not imply (6a).

Theorem 2. *Let Γ be a strategic game such that each strategy set X_i is simultaneously a compact topological space and a distributive lattice. Let each utility function u_i satisfy the condition (2a) with $X := X_i$, $S := X_{-i}$, and $\mathcal{C} := \mathcal{C}_i$. Let every function $u_i(\cdot, x_{-i})\colon X_i \to \mathcal{C}_i$ ($i \in N$, $x_{-i} \in X_{-i}$) satisfy the condition (5a). If Γ has the P-finite deviation property, then it has the very weak FIP. If Γ has the Q-finite deviation property, then it has the very-very weak FIP. In particular, Γ possesses a Nash equilibrium in either case.*

Essentially, this theorem follows from Proposition 3.4 above and Theorem 1 of Kukushkin et al. (2005). Since the assumptions of the latter theorem were somewhat stronger than those made here, a complete proof is given in Appendix, Section C.

Theorem 2′. *Let Γ be a strategic game such that each strategy set X_i is simultaneously a compact topological space and a distributive lattice. Let each utility function u_i satisfy the condition (2b) with $X := X_i$, $S := X_{-i}$, and $\mathcal{C} := \mathcal{C}_i$. Let every function $u_i(\cdot, x_{-i})\colon X_i \to \mathcal{C}_i$ ($i \in N$, $x_{-i} \in X_{-i}$) satisfy the condition (5b). If Γ has the P-finite deviation property, then it has the very weak FIP. If Γ has the Q-finite deviation property, then it has the very-very weak FIP. In particular, Γ possesses a Nash equilibrium in either case.*

The proof is dual to that of Theorem 2.

To formulate a weaker set of assumptions sufficient for the existence of a Nash equilibrium, we need a number of auxiliary definitions. Let X be a lattice, S be a poset, $\mathcal{C}$ be a chain, and u be a function

$u\colon X \times S \to \mathcal{C}$. We say that $s \in S$ is *upward-looking* iff (at least) one of the following conditions holds:

$$(2b) \text{ holds for all } x, y \in X \text{ and } s' \in S, \\ \text{and the function } u(\cdot, s)\colon X \to \mathcal{C} \text{ satisfies (6a) for all } x, y \in X; \quad (7a)$$

$$(3) \text{ holds for all } x, y \in X \text{ and } s' \in S, \\ \text{and the function } u(\cdot, s)\colon X \to \mathcal{C} \text{ satisfies (5a) for all } x, y \in X. \quad (7b)$$

Dually, $s' \in S$ is *downward-looking* iff (at least) one of the following conditions holds:

$$(2a) \text{ holds for all } x, y \in X \text{ and } s \in S, \\ \text{and the function } u(\cdot, s')\colon X \to \mathcal{C} \text{ satisfies (6b) for all } x, y \in X; \quad (8a)$$

$$(3) \text{ holds for all } x, y \in X \text{ and } s \in S, \\ \text{and the function } u(\cdot, s')\colon X \to \mathcal{C} \text{ satisfies (5b) for all } x, y \in X. \quad (8b)$$

For every $s \in S$, we denote $R(s) := \operatorname{Argmax}_{x \in X} u(x, s)$, the set of *best responses*. A *monotone pseudopartition* of S consists of two subsets $S^{\uparrow}, S^{\downarrow} \subseteq S$ such that, whenever $s < s'$, there holds either $s \in S^{\uparrow}$ or $s' \in S^{\downarrow}$. Clearly, any two points from $S \setminus (S^{\uparrow} \cup S^{\downarrow})$ must be incomparable in the order on S.

Proposition 5.1. *Let X be a finite lattice, S be a poset, $\mathcal{C}$ be a chain, and u be a function $u\colon X \times S \to \mathcal{C}$. Let there be a monotone pseudopartition $\langle S^{\uparrow}, S^{\downarrow} \rangle$ of S such that every $s \in S^{\uparrow}$ is upward-looking while every $s' \in S^{\downarrow}$ is downward-looking. Then there exists an increasing mapping $r\colon S \to X$ such that $r(s) \in R(s)$ for every $s \in S$.*

Proof. Since X is finite, $R(s) \neq \emptyset$ for every $s \in S$. The existence of an increasing selection r from R will immediately follow from Theorem 3.2 of Veinott (1989), or, easier to find, Proposition 2.5 from Kukushkin (2013a), once we show that the correspondence R is *weakly ascending* in the sense of Veinott (1989). That property means that $R(s') \geq^{\mathrm{wV}} R(s)$ whenever $s < s'$, where

$$Y \geq^{\mathrm{wV}} Z \rightleftharpoons \forall y \in Y \, \forall z \in Z \, [y \vee z \in Y \;\text{ or }\; y \wedge z \in Z].$$

Let $s < s'$; by our assumption, either $s \in S^{\uparrow}$ or $s' \in S^{\downarrow}$. In the first case, s is upward-looking and hence $R(s') \geq^{\mathrm{wV}} R(s)$ either by (7a) and Proposition 20 from Kukushkin (2013b), or by (7b) and Proposition 12 from Kukushkin (2013b). In the second case, s' is downward-looking and hence $R(s') \geq^{\mathrm{wV}} R(s)$ either by (8a) and Proposition 21 from Kukushkin (2013b), or by (8b) and Proposition 16 from Kukushkin (2013b). ∎

Proposition 5.2. *Let Γ be a strategic game such that each strategy set X_i is simultaneously a compact topological space and a distributive lattice. Let Γ have the Q-finite deviation property. Let, for each $i \in N$, the assumptions of Proposition 5.1, except for the finiteness of X, hold for $X := X_i$, $S := X_{-i}$, $\mathcal{C} := \mathcal{C}_i$, and $u := u_i$. Then Γ possesses a Nash equilibrium.*

Proof. Supposing the contrary, we may apply Proposition 3.3 to the whole X_N and obtain a finite set Z of pairs $\langle i \in N, y_i \in X_i \rangle$ such that for every $x_N \in X_N$ there holds $(y_i, x_{-i}) \rhd_i^{\mathrm{Ind}} x_N$ for (at least) one $\langle i, y_i \rangle \in Z$.

Fixing an arbitrary $x_N^0 \in X_N$, we, for each $i \in N$, define X_i' as $\{x_i^0\} \cup \{y_i \mid \langle i, y_i \rangle \in Z\} \subseteq X_i$ and X_i'' as the minimal sublattice of X_i containing X_i'. Since X_i is distributive, X_i'' is still finite. The subsets X_i'' define a subgame $\Gamma'' \leq \Gamma$, which inherits appropriate conditions (2), (3), (5), or (6) from Γ. Therefore, the assumptions of Proposition 5.1 hold with $X := X_i''$, $S := X_{-i}''$, and $\mathcal{C} := \mathcal{C}_i$ as well, and hence there are increasing selections r_i from the best responce correspondences in Γ'' for all $i \in N$.

Applying Tarski's fixed point theorem to the Cartesian product of r_i's, we obtain that $\mathrm{E}(\Gamma'') \neq \emptyset$. On the other hand, the definition of X_i' ensures that, for every $x_N \in X_N''$, there is $y_N \in X_N''$ such that $y_N \rhd^{\mathrm{Ind}} x_N$, i.e., $\mathrm{E}(\Gamma'') = \emptyset$. The contradiction proves that $\mathrm{E}(\Gamma) \neq \emptyset$. □

If a strategic game satisfies the assumptions of Theorem 2, then every $x_{-i} \in X_{-i}$ is upward-looking by (7b); similarly, the assumptions of Theorem 2′ imply that every $x_{-i} \in X_{-i}$ is downward-looking by (8b). Whether (and how) the assumptions of Theorem 2 or Theorem 2′ could be weakened remains an open question. What *is* certain is that the assumptions of Proposition 5.2 do not ensure even the weak FIP in a finite game.

Example 5.1. Let us consider a strategic game Γ where $N := \{1, 2\}$, $X_1 := X_2 := \{(0,0), (0,1), (1,0), (1,1)\} \subset \mathbb{R}^2$ (with the natural order), and the utilities are defined by the following matrices (player 1 chooses a position within a matrix, player 2 matrix itself; the axes are directed from left to right and from bottom to top):

$$\begin{bmatrix} (0,2) & (0,1) \\ (0,1) & (2,0) \end{bmatrix} \quad \begin{bmatrix} (1,0) & (2,2) \\ (0,0) & (1,0) \end{bmatrix}$$

$$\begin{bmatrix} (1,0) & (0,0) \\ (2,2) & (1,0) \end{bmatrix} \quad \begin{bmatrix} (2,0) & (0,1) \\ (0,1) & (0,2) \end{bmatrix}.$$

Conditions (2) and (6) hold everywhere. There are two Nash equilibria: the leftmost bottom and the rightmost top. However, no improvement path started from the leftmost top or the rightmost bottom ever reaches either of them. In other words, single crossing plus weak quasisupermodularity are sufficient for the existence of a Nash equilibrium, in accordance with Proposition 5.2, but do not ensure the (very) weak FIP.

Example 5.2. Let us consider a strategic game Γ where $N := \{1, 2, 3\}$, $X_i := \{0, 1\} \subset \mathbb{R}$ (with the natural order) for each $i \in N$, and the utilities are defined by the following matrices (player 1 chooses rows, player 2 columns, and player 3 matrices; the axes are directed from left to right and from bottom to top):

$$\begin{bmatrix} (1,0,1) & (0,1,1) \\ (1,1,1) & (1,1,0) \end{bmatrix} \quad \begin{bmatrix} (1,1,0) & (1,1,1) \\ (0,1,1) & (1,0,1) \end{bmatrix}.$$

Conditions (3) and (4) hold everywhere (the latter, trivially). There are two Nash equilibria: the leftmost bottom and the rightmost top. However, no improvement path started anywhere else ever reaches either of them. In other words, weak single crossing plus quasisupermodularity are sufficient for the existence of a Nash equilibrium, in accordance with Proposition 5.2, but do not ensure the (very) weak FIP.

Example 5.3. Let us consider a strategic game Γ where $N := \{1,2\}$, $X_1 := \{(0,0),(0,1),(1,0),(1,1)\} \subset \mathbb{R}^2$, $X_2 := \{0,1\} \subset \mathbb{R}$ (both with the natural order), and the utilities are defined by the following matrices (player 1 chooses a position within a matrix, player 2 matrix itself; the axes are directed from left to right and from bottom to top):

$$\begin{bmatrix} (0,1) & (0,0) \\ (0,1) & (1,0) \end{bmatrix} \quad \begin{bmatrix} (1,0) & (0,1) \\ (0,0) & (0,1) \end{bmatrix}.$$

The utility function of player 1 satisfies (3) and (6) everywhere. The utility function of player 2 satisfies (2) and, trivially, (4). However, there is no Nash equilibrium. In other words, combining weak versions of both single crossing and quasisupermodularity, we do not obtain even the mere existence of an equilibrium.

6 Aggregative games

We call a strategic game *aggregative* iff there are mappings $\sigma_i \colon X_{-i} \to \mathbb{R}$ $(i \in N)$, *aggregation rules*, and $U_i \colon \sigma_i(X_{-i}) \times X_i \to \mathcal{C}_i$ $(i \in N)$ such that

$$u_i(x_N) = U_i(\sigma_i(x_{-i}), x_i) \tag{9}$$

for all $i \in N$ and $x_N \in X_N$. For each $i \in N$, we denote $S_i := \sigma_i(X_{-i}) \subseteq \mathbb{R}$. An aggregative game is *J-aggregative* iff each strategy set X_i is a poset, while there are mappings $g \colon X_N \to \mathbb{R}$, $F_i \colon S_i \times X_i \to \mathbb{R}$ and $v_i \colon X_{-i} \to \mathbb{R}$ $(i \in N)$ satisfying the following conditions.

First, for all $i \in N$ and $x_N \in X_N$,

$$g(x_N) = F_i(\sigma_i(x_{-i}), x_i) + v_i(x_{-i}). \tag{10}$$

Second, each F_i has the *strictly increasing differences* property (Topkis, 1978):

$$\forall s_i, s_i' \in S_i \, \forall y_i, x_i \in X_i \left[[y_i > x_i \;\&\; s_i' > s_i] \Rightarrow F_i(s_i', y_i) - F_i(s_i', x_i) > F_i(s_i, y_i) - F_i(s_i, x_i) \right]. \tag{11}$$

The intuition behind condition (10) can be explained as follows. Since the utility of each player i only depends on $\sigma_i(x_{-i})$, we may assume that she only observes the aggregate, without knowing who chose what. Then (10) implies that she knows $g(x_N)$ up to an additive term, which is beyond her influence anyway. It seems impossible to explain why (11) is needed without studying the proof of Theorem 3 in detail.

Remark. Jensen (2010) called a game satisfying conditions (9) and (10) "generalized quasi-aggregative"; a motivation for this terminology is given in Footnote 4 on p. 48 of that paper, see also the paragraph following Definition 2 on p. 49.

Example 6.1. Let $X_i \subseteq \mathbb{R}$ and $\sigma_i(x_{-i}) := \sum_{j \neq i} \alpha_{ij} x_j$, where $\alpha_{ij} \in \mathbb{R}$ and $\alpha_{ij} = \alpha_{ji}$ for all $i \neq j$. Then $g(x_N) := (1/2) \sum_{i \neq j} \alpha_{ij} x_i x_j$, $F_i(s_i, x_i) := s_i x_i$, and $v_i(x_{-i}) := (1/2) \sum_{k \neq i \neq j} \alpha_{kj} x_k x_j$ satisfy both (10) and (11).

Jensen (2010) provides a number of other examples.

Theorem 3. *Let Γ be a J-aggregative game such that each strategy set X_i is simultaneously a compact topological space and a distributive lattice. Let, for each $i \in N$, the assumptions of Proposition 5.1, except for the finiteness of X, hold for $X := X_i$, $S := S_i$, $\mathcal{C} := \mathcal{C}_i$, and $u := U_i$. If Γ has the P-finite deviation property, then it has the very weak FIP. If Γ has the Q-finite deviation property, then it has the very-very weak FIP. In particular, Γ possesses a Nash equilibrium in either case.*

The proof, based on Proposition 3.4 and a combination of ideas from Jensen (2010) and Kukushkin (2016), is deferred to Appendix, Section D.

Since no monotonicity assumptions were imposed on σ_i's, the same conditions (2), (3), (5), and (6) have a different, more general meaning here than in Section 5. For instance, the signs of α_{ij}'s in Example 6.1 may be arbitrary, so such a game may exhibit strategic complements, or strategic substitutes, or a combination of both ("strategic supplements"). In particular, if $\alpha_{ij} = -1$ for all $i \neq j$ in Example 6.1, we obtain a game with strategic substitutes and additive aggregation, and hence our Theorem 3 immediately implies Theorem 3 of Novshek (1985).

Even in the case of strategic complements, i.e., when all σ_i's are increasing, Theorem 3 adds something to the results of Section 5, giving the assertion of Theorem 2 under the assumptions of Proposition 5.2. Note that the games in Examples 5.1 and 5.2 are not aggregative, whereas that of Example 5.3 *is*.

7 Concluding remarks

7.1. The description of the preferences of the players with "generalized" utility functions is equivalent to the description with complete binary relations as in Reny (2016). An even more general description would emerge if each $\mathcal{C}_i$ were just a poset. Theorem 1 would remain valid in this case with the same proof, cf. Kukushkin (2011, Section 4.5). Whether Theorems 2 and 3 allow such a broad generalization is not clear at the moment; most likely, additional assumptions would be needed.

7.2. The compactness assumption in Proposition 3.2 cannot simply be dropped. If each $\mathcal{C}_i$ is just $\mathbb{R}$, it boils down to the condition that each u_i is bounded, both above and below. The fact that the proposition may become wrong without an upper bound on utilities may be demonstrated with a one-person game. As to the lower bound, two players are needed, but one of them may be a dummy.

Example 7.1. Let us consider a game where $N := \{1, 2\}$, $X_1 := [0, 1]$, $X_2 := \{0\}$, and the utility mapping is this:

$$u_N(x_N) := \begin{cases} (1 - x_1, -1/x_1), & \text{if } x_1 > 0, \\ (0, 0), & \text{if } x_1 = 0. \end{cases}$$

The game is better-reply secure since the graph G of the utility mapping u_N is closed and a payoff strictly above $u_1(x_N)$ is secured by any $y_1 \in]0,1[$ if $x_1 = 0$, or by any $y_1 \in]0, x_1[$ if $x_1 > 0$. Thus, all assumptions of Proposition 3.2 are satisfied except that u_2 is not bounded below. On the other hand, no open neighborhood of $(0,0)$ is finitely dominated; moreover, there is no Nash equilibrium.

As suggested by Reny (1999) himself, the proposition can be made applicable to unbounded utilities via a re-interpretation of better-reply security. Namely, we could perceive u_N as a mapping $X_N \to \bar{\mathbb{R}}^N$, where $\bar{\mathbb{R}} := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$, and require the inequality in the definition to hold for vectors in $\bar{G}(x_N)$ with infinite coordinates as well. (It should be noted that $\bar{\mathbb{R}}$ is compact in *its* order interval topology.) In Example 7.1, G will no longer be closed under this interpretation, $\bar{G}(0,0) = \{(0,0), (1, -\infty)\}$, and player 1 cannot secure any payoff above 1. In other words, another assumption will fail and there will be no surprise in the absence of an equilibrium.

One could suspect the compactness assumption to imply that the preferences can actually be described with a *real-valued* utility function. However, this is not the case: if, e.g., $\mathcal{C}_i$ is $\mathbb{R} \times \{0,1\}$ with the lexicographic order, then the closure of every bounded subset of $\mathcal{C}_i$ is compact, but its embedding into the real line may be impossible (Wakker, 1988, Lemma 3.1).

7.3. Exactly as the main result of Kukushkin (2011), our Theorem 1 is an extension of the theorem of Bergstrom (1975) and Walker (1977) to strategic games. It even suggests a new generalization of that old theorem: An acyclic binary relation $\rhd$ on a compact topological space X admits a maximizer if, whenever $y \rhd x$, there is an open neighborhood O of x and a finite set $\{z^1, \ldots, z^m\} \subset X$ such that for every $x' \in O$ there is k for which $z^k \rhd x'$. Funnily, this particular result seems to have never been published although there are quite a few even more straightforward generalizations in the literature.

7.4. It is interesting to note that no consistency between topology and order is needed in Theorem 2 and Theorem 3, which fact contrasts with Kukushkin (2016, Section 3). Moreover, the topology on each X_i need not even be Hausdorff and the lattices need not be complete. On the other hand, we have to assume the lattices X_i to be *distributive* in either theorem because we could not assert that Γ'' is finite otherwise. There may be weaker assumptions with the same implication, but, to my knowledge, all lattices in economics models are distributive.

7.5. There is some superficial similarity between our Proposition 5.2 and Theorem 2 of Prokopovych and Yannelis (2017); however, that similarity should not be overestimated. Of the four principal assumptions of the latter theorem, two are stronger than corresponding assumptions here: X_i's are chains rather than lattices and the game has to be "better reply secure" (under a stronger, cardinal, interpretation of the property) rather than have Q-finite deviation property. The assumption of "upward or downward upper semicontinuity" has no counterpart here. Finally, the "approximate downward (or upward) transfer single-crossing" assumption, which presumes cardinal utility functions, is simply incomparable with our versions of the single crossing: it is not implied by (2) and does not imply (3). Unlike our conditions (2), (3), (4), (5), and (6), that last assumption need not be inherited by subgames; therefore, a proof of Theorem 2 of Prokopovych and Yannelis (2017) in the style of our Proposition 5.2 seems impossible. It remains unclear, without a further study, whether their assumptions imply the very(-very) weak FIP.

7.6. The key role in the proof of Theorem 3 is played by a construction essentially invented by Jensen (2010), who built on Huang (2002), Dubey et al. (2006), and Kukushkin (2005). Unfortunately, there were technical oversights in Jensen (2010): the proof needed stronger topological assumptions than were made explicitly (Jensen, 2012). In a personal communication, Jensen conjectured that his main theorem is nonetheless valid as stated. Our Theorem 3 makes a significant step towards the vindication of his position.

Appendix: Proofs

A Proof of Proposition 3.2

Let $\bar{x}_N \in X_N$. Since Γ is better-reply secure, for every $v_N \in \bar{G}(\bar{x}_N)$, there are $j(v_N) \in N$, $\alpha(v_N) \in \mathcal{C}_{j(v_N)}$, $y_{j(v_N)} \in X_{j(v_N)}$, and $V_{-j(v_N)}(v_N) \subseteq X_{-j(v_N)}$ such that $V_{-j(v_N)}(v_N)$ is open, $\bar{x}_{-j(v_N)} \in V_{-j(v_N)}(v_N)$, $\alpha(v_N) > v_{j(v_N)}$, and, whenever $x_{-j(v_N)} \in V_{-j(v_N)}(v_N)$, there holds $u_{j(v_N)}(y_{j(v_N)}, x_{-j(v_N)}) \geq \alpha(v_N)$. Denoting $W(v_N) := \{w_N \in \mathcal{C}_N \mid \alpha(v_N) > w_{j(v_N)}\}$, we have $v_N \in W(v_N)$ and hence $\bar{G}(\bar{x}_N) \subseteq \bigcup_{v_N \in \bar{G}(\bar{x}_N)} W(v_N)$. Since every $W(v_N)$ is open while $\bar{G}(\bar{x}_N)$ is compact, there are $v_N^1, \ldots, v_N^m \in \bar{G}(\bar{x}_N)$ such that $\bar{G}(\bar{x}_N) \subseteq \bigcup_{h=1}^m W(v_N^h)$.

Whenever $j(v_N^k) = j(v_N^h)$ and $\alpha(v_N^k) \geq \alpha(v_N^h)$, we have $W(v_N^k) \supseteq W(v_N^h)$ and hence $W(v_N^h)$ is not needed to provide an open cover of $\bar{G}(\bar{x}_N)$. Deleting such superfluous subsets, we obtain a subset $M \subseteq N$ and, for each $i \in M$, a utility level $\alpha_i \in \mathcal{C}_i$, a strategy $y_i \in X_i$, and an open neighborhood V_{-i} of $\bar{x}_{-i}$ in X_{-i} such that $u_i(y_i, x_{-i}) \geq \alpha_i$ whenever $x_{-i} \in V_{-i}$, and $\bar{G}(\bar{x}_N) \subseteq \bigcup_{i \in M}\{w_N \in \mathcal{C}_N \mid \alpha_i > w_i\} =: \tilde{W}$.

Claim A.1. *There is an open neighborhood V of $\bar{x}_N$ such that $u_N(x_N) \in \tilde{W}$ whenever $x_N \in V$.*

Remark. In principle, this claim belongs to textbook material. Since our assumptions are broader than usual, a complete proof is given.

Proof. We set $F := (\operatorname{cl} u_N(X_N)) \setminus \tilde{W} \subset \mathcal{C}_N$; F is compact. For every $w_N \in F$, we have $(\bar{x}_N, w_N) \notin \bar{G}$. Since $\bar{G}$ is closed, there is an open neighborhood $V'(w_N)$ of $(\bar{x}_N, w_N)$ in $X_N \times \mathcal{C}_N$ such that $V'(w_N) \cap \bar{G} = \emptyset$; without restricting generality, we have $V'(w_N) = V'_X(w_N) \times V'_C(w_N)$, where $V'_X(w_N)$ is open in X_N, while $V'_C(w_N)$ is open in $\mathcal{C}_N$. Since $\{\bar{x}_N\} \times F$ is compact, it is covered by a finite number of such neighborhoods: $V'(w_N^1)$, ..., $V'(w_N^{m'})$. We define $V := \bigcap_{h=1}^{m'} V'_X(w_N^h)$; V is open and $\bar{x}_N \in V$.

Now if $x_N \in V$ and $u_N(x_N) \notin \tilde{W}$, we would have $u_N(x_N) \in F$; therefore, $\langle \bar{x}_N, u_N(x_N)\rangle \in V'(w_N^h)$ for some h, and hence $u_N(x_N) \in V'_C(w_N^h)$. Since $x_N \in V'_X(w_N^h)$, we have $\langle x_N, u_N(x_N)\rangle \in V'(w_N^h)$ as well. Therefore, $\langle x_N, u_N(x_N)\rangle \notin G$, which is impossible. □

Picking such an open neighborhood V, we define $O := V \cap \bigcap_{i \in M}[X_i \times V_{-i}]$. Again, O is open and $\bar{x}_N \in O$. Let us show that O is dominated by $\{\langle i, y_i\rangle\}_{i \in M}$. Let $x_N \in O$; hence $u_N(x_N) \in \tilde{W}$ by Claim A.1 and hence $\alpha_i > u_i(x_N)$ for some $i \in M$. Since $x_{-i} \in V_{-i}$, we have $u_i(y_i, x_{-i}) \geq \alpha_i > u_i(x_N)$, i.e., $(y_i, x_{-i}) \rhd_i^{\mathrm{Ind}} x_N$ indeed.

B Proof of Propositions 4.1 and 4.2

In addition to $M(x_N) := \operatorname{Argmin}_i x_i$ defined in Section 4, we introduce these notations: $m(x_N) := \min_i x_i$; $m_2(x_N) := \sup\{p \in \mathbb{R}_+ \mid \#\{i \in N \mid x_i < p\} < 2\} = \inf\{p \in \mathbb{R}_+ \mid \#\{i \in N \mid x_i \le p\} \ge 2\}$; $M_2(x_N) := \{i \in N \mid x_i \le m_2(x_N)\}$; plainly speaking, $m_2(x_N)$ is the second price from the bottom ("Vickrey price"). Clearly, $\#M(x_N) > 1 \iff m(x_N) = m_2(x_N) \iff M(x_N) = M_2(x_N)$.

B.1 Acyclicity

As a first, trivial observation, we notice that $u_i(x_N) \ge 0$ for all $i \in N$ and $x_N \in X_N$, and hence the improving player must obtain a strictly positive profit. In particular, $y_i > c_i$ and $i \in M(y_N)$ whenever $y_N \rhd_i^{\mathrm{Ind}} x_N$.

Claim B.1.1. *If $y_N \rhd_i^{\mathrm{Ind}} x_N$ and $i \in M(x_N)$, then $m_2(y_N) = m_2(x_N)$ and $M_2(y_N) = M_2(x_N)$.*

Proof. If $M(x_N) = \{i\}$, then $y_i \le m_2(x_N)$ and hence both statements hold. If $\#M(x_N) > 1$, then $y_i < x_i$ and hence $m(y_N) < m(x_N) = m_2(x_N) = m_2(y_N)$ and $M_2(y_N) = M_2(x_N)$. □

Claim B.1.2. *If $y_N \rhd_i^{\mathrm{Ind}} x_N$ and $i \notin M(x_N)$, then either $m_2(y_N) < m_2(x_N)$, or $m_2(y_N) = m_2(x_N)$ and $M_2(y_N) \supset M_2(x_N)$.*

Proof. We have $y_i \le m(x_N) < x_i$. If $\#M(x_N) > 1$, then $m_2(y_N) = m_2(x_N) = m(x_N)$ and $M_2(y_N) = M_2(x_N) \cup \{i\} \supset M_2(x_N)$. If $M(x_N) = \{j\}$, then $M_2(y_N) = \{i, j\}$ and $m_2(y_N) = m(x_N) < m_2(x_N)$. □

Now suppose, to the contrary, that there is an improvement cycle, i.e., a sequence $x_N^0, \dots, x_N^m$ such that $m > 0$, $x_N^m = x_N^0$, and $x_N^{k+1} \rhd_{i(k)}^{\mathrm{Ind}} x_N^k$ for each $k = 0, 1, \dots, m-1$. Clearly, $i(k)$ cannot be the same for all k; moreover, we may, without restricting generality, assume $i(k+1) \ne i(k)$ for all k. Claims B.1.1 and B.1.2 imply that $m_2(x_N^k) = p^0$ and $M_2(x_N^k) = M^0$ for all $k = 0, 1, \dots, m$. Claim B.1.2 implies that $i(k) \in M(x_N^k) \subseteq M^0$ for all k. Now a contradiction is obvious: If $i(0) \in M(x_N^0)$ and $x_N^1 \rhd_{i(0)}^{\mathrm{Ind}} x_N^0$, then $M(x_N^1) = \{i(0)\}$ and hence we cannot have both $i(1) \in M(x_N^1)$ and $i(1) \ne i(0)$.

Remark. Neither upper semicontinuity, nor monotonicity of D are needed in the proof.

B.2 Single deviation: Sufficiency

First of all, for every $x_N \in X_N$ and $p \in \mathbb{R}_+$, we fix and denote $O^*(x_N, p)$ an open neighborhood of x_N where all strict inequalities between components x_i, as well as between them and p, are preserved, i.e., whenever $x_i > x_j$ (or $x_i > p$ or $x_i < p$), there holds $x'_i > x'_j$ ($x'_i > p$, $x'_i < p$) for all $x'_N \in O^*(x_N, p)$.

Now let $\bar{x}_N \in X_N$. We consider several alternatives.

A. There is $i \in N \setminus M(\bar{x}_N)$ for which $c_i < m(\bar{x}_N)$. Picking an arbitrary $y_i \in \,]c_i, m(\bar{x}_N)[$, we immediately obtain $u_i(y_i, x_{-i}) > 0 = u_i(x_N)$ for every $x_N \in O^*(\bar{x}_N, y_i)$. Therefore, $\bar{x}_N \notin \mathrm{E}(\Gamma)$ and there is no problem with the R-single deviation property at this profile.

Henceforth, we assume **A** not to be the case, i.e., $c_i \ge m(\bar{x}_N)$ whenever $i \notin M(\bar{x}_N)$. It follows immediately that $\operatorname{Argmin}_{i \in N} c_i \subseteq M(\bar{x}_N)$

B. $M(\bar{x}_N) = \{i\}$; then $\mathrm{Argmin}_{j\in N}\, c_j = \{i\}$ as well and $m_2(\bar{x}_N) \geq \bar{c} := \min_{j\neq i} c_j \geq \bar{x}_i$, with at least one of the inequalities strict. We consider two alternatives.

B1. There is $y_i \in [c_i, m_2(\bar{x}_N)]$ such that $(y_i - c_i)\cdot D(y_i) > u_i(\bar{x}_N)$. Being upper semicontinuous and decreasing, D is left continuous. Therefore, we may pick $y_i < m_2(\bar{x}_N)$ such that $u_i(y_i, \bar{x}_{-i}) = (y_i - c_i)\cdot D(y_i) > u_i(\bar{x}_N)$; moreover, $(y_i - c_i)\cdot D(y_i) > (x_i - c_i)\cdot D(x_i)$ for all x_N from an open neighborhood O of $\bar{x}_N$. Thus, $\bar{x}_N \notin \mathrm{E}(\Gamma)$, $\{\langle i, y_i\rangle\}$ dominates $O \cap O^*(\bar{x}_N, y_i)$, and hence there is no problem with the R-single deviation property at this profile.

B2. $u_i(y_i, \bar{x}_{-i}) \leq u_i(\bar{x}_N)$ for all $y_i \in [c_i, m_2(\bar{x}_N)]$; in other words, $\bar{x}_i \in \mathrm{Argmax}_{x_i\in[c_i,m_2(\bar{x}_N)]}(x_i - c_i)\cdot D(x_i)$. Then, obviously, $\bar{x}_N \in \mathrm{E}(\Gamma)$; therefore, nothing is required even for the R-single deviation property.

C. $\#M(\bar{x}_N) > 1$. We partition $M(\bar{x}_N)$ into $M^0 \cup M^+$, where $M^0 := \{i \in M(\bar{x}_N) \mid c_i = m(\bar{x}_N)\, [= \bar{x}_i]\}$ and $M^+ := \{i \in M(\bar{x}_N) \mid c_i < m(\bar{x}_N)\, [= \bar{x}_i]\}$, and consider three alternatives.

C1. $M^+ = \emptyset$; then $m(\bar{x}_N) = \min_{i\in N} c_i$ and $\bar{x}_N \in \mathrm{E}(\Gamma)$; again, nothing is required even for the R-single deviation property.

C2. $\#M^+ > 1$. We fix an $i \in M^+$. Since D is left continuous (see **B1**), there is $y_i < \bar{x}_i$ and an open neighborhood O' of $\bar{x}_N$ such that $(y_i - c_i)\cdot D(y_i) > (x_i - c_i)\cdot D(x_i)/(\#M^+)$ for all $x_N \in O'$. In particular, $\bar{x}_N \notin \mathrm{E}(\Gamma)$. We set $\hat{c} := \max_{j\in M^+} c_j\, [< m(\bar{x}_N)]$ and pick $p \in]\max\{y_i, \hat{c}\}, m(\bar{x}_N)[$. Let us show that $\{\langle i, y_i\rangle\} \cup \{\langle j, p\rangle\}_{j\in M^+\setminus\{i\}}$ dominates $O := O' \cap O^*(\bar{x}_N, p)$.

Let $x_N \in O$; then $M(x_N) \subseteq M(\bar{x}_N)$ since $x_N \in O^*(\bar{x}_N, y_i)$. If $i \notin M(x_N)$, then $u_i(x_N) = 0$ while $u_i(y_i, x_{-i}) > 0$ since $x_N \in O^*(\bar{x}_N, y_i)$; hence $(y_i, x_{-i}) \rhd_i^{\mathrm{Ind}} x_N$. If there is $j \in M^+ \setminus M(x_N)$ such that $j \neq i$, then $u_j(x_N) = 0$ and, defining y_N by $y_j := p$, $y_{-j} := x_{-j}$, we obtain $y_N \rhd_j^{\mathrm{Ind}} x_N$ since $x_N \in O^*(\bar{x}_N, y_i)$. Finally, if $M^+ \subseteq M(x_N)$, then $u_i(x_N) = (x_i - c_i)\cdot D(x_i)/\#M(x_N) \leq (x_i - c_i)\cdot D(x_i)/\#M^+ < (y_i - c_i)\cdot D(y_i)$ since $x_N \in O'$; therefore, $(y_i, x_{-i}) \rhd_i^{\mathrm{Ind}} x_N$ again. Thus, there is no problem with the R-single deviation property at this profile.

C3. $M^+ = \{i\}$. Similarly to the case **B** above, we have $\mathrm{Argmin}_{j\in N}\, c_j = \{i\}$ as well and $\bar{x}_i = m(\bar{x}_N) = m_2(\bar{x}_N) = \bar{c} := \min_{j\neq i} c_j$. Since D is left continuous, $\bar{x}_N \notin \mathrm{E}(\Gamma)$. We consider two alternatives.

C3a. Let (1a) hold. Picking $y_i \in \mathrm{Argmax}_{x_i\in[c_i,\bar{c}]}(x_i - c_i)\cdot D(x_i)$, we have $y_i < \bar{x}_i$ and $(y_i - c_i)\cdot D(y_i) > (\bar{x}_i - c_i)\cdot D(\bar{x}_i)$. Since D is upper semicontinuous, we have $(y_i - c_i)\cdot D(y_i) > (x_i - c_i)\cdot D(x_i)$ for all x_N from an open neighborhood O of $\bar{x}_N$. Thus, $\{\langle i, y_i\rangle\}$ dominates $O \cap O^*(\bar{x}_N, y_i)$, and hence there is no problem with the R-single deviation property at this profile.

C3b. Let (1a) fail, i.e., $\bar{c} \in \mathrm{Argmax}_{x_i\in[c_i,\bar{c}]}(x_i - c_i)\cdot D(x_i)$. We consider further alternatives.

C3bI. Let both (1b) and (1e) hold. Denoting $V_i^+ := \max_{x_i\in[c_i,\bar{c}]}(x_i - c_i)\cdot D(x_i)$, we, invoking (1b), pick $c > \bar{c}$ such that $(x_i - c_i)\cdot D(x_i) < V_i^+$ for every $x_i \in]\bar{c}, c]$, and, invoking (1e), pick $y_i \in [c_i, \bar{c}[$ for which $(y_i - c_i)\cdot D(y_i) = V_i^+$. Now we define $O := O^*(\bar{x}_N, c) \cap O^*(\bar{x}_N, y_i)$ and show that $\{\langle i, y_i\rangle\}$ dominates $O \setminus \mathrm{E}(\Gamma)$.

Indeed, let $x_N \in O$. First, we have $u_i(y_i, x_{-i}) = V_i^+$. If $u_i(x_N) < V_i^+$, then we are home. If $u_i(x_N) = V_i^+$, then $x_i \leq \bar{c}$ and hence $x_N \in \mathrm{E}(\Gamma)$. Thus, there is no problem with the P-single deviation property at this profile.

C3bII. Let (1c) hold. Then we pick $c > \bar{c}$ for which $\bar{c} \in \mathrm{Argmax}_{x_i\in[c_i,c]}(x_i - c_i)\cdot D(x_i)$. Now every x_N such that $x_i = \bar{x}_i = \bar{c}$ and $x_j \in]\bar{c}, c]$ for all $j \in M^+ \setminus \{i\}$ is a Nash equilibrium. Therefore, $\bar{x}_N \in \operatorname{cl} \mathrm{E}(\Gamma)$ and nothing is required of it if we want to establish the Q-simple deviation property.

C3bIII. Let (1d) hold. For every $\varepsilon > 0$, we pick $x_i^\varepsilon \in [\bar{c} - \varepsilon, \bar{c}[\cap \operatorname{Argmax}_{x_i \in [c_i, \bar{c}]} (x_i - c_i) \cdot D(x_i)$ and define $x_N^\varepsilon := (x_i^\varepsilon, \bar{x}_{-i})$. Clearly, $x_N^\varepsilon \in \mathrm{E}(\Gamma)$ for every $\varepsilon > 0$, and hence $\bar{x}_N \in \operatorname{cl} \mathrm{E}(\Gamma)$ with the same implications as in the case **C3bII**.

Let us summarize our findings. If $\# \operatorname{Argmin}_{i \in N} c_i > 1$, then every $\bar{x}_N \in X_N$ belongs to one of the cases **A**, **C1**, or **C2**; in each of them, the R-simple deviation property has been established. Let $\operatorname{Argmin}_{j \in N} c_j = \{i\}$. If (1a) holds, then, additionally, the cases **B1**, **B2**, and **C3a** become possible, in which, again, the R-simple deviation property has been established. If (1a) fails, but (1b) and (1e) hold, then **C3a** is replaced in the list with **C3bI** and we have the P-simple deviation property. Finally, if (1a) fails, but (1c) or (1d) holds, then **C3a** is replaced with **C3bII** or **C3bIII**, and in either case we have the Q-simple deviation property.

B.3 Necessity

Let $\operatorname{Argmin}_{j \in N} c_j = \{i\}$, and let (1a) fail, i.e., $\bar{c} \in \operatorname{Argmax}_{x_i \in [c_i, \bar{c}]} (x_i - c_i) \cdot D(x_i)$. We define $\bar{x}_N \in X_N$ by $\bar{x}_i := \bar{c}$ and $\bar{x}_j := c_j$ for all $j \neq i$. In the taxonomy of Section B.2, this profile belongs to the case **C3**; hence, $\bar{x}_N \notin \mathrm{E}(\Gamma)$. Let $O \subseteq X_N$ be an open neighborhood of $\bar{x}_N$ and Z be a finite set of pairs $\langle j, y_j \rangle$ ($j \in N$, $y_j \in X_j$). For each $j \in N$, we denote $Y_j := \{y_j \in X_j \mid \langle j, y_j \rangle \in Z\}$. Then we set $Y^+ := (]\bar{c}, +\infty[\cap \bigcup_{j \in N} Y_j)$ and $p^+ := \min Y^+ \, [> \bar{c}]$.

Claim B.3.1. *O is not dominated with Z.*

Proof. The open neighborhood $O \cap O^*(\bar{x}_N, p^+)$ of $\bar{x}_N$ (with $O^*(\bar{x}_N, p^+)$ defined in the beginning of Section B.2) contains x_N for which $x_i = \bar{c}$ and $x_j > \bar{c}$ for all $j \neq i$. Whenever $j \neq i$ and $y_j \in Y_j$, we have $m(y_j, x_{-j}) = \bar{c}$ and hence $u_j(y_j, x_{-j}) = 0 = u_j(x_N)$. If $y_i \in Y_i$ and $y_i > \bar{c}$, then $m(y_i, x_{-i}) < p^+ \leq y_i$ and hence $u_i(y_i, x_{-i}) = 0 < u_i(x_N)$. Finally, if $y_i \in Y_i$ and $y_i < \bar{c}$, then we have $u_i(x_N) = (\bar{c} - c_i) \cdot D(\bar{c}) \geq (y_i - c_i) \cdot D(y_i) = u_i(y_i, x_{-i})$, the inequality in the middle following from the negation of (1a). □

Thus, there is no R-finite deviation property without (1a).

Claim B.3.2. *Let $\bar{p} \in \mathbb{R}_+$ be such that $\bar{p} > c_i$ and $(\bar{p} - c_i) \cdot D(\bar{p}) > (p - c_i) \cdot D(p)$ for all $p \in [p^-, \bar{p}[$ (with $c_i \leq p^- < \bar{p}$). Then for every $p \in [p^-, \bar{p}[$ and $p' \in]p, \bar{p}[$, there is $p'' \in]p', \bar{p}[$ such that $(p'' - c_i) \cdot D(p'') > (p - c_i) \cdot D(p)$.*

A straightforward proof, based on the left continuity of D, is omitted.

Claim B.3.3. *If either (1b) or (1e) does not hold, then $O \setminus \mathrm{E}(\Gamma)$ is not dominated with Z.*

Proof. Let (1b) fail; then $O \cap O^*(\bar{x}_N, p^+)$ contains x_N such that $\bar{c} < x_i < x_j$ for all $j \neq i$ and $(x_i - c_i) \cdot D(x_i) \geq (\bar{c} - c_i) \cdot D(\bar{c})$. We have $u_j(y_j, x_{-j}) \leq u_j(x_N)$ for all $\langle j, y_j \rangle \in Z$ for the same reason as in the proof of Claim B.3.1. Meanwhile, $x_N \notin \mathrm{E}(\Gamma)$ because this profile belongs to the case **A** in the taxonomy of Section B.2.

Let (1e) fail; then Claim B.3.2 is applicable to $\bar{p} := \bar{c}$. Therefore, $x_N \notin \mathrm{E}(\Gamma)$ whenever $x_i < \bar{c}$. Moreover, by the same claim, we can pick x_i in such a way that $x_N := (x_i, \bar{x}_{-i}) \in O \cap O^*(\bar{x}_N, p^+)$ while $(x_i - c_i) \cdot D(x_i) > (y_i - c_i) \cdot D(y_i)$ for all $y_i \in Y_i \cap [c_i, \bar{c}[$. Again, $u_j(y_j, x_{-j}) \le u_j(x_N)$ for all $\langle j, y_j \rangle \in Z$. □

Thus, there is no P-finite deviation property without (1b) *and* (1e).

Claim B.3.4. *If neither (1c) nor (1d) holds, then $\bar{x}_N \notin \operatorname{cl} \mathrm{E}(\Gamma)$.*

Proof. The negation of (1d) implies the existence of $p^- \in [c_i, \bar{c}[$ such that Claim B.3.2 is applicable to $\bar{p} := \bar{c}$ and p^-. Let us show that $O^*(\bar{x}_N, p^-) \cap \mathrm{E}(\Gamma) = \emptyset$.

Let $x_N \in O^*(\bar{x}_N, p^-)$. If $m(x_N) < \bar{c}$, and hence $M(x_N) = \{i\}$, then Claim B.3.2 immediately implies that $x_N \notin \mathrm{E}(\Gamma)$. If $m(x_N) > \bar{c}$, then this profile belongs to the case **A** or **C2** in the taxonomy of Section B.2; hence $x_N \notin \mathrm{E}(\Gamma)$ again. If $m(x_N) = \bar{c}$ and $i \notin M(x_N)$ or $\# M(x_N) > 1$, then x_N belongs to the case **A** or **C3** with the same implication. Finally, if $m(x_N) = \bar{c}$ and $M(x_N) = \{i\}$, then x_N belongs to the case **B1** since (1c) does not hold. □

Taking into account Claim B.3.1, we see that there is no Q-finite deviation property without (1c) *or* (1d).

Claim B.3.5. *Let condition (1e) not hold, let $x_N^0 \in X_N$ be such that $x_i^0 < \bar{c}$ and $x_j^0 = \bar{c}$ for at least one $j \neq i$, and let $Y \subset X_N$ denote the set of strategy profiles y_N for which there exists an improvement path starting at x_N^0 and ending at y_N. Then there is an open neighborhood O of $\mathrm{E}(\Gamma)$ such that $Y \cap O = \emptyset$.*

Proof. Obviously, only player i is capable of improvements at x_N^0. The negation of (1e) means that $(\bar{c} - c_i) \cdot D(\bar{c}) > (x_i - c_i) \cdot D(x_i)$ for every $x_i < \bar{c}$, and hence Claim B.3.2 applies with $\bar{p} := \bar{c}$. A straightforward inductive argument shows that, at every $y_N \in Y$, the inequality $y_i \le \bar{c}$ holds and only player i is capable of improvements. Therefore, $Y \cap \mathrm{E}(\Gamma) = \emptyset$. Denoting $O := \{x_N \in X_N \mid x_j > \bar{c}\}$, we immediately see that O is open, $\mathrm{E}(\Gamma) \subseteq O$, and $Y \cap O = \emptyset$; actually, even $\operatorname{cl} Y \cap O = \emptyset$. □

Thus, there is no very weak FIP property without (1e).

Claim B.3.6. *If none of conditions (1) holds, then $\mathrm{E}(\Gamma) = \emptyset$.*

Proof. Let $x_N \in X_N$. If $m(x_N) < \bar{c}$, then $M(x_N) = \{i\}$ and $x_N \notin \mathrm{E}(\Gamma)$ as shown in the proof of Claim B.3.5. If $m(x_N) \ge \bar{c}$, then $x_N \notin \mathrm{E}(\Gamma)$ as shown in the proof of Claim B.3.4 (the assumption $x_N \in O^*(\bar{x}_N, p^-)$ was not needed in that part of the proof). □

B.4 Very weak FIP: Sufficiency

Let $\mathrm{Argmin}_{j \in N} c_j = \{i\}$, and let condition (1e) hold. If (1b) also holds, then Γ has the P-single deviation property and hence the very weak FIP as well; so let (1b) fail, and hence (1a) fail too. We want to show that the set $\mathrm{E}(\Gamma)$ can be approached starting from any $x_N^0 \in X_N$. If $x_N^0 \in \mathrm{E}(\Gamma)$, we are home immediately. For every $x_N \in X_N$, we denote $m_{-i}(x_N) := \min_{j \neq i} x_j \, [\ge \bar{c}]$. Then we pick $x_i^+ \in [c_i, \bar{c}[\cap \mathrm{Argmax}_{x_i \in [c_i, \bar{c}]} \big((x_i - c_i) \cdot D(x_i) \big)$, which is possible because of (1e), and set $Y^0 := \{x_N \in X_N \mid \bar{c} < x_i < m_{-i}(x_N)\}$. Every $x_N \in Y^0$ belongs to the case **A** in the taxonomy of Section B.2; hence $Y^0 \cap \mathrm{E}(\Gamma) = \emptyset$.

Claim B.4.1. *For every $x_N \in X_N \setminus (Y^0 \cup \mathrm{E}(\Gamma))$, there is $y_N \in Y^0 \cup \mathrm{E}(\Gamma)$ such that $y_N \rhd_i^{\mathrm{Ind}} x_N$.*

Proof. We note first that $m_{-i}(x_N) \geq \bar{c}$ in any case. Then we consider four alternatives. (i) Let $m_{-i}(x_N) = \bar{c}$. We define $y_N := (x_i^+, x_{-i})$; obviously, $y_N \in \mathrm{E}(\Gamma)$. Since $x_N \notin \mathrm{E}(\Gamma)$, we have $y_N \rhd_i^{\mathrm{Ind}} x_N$. (ii) Let $\bar{c} < m_{-i}(x_N) < x_i$. Then x_N belongs to the case **A** in the taxonomy of Section B.2; picking $y_i \in]\bar{c}, m_{-i}(x_N)[$ and setting $y_N := (y_i, x_{-i})$, we obtain $y_N \in Y^0$ such that $y_N \rhd_i^{\mathrm{Ind}} x_N$. (iii) Let $\bar{c} < m_{-i}(x_N) = x_i$. Then x_N belongs to the case **C2** in the same taxonomy, and we are home picking $y_i \in]\bar{c}, x_i[$ close enough to x_i. (iv) Let $x_i \leq \bar{c} < m_{-i}(x_N)$. The negation of (1b) ensures that we can pick $y_i \in]\bar{c}, m_{-i}(x_N)[$ for which $(y_i, x_{-i}) \rhd_i^{\mathrm{Ind}} x_N$; obviously, $(y_i, x_{-i}) \in Y^0$. □

We pick $j \neq i$ for which $c_j = \bar{c}$ and define x_N^* by $x_i^* := x_i^+$, $x_j^* := \bar{c}$ and $x_{-ij}^* := x_{-ij}^0$; obviously, $x_N^* \in \mathrm{E}(\Gamma)$. Now let $x_N^0 \in Y^0$. We define an infinite improvement path $\langle x_N^k \rangle_{k \in \mathbb{N}}$ by $x_i^{2k+1} := x_i^{2k}$, $x_i^{2k+2} := (\bar{c} + x_j^{2k+1})/2$, $x_j^{2k+1} := (\bar{c} + x_i^{2k})/2$, and $x_j^{2k+2} := x_j^{2k+1}$. Clearly, the path converges to x_N^ω where $x_i^\omega = x_j^\omega = \bar{c}$ and $x_{-ij}^\omega = x_{-ij}^0$. For every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $0 < x_j^{2k+1} - \bar{c} < \varepsilon$. Replacing x_i^{2k+1} with x_i^+, player i makes an improvement and obtains a strategy profile whose distance to x_N^* is less than ε. (*That* profile need not be an equilibrium!)

C Proof of Theorem 2

In light of Proposition 3.4, it is enough to show that Γ has the quasi weak FIP. Let $\Gamma' \leq \Gamma$ be finite. Similarly to the proof of Proposition 5.2, we define X_i'', for each $i \in N$, as the minimal sublattice of X_i containing X_i'. The subsets X_i'' define a finite subgame $\Gamma'' \leq \Gamma$, which inherits conditions (2a) and (5a) from Γ.

Now, exactly as in the proof of Theorem 1 of Kukushkin et al. (2005), we define

$$X^{\uparrow} := \{x_N \in X_N'' \mid \exists y_N \in X_N'' \, [y_N > x_N \;\&\; y_N \rhd^{\mathrm{Ind}} x_N]\}; \quad X^{\downarrow} := X_N'' \setminus X^{\uparrow};$$

$$y_N \succ x_N \rightleftharpoons \big[[y_N \in X^{\downarrow} \;\&\; x_N \in X^{\uparrow}] \text{ or } [x_N, y_N \in X^{\uparrow} \;\&\; y_N > x_N] \text{ or } [x_N, y_N \in X^{\downarrow} \;\&\; y_N < x_N] \big]. \quad (12)$$

Clearly, $\succ$ is irreflexive and transitive.

Claim C.1. *If $x_N \in X_N'' \setminus \mathrm{E}(\Gamma'')$, then there exists $y_N \in X_N''$ such that $y_N \rhd^{\mathrm{Ind}} x_N$ and $y_N \succ x_N$.*

Proof. If $x_N \in X^{\uparrow}$, then we pick $y_N \in X_N''$ such that $y_N \rhd^{\mathrm{Ind}} x_N$ and $y_N > x_N$. If $y_N \in X^{\downarrow}$, then $y_N \succ x_N$ by the first disjunctive term in (12). If $y_N \in X^{\uparrow}$, then $y_N \succ x_N$ by the second disjunctive term in (12).

Let $x_N \in X^{\downarrow}$. We pick $i \in N$ and $y_N \in X_N''$ such that $y_N \rhd_i^{\mathrm{Ind}} x_N$. Denoting $Y_i := \{z_i \in X_i'' \mid z_i \leq x_i\}$, we pick $\bar{z}_i \in \mathrm{Argmax}_{z_i \in Y_i} u_i(z_i, x_{-i})$, which is possible because Y_i is finite. Since $x_N \in X^{\downarrow}$, $y_i > x_i$ is impossible. If $y_i < x_i$, then $u_i(\bar{z}_i, x_{-i}) \geq u_i(y_i, x_{-i})$; hence $u_i(\bar{z}_i, x_{-i}) > u_i(x_N)$ and hence $\bar{z}_i < x_i$. If y_i and x_i are incomparable in the order, then $y_i \vee x_i > x_i$ and $y_i \wedge x_i < x_i$. An assumption that $u_i(x_N) \geq u_i(y_i \wedge x_i, x_{-i})$ would imply $u_i(y_i, x_{-i}) > u_i(y_i \wedge x_i, x_{-i})$, and hence $u_i(y_i \vee x_i, x_{-i}) > u_i(x_N)$

by (5a), contradicting our assumption that $x_N \in X^{\downarrow}$. Therefore, $u_i(y_i \wedge x_i, x_{-i}) > u_i(x_N)$; hence $u_i(\bar{z}_i, x_{-i}) > u_i(x_N)$ and $\bar{z}_i < x_i$ again. Denoting $z_N := (\bar{z}_i, x_{-i})$, we see that $z_N \rhd^{\mathrm{Ind}} x_N$ and $z_N < x_N$. To show that $z_N \succ x_N$, we only have to show that $z_N \in X^{\downarrow}$.

Suppose the contrary: there are $j \in N$ and $y_j > z_j$ such that

$$u_j(y_j, z_{-j}) > u_j(z_N). \tag{13}$$

Let us consider two alternatives.

If $j = i$ (hence $z_{-j} = x_{-i}$), $y_i > x_i$ would contradict $x_N \in X^{\downarrow}$ while $y_i < x_i$ would contradict the choice of $\bar{z}_i$; therefore, we have to assume that y_i and x_i are incomparable, hence $y_i \vee x_i > x_i$. The choice of $\bar{z}_i$ implies $u_i(\bar{z}_i, x_{-i}) \geq u_i(y_i \wedge x_i, x_{-i})$ and hence, by (13) and (5a), $u_i(y_i \vee x_i, x_{-i}) > u_i(x_N)$, contradicting the assumption $x_N \in X^{\downarrow}$.

Thus, we are led to $j \neq i$; hence $y_j > z_j = x_j$ and $z_{-j} < x_{-j}$. Now (13) and (2a) imply $u_j(y_j, x_{-j}) > u_j(x_N)$, again contradicting the assumption $x_N \in X^{\downarrow}$. □

Finally, having $x_N^0 \in X''_N \setminus \mathrm{E}(\Gamma'')$, we start building an improvement path, applying Claim C.1 at each step, i.e., picking $x_N^{k+1} \in X''_N$ such that $x_N^{k+1} \rhd^{\mathrm{Ind}} x_N^k$ and $x_N^{k+1} \succ x_N^k$, as long as $x_N^k \notin \mathrm{E}(\Gamma'')$. Since $\succ$ is an order, we cannot return back. Since X''_N is finite, we reach $\mathrm{E}(\Gamma'')$ at some stage.

D Proof of Theorem 3

In light of Proposition 3.4, it is enough to show that Γ has the quasi weak FIP. Let $\Gamma' \leq \Gamma$ be finite. Exactly as in the proof of Theorem 2, we define X''_i, for each $i \in N$, as the minimal sublattice of X_i containing X'_i. The subsets X''_i define a finite subgame $\Gamma'' \leq \Gamma$; we denote $S''_i := \sigma_i(X''_{-i})$.

To establish that Γ'' has the weak FIP, we argue similarly to Jensen (2010) or rather Kukushkin (2016). For each $i \in N$, Proposition 5.1 implies the existence of an increasing selection r_i from the best response correspondence R_i (in Γ''). Henceforth, we fix such a selection for each $i \in N$ and denote $X_i^0 := r_i(S''_i)$. Clearly, $X_i^0 \subseteq X''_i$ is a chain.

Now, we introduce this *admissible best response improvement relation* $\rhd^{\mathrm{BR}}$ on X''_N ($i \in N$, y_N, $x_N \in X''_N$):

$$y_N \rhd_i^{\mathrm{BR}} x_N \rightleftharpoons [y_N \rhd_i^{\mathrm{Ind}} x_N \;\&\; y_i = r_i(x_{-i})];$$
$$y_N \rhd^{\mathrm{BR}} x_N \rightleftharpoons \exists i \in N\, [y_N \rhd_i^{\mathrm{BR}} x_N].$$

Since $r_i(x_{-i})$ is defined for every $x_{-i} \in X''_{-i}$, every maximizer of $\rhd^{\mathrm{BR}}$ on X''_N is a Nash equilibrium in Γ''. Since X''_N is finite, it is sufficient to show that $\rhd^{\mathrm{BR}}$ is acyclic. We achieve this objective by producing an *order potential* of $\rhd^{\mathrm{BR}}$, i.e., an irreflexive and transitive binary relation $\succ$ on X''_N such that

$$\forall x_N, y_N \in X''_N \left[y_N \rhd^{\mathrm{BR}} x_N \Rightarrow y_N \succ x_N\right].$$

For each $i \in N$, we, henceforth, assume that $S''_i = \{s_i^0, s_i^1, \ldots, s_i^m\}$ (m may depend on i, naturally) with $s_i^k > s_i^h$ whenever $k > h$. For each $x_i \in X_i^0$, we define $\varkappa_i(x_i) := \min\{k \mid x_i = r_i(s_i^k)\}$ and

$$\Phi_i(x_i) := -F_i(s_i^{\varkappa_i(x_i)}, x_i) + \sum_{k<\varkappa_i(x_i)} [F_i(s_i^{k+1}, r_i(s_i^k)) - F_i(s_i^k, r_i(s_i^k))]. \tag{14}$$

For $x_i \in X_i'' \setminus X_i^0$, we define $\Phi_i(x_i)$ arbitrarily, e.g., $\Phi_i(x_i) := 0$. For every $x_N \in X_N''$, we define a set $N^0(x_N) := \{i \in N \mid x_i \in X_i^0\}$ and a function

$$H(x_N) := g(x_N) + \sum_{i \in N} \Phi_i(x_i). \tag{15}$$

Now, we are ready to define our potential, a binary relation on X_N'':

$$y_N \succ x_N \rightleftharpoons \big[N^0(y_N) \supset N^0(x_N) \ \text{ or } \ [N^0(y_N) = N^0(x_N) \ \& \ H(y_N) > H(x_N)] \ \text{ or } \\ [N^0(y_N) = N^0(x_N) \ \& \ H(y_N) = H(x_N) \ \& \ y_N > x_N]\big]. \tag{16}$$

Obviously, $\succ$ is irreflexive and transitive.

Claim D.1. *If $x_N, y_N \in X_N''$ and $y_N \rhd^{\mathrm{BR}} x_N$, then $y_N \succ x_N$.*

Proof. Let $y_N \rhd_i^{\mathrm{BR}} x_N$ and $\sigma_i(x_{-i}) = s_i^{\bar{k}}$. We have $y_i = r_i(s_i^{\bar{k}}) \neq x_i$ by definition; hence $y_i \in X_i^0$ and $N^0(y_N) \supseteq N^0(x_N)$. If the inclusion is strict, we have $y_N \succ x_N$ by the first term in (16).

Let us assume $N^0(y_N) = N^0(x_N)$, i.e., $x_i \in X_i^0$. Taking into account (14), we can rewrite (15) as

$$H(x_N) = \sum_{k<\varkappa_i(x_i)} [F_i(s_i^{k+1}, r_i(s_i^k)) - F_i(s_i^k, r_i(s_i^k))] + F_i(s_i^{\bar{k}}, x_i) - F_i(s_i^{\varkappa_i(x_i)}, x_i) + C(x_{-i}); \tag{17a}$$

$$H(y_N) = \sum_{k<\varkappa_i(y_i)} [F_i(s_i^{k+1}, r_i(s_i^k)) - F_i(s_i^k, r_i(s_i^k))] + F_i(s_i^{\bar{k}}, y_i) - F_i(s_i^{\varkappa_i(y_i)}, y_i) + C(x_{-i}). \tag{17b}$$

Let us assume that $x_i > y_i$; then $\varkappa_i(y_i) \le \bar{k} < \varkappa_i(x_i)$. Subtracting (17a) from (17b), we obtain

$$H(y_N) - H(x_N) = [F_i(s_i^{\varkappa_i(x_i)}, x_i) - F_i(s_i^{\bar{k}}, x_i)] - \sum_{\bar{k} \le k < \varkappa_i(x_i)} [F_i(s_i^{k+1}, r_i(s_i^k)) - F_i(s_i^k, r_i(s_i^k))] = \\ \sum_{\bar{k} \le k < \varkappa_i(x_i)} \Big([F_i(s_i^{k+1}, x_i) - F_i(s_i^k, x_i)] - [F_i(s_i^{k+1}, r_i(s_i^k)) - F_i(s_i^k, r_i(s_i^k))] \Big).$$

By (11), the difference is strictly positive. Therefore, $y_N \succ x_N$ by the second term in (16).

Now let us assume that $x_i < y_i$; then $\varkappa_i(x_i) < \varkappa_i(y_i) \le \bar{k}$. Subtracting (17a) from (17b), we obtain

$$H(y_N) - H(x_N) = \\ \sum_{\varkappa_i(x_i) \le k < \varkappa_i(y_i)} [F_i(s_i^{k+1}, r_i(s_i^k)) - F_i(s_i^k, r_i(s_i^k))] + [F_i(s_i^{\bar{k}}, y_i) - F_i(s_i^{\varkappa_i(y_i)}, y_i)] - [F_i(s_i^{\bar{k}}, x_i) - F_i(s_i^{\varkappa_i(x_i)}, x_i)] \\ = \sum_{\varkappa_i(x_i) \le k < \varkappa_i(y_i)} \Big([F_i(s_i^{k+1}, r_i(s_i^k)) - F_i(s_i^k, r_i(s_i^k))] - [F_i(s_i^{k+1}, x_i) - F_i(s_i^k, x_i)] \Big) + \\ \Big([F_i(s_i^{\bar{k}}, y_i) - F_i(s_i^{\varkappa_i(y_i)}, y_i)] - [F_i(s_i^{\bar{k}}, x_i) - F_i(s_i^{\varkappa_i(y_i)}, x_i)] \Big).$$

By (11), the difference is non-negative; it can only be zero if $\varkappa_i(y_i) = \bar{k} = \varkappa_i(x_i) + 1$. Thus, $y_N \succ x_N$ by the second or the third term in (16). $\square$

To summarize, we established that the admissible best response improvement relation $\rhd^{\mathrm{BR}}$ is acyclic on X''_N. Starting from $x^0_N \in X''_N$ an admissible best response improvement path in Γ'', we inevitably reach a Nash equilibrium at some stage. Therefore, Γ'' has the weak FIP.

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