Novshek's trick as a polynomial algorithm

Nikolai S. Kukushkin*

June 9, 2013

Abstract

An algorithm is described that constructs, in a polynomial time, a Nash equilibrium in a finite game with additive aggregation and decreasing best responses.

MSC2010 Classifications: 91A10; 91A04.

Key words: polynomial algorithm; fixed point; additive aggregation; decreasing best responses

1 Introduction

In games with increasing best responses ("strategic complements"), the existence of a Nash equilibrium can be easily derived [12] from Tarski's fixed point theorem [11]. Moreover, an equilibrium in a finite game can be found, by Algorithms I or II of [12], in a polynomial time (w.r.t. the number of players and the maximal number of strategies of one player).

In a game with *decreasing* best responses ("strategic substitutes"), there may be no equilibrium at all, the straightforward analog of Tarski's theorem for decreasing mappings being plainly wrong. Nonetheless, Novshek [10] showed that this property of the best responses ensures the existence of an equilibrium in the Cournot oligopoly model [2]. Although he worked with a continuous model and argued by continuity, the key role was played by a purely discrete trick. The fact was shown in [6], where an algorithm constructing a Nash equilibrium in every finite game with integer strategies, additive aggregation, and decreasing best responses was explicitly defined. Unfortunately, the necessity to scan virtually the whole set of strategy profiles could not be excluded, hence the time taken could only be estimated as an exponential.

*Russian Academy of Sciences, Dorodnicyn Computing Center, 40, Vavilova, Moscow 119333 Russia
E-mail: ququns@inbox.ru HomePage

The objective of this paper is to present a modification of the algorithm of [6] that is guaranteed to find a Nash equilibrium in a polynomial time.

Quite recently, Bashlaeva and Lebedev [1] described an algorithm that checks the existence of a Nash equilibrium in a finite game with additive aggregation in a polynomial time, and produces one if it exists. However, that algorithm, being of a dynamic programming type, demands huge memory. Here, we just move through the set of strategy profiles, the next profile at each step being determined by the current one, without any need to remember the history.

2 Fixed-point framework

We start in a fixed-point framework, essentially the same as in [6, Proposition 1]. There is a finite set N (of *players*); for each $i \in N$, there is a set $X_i = \{0, \dots, m_i\}$ (of *strategies*) and a multi-function $R_i \colon S_i \to 2^{X_i} \setminus \{\emptyset\}$, where $S_i := \{0, \dots, \sum_{j \neq i} m_j\}$ (*the best responses*). A strategy profile $x^0_N \in X_N := \prod_{i \in N} X_i$ is a *fixed point* if, for each $i \in N$,

$$x^0_i \in R_i\big(\sum_{j \neq i} x^0_j\big).$$

Assuming each R_i *decreasing* in the sense that $x'_i \leq x_i$ whenever $x'_i \in R_i(s'_i)$, $x_i \in R_i(s_i)$, and $s'_i > s_i$, we describe an algorithm that constructs a fixed point, thereby proving its existence.

We set $T := \sum_{i \in N} X_i = \{0, \dots, \bar{m}\}$, where $\bar{m} := \sum_{i \in N} m_i$, and extend each R_i to a mapping $T \to 2^{X_i} \setminus \{\emptyset\}$ by setting $R_i(s_i) := \{0\}$ when $s_i \in T \setminus S_i$. For every $i \in N$ and $t \in T$, we define

$$B_i(t) := \{x_i \in X_i \mid x_i \in R_i(t - x_i)\}.$$

Then we assume the set N linearly ordered, say, $N = \{1, \dots, n\}$, and consider $T \times N$ with the lexicographic order where the t-component matters first, so $(0, 1)$ is the minimum, followed by $(0, 2)$, etc., while $(\bar{m}, n)$ is the maximum. We define $C^* := (T \times N) \cup \{*\}$ assuming $* > (t, i)$ for every $(t, i) \in T \times N$. For every $(t, i) \in T \times N$, its *successor* $(t, i)'$ is uniquely defined: $(t, i)' = (t, i + 1)$ if $i < n$, $(t, n)' = (t + 1, 1)$ if $t < \bar{m}$, and $(\bar{m}, n)' = *$.

We define a mapping ξ_N from (a subset of) C^* to X_N, i.e., a sequence of strategy profiles indexed by pairs (t, i), by backward recursion. First, we set $\xi_j(*) := 0$ for all $j \in N$. Then we apply a uniform procedure ensuring the following properties whenever $\xi_N(t, i)$ is defined:

$$\xi_i(t, i) \in B_i(t); \tag{1a}$$

$$\sum_{j \in N} \xi_j(t, i) \leq t; \tag{1b}$$

$$\forall j \in N\left[\xi_j(t,i) \geq \xi_j((t,i)')\right]. \tag{1c}$$

Having $\xi_N((t,i)')$ already defined, we set

$$Q_i(t) := \{x_i \in B_i(t) \mid x_i \geq \xi_i((t,i)') \;\&\; x_i + \sum_{j \neq i} \xi_j((t,i)') \leq t\}.$$

If $Q_i(t) = \emptyset$, the process stops with a diagnosis "failure." Otherwise, we define $\xi_j(t,i) := \xi_j((t,i)')$ for all $j \neq i$ and $\xi_i(t,i) := \max Q_i(t)$; conditions (1) are easy to check. If $\sum_{j \in N} \xi_j(t,i) = t$, the process stops with a diagnosis "success"; otherwise, we move further.

Theorem. *The process just described ends at some stage (t^+, i^+) with a diagnosis "success," and $\xi_N(t^+, i^+)$ is a fixed point.*

Proof. To start with, $0 \in Q_i(\bar{m}) \neq \emptyset$ for each $i \in N$, so a failure could only happen when $t < \bar{m}$.

Claim 1. *If $\xi_N(t,i)$ is defined and $\sum_{j \in N} \xi_j(t,i) < t$, then $\xi_i(t,i) \in B_i(t-1)$.*

Proof of Claim 1. We denote $s_i := t - \xi_i(t,i) > \sum_{j \neq i} \xi_j(t,i)$. Since $\xi_i(t,i) \in B_i(t)$, we have $\xi_i(t,i) \in R_i(s_i)$. If $\xi_i(t,i) \in R_i(s_i - 1)$ as well, we are home. Supposing the contrary, we must have $x_i > \xi_i(t,i)$ for every $x_i \in R_i(s_i - 1) \neq \emptyset$. Let us fix an $x_i \in R_i(s_i - 1)$ and define $\tau := x_i + s_i - 1$; clearly, $\tau - t = x_i - \xi_i(t,i) - 1 \geq 0$. Now we have $x_i \in B_i(\tau)$; besides, $x_i + \sum_{j \neq i} \xi_j((\tau,i)') \leq \tau$ since $\xi_j((\tau,i)') \leq \xi_j((t,i)')$ for every $j \in N$. Thus, $x_i \in Q_i(\tau)$ and $x_i > \xi_i(\tau,i)$, which contradicts the choice of $\xi_i(\tau,i)$. □

Claim 2. *Whenever $\xi_N((t,i)')$ is defined, we have $Q_i(t) \neq \emptyset$.*

Proof of Claim 2. The assumptions of Claim 1 hold for $(t+1,i)$, hence $\xi_i(t+1,i) \in B_i(t)$. Moreover, since $\xi_i((t,i)') = \xi_i(t+1,i)$, we have $\xi_i(t+1,i) \in Q_i(t)$ as well. □

Note that if $\xi_N(0,n)$ is defined, then conditions (1b) and (1c) immediately imply that $\xi_i(0,n) = 0$ for each $i \in N$, which means a success. Thus, Claim 2 implies that our process cannot end in failure. Since C^* is finite, it must end; therefore, it ends with success.

Claim 3. *If the process stops at (t^+, i^+) with a diagnosis "success," then $\xi_N(t^+, i^+)$ is a fixed point.*

Proof of Claim 3. It is obviously sufficient to prove that $\xi_i(t^+, i^+) \in B_i(t^+)$ for all $i \in N$. For $i \geq i^+$, this immediately follows from the description of the process. For $i < i^+$, we have $\xi_i(t^+, i^+) \in B_i(t^+ + 1)$ from the same description; then we invoke Claim 1. □

The theorem is proven. □

Denoting $m := \max_{i \in N} m_i$, we see that $\#C^* \le O(mn^2)$. At each step (t,i), the set $\{x_i \in X_i \mid \xi_i((t,i)') \le x_i \le t - \sum_{j \ne i} \xi_j((t,i)')\}$ is scanned to determine which strategies belong to $Q_i(t)$. Altogether, the algorithm requires no more than $O(m^2n^2)$ "elementary" steps.

Remark. The number of steps required in [1] is bounded above by $O(m^3n^3)$.

3 Strategic games

Let Γ be a strategic game with a finite set N of players and strategy sets $X_i = \{0, \dots, m_i\}$ for each $i \in N$; let the utility functions of the players be $u_i(x_N) = U_i(\sum_{j \ne i} x_j, x_i)$. Denoting $S_i := \{0, \dots, \sum_{j \ne i} m_j\}$, we define *the best response correspondence* of each player $i \in N$, $R_i \colon S_i \to 2^{X_i} \setminus \{\emptyset\}$, by $R_i(s_i) := \operatorname{Argmax}_{x_i \in X_i} U_i(s_i, x_i)$. A strategy profile $x_N^0 \in X_N := \prod_{i \in N} X_i$ is a *Nash equilibrium* if, for each $i \in N$,

$$x_i^0 \in R_i\bigl(\sum_{j \ne i} x_j^0\bigr).$$

The constructions from the previous section apply whenever each R_i is decreasing. The only difference is that we no longer can view checking whether $x_i \in R_i(s_i)$ as an elementary operation. However, such a check requires no more than m_i comparisons of utilities, hence our algorithm remains polynomial, with time taken not exceeding $O(m^3n^2)$.

If there are no *a priori* grounds to believe that the best responses are decreasing, we can invoke well-known sufficient conditions: *strictly decreasing differences* (Topkis [12]),

$$\forall x_i', x_i \in X_i \, \forall s_i', s_i \in S_i \, \bigl[[x_i' > x_i \ \& \ s_i' > s_i] \Rightarrow \\ U_i(s_i', x_i') - U_i(s_i', x_i) < U_i(s_i, x_i') - U_i(s_i, x_i)\bigr], \quad (2)$$

or, more generally, *strict single crossing* (Milgrom and Shannon [9]),

$$\forall x_i', x_i \in X_i \, \forall s_i', s_i \in S_i \, \bigl[[x_i' > x_i \ \& \ s_i' > s_i \ \& \\ U_i(s_i', x_i') \ge U_i(s_i', x_i)] \Rightarrow U_i(s_i, x_i') > U_i(s_i, x_i)\bigr]. \quad (3)$$

The verification of either condition can be done in a polynomial time. In both cases, we may restrict ourselves to $s_i' = s_i + 1$; in the first case, to $x_i' = x_i + 1$ as well. Thus, no more than $O(m^2n^2)$ comparisons of utilities are required to check (2), and no more than $O(m^3n^2)$ to check (3).

Remark. Instead of checking conditions (2) or (3), we can just run the algorithm. If the process ends successfully, we have a Nash equilibrium; if it ends in failure, we, at least, know that the best responses are not decreasing.

If each X_i is an arbitrary finite set of integers, a straightforward modification of our constructions is needed, cf. [6, Proposition 2]; the algorithm remains polynomial w.r.t. the number of players and the maximal "range" of a strategy set, $\max X_i - \min X_i$.

If each X_i is an arbitrary finite set of real numbers, the existence of a Nash equilibrium still holds because we can approximate the game with an integer one; however, the algorithm can hardly be called polynomial in any reasonable sense. By [7, Theorem 2], a Nash equilibrium will be reached in a finite number of steps if we iterate the best responses starting from any strategy profile. Unfortunately, the number of steps needed admits no polynomial estimate.

Remark. The sequence $\xi_N(t,i)$ in the above proof is *not* generated by iteration of the best responses.

A completely unrelated trick invented by Huang [4] for studying fictitious play was made applicable to best response dynamics in [3]. Further development [8, 5] showed that additive aggregation can be replaced with, e.g., general linear one, $u_i(x_N) = U_i(\sum_{j\neq i} a_{ij}x_j, x_i)$ provided $a_{ij} = a_{ji}$ for all $j \neq i$. If (3) holds, then a Nash equilibrium exists and is reached from any strategy profile, in a finite number of steps, by iterations of the best responses. Unfortunately, the number of steps needed again admits no polynomial estimate, and there is no analog of Novshek's trick in this, more general, situation.

Remark. Since there are no restrictions on the signs of a_{ij}, the best responses here may be increasing in some players' strategies and decreasing in others'.

Finally, let us return to games with strategic complements. Both Algorithms I and II of [12] produce increasing sequences in each player's strategy set; therefore, the total number of steps cannot exceed mn. Adding the comparisons of utilities, we obtain $O(m^2n^2)$. It should be stressed that there is no need for any aggregation in the utilities and that the strategy sets may be arbitrary finite lattices. Although the standard sufficient conditions for increasing best responses, similar to (2) or (3), apparently cannot be verified in polynomial time in the general case, we can run either algorithm anyway and, as above, either find an equilibrium, or learn that the best responses are *not* increasing.

Acknowledgments

Financial support from the Russian Foundation for Basic Research (projects 11-07-00162 and 11-01-12136) is acknowledged. I have benefitted from fruitful discussions with Vasiliy Lebedev, who stirred up my interest in the topic, and Vladimir Gurvich. Helpful suggestions by an anonymous referee are also appreciated.

References

[1] Bashlaeva, I.A., and V.N. Lebedev, 2012. Search for a fixed point of a discrete decreasing operator. (in Russian) Matematicheskaya Teoriya Igr i ee Prilozheniya (Mathematical Game Theory and Its Applications) 4(3), 21–32.

[2] Cournot, A.A., 1838. Recherches sur les principes mathématiques de la théorie des richesses. Hachette, Paris.

[3] Dubey, P., O. Haimanko, and A. Zapechelnyuk, 2006. Strategic complements and substitutes, and potential games. Games and Economic Behavior 54, 77–94.

[4] Huang, Z., 2002. Fictitious play in games with a continuum of strategies. Ph.D. Thesis, State University of New York at Stony Brook, Department of Economics.

[5] Jensen, M.K., 2010. Aggregative games and best-reply potentials. Economic Theory 43, 45–66.

[6] Kukushkin, N.S., 1994. A fixed-point theorem for decreasing mappings. Economics Letters 46, 23–26.

[7] Kukushkin, N.S., 2004. Best response dynamics in finite games with additive aggregation. Games and Economic Behavior 48, 94–110.

[8] Kukushkin, N.S., 2005. "Strategic supplements" in games with polylinear interactions. EconWPA Paper #0411008.

[9] Milgrom, P., and C. Shannon, 1994. Monotone comparative statics. Econometrica 62, 157–180.

[10] Novshek, W., 1985. On the existence of Cournot equilibrium. Review of Economic Studies 52, 85–98.

[11] Tarski, A., 1955. A lattice-theoretical fixpoint theorem and its applications. Pacific Journal of Mathematics 5, 285–309.

[12] Topkis, D.M., 1979. Equilibrium points in nonzero-sum n-person submodular games. SIAM Journal on Control and Optimization 17, 773–787.