Acyclicity of Monotonic Endomorphisms*

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Abstract

An abstract theory of improvement dynamics for binary relations in topological spaces is developed, providing a general framework for studying various improvement relations and tâtonnement processes in strategic games. It has already been established that the presence of aggregation (i.e., of a complete ordering on the set of strategy profiles) of certain kinds is conducive to the acyclicity of best response improvement paths. Here the connection between acyclicity and aggregation is studied in the context of a monotonic endomorphism, where more definite results prove obtainable. A general question of "minimal concord" between topology and order, sufficient for meaningful conclusions, is also addressed.

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1 Introduction

It has long been recognized in the (individual or social) choice theory that an acyclic binary relation on a finite set always has a maximal element. However, the fact that this simple theorem can play a rather important rôle in strategic game theory seems not to have been recognized till the seminal paper of Monderer and Shapley (1996). Many equilibrium concepts in game theory are defined through the absence of "objections" or "blocking." For instance, Nash equilibrium assumes the absence of profitable individual deviations; strong equilibrium, of profitable coalition deviations. Since the "blocking" defines a binary relation on the set of strategy profiles, equilibria themselves can be perceived as maximal elements. Therefore, the acyclicity of the underlying relation ensures the existence of an equilibrium as well as the convergence of all improvement paths.

In Kukushkin (2000) a framework was developed for studying various tâtonnement processes in strategic games, including best response dynamics in the style of Milchtaich (1996). The main results concerned "systems of reactions," a natural generalization of the best response correspondences in strategic games. Roughly speaking, the acyclicity of best response dynamics in games with strategic complements or substitutes (Bulow et al., 1985) was established, provided that each player is only affected by an appropriate aggregate of the strategies of the partners. (To be more precise, the uniqueness of best responses was also assumed in most theorems.) Naturally, a question arises of whether the presence of such aggregates is necessary (in a sense), or there may exist absolutely different sufficient conditions.

The research reported here was motivated by this question although nothing more than a certain advance in understanding can be claimed. Necessary and sufficient conditions for acyclicity are obtained for endomorphisms, i.e., for order-preserving mappings of a partially ordered set into itself; unlike systems of reactions, this context is related to strategic games only metaphorically. Besides, the approach to acyclicity of binary relations on topological spaces outlined in Kukushkin (2000) is developed in a more systematic way.

When relations on infinite sets are considered, the absence of finite improvement cycles does not ensure even the existence of a maximizer on a compact space, to say nothing of the convergence of improvement paths. Improvement paths parameterized with transfinite numbers suggest themselves strongly: if an infinite number of steps have been made, a limit point can be taken and, if the point is still not a maximizer, the process can continue further; it is essential whether it may or may not return back after a "transfinite number" of steps. Under closer inspection, acyclicity splits into two different properties: Ω-*acyclicity* and *weak* Ω-*acyclicity* (roughly speaking, the difference between them corresponds to that between a limit point and a limit), one of them being sufficient, and the other necessary, for the existence of maximizers over every compact subset.

Some readers may feel that transfinite dynamics cannot make any sense: it will take "all the time" to reach the first limit. However, this objection would be equally valid with respect to any theorem stating, e.g., the convergence of a dynamic process to an equilibrium: the convergence takes all the time, hence the agents won't be able to enjoy being in an equilibrium. Actually, the relationship between mathematical statements involving infinity and statements about the real world is not at all straightforward and has been a subject of philosophical debate for centuries; no attempt to add anything to the debate is made here.

A view is acceptable to us that an infinite or transfinite process "really" means a finite one with indefinite (perhaps very long) duration. Anyway, what matters is logical consistency of mathematical constructions. Besides, the properties of binary relations defined in terms of improvement paths can bear on the problem of the relevance of maximizers and make sense regardless of whether such paths are perceived as adequate to describe actual dynamics (Kukushkin, 2002).

It should be noted that the use of improvement paths in the general theory of binary relations was not unpopular in the 1970s (Smith, 1974; Mukherji, 1977); later on, it somehow went out of fashion. This work argues that the potential of improvement paths as a means of understanding the structure of binary relations is far from being exhausted.

What is relevant to an abstract binary relation is relevant to an endomorphism, i.e., to a mapping of a set into itself, as well: fixed points are maximizers; iteration of the endomorphism generates improvement paths. Tarski's (1955) fixed point theorem establishes the existence of a fixed point for an increasing (i.e., order-preserving) endomorphism of a complete lattice; however, the convergence of iterations is not ensured [to be more precise, it only takes place for paths starting "near the top" or "near the bottom" (Topkis, 1979; Vives, 1990)].

Acyclicity presents itself when it comes to increasing endomorphisms of sets with orderings (i.e., reflexive, transitive, and complete relations), provided there is a certain degree of concord between topology and order. The usual assumption of continuity (cf. Nachbin, 1965; Birkhoff, 1967) proves too stringent. For instance, if an endomorphism preserves a lexicographic ordering constructed of continuous orderings, the convergence of iterations to a fixed point is ensured; the fact may seem exotic, but monotonicity w.r.t. a lexicographic order emerges in important economic models (Milgrom and Shannon, 1994; Edlin and Shannon, 1998).

Section 2 starts with a "folk" Theorem 0 stating the equivalence of several properties of binary relations on finite sets to acyclicity. Then Theorems 1 and 2 establish a hierarchy of mostly nonequivalent properties of binary relations on topological spaces, each of which having something to do with acyclicity.

Section 3 considers orderings on a topological space. A general question of "minimal concord" between topology and order, sufficient for meaningful conclusions, is addressed, leading to the formulation of two properties: *pseudocontinuity* and *quasi-continuity*. Corollary to Theorem 3 shows that the latter property of a complete order is equivalent to the condition that every compact subset is compact in its intrinsic (as a chain) topology. In Subsection 3.2 special attention is paid to lexicography; a lexicographic combination of quasicontinuous orderings is proved to be quasicontinuous as well (Theorem 4).

The key rôle is played by Section 4: Theorems 5 and 6 show that the presence of an appropriate ordering is necessary and sufficient for the acyclicity of all monotonic endomorphisms.

Section 5 adds to Kukushkin (2000) quantitatively: Theorems 7 and 8 show that another kind of aggregation, that defined by the maximum (minimum) function, ensures acyclicity under the strategic complements (substitutes) condition. There are also some achievements in purely technical terms; they are discussed in Subsection 5.4.

2 Acyclicity

2.1 Improvement Paths

A *binary relation* on a set X is a Boolean function $X \times X \to \{\top, \bot\}$. It is a usual practice to write yRx instead of $R(y,x) = \top$, and $zRyRx$ instead of zRy and yRx. We will usually denote relations with symbols such as $\triangleright$ or $\succ$ rather than with letters.

A binary relation $\triangleright$ is *reflexive* if $x \triangleright x$ for all $x \in X$, *irreflexive* if $x \triangleright x$ for no $x \in X$, and *transitive* if $z \triangleright y \triangleright x \Rightarrow z \triangleright x$. If $y \triangleright x \Rightarrow y \mathrel{\triangleright\!\!\triangleright} x$ for all $y, x \in Y \subseteq X$, we say that $\triangleright$ is *coarser* than $\mathrel{\triangleright\!\!\triangleright}$ on Y, or that $\mathrel{\triangleright\!\!\triangleright}$ is *finer* than $\triangleright$ on Y (for equivalence relations, the opposite terminology appears more reasonable, but we mostly have in mind irreflexive relations). For a given relation $\triangleright$, its *transitive closure* $\succeq$ on Y is defined by: $y \succeq x$ if and only if $y \mathrel{\triangleright\!\!\triangleright} x$ for every $\mathrel{\triangleright\!\!\triangleright}$ which is transitive and finer than $\triangleright$ on Y. Thus, $\succeq$ is "the coarsest transitive refinement of $\triangleright$."

A *simple improvement path* for $\triangleright$ in $Y \subseteq X$ (the relation and the set will not be mentioned when clear from the context) is a (finite or infinite) sequence $\{x^k\}_{k=0,1,\ldots}$ such that $x^k \in Y$ and $x^{k+1} \triangleright x^k$ for all relevant k. Obviously, $y \succeq x$, where $\succeq$ is the transitive closure of $\triangleright$ on Y, if and only if there exists a finite improvement path $\{x^0, \ldots, x^m\}$ in Y such that $x^0 = x$, $m > 0$, and $x^m = y$. A *finite improvement cycle* is a finite improvement path such that $x^0 = x^m$ and $m > 0$. A relation is *acyclic* if it admits no finite improvement cycle; obviously, $\triangleright$ is acyclic if and only if its transitive closure is irreflexive.

A *maximizer for* $\triangleright$ *over* $Y \subseteq X$ is $x \in Y$ such that $y \triangleright x$ does not hold for any $y \in Y$.

$\rhd$ has the *finite improvement path (FIP)* property on $Y \subseteq X$ if there exists no infinite improvement path for $\rhd$ in Y. $\rhd$ has the *weak FIP* property on Y if, for every $x \in Y$, there exists a finite improvement path $\{x^0, \ldots, x^m\}$ for $\rhd$ in Y such that $x^0 = x$ and x^m is a maximizer for $\rhd$ over Y. $\rhd$ has the *von Neumann-Morgenstern (NM)* property on Y if the previous condition holds with $m \leq 1$; this actually means that the set of all maximizers for $\rhd$ over Y is a von Neumann-Morgenstern solution (hence the unique NM-solution) on Y.

Remark. The FIP property was introduced for the individual improvement relation in a strategic game by Monderer and Shapley (1996); the same property of an abstract binary relation was called "strict acyclicity" by von Neumann and Morgenstern (1953). The term "weak FIP" was used by Friedman and Mezzetti (2001) for the individual improvement relation in strategic games; the same property of the best response improvement relation in a strategic game was called "weak acyclicity" by Young (1993).

Clearly, each of the three properties implies the existence of a maximizer over a nonempty Y. Either of NM and FIP implies the weak FIP property; generally, they do not imply each other. The following important theorem hardly deserves a formal proof.

Theorem 0. *Let $\rhd$ be a binary relation on a set X; then the following statements are equivalent:*

$\rhd$ *is acyclic on X;*

$\rhd$ *has the FIP property on every finite subset $Y \subseteq X$;*

$\rhd$ *has the weak FIP property on every finite subset $Y \subseteq X$;*

$\rhd$ *admits a maximizer over every nonempty finite subset $Y \subseteq X$.*

Our first objective is to study whether and how Theorem 0 could be extended to topological spaces, replacing "finite" with "compact." Throughout the paper, whenever a topological term is mentioned, X is assumed a Hausdorff topological space with a countable base of open sets (as is well known, topology on X is then adequately described by convergent sequences). Improvement paths will be parameterized by countable ordinal numbers. Some of the previous definitions also need modification in this case.

A partially ordered set is *well ordered* if every subset contains a least point (then the set obviously must be a chain). *Ordinal numbers*, or just *ordinals*, are types of well ordered sets; Natanson (1974, Chapter XIV), can be used as a reference book. The set of all countable ordinals, denoted K, is well ordered (but uncountable) itself. We denote $[0, \alpha[$ the set $\{\beta \in K \mid \beta < \alpha\}$; note that $\alpha \notin [0, \alpha[$. Each $\alpha \in K$ is the type of $[0, \alpha[$ (Theorem 3, Section 3, Chapter XIV of Natanson). For each $\alpha \in K$, its *successor*, denoted $\alpha+1$, is uniquely defined as the type of the set $\{\beta \in K | \beta \leq \alpha\}$. An ordinal $\alpha \in K \setminus \{0\}$ is called *isolated* if $\alpha = \beta+1$; otherwise, α is called a *limit ordinal number*. The least limit ordinal is ω: the type of the chain of all natural numbers. ω and greater ordinals are called *transfinite numbers*. It is sometimes convenient to consider a partial function $\alpha - 1$ defined by the equality $\alpha = (\alpha - 1) + 1$ for isolated α and not defined at all for limit ordinals. Every countable subset of K has a least upper bound in K (Theorem 2, Section 5, Chapter XIV of Natanson). Every limit ordinal $\alpha \in K$ is the least upper bound of a strictly increasing infinite sequence in K (Theorem 4, Section 5, Chapter XIV of Natanson).

Let $\rhd$ be a binary relation on X. An *improvement path* for $\rhd$ in $Y \subseteq X$ (the relation and the set will not be mentioned when clear from the context) is a mapping $\pi : \mathrm{Dom}(\pi) \to Y$, where $\mathrm{Dom}(\pi)$ is either K or $[0, \mu[$ for $\mu \in K$, satisfying these two conditions:

1. $\pi(\alpha + 1) \rhd \pi(\alpha)$ whenever $\alpha, \alpha + 1 \in \mathrm{Dom}(\pi)$;
2. if $\alpha \in \mathrm{Dom}(\pi)$ and α is a limit ordinal, there exists a sequence $\{\beta^k\}_k$ for which $\beta^{k+1} > \beta^k$ for all $k = 0, 1, \ldots$, $\alpha = \sup_k \beta^k$, and $\pi(\alpha) = \lim_{k\to\infty} \pi(\beta^k)$.

$\rhd$ has the *countable improvement path (CIP)* property on Y if there exists no improvement path π for $\rhd$ in Y with $\mathrm{Dom}(\pi) = K$. $\rhd$ has the *weak CIP* property on Y if, for every $x \in Y$, there exists an improvement path π in Y such that $\pi(0) = x$, $\mathrm{Dom}(\pi) = [0, \alpha + 1[$, and $\pi(\alpha)$ is a maximizer for $\rhd$ over Y.

A discrepancy in our terminology can easily be observed: generally, CIP does not imply weak CIP (consider, e.g., the real line with the standard order). On a compact space, however, where the only obstacle to extending an improvement path further is the fact that it has reached a maximizer, CIP means that every improvement path, if continued whenever possible, ends at a maximizer, thus implying the weak CIP. A formal proof is to be found in Theorem 2 ([2.4] $\Rightarrow$ [2.5]) below.

2.2 Transitive Relations

A binary relation $\succ$ on X is called *ω-transitive* if it is transitive and the conditions $x^\omega = \lim_{k\to\infty} x^k$ and $x^{k+1} \succ x^k$ for all $k = 0, 1, \ldots$ always imply $x^\omega \succ x^0$.

Remark. It is worth noting that $x^\omega \succ x^k$ is valid for all $k = 0, 1, \ldots$ in the above situation, once $\succ$ is ω-transitive.

A mapping $\nu : \mathrm{Dom}(\nu) \to X$, where $\mathrm{Dom}(\nu)$ is either K or $[0, \mu[$ for $\mu \in K$, is called a *monotonic path* for $\succ$ if it satisfies the condition: $\alpha > \beta \Rightarrow \nu(\alpha) \succ \nu(\beta)$ for all $\alpha, \beta \in \mathrm{Dom}(\nu)$.

Proposition 2.1. *Let $\succ$ be an irreflexive and ω-transitive relation on X; then there exists no monotonic path ν for $\succ$ with* $\mathrm{Dom}(\nu) = K$.

Proof. Supposing the contrary, we denote $F(\alpha) = \mathrm{cl}\,\nu(\{\beta \in K | \ \beta > \alpha\})$ $[\neq \emptyset]$ for every $\alpha \in K$, and $F = \bigcap_{\alpha \in K} F(\alpha)$; clearly, all the sets $F(\alpha)$ contain one another. Since X has a countable base of open sets, there exists a countable subset $L \subseteq K$ such that $F = \bigcap_{\alpha \in L} F(\alpha)$ (the Lindelöf theorem, see, e.g., Kuratowski, 1966, p. 54); clearly, L cannot have a maximum. Now we denote $\alpha^* = \sup L$ (it exists in K by the Statement 2 of Theorem 2, Section 5, Chapter XIV of Natanson, 1974) and $x^* = \nu(\alpha^*)$; $\alpha^* > \alpha$ for every $\alpha \in L$, hence $x^* \in F$.

Let $U_1, \ldots, U_k, \ldots$ be a countable base of open neighbourhoods of x^*; without restricting generality, $U_{k+1} \subseteq U_k$ for all k. Let us pick $\alpha^0 > \alpha^*$ arbitrarily and then define, by induction, a sequence $\{\alpha^k\}_{k=1,2,\ldots}$ such that $\alpha^k > \alpha^{k-1} > \alpha^*$ and $\nu(\alpha^k) \in U_k$ for each k: since $x^* \in F \subseteq F(\alpha^{k-1})$, there exists $\alpha^k > \alpha^{k-1}$ such that $\nu(\alpha^k) \in U_k$. Now $\nu(\alpha^k) \to x^*$ and $\nu(\alpha^k) \succ \nu(\alpha^{k-1})$ for all k; since $\succ$ is ω-transitive, $x^* \succ \nu(\alpha^k) \succ \nu(\alpha^*) = x^*$, which contradicts the irreflexivity of $\succ$. □

Let Π be a set of monotonic paths for $\succ$; we call Π *closed w.r.t. extension* if, for any monotonic path ν for $\succ$, the condition that each restriction of ν to $[0, \mu+1[$ $\big(\mu \in \mathrm{Dom}(\nu)\big)$ belongs to Π implies that $\nu \in \Pi$ too. The set of $\nu \in \Pi$ with $\mathrm{Dom}(\nu) \subset K$ will be denoted Π^*. A *continuation rule* is a mapping $\vartheta : \Pi^* \to \Pi^*$ such that every $\nu \in \Pi^*$ is the restriction of $\vartheta(\nu)$ to $\mathrm{Dom}(\nu)$ [i.e., $\mathrm{Dom}(\nu) \subseteq \mathrm{Dom}(\vartheta(\nu))$ and $\beta \in \mathrm{Dom}(\nu) \Rightarrow \vartheta(\nu)(\beta) = \nu(\beta)$].

Proposition 2.2. *Let $\succ$ be an irreflexive and ω-transitive relation on X, Π be a nonempty set of monotonic paths for $\succ$, closed w.r.t. extension, and $\vartheta : \Pi^* \to \Pi^*$ be a continuation rule. Then there exists $\nu \in \Pi^*$ such that $\vartheta(\nu) = \nu$.*

Proof. By Proposition 2.1, $\Pi^* = \Pi$. We introduce a strict order on Π:

$$\nu'' > \nu' \iff \mathrm{Dom}(\nu') \subset \mathrm{Dom}(\nu'') \,\&\forall \beta \in \mathrm{Dom}(\nu') \ [\nu'(\beta) = \nu''(\beta)].$$

If ν is a maximizer for the order over Π, then $\vartheta(\nu) = \nu$. Let us establish the existence of a maximizer by Zorn's Lemma. If $C \subseteq \Pi$ is a chain, we define $\Delta = \bigcup_{\nu \in C} \mathrm{Dom}(\nu)$. For every $\beta \in \Delta$, we define $\nu^*(\beta) = \nu(\beta)$ for $\nu \in C$ and $\beta \in \mathrm{Dom}(\nu)$ (the fact that C is a chain ensures that it does not matter which ν satisfying the conditions is used), obtaining $\nu^* \in \Pi$ (because Π is closed w.r.t. extension) such that $\nu^* \geq \nu$ for all $\nu \in C$. □

Lemma 2.3. *Let $\succeq$ be an ω-transitive relation on X and π be an improvement path for $\succeq$; then π is a monotonic path for $\succeq$.*

Proof. By transfinite recursion in α, we prove the following statement: $\pi(\beta') \succeq \pi(\beta)$ whenever $\alpha \geq \beta' > \beta$ and $\pi(\beta')$ is defined. If $\alpha = 1$, the definition of an improvement path suffices; when considering an induction step, we only have to check the statement for $\beta' = \alpha$. If α is isolated, then $\pi(\alpha) \succeq \pi(\alpha-1)$ and $\pi(\alpha-1) \succeq \pi(\beta)$ (by the induction hypothesis) imply $\pi(\alpha) \succeq \pi(\beta)$ by the transitivity of $\succeq$. If α is a limit ordinal, then, by the definition of an improvement path, $\pi(\alpha) = \lim_{k\to\infty} \pi(\beta^k)$, where $\beta^{k+1} > \beta^k$ for all $k = 0, 1, \ldots$, and $\alpha = \sup_k \beta^k$; without restricting generality, we may assume $\beta^0 > \beta$. Thus, we have $\pi(\beta^0) \succeq \pi(\beta)$ and $\pi(\beta^{k+1}) \succeq \pi(\beta^k)$ for all k by the induction hypothesis, hence $\pi(\alpha) \succeq \pi(\beta^0)$ by the ω-transitivity of $\succeq$, hence $\pi(\alpha) \succeq \pi(\beta)$. □

Theorem 1. *A binary relation $\succ$ on X has the NM property on every finite subset $Y \subseteq X$ if and only if $\succ$ is irreflexive and transitive on X. A binary relation $\succ$ on X has the NM property on every compact subset $Y \subseteq X$ if and only if $\succ$ is irreflexive and ω-transitive on X.*

Proof. The sufficiency in the first statement is straightforward. It was noticed first by von Neumann and Morgenstern (1953, (65:I)); however, they did not mention the necessity of the condition, which is also rather simple: If $x \succ x$, there is no maximizer over $\{x\}$; if $z \succ y \succ x$, then neither x nor y can be maximizers for $\succ$ over $\{x, y, z\}$, hence $z \succ x$ is obligatory. The necessity in the second statement needs just one similar step more: If $x^k \to x^\omega$ and $x^{k+1} \succ x^k$ for all k, then none of x^k can be a maximizer for $\succ$ over $\{x^k\}_k \cup \{x^\omega\}$, which is compact, hence $x^\omega \succ x^0$ is obligatory.

Finally, let $\succ$ be irreflexive and ω-transitive on X, $Y \subseteq X$ be compact, and $x^0 \in Y$. We denote Π the set of improvement paths π for $\succ$ in Y with $\pi(0) = x^0$; obviously, $\Pi = \emptyset$ if and only if x^0 is a maximizer for $\succ$ over Y, in which case there is nothing to prove. Taking into account Lemma 2.3, we easily see that Π is a nonempty set of monotonic paths for $\succ$ closed w.r.t. extension. As can easily be seen, each $\pi \in \Pi$ with $\mathrm{Dom}(\pi) = [0, \alpha[$ satisfies just one of the following conditions:

$$\alpha \text{ is a limit ordinal;} \tag{2.1a}$$

$$\alpha - 1 \text{ is defined and } \pi(\alpha - 1) \text{ is not a maximizer for } \succ \text{ over } Y; \tag{2.1b}$$

$$\alpha - 1 \text{ is defined and } \pi(\alpha - 1) \text{ is a maximizer for } \succ \text{ over } Y. \tag{2.1c}$$

Let us consider a continuation rule $\vartheta : \Pi \to \Pi$ such that $\vartheta(\pi) = \pi$ if (2.1c) holds, and $\mathrm{Dom}(\vartheta(\pi)) = [0, \alpha + 1[$ otherwise: if (2.1a) holds, then $\vartheta(\pi)(\alpha)$ is a limit point of π, which always exists because of Theorem 4, Section 5, Chapter XIV of Natanson (1974) and the assumed compactness of Y; if (2.1b) holds, then $\vartheta(\pi)(\alpha) \succ \pi(\alpha - 1)$. Now Proposition 2.2 and the definition of ϑ immediately imply the existence of π starting at x^0 and ending at a maximizer $\pi(\alpha)$. Since π is a monotonic path for $\succ$, $\pi(\alpha) \succ x$. □

For a given relation $\rhd$, its *ω-transitive closure $\succeq$ on $Y \subseteq X$* is defined by: $y \succeq x$ if and only if $y \rhd\!\!\rhd x$ for every $\rhd\!\!\rhd$ which is ω-transitive and finer than $\rhd$ on Y. Thus, $\succeq$ is "the coarsest ω-transitive refinement of $\rhd$."

Lemma 2.4. *Let $\rhd$ be a binary relation on X, $\succeq$ be its ω-transitive closure on $Y \subseteq X$, and $y, x \in Y$. Then $y \succeq x$ if and only if there exist an improvement path π for $\rhd$ in Y and $\alpha \in K$ satisfying $\pi(0) = x$, $\alpha > 0$, and $\pi(\alpha) = y$.*

Proof. Let there be an improvement path π such that $x = \pi(0)$, $y = \pi(\alpha)$, and $\alpha > 0$. If $\rhd\!\!\rhd$ is an ω-transitive relation finer than $\rhd$, then π is an improvement path for $\rhd\!\!\rhd$ too, hence, by Lemma 2.3, a monotonic path for $\rhd\!\!\rhd$, hence $y \rhd\!\!\rhd x$. Since $\rhd\!\!\rhd$ was arbitrary, $y \succeq x$.

The opposite implication is proved by noticing that the relation "there is an improvement path of a strictly positive length from x to y" is ω-transitive (a formal proof consists in a reference to Theorem 7, Section 2, Chapter XIV of Natanson (1974), about lexicographic sums of well ordered sets). □

2.3 Acyclic Relations

An *improvement cycle for* $\rhd$ is an improvement path π such that $\mathrm{Dom}(\pi) = [0, \alpha+1[$, $\alpha > 0$, and $\pi(\alpha) = \pi(0)$. $\rhd$ is called Ω-*acyclic* if there is no improvement cycle for $\rhd$. An improvement path $\pi : \mathrm{Dom}(\pi) \to X$ is called *narrow* if $\pi(\beta^k) \to \pi(\alpha)$ for every limit ordinal $\alpha \in \mathrm{Dom}(\pi)$ and every strictly increasing sequence $\{\beta^k\}_{k=0,1,\ldots}$ such that $\alpha = \sup_k \beta^k$; in other words, if each $\pi(\alpha)$ is the limit of the preceding path rather than a limit point. $\rhd$ is called *weakly* Ω-*acyclic* if there is no narrow improvement cycle for $\rhd$; it is called ω-*acyclic* if it is acyclic and the conditions $x^{k+1} \rhd x^k$ for all $k = 0, 1, \ldots$ and $x^0 = \lim_{k\to\infty} x^k$ are incompatible.

Remark. The notion of ω-acyclicity was first introduced (for complete relations) by Smith (1974) under the name of "weak σ-transitivity"; however, "acyclicity" seems a more appropriate term here. It is worth noting that $x^k \rhd x^\omega$ is impossible for any ω-acyclic relation $\rhd$, any improvement path $x^k \to x^\omega$, and any $k = 0, 1, \ldots$

A *potential* for $\rhd$ (on X) is an irreflexive and ω-transitive relation $\succ$ finer than $\rhd$, i.e., satisfying $y \rhd x \Rightarrow y \succ x$ for all $y, x \in X$.

The equivalence of all conditions in Theorem 0 is replaced with a chain of implications:

Theorem 2. *For a binary relation $\rhd$ on X, let us consider the following conditions:*

2.1. $\rhd$ *is Ω-acyclic on X;*

2.2. $\rhd$ *admits a potential on X;*

2.3. $\rhd$ *has the CIP property on X;*

2.4. $\rhd$ *has the CIP property on every compact $Y \subseteq X$;*

2.5. $\triangleright$ *has the weak CIP property on every compact* $Y \subseteq X$*;*

2.6. $\triangleright$ *admits a maximizer over every nonempty compact* $Y \subseteq X$*;*

2.7. $\triangleright$ *is weakly* Ω*-acyclic on* X*;*

2.8. $\triangleright$ *is* ω*-acyclic on* X*.*

Then this chain of implications holds:

$$[2.1] \iff [2.2] \iff [2.3] \Rightarrow [2.4] \Rightarrow [2.5] \iff [2.6] \Rightarrow [2.7] \Rightarrow [2.8].$$

Proof. [2.1] $\Rightarrow$ [2.2]: By Lemma 2.4, Ω-acyclicity of $\triangleright$ implies that its ω-transitive closure is irreflexive, hence is a potential for $\triangleright$.

[2.2] $\Rightarrow$ [2.3]: Let $\succ$ be a potential, and π an improvement path, for $\triangleright$. By definition, π is an improvement path for $\succ$, hence, by Lemma 2.3, is a monotonic path for $\succ$; therefore, Proposition 2.1 states that $\mathrm{Dom}(\pi) \subset K$.

[2.3] $\Rightarrow$ [2.1] is, in a sense, straightforward: If there were an improvement cycle, we could move along it forever. However, a rigorous proof needs technical details.

Let us suppose the contrary: there are $\alpha \in K$ and an improvement path π for $\triangleright$ such that $\alpha > 0$ and $\pi(0) = \pi(\alpha)$. By transfinite recursion we prove, for each $\beta \in K$, the existence of $\sigma(\beta) \in K$ such that $\sigma(\beta) < \alpha$ and the superposition $\pi \circ \sigma$ is an improvement path for $\triangleright$. For $\beta < \alpha$, we define $\sigma(\beta) = \beta$; then $\pi \circ \sigma$ coincides with π. If $\sigma(\beta - 1) + 1 < \alpha$, we define $\sigma(\beta) = \sigma(\beta - 1) + 1$, obtaining $\pi(\sigma(\beta)) \triangleright \pi(\sigma(\beta - 1))$. If $\sigma(\beta - 1) + 1 = \alpha$, we define $\sigma(\beta) = 0$; the requirement from the definition of an improvement path is satisfied for $\pi \circ \sigma$ at β because $\pi(\sigma(\beta)) = \pi(0) = \pi(\alpha)$ and $\pi(\alpha) \triangleright \pi(\alpha - 1) = \pi(\sigma(\beta - 1))$. Finally, if β is a limit ordinal, we denote $\gamma = \sup_{\beta' < \beta} \sigma(\beta')$, obtaining $\gamma \leq \alpha$; if $\gamma < \alpha$, we define $\sigma(\beta) = \gamma$; if $\gamma = \alpha$, $\sigma(\beta) = 0$. When $\sigma(\beta)$ is defined for all $\beta \in K$, $\pi \circ \sigma$ becomes an improvement path defined on K, which contradicts CIP.

[2.3] $\Rightarrow$ [2.4] is obvious.

[2.4] $\Rightarrow$ [2.5]: Let $Y \subseteq X$ be compact and $x^0 \in Y$. We denote $\succ$ the ω-transitive closure of $\triangleright$ on Y; it is irreflexive because of Lemma 2.4 and [2.3] $\Rightarrow$ [2.1] applied to Y. If x^0 is a maximizer for $\succ$ over Y, then it is a maximizer for $\triangleright$ as well and we are home. Otherwise, Theorem 1 implies the existence of a maximizer y for $\succ$ over Y such that $y \succ x^0$; by Lemma 2.4, x^0 is connected to y with an improvement path for $\triangleright$.

[2.5] $\Rightarrow$ [2.6] is obvious.

[2.6] $\Rightarrow$ [2.5]: Let $\triangleright$ be a binary relation on X admitting a maximizer over every nonempty compact subset $Y \subseteq X$, and let us fix Y. We denote $\succeq$ the ω-transitive closure of $\triangleright$ on Y, $\succ$ the asymmetric, and $\sim$ symmetric, components of $\succeq$; both $\succ$ and

$\sim$ are obviously ω-transitive too. By Lemma 2.4, $y \succeq x$ implies that x is connected to y with an improvement path for $\rhd$; therefore, by Theorem 1, every $x \in Y$ is connected with an improvement path for $\rhd$ to a maximizer for $\succ$ over Y.

Let x^0 be a maximizer for $\succ$ over Y; if we show that x^0 is connected with an improvement path for $\rhd$ to a maximizer for $\rhd$ over Y, our proof will be completed. We denote $Z = \{y \in Y|\ y \sim x^0\}$, $Z^* = \operatorname{cl} Z$; if $Z^* = \emptyset$, then $Z = \emptyset$ and x^0 is a maximizer for $\rhd$ over Y. If $Z^* \neq \emptyset$, there exists a maximizer x^* for $\rhd$ over Z^*; let us show $x^* \in Z$. Otherwise, there is an infinite sequence $x^k \to x^*$ such that $x^k \in Z$ for all k, hence $x^k \sim x^{k+1}$; therefore, there exist improvement paths from x^0 to x^1 to x^2 etc. Denoting π the "superposition" of the paths, we obtain a path π such that $\pi(0) = x^0$ and $\pi(\beta^k) = x^k$ for $k = 1, 2, \ldots$ (formally, Theorem 7, Section 5, Chapter XIV of Natanson (1974) should be invoked); denoting $\alpha = \sup_k \beta^k$ and $\pi(\alpha) = x^*$, we see that $x^* \succeq x^0$, hence $x^* \sim x^0$ [because x^0 is a maximizer for $\succ$], hence $x^* \in Z$. Suppose $y \rhd x^*$ for $y \in Y$; then we may define $\pi(\alpha + 1) = y$, so $y \succeq x^0$, hence $y \sim x^0$, hence $y \in Z$, contradicting the choice of x^*.

[2.6] $\Rightarrow$ [2.7]: Let π be a narrow improvement path such that $\pi(\alpha^*) = \pi(0)$ for some $\alpha^* > 0$. We denote $Y = \{\pi(\alpha)\}_{\alpha \in [0,\alpha^*[}$; obviously, there is no maximizer for $\rhd$ over Y. Let us show that Y is compact. Let $y^k \in Y$ for $k = 0, 1, \ldots$; by the definition of Y, $y^k = \pi(\alpha^k)$, $\alpha \in [0, \alpha^*[$. Denote $\beta^k = \min_{h \geq k} \alpha^h$ (it exists because $[0, \alpha^*[$ is well ordered); obviously, $\beta^{k+1} \geq \beta^k$ for all k. Without restricting generality, either $\beta^{k+1} = \beta^k$ for all k, or $\beta^{k+1} > \beta^k$ for all k. In the first case, $\{\pi(\beta^k)\}_k$ is obviously convergent; in the second, denoting $\beta^\omega = \sup_k \beta^k$, we have $\beta^\omega \leq \alpha^*$ and $y^k \to \pi(\beta^\omega) \in Y$ because π is narrow.

[2.7] $\Rightarrow$ [2.8] is obvious. □

Remark. Mukherji (1977, Corollary 3) proved the implication [2.6] $\Rightarrow$ [2.8].

Example 2.1. The set X is a circle parameterized with a real number modulo 2π. Denoting $\oplus$ addition modulo 2π, we define a binary relation as follows:

$$y \rhd x \iff y = x \oplus \psi,$$

where ψ is incommensurable with 2π.

Example 2.2. The set X is the same as in the previous example; the relation $\rhd$ is almost the same, but exactly one pair, $(\psi, 0)$, has been deleted from the graph of the relation.

Example 2.3. The set X is almost the same as in the previous examples, but one point, ψ, has been deleted (thus X is topologically an open interval). The relation $\rhd$ on the remaining points is the same as in Example 2.2.

In each of the examples, the relation is weakly Ω-acyclic (because no infinite improvement path is narrow), but not Ω-acyclic (because the initial point of every infinite improvement path is among its limit points). In Example 2.1, the set X is compact, but there is no maximizer over X. In Example 2.2, there is the weak CIP property on every compact subset $Y \subseteq X$: if there is an infinite improvement path in Y, then $0 \in Y$ is a limit point; otherwise, every improvement path in Y reaches a maximizer at a finite step. In Example 2.3, there is even the FIP property on every compact subset $Y \subseteq X$.

Example 2.4. The set X is the same as in Example 2.1, but parameterized with $x \in [-\pi, \pi[$ ("$-\pi = \pi$"). The relation $\triangleright$ is defined by a mapping $f : X \to X$, $y \triangleright x \iff y = f(x)$, where

$$f(x) = (x + \pi)/2, \text{ if } 0 \leq x < \pi;$$
$$f(x) = x/2, \text{ if } -\pi \leq x < 0$$

(f is discontinuous at 0 and $\pm\pi$). Any simple improvement path with $x^0 \geq 0$ converges to $x^\omega = \pi[= -\pi]$; any simple improvement path with $x^0 < 0$ converges to $x^\omega = 0$. Thus the relation is ω-acyclic, but not weakly Ω-acyclic because $x^0 = 0$ implies $x^{\omega+\omega} = x^0$ (and the path is narrow).

Thus, none of the one-sided implications in Theorem 2 can be reversed.

Comparing the formulations of Theorems 0 and 2, we see that in the new situation the acyclicity condition "splits" into several properties, each of which deserving some attention. Logically speaking, a condition (formulated without mentioning maximizers) might exist, intermediate between Ω-acyclicity and weak Ω-acyclicity and equivalent to the existence of maximizers on compact subsets. It seems instructive, however, to try to imagine a general sufficient condition for the existence, valid for both Example 2.3 and the standard order on an open interval.

3 Aggregation

3.1 Orderings

A *preorder* is a reflexive and transitive binary relation; an *ordering* is a complete preorder (the term "weak order" is used as often, but it is grammatically more convenient to use one word rather than two). With every preorder $\succeq$, asymmetric relations $\succ$ and $\prec$, as well as an equivalence relation $\sim$, are naturally associated.

A *maximum* of $Y \subseteq X$ w.r.t. an ordering $\succeq$ is $x \in Y$ such that $x \succeq y$ for all $y \in Y$. A *maximizing sequence* on $Y \subseteq X$ w.r.t. an ordering $\succeq$ is an infinite sequence $\{x^k\}_{k=0,1,\ldots}$ such that $x^k \in Y$ and $x^{k+1} \succ x^k$ for all k, and for every $y \in Y$ there is k for which $x^k \succ y$. If $Z \subseteq Y \subseteq X$, a *least upper bound*, or a *supremum*, of Z in Y w.r.t. an ordering $\succeq$ is $x \in Y$ for which $x \succeq z$ for all $z \in Z$, and $y \succeq x$ whenever $y \in Y$

and $y \succeq z$ for all $z \in Z$. Dually are defined a *minimum*, a *minimizing sequence*, and a *greatest lower bound* or an *infimum*.

Proposition 3.1. *If $\succeq$ is an ordering on X, then $\succ$ is ω-transitive if and only if every nonempty compact $Y \subseteq X$ contains a maximum w.r.t. $\succeq$.*

Proof. If $\succ$ is ω-transitive, then, by Theorem 1, there exists a maximizer for $\succ$ over Y, which is obviously a maximum of Y w.r.t. $\succeq$. If $\succ$ is not ω-transitive, i.e., $x^k \to x^\omega$, $x^{k+1} \succ x^k$ for all k, but $x^0 \succeq x^\omega$, then the sequence $y^0 = x^\omega$, $y^k = x^k$ for $k = 1, 2, \ldots$ violates the ω-acyclicity condition. $\square$

Remark. The statement is due to Smith (1974, Theorem 4.1).

Proposition 3.2. *If $\succeq$ is an ordering with an ω-transitive component $\succ$, then every nonempty $Y \subseteq X$ contains either a maximum or a maximizing sequence w.r.t. $\succeq$.*

Proof. Let Π be the set of monotonic paths for $\succ$ in Y. We define a continuation rule $\vartheta : \Pi \to \Pi$ as follows: If there exist $y^* \in Y$ such that $y^* \succ \nu(\beta)$ for all $\beta \in \mathrm{Dom}(\nu) = [0, \mu[$, then $\mathrm{Dom}(\vartheta(\nu)) = [0, \mu + 1[$ and $\vartheta(\nu)(\mu)$ is one such y^*; otherwise, $\vartheta(\nu) = \nu$. Since Π is nonempty and closed w.r.t. extension, Proposition 2.2 implies the existence of a "fixed point," $\vartheta(\nu^*) = \nu^*$; let $\mathrm{Dom}(\nu^*) = [0, \mu^*[$. If $\mu^* - 1$ is defined, then $\nu^*(\mu^* - 1)$ is a maximum of Y; otherwise, there exists an increasing sequence $\{\beta^k\}_k$ in K such that $\mu^* = \sup_k \beta^k$, in which case $\{\nu(\beta^k)\}_k$ is a maximizing sequence on Y. $\square$

If $\succeq$ is represented by a real function $\varphi : X \to I\!R$ ($y \succeq x \iff \varphi(y) \geq \varphi(x)$), the statement of Proposition 3.2 holds without any restriction on φ. Generally, the ω-transitivity condition cannot be dropped.

Example 3.1. Let $X = [0, 1]$. By the Axiom of Choice, X can be well ordered; let $\succ$ be such an order. We denote Y the set of $x \in X$ such that $\{y \in X | \ x \succ y\}$ is countable. Y cannot be countable itself: otherwise, the minimal $x \in X$ such that $x \succ y$ for all $y \in Y$ would belong to Y. On the other hand, if Y contained a maximum or a maximizing sequence, it would be countable.

An ordering $\succeq$ on X is *continuous* if every lower contour $\{x \in X | \ y \succ x\}$ ($y \in X$) and every upper contour $\{y \in X | \ y \succ x\}$ ($x \in X$) are open. We call an ordering $\succeq$ on X *pseudocontinuous* if both $\succ$ and $\prec$ are ω-transitive. We call an ordering $\succeq$ on X *quasicontinuous* if it is pseudocontinuous and there are no two infinite sequences $\{x^k\}_k$ and $\{y^h\}_h$ such that $x^k \to x^\omega$, $y^h \to y^\omega$, and

$$\forall k \ \forall h \ \left[x^\omega \succ y^h \succ y^{h+1} \succ x^{k+1} \succ x^k \succ y^\omega\right] \tag{3.1}$$

Theorem 3. *An ordering $\succeq$ on X is quasicontinuous if and only if for every nonempty compact $Y \subseteq X$ and every $Z \subseteq Y$ there exists a supremum of Z in Y w.r.t. $\succeq$.*

Proof. Let $\succeq$ be quasicontinuous, $Z \subseteq Y \subseteq X$, and Y be nonempty and compact; we denote $Z^+ = \{y \in Y | \ \forall z \in Z[y \succ z]\}$. If $Z = \emptyset$, then $Z^+ = Y$ is compact, hence contains a maximizer for $\prec$, which is obviously a supremum of Z. Let $Z \neq \emptyset$; by Proposition 3.2, it contains either a maximum or a maximizing sequence w.r.t. $\succ$. In the first case, the maximum is obviously a supremum. In the second case, since Y is compact, the sequence may be assumed convergent, $x^k \to x^\omega \in Y$; by the ω-transitivity, $x^\omega \succ x^k$ for all k, hence $x^\omega \in Z^+ \neq \emptyset$. By the dual to Proposition 3.2, Z^+ contains either a minimum, which is again the supremum needed, or a minimizing sequence, $\{y^h\}_h$, which may be assumed convergent, $y^h \to y^\omega \in Y$ with $y^\omega \prec y^h$ for all h. Now $y^\omega \prec x^m$ for some m would imply that the sequences $\{x^k\}_{k=m,m+1,\dots}$ and $\{y^h\}_h$ satisfy the prohibited condition (3.1). Therefore, $y^\omega \in Z^+$; but then we must have $y^\omega \succ y^h$ for some h, which is impossible.

Now let $\succeq$ not be quasicontinuous. If it is not even pseudocontinuous, e.g., if $x^k \to x^\omega$ and $x^{k+1} \succ x^k$ for all k, but $x^0 \succeq x^\omega$, we define $Y = \{x^k\}_{k=0,1,\dots} \cup \{x^\omega\}$. Y is obviously compact but has no supremum in itself. If $\succeq$ is pseudocontinuous, but (3.1) holds, we define $Y = \{x^k\}_{k=0,1,\dots} \cup \{x^\omega\} \cup \{y^h\}_{h=0,1,\dots} \cup \{y^\omega\}$ and $Z = \{x^k\}_k \subset Y$. It is easy to see that upper bounds for Z in Y form the set $\{x^\omega\} \cup \{y^h\}_h$, which contains no minimum. $\square$

In the light of the well known characterization of chains compact in their intrinsic topology (Birkhoff, 1967), we immediately obtain the following

Corollary. *A complete order on a Hausdorff topological space X with a countable base of open sets is quasicontinuous if and only if every compact subspace $Y \subseteq X$ is compact in its intrinsic (as a chain) topology.*

Besides continuous orderings, obviously quasicontinuous are those with a finite number of equivalence classes. A rather general class of quasicontinuous orderings is described in the following subsection.

3.2 Lexicography

In this subsection, a lexicographic combination of preorders is defined; in a sense, the concept is less general than that of Fishburn (1974): we always obtain a preorder. On the other hand, our concept does not presuppose constituent relations to be defined on the whole X.

An *arboreous poset* is a partially ordered set A such that (1) there exists $\alpha^{\min} \in \mathrm{A}$ such that $\alpha \geq \alpha^{\min}$ for all $\alpha \in \mathrm{A}$, (2) every set $\overleftarrow{\alpha} = \{\beta \in \mathrm{A} | \ \beta \leq \alpha\}$ (for $\alpha \in \mathrm{A}$) is well ordered, and (3) every chain $\Delta \subseteq \mathrm{A}$ has a least upper bound in A.

Remark. In a sense, a finite arboreous poset is the same thing as a finite tree.

We denote B the set of non-maximal elements of A, and $N(\beta)$ (for every $\beta \in \mathrm{B}$), the set of $\alpha \in \mathrm{A}$ immediately following β (i.e., $\alpha > \beta$, and $\alpha > \gamma > \beta$ is impossible).

Lemma 3.3. *For every $\alpha, \beta \in \mathrm{A}$, there exists their greatest lower bound. If $\alpha > \beta$, then there exists $\alpha' \in N(\beta)$ such that $\alpha \geq \alpha'$.*

Proof. We denote $\Delta = \{\gamma \in \mathrm{A} | \; \gamma \leq \alpha \;\&\; \gamma \leq \beta\}$; $\Delta \subseteq \overleftarrow{\alpha}$ hence is a chain. By the condition (3) of the above definition, there exists $\gamma^* = \sup \Delta$. The definition of Δ implies $\gamma^* \in \Delta$, hence γ^* is the greatest lower bound needed. If $\alpha > \beta$, we denote $\Delta' = \{\gamma \in \overleftarrow{\alpha} | \; \gamma > \beta\}$. By the condition (2) above, Δ' has the least element α', which obviously belongs to $N(\beta)$. □

For every $\alpha, \beta \in \mathrm{A}$, their greatest lower bound will be denoted $\alpha \wedge \beta \in \mathrm{A}$. For every $\alpha \in \mathrm{A}$, we denote $\mu(\alpha)$ the type of $\overleftarrow{\alpha}$. $\alpha \in \mathrm{A}$ is called an *isolated vertex* if $\mu(\alpha)$ is an isolated ordinal; $\alpha \in \mathrm{A}$ is called a *limit vertex* if $\mu(\alpha)$ is a limit ordinal. For every isolated $\alpha \in \mathrm{A}$, $\alpha - 1 \in \mathrm{A}$ is defined in an obvious way. A is *countably arboreous* if $\mu(\alpha) \in K$ for every $\alpha \in \mathrm{A}$; A itself need not be countable in this case.

An *arboreous partitioning* of a set X with an arboreous poset of indices A is a family of nonempty subsets $C(\alpha) \subseteq X$ ($\alpha \in \mathrm{A}$) such that (1) $C(\alpha^{\min}) = X$, (2) $\{C(\alpha)\}_{\alpha \in N(\beta)}$ is a partitioning of $C(\beta)$ for each $\beta \in \mathrm{B}$, and (3) for every limit vertex α, $C(\alpha) = \bigcap_{\alpha' < \alpha} C(\alpha')$.

Clearly, $C(\alpha) \subseteq C(\beta)$ whenever $\alpha \geq \beta$, and $C(\alpha) \cap C(\beta) = \emptyset$ whenever α and β are incomparable. If $\# N(\alpha) = 1$ for some $\alpha \in \mathrm{B}$, this α can be deleted from A without changing the partitioning; for technical simplicity, we assume that $\# N(\alpha) > 1$ for all $\alpha \in \mathrm{B}$.

Lemma 3.4. *For every $x \in X$, there exists $\alpha \in \mathrm{A} \setminus \mathrm{B}$ such that $x \in C(\alpha)$.*

Proof. Let $x \in X$; we denote $\Delta = \{\alpha \in \mathrm{A} | \; x \in C(\alpha)\}$. Since Δ is a chain, there exists its least upper bound α^* in A. Let us show $\alpha^* \in \Delta$: otherwise, α^* must be a limit vertex, but then $C(\alpha^*) = \bigcap_{\alpha \in \Delta} C(\alpha) \ni x$. Finally, if $\alpha^* \in \mathrm{B}$, then x must belong to $C(\beta)$ for some $\beta \in N(\alpha^*)$ — a contradiction. □

Thus, the family $\{C(\alpha)\}_{\alpha \in \mathrm{A} \setminus \mathrm{B}}$ is a partitioning of X; for every $x \in X$, $\tau(x) \in \mathrm{A} \setminus \mathrm{B}$ is uniquely defined by $x \in C(\tau(x))$.

Finally, we define an *arboreous lexicographic construction.* Let there be a countably arboreous partitioning of a set X with a set of indices A, and let a preorder $\succeq_\beta$ on $C(\beta)$ be given for each $\beta \in \mathrm{B}$, equivalence classes of which are exactly the sets $C(\alpha)$ for $\alpha \in N(\beta)$. We set

$$y \sim x \iff \forall \alpha \in \mathrm{A}\, [y \in C(\alpha) \iff x \in C(\alpha)],$$

$$y \succ x \iff \exists \alpha \in \mathrm{A}\, [y \succ_\alpha x],$$

and $y \succeq x \iff [y \succ x \text{ or } y \sim x]$.

An alternative definition of $\succeq$ is sometimes more convenient.

Lemma 3.5. *For every $x, y \in X$, $y \sim x$ if and only if $\tau(y) = \tau(x)$, whereas $y \succ x$ if and only if $y \succ_{\tau(y)\wedge\tau(x)} x$.*

Proof. If $y \succ_\alpha x$, then $y, x \in C(\alpha)$ and there exist β'', $\beta' \in N(\alpha)$ such that $\beta'' \neq \beta'$, $y \in C(\beta'')$ and $x \in C(\beta')$; it follows immediately that $\alpha = \tau(y) \wedge \tau(x)$. If $\tau(y) = \tau(x)$ and $x \in C(\alpha)$, then $\alpha \leq \tau(x) = \tau(y)$ hence $y \in C(\alpha)$ too. □

Proposition 3.6. *If $\succeq$ is defined by an arboreous lexicographic construction, then it is a preorder.*

Proof. Only transitivity deserves attention. Let $z \succ y \succ x$; we denote $\alpha_1 = \tau(z)\wedge\tau(y)$ and $\alpha_2 = \tau(y)\wedge\tau(x)$. α_1 and α_2 both belong to the same chain $\overleftarrow{\tau(y)}$, hence $\alpha_1\wedge\alpha_2 = \alpha_1$ or $\alpha_1 \wedge \alpha_2 = \alpha_2$. In the first case, we have $z \succ_{\alpha_1} y \succeq_{\alpha_1} x$; in the second case, $z \succeq_{\alpha_1} y \succ_{\alpha_1} x$; in either case, $z \succ_{\alpha_1} x$, hence $z \succ x$. If an equivalence is present, the corresponding α_i is maximal, making one of the above cases obligatory. □

Proposition 3.7. *If $\succeq$ is defined by an arboreous lexicographic construction and every $\succeq_\alpha$ ($\alpha \in \mathrm{B}$) is ω-transitive, then $\succeq$ is ω-transitive too.*

Proof. First of all, it is easy to see that both $\succ_\alpha$ and $\sim_\alpha$ ($\alpha \in \mathrm{B}$) are ω-transitive; therefore, each $C(\alpha)$ ($\alpha \in \mathrm{A}$) is closed. Let $x^k \to x^\omega$ and $x^{k+1} \succeq x^k$ for all $k = 0, 1, \ldots$ Without restricting generality, either $x^{k+1} \sim x^k$ for all $k = 0, 1, \ldots$ or $x^{k+1} \succ x^k$ for all $k = 0, 1, \ldots$ In the first case, $x^\omega \sim x^0$ is obvious; in the second one, we denote $\alpha_k = \tau(x^{k+1}) \wedge \tau(x^k)$, $(k = 0, 1, \ldots)$ and $\alpha_\omega = \sup_k \alpha_k$. Without restricting generality, either $\alpha_{k+1} = \alpha_k$ for all $k = 0, 1, \ldots$ (hence $\alpha_\omega = \alpha_0 \in \mathrm{B}$) or $\alpha_{k+1} > \alpha_k$ for all $k = 0, 1, \ldots$ (hence α_ω is a limit vertex). In the first case, we have $x^\omega \succ_{\alpha_0} x^0$ because $\succ_{\alpha_0}$ is ω-transitive; in the second case, we have $x^\omega \succ_{\alpha_0} x^0$ because all x^k for $k = 1, 2, \ldots$ belong to the same equivalence class of $\succeq_{\alpha_0}$, which is closed. Thus, $x^\omega \succ x^0$. □

Proposition 3.8. *If $\succeq$ is defined by an arboreous lexicographic construction and every $\succeq_\alpha$ ($\alpha \in \mathrm{B}$) is an ordering, then $\succeq$ is an ordering too.*

The proof is straightforward.

Proposition 3.9. *If $\succeq$ is defined by an arboreous lexicographic construction, every $\succeq_\alpha$ is a pseudocontinuous ordering and every $C(\alpha)$, $\alpha \in \mathrm{B}$, is closed, then $\succeq$ is a pseudocontinuous ordering too.*

Proof. Obvious modifications in the proof of Proposition 3.7 are sufficient. □

Theorem 4. *If $\succeq$ is defined by an arboreous lexicographic construction, every $\succeq_\alpha$ is a quasicontinuous ordering and each $C(\alpha)$, $\alpha \in \mathrm{B}$, is closed, then $\succeq$ is a quasicontinuous ordering too.*

Proof. By Proposition 3.9, we only have to prove the impossibility of Equality (3.1). Supposing the contrary, we, exactly as in the proof of Proposition 3.7, denote $\alpha_k = \tau(x^{k+1}) \wedge \tau(x^k)$ $(k = 0, 1, \dots)$ and $\alpha_\omega = \sup_k \alpha_k$. Without restricting generality, either $\alpha_{k+1} = \alpha_k$ for all $k = 0, 1, \dots$ (hence $\alpha_\omega = \alpha_0 \in \mathrm{B}$) or $\alpha_{k+1} > \alpha_k$ for all $k = 0, 1, \dots$

In the first case, we have $x^k \in C(\alpha_0)$ for all k, hence $x^\omega \in C(\alpha_0)$ because $C(\alpha_0)$ is closed, hence $y^h \in C(\alpha_0)$ for all h, hence $y^\omega \in C(\alpha_0)$. Without restricting generality, either $y^{h+1} \prec_{\alpha_0} y^h$ for all h, or $y^{h+1} \sim_{\alpha_0} y^h$ for all h. The first relation implies (3.1) for $\succ_{\alpha_0}$, contradicting our assumption; the second would imply $y^\omega \sim_{\alpha_0} y^h$ for all h, contradicting $y^h \succ x^k \succ y^\omega$.

In the second case, we have $C(\alpha_\omega) = \bigcap_k C(\alpha_k) \ni x^\omega$. If $y^h \notin C(\alpha_\omega)$ for some h, then $\tau(y^h) \wedge \tau(x^\omega) < \alpha_\omega$, hence $\tau(y^h) \wedge \tau(x^\omega) \le \alpha_k$ for some k, hence $y^h \in C(\alpha_k)$, hence $y^h \prec x^{k+1}$; therefore, $y^h \in C(\alpha_\omega)$ for all h. Now $x^\omega \succ y^h$ implies $\alpha_\omega \in B$, hence $C(\alpha_\omega)$ is closed, hence $y^\omega \in C(\alpha_\omega)$, hence $y^\omega \succ x^k$ for all k, contradicting the supposed (3.1). (Interestingly, there was no need for the quasicontinuity of any $\succ_\alpha$ in this case). $\square$

Corollary. *If $\succeq$ is defined by an arboreous lexicographic construction and every $\succeq_\alpha$ ($\alpha \in \mathrm{B}$) is a continuous ordering, then $\succeq$ is a quasicontinuous ordering.*

Proposition 3.10. *If every $C(\alpha)$, $\alpha \in \mathrm{A}$, is closed, then* A *is countably arboreous.*

Proof. Otherwise, there must be $\alpha \in \mathrm{A}$ for which the type of $\overleftarrow{\alpha}$ is K (hence α is a limit vertex). By the condition (3) of the definition of an arboreous partitioning, $C(\alpha) = \bigcap_{\alpha' < \alpha} C(\alpha')$. Invoking the Lindelöf theorem (Kuratowski, 1966, p. 54), we obtain the existence of a countable set $\Delta \subset \overleftarrow{\alpha}$ such that $C(\alpha) = \bigcap_{\alpha' \in \Delta} C(\alpha')$. Denoting $\alpha^* = \sup \Delta$ (existing in $\overleftarrow{\alpha} = K$ by the Statement 2 of Theorem 2, Section 5, Chapter XIV of Natanson, 1974), we obtain $C(\alpha^*) = C(\alpha)$, hence $C(\alpha^* + 1) = C(\alpha^*)$, contradicting our assumption $\# N(\alpha^*) > 1$. $\square$

Example 3.2. A (very) partial case of the above construction is lexicography defined (in a commonly accepted way) by a finite list of continuous functions $\varphi_i : X \to I\!R$ $(i = 1, \dots n)$; A then consists of corteges $\langle v_1, \dots v_k \rangle$ of feasible values of the functions $\varphi_1, \dots \varphi_k$ $(k \le n)$, including the empty cortege, $\alpha^{\min}$, and ordered by the inclusion. The partitioning is defined by $C(\langle v_1, \dots v_k \rangle) = \{x \in X | \ \forall h \le k \, [\varphi_h(x) = v_h]\}$. The Corollary to Theorem 4 implies that the lexicographic ordering is quasicontinuous. We also see that an arboreous lexicographic construction need not preserve continuity.

Example 3.3. Let φ_1 and φ_2 be continuous functions $X \to I\!R$; we set $y \succeq x \iff \big[\varphi_1(y) > \varphi_1(x)$ or $\big([\varphi_1(y) = \varphi_1(x)$ and is rational$]$ & $\varphi_2(y) \ge \varphi_2(x)\big)$ or $\big([\varphi_1(y) =$

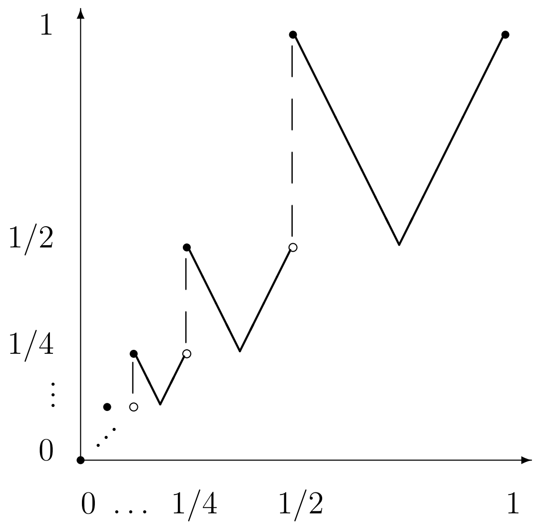

Figure 1: Function representing a quasicontinuous, but not arboreously lexicographic, ordering

$\varphi_1(x)$ and is irrational] & $\varphi_2(y) \leq \varphi_2(x))\big]$. The arboreous poset A is the same as in the previous example (for $n = 2$), but the orderings $\succeq_{v_1}$ with irrational v_1 differ. Theorem 4 implies that this exotic ordering is quasicontinuous as well.

Example 3.4. Let $X = [0, 1]$ and the ordering $\succeq$ be represented by the following numeric function ($y \succeq x \iff \varphi(y) \geq \varphi(x)$):

$\varphi(1) = 1$;

$\varphi(x) = 2x - 1/2^{k-1}$ for $3/2^{k+1} \leq x < 1/2^{k-1}$ ($k = 1, 2, \dots$);

$\varphi(x) = -2x + 1/2^{k-2}$ for $1/2^k \leq x \leq 3/2^{k+1}$ ($k = 1, 2, \dots$);

$\varphi(0) = 0$.

It is easy to check that $\varphi(x)$ is continuous everywhere except for points $x = 1/2^k$, where it is upper semicontinuous (see Fig. 1). (3.1) is impossible for trivial reason: $\varphi(y^{h+1}) < \varphi(y^h)$ for all h and $y^h \to y^\omega$ imply $\varphi(y^\omega) = \inf \varphi(y^h)$.

If the ordering could be generated by an arboreous lexicographic construction with continuous orderings, we would have a nontrivial continuous ordering on X, $\succeq_{\alpha^{\min}}$, such that

$$y \succ_{\alpha^{\min}} x \Rightarrow y \succ x \tag{3.2}$$

for all $y, x \in X$. Suppose for the moment that such an ordering exists. Let us consider the sequence $x^k = 1/2 - 1/2^{k+3}$ ($k = 0, 1, \dots$); we have $\varphi(x^{k+1}) > \varphi(x^k)$, hence $x^{k+1} \succ x^k$ for all k; obviously, $x^k \to x^\omega = 1/2$. Now we have $x^k \prec x \preceq x^\omega$ for all k and all $x \in [1/2, 1]$; the continuity of $\succeq_{\alpha^{\min}}$ and (3.2) imply that $y \sim_{\alpha^{\min}} x$ for all $y, x \in [1/2, 1]$. Considering the sequence $x^k = 1/4 - 1/2^{k+4}$ ($k = 0, 1, \dots$), which is also increasing and converges to $1/4$, we, quite similarly, obtain $x^k \prec x \preceq x^\omega$ for all k and all $x \in [1/4, 1/2]$, hence $y \sim_{\alpha^{\min}} x$ for all $y, x \in [1/4, 1/2]$. Continuing the process, we obtain $y \sim_{\alpha^{\min}} x$ for all $y, x \in]0, 1]$, hence for all $y, x \in [0, 1]$. But this contradicts the condition $\#N(\alpha^{\min}) > 1$.

Rather informally, the ordering can be interpreted as generated by a lexicographic construction (not arboreous!) with continuous orderings on components. Let us define $C(-k) = \{1/2^{k+1}\} \cup [1/2^k, 1]$ $(k = 0, 1, \dots)$, $C(-\infty) = [0, 1]$, $\varphi_0(x) = 1$ for $x \in C(0)\,[= \{1/2, 1\}]$, $\varphi_k(x) = 1/2^{k-1}$ for $x \in C(-k + 1)$, and $\varphi_k(x) = \varphi(x)$ for $x \in C(-k) \setminus C(-k+1)$, $(k = 1, 2 \dots)$. For each $x \in X$, $k(x) = \min\{k |\ x \in C(-k)\}$ is uniquely defined (assuming $k(0) = \infty$); it is easy to check that each φ_k is continuous on $C(-k)$ and $y \succeq x \iff \big(k(y) > k(x)$ or $[k(y) = k(x)$ & $\varphi_{k(x)}(y) \geq \varphi_{k(x)}(x)]\big)$. It remains an open problem whether quasicontinuous orderings with closed equivalence classes could be characterized as generated by lexicographic constructions (suitably interpreted) with continuous orderings on components.

4 Monotonic Endomorphisms

As in Kukushkin (2000), we associate with every mapping $F : X \to 2^X \setminus \{\emptyset\}$ (a correspondence $X \to X$) a binary relation $\rhd^F$: $y \rhd^F x \iff x \notin F(x) \ni y$; for the particular case of a mapping $f : X \to X$, we have $y \rhd^f x \iff y = f(x) \neq x$. Maximizers for $\rhd^F$ ($\rhd^f$) are exactly the fixed points of F (or f), while improvement paths for $\rhd^F$ ($\rhd^f$) combine iterating F (or f) and picking limit points. We call a mapping F (or f) acyclic, ω-acyclic, or Ω-acyclic, if so is $\rhd^F$ ($\rhd^f$).

Let $\succeq$ be a preorder on a set X; we define a preorder $\succeq^*$ on $2^X \setminus \{\emptyset\}$:

$$Y \succeq^* Z \iff [\forall y \in Y \setminus Z\ \forall x \in Y \cap Z\ \forall z \in Z \setminus Y \quad (y \succ x \succ z)].$$

It is easy to see that $Y \sim^* Z$ iff $Y = Z$ (i.e., $\succeq^*$ is a partial order) and that $\{y\} \succ^* \{z\}$ iff $y \succ z$.

Remark. $\succeq^*$ is a natural adaptation of Veinott's order on sublattices (Topkis, 1978) to preorders.

A mapping $F : X \to 2^X \setminus \{\emptyset\}$ is *increasing* w.r.t. $\succeq$ if $y \succeq x \Rightarrow F(y) \succeq^* F(x)$, and *decreasing* (w.r.t. $\succeq$) if $y \succeq x \Rightarrow F(x) \succeq^* F(y)$. For the important particular case of a single-valued mapping we easily obtain that $f : X \to X$ is increasing w.r.t. $\succeq$ iff $y \succeq x \Rightarrow [f(y) \succ f(x)$ or $f(y) = f(x)]$; in particular, $y \sim x \Rightarrow f(y) = f(x)$. A set X with a preorder $\succeq$ has the fixed point property if every increasing mapping $f : X \to X$ has a fixed point.

Theorem 5. *Let $\succeq$ be a preorder on a set X; then the following conditions are equivalent:*

5.1. $\succeq$ *is an ordering;*

5.2. *every increasing mapping* $F : X \to 2^X \setminus \{\emptyset\}$ *is acyclic;*

5.3. *every increasing mapping* $f : X \to X$ *is acyclic;*

5.4. *every finite* $Y \subseteq X$ *has the fixed point property.*

Proof. [5.1] $\Rightarrow$ [5.2]: Let $\{x^k\}_{k=0,1,\dots}$ be an improvement path for $\rhd_F$; without restricting generality, $x^1 \succ x^0$. Let us show, by induction, that $x^{k+1} \succ x^k$ whenever x^{k+1} is defined. Indeed, we have $x^k \succ x^{k-1}$, $x^k \in F(x^{k-1}) \setminus F(x^k)$, and $x^{k+1} \in F(x^k)$; since F is increasing, $x^{k+1} \succ x^k$. Now the impossibility of $x^m = x^0$ with $m > 0$ is obvious.

[5.2] $\Rightarrow$ [5.3] is obvious.

[5.3] $\Rightarrow$ [5.1]. Suppose the contrary: there exist incomparable $a, b \in X$. If there is $c \in X$ such that $c \succ a$ and $c \succ b$ (while a and b are incomparable), we define $f : X \to X$ as follows: $f(x) = x$ if $a \succeq x$ and $b \succeq x$; $f(x) = b$ if $a \succeq x$, but not $b \succeq x$; $f(x) = a$ if $b \succeq x$, but not $a \succeq x$; $f(x) = c$ otherwise. It is easily checked that f is increasing, $f(a) = b$ and $f(b) = a$. If there are $a, b, c \in X$ such that $a \succ c$, $b \succ c$, and a and b are incomparable, we define $f : X \to X$ in a dual way. Finally, if whenever a and b are incomparable, there is no c comparable with both a and b, we define $f(x) = b$ if x is comparable with a, and $f(x) = a$ otherwise.

[5.1] $\Rightarrow$ [5.4]: An ordering on X is an ordering on every $Y \subseteq X$ and an acyclic mapping of a finite set into itself obviously has a fixed point.

[5.4] $\Rightarrow$ [5.1]: If $a, b \in X$ are incomparable, we define $f : \{a, b\} \to \{a, b\}$ by $f(a) = b$ and $f(b) = a$. □

Proposition 4.1. *Let* $\succeq$ *be a preorder on a space* X*; then the following conditions are equivalent:*

4.1.1. $\succeq$ *is a pseudocontinuous ordering;*

4.1.2. *every increasing mapping* $F : X \to 2^X \setminus \{\emptyset\}$ *is* ω*-acyclic;*

4.1.3. *every increasing mapping* $f : X \to X$ *is* ω*-acyclic;*

Proof. [4.1.1] $\Rightarrow$ [4.1.2]: Let $\{x^k\}_{k=0,1,\dots}$ be an improvement path for $\rhd^F$ and $x^k \to x^\omega$; without restricting generality, $x^1 \succ x^0$. As was shown in the previous theorem, $x^{k+1} \succ x^k$ for all k, hence, by pseudocontinuity, $x^\omega \succ x^0$, hence $x^\omega = x^0$ is impossible.

[4.1.2] $\Rightarrow$ [4.1.3] is obvious.

[4.1.3] $\Rightarrow$ [4.1.1]. By the previous theorem, $\succeq$ is an ordering; we only have to prove the ω-transitivity. Let a convergent sequence $x^k \to x^\omega$ violate the condition. Without restricting generality, we may assume that $x^{k+1} \succ x^k$ for all k, but $x^\omega \prec x^0$. Denoting

$X^+ = \{x \in X |\ \forall k\, [x \succeq x^k]\}$, we define, for every $x \in X \setminus X^+$, $\varkappa(x) = \min\{k |\ x \prec x^k\}$; then $y \succeq x$ implies $\varkappa(y) \geq \varkappa(x)$. Now we define $f(x) = x$ for $x \in X^+$ and $f(x) = x^{\varkappa(x)}$ otherwise; obviously, $f : X \to X$ is increasing, $f(x^k) = x^{k+1}$, and $f(x^\omega) = x^0$. Therefore, $x^\omega = \lim_{k\to\infty} f^k(x^\omega)$. $\square$

Theorem 6. *Let $\succeq$ be a preorder on a set X; then the following conditions are equivalent:*

6.1. *$\succeq$ is a quasicontinuous ordering;*

6.2. *every increasing mapping $F : X \to 2^X \setminus \{\emptyset\}$ is Ω-acyclic;*

6.3. *every increasing mapping $F : X \to 2^X \setminus \{\emptyset\}$ is weakly Ω-acyclic;*

6.4. *every increasing mapping $f : X \to X$ is Ω-acyclic;*

6.5. *every increasing mapping $f : X \to X$ is weakly Ω-acyclic;*

6.6. *every compact $Y \subseteq X$ has the fixed point property.*

Proof. [6.1] $\Rightarrow$ [6.2] is proven in Kukushkin (2000; Theorem 4.2).

[6.2] $\Rightarrow$ [6.3], [6.2] $\Rightarrow$ [6.4], [6.3] $\Rightarrow$ [6.5], and [6.4] $\Rightarrow$ [6.5] are obvious.

[6.5] $\Rightarrow$ [6.1]: By Proposition 4.1, $\succeq$ is a pseudocontinuous ordering. Let two sequences $x^k \to x^\omega$ and $y^h \to y^\omega$ satisfy (3.1); without restricting generality, we may assume that $x^0 = y^\omega$ and $y^0 = x^\omega$. We define: $X^+ = \{x \in X |\ \forall k\, [x \succeq x^k]\}$, $X^- = \{x \in X |\ \forall h\, [x \preceq y^h]\}$, $Y^- = X \setminus X^+$, $Y^+ = X \setminus X^-$, and $Y^0 = X^+ \cap X^-$; it is easy to see that $\langle Y^-, Y^0, Y^+\rangle$ is a partitioning of X consistent with $\succeq$. For $x \in Y^-$, we define $\varkappa(x) = \min\{k |\ x \prec x^k\}$; for $x \in Y^+$, $\eta(x) = \min\{h |\ x \succ y^h\}$. Finally, we define $f(x) = x^{\varkappa(x)}$ if $x \in Y^-$, $f(x) = x$ if $x \in Y^0$, and $f(x) = y^{\eta(x)}$ if $x \in Y^+$. It is easy to check that $f : X \to X$ is increasing, $x^k \in Y^-$ and $f(x^k) = x^{k+1}$ for all k, and $y^h \in Y^+$ and $f(y^h) = y^{h+1}$ for all h. Now a narrow cycle for $\rhd^f$ is evident: $\pi(0) = y^\omega = x^0$, $\pi(k) = x^k$ $(k = 1, 2, \dots)$, $\pi(\omega + h) = y^h$ $(h = 0, 1, \dots)$, and $\pi(\omega \cdot 2) = y^\omega = \pi(0)$.

[6.1] $\Rightarrow$ [6.6]: A quasicontinuous ordering on X is a quasicontinuous ordering on every $Y \subseteq X$, and an Ω-acyclic mapping of a compact space into itself has a fixed point by Theorem 2.

[6.6] $\Rightarrow$ [6.1]: $\succeq$ must be an ordering by [5.4] $\Rightarrow$ [5.1]. If $\succeq$ is not pseudocontinuous, we note that $Y = \{x^k\}_k \cup \{x^\omega\}$ from the proof of [4.1.3] $\Rightarrow$ [4.1.1] is compact and f from the same proof maps Y into Y without fixed points. Finally, if $\succeq$ is not quasicontinuous, $Y = \{x^k\}_k \cup \{y^h\}_h$ from the proof of [6.5] $\Rightarrow$ [6.1] is compact and f from the same proof maps Y into Y without fixed points. $\square$

Remark. Theorem 6 demonstrates that ω-acyclicity is of much less importance in this context than for general relations.

Remark. When [6.6] $\Rightarrow$ [6.1] and the necessity in Theorem 3 are put together, we clearly see an analogy with the main result of Davis (1955).

In the standard usage, the term "monotonic" applies to increasing and decreasing mappings alike. Here we generally follow the tradition, but it must be kept in view that the two classes often behave differently.

Proposition 4.2. *Let X be a set with a fixed preorder $\succeq$. Then the following statements are equivalent:*

4.2.1. *the preorder is degenerate, i.e., $y \sim x$ for all $y, x \in X$;*

4.2.2. *every decreasing mapping $f : X \to X$ is acyclic;*

4.2.3. *every decreasing mapping $f : X \to X$ has a fixed point.*

Proof. Suppose [4.2.1] does not hold. If there exist $a, b \in X$ such that $b \succ a$, we define $f(x) = a$ for $x \succeq b$ and $f(x) = b$ otherwise; f is obviously decreasing, $f(a) = b$, $f(b) = a$, and there is no fixed point. If no such pair exists, there must be $a, b \in X$ such that a and b are incomparable. Now we define $f(x) = b$ if $x \sim a$; otherwise, $f(x) = a$. Again, f is decreasing, $f(a) = b$, $f(b) = a$, and there is no fixed point. $\square$

5 Maximum Aggregation in Systems of Reactions

5.1 Basic Definitions

A *system of reactions* $\mathcal{S}$ is defined by a finite set of indices N, and sets X_i and mappings $R_i : X_{-i} \to 2^{X_i} \setminus \{\emptyset\}$ for all $i \in N$. $x^0 \in X = \prod_{i \in N} X_i$ is called a *fixed point* of $\mathcal{S}$ if $x_i^0 \in R_i(x_{-i}^0)$ for all $i \in N$. For technical convenience, we allow singleton N, in which case $R_i(\,)$ (for $N = \{i\}$) is just a nonempty subset of $X_i = X$.

With every system $\mathcal{S}$, one can associate binary relations on X: $y \rhd_i^{\mathcal{S}} x \iff [y_{-i} = x_{-i} \;\&\; x_i \notin R_i(x_{-i}) \ni y_i]$, $y \rhd^{\mathcal{S}} x \iff \exists i \in N\, [y \rhd_i^{\mathcal{S}} x]$. Clearly, $x \in X$ is a maximizer for $\rhd^{\mathcal{S}}$ if and only if x is a fixed point of $\mathcal{S}$; improvement paths for $\rhd^{\mathcal{S}}$ are generated by iterating R_i's and picking limit points. We call $\mathcal{S}$ (Ω-)*acyclic* if so is $\rhd^{\mathcal{S}}$. As a rule, we omit the superscript $^{\mathcal{S}}$ at $\rhd$. By Theorem 2, an Ω-acyclic system of reactions with compact sets X_i has a fixed point.

With every reaction R_i ($i \in N$), a mapping $\tilde{R}_i : X \to 2^X \setminus \{\emptyset\}$ can be associated, defined by $\tilde{R}_i(x) = \{x_{-i}\} \times R_i(x_{-i})$. It is easy to see that $\rhd_i^{\mathcal{S}}$ coincides with the relation $\rhd^{\tilde{R}_i}$ as defined in the beginning of Section 4. In this sense, the (Ω-)acyclicity of a system of reactions is equivalent to the "joint" (Ω-)acyclicity of several endomorphisms.

Let $\mathcal{S}$ be a system of reactions, $I \subseteq N$ and $x_I \in X_I$. A *reduced system of reactions* $\mathcal{S}(x_I)$ is characterized by the set of indices $N \setminus I$ and, for each $i \in N \setminus I$, the same X_i and $R'_i(x_{N\setminus I\setminus\{i\}}) = R_i(x_I, x_{N\setminus I\setminus\{i\}})$. If $I = N \setminus \{i\}$ ($i \in N$), then the reduced system has a single index, and our convention about the meaning of $R_i(\,)$ in such systems proves adequate. A class $\mathfrak{S}$ of systems of reactions is *hereditary* if $\mathcal{S} \in \mathfrak{S} \Rightarrow \forall I \subseteq N\, \forall x_I \in X_I\, [\mathcal{S}(x_I) \in \mathfrak{S}]$.

A *quasipotential* of a system of reactions is a preorder $\succeq$ on X such that

1. $y \triangleright x \Rightarrow y \succeq x$;
2. its asymmetric component $\succ$ is ω-transitive;
3. for each $x \in X$, there is $V(x) \subseteq N$ such that $y \triangleright_i x$ implies
   (a) $i \in V(x) \Rightarrow y \succ x$ and
   (b) $x \succeq y \Rightarrow V(y) = V(x)$.

Let $\mathfrak{S}$ be a hereditary class of systems of reactions. A collection of quasipotentials (one for each $\mathcal{S} \in \mathfrak{S}$) is called *consistent* if, for each $\mathcal{S} \in \mathfrak{S}$,

$$[x, y \in X \;\&\; C = V(x) \;\&\; x_C = y_C \;\&\; y_{-C} \succeq_{-C} x_{-C}] \Rightarrow \\ \big(y \succeq x \;\&\; [y \succ x \text{ or } V(y) = V(x)]\big), \quad (5.1)$$

where $\succeq_{-C}$ denotes the quasipotential for the reduced system defined by $x_C = y_C$.

Theorem 5.4 of Kukushkin (2000) shows that the existence of a consistent collection of quasipotentials for a hereditary class $\mathfrak{S}$ implies the existence of a potential for (hence the Ω-acyclicity of) every $\mathcal{S} \in \mathfrak{S}$.

In the proofs to follow, it is convenient to consider $\mathbb{R}^* = \mathbb{R} \times \{0, 1\}$ ordered in a lexicographic way: $\langle s', \vartheta' \rangle > \langle s, \vartheta \rangle \iff [s' > s \text{ or } (s' = s \;\&\; \vartheta' > \vartheta)]$. We assume that $\mathbb{R}$ is embedded into $\mathbb{R}^*$ by $s \mapsto \langle s, 0 \rangle$. $\mathbb{R}^*$ is an interesting object in itself (Wakker, 1988), but we only use it as an auxiliary construction. The following properties are essential: if $s \in \mathbb{R}$, $s' \in \mathbb{R}^*$, and $s' > \langle s, 1 \rangle$, then there exists $s'' \in \mathbb{R}$ such that $s' > s'' > s$; if $s^k \in \mathbb{R}$ for $k = 0, 1, \ldots$, $s^k \to s^\omega\, [\in \mathbb{R}]$, $s^* \in \mathbb{R}^*$, and $s^\omega > s^*$, then $s^k > s^*$ for all k large enough (if $s^\omega < s^*$, then $s^k > s^*$ for all k is possible).

5.2 Increasing Reactions

Theorem 7. *Let $\mathcal{S}$ be a system of reactions such that each X_i ($i \in N$) is a compact subset of $\mathbb{R}$ and every $R_i(x_{-i})$ ($i \in N$, $x_{-i} \in X_{-i}$) is closed in X_i; let, for each $i \in N$, a subset $I(i) \subseteq N$ be given such that $j \in I(i) \iff i \in I(j)$; let every reaction R_i be* *increasing w.r.t. the ordering on X_{-i} defined by the function $\psi_i(x_{-i}) = \max_{j\in I(i)} x_j$ and the partial order on $2^{X_i} \setminus \{\emptyset\}$ defined in Section 4. Then $\mathcal{S}$ is Ω-acyclic.*

Remark. In principle, we should allow for the case $I(i) = \emptyset$ for some $i \in N$. It seems natural to treat it exactly as the case of $N = \{i\}$ in the general setting, assuming that $R_i()$ is just a subset of X_i; then x_i can change only once along a path and cannot affect any other x_j. Therefore, we may argue as if $I(i) \neq \emptyset$ for every $i \in N$.

Proof. For each $i \in N$, we define $\Psi_i = \psi_i(X_{-i})$; clearly, there exists an increasing mapping $\rho_i : \Psi_i \to 2^{X_i} \setminus \{\emptyset\}$ such that $R_i(x_{-i}) = \rho_i(\psi_i(x_{-i}))$ for every $x_{-i} \in X_{-i}$. For $x_i \in X_i$, we denote $S_i(x_i) = \{s_i \in \Psi_i |\ x_i \in \rho_i(s_i) \text{ or } \forall \xi_i \in \rho_i(s_i)\,[\xi_i > x_i]\}$, $\sigma_i(x_i) = \inf S_i(x_i) \in \Psi_i$, $\vartheta_i(x_i) = 0$ if $\sigma_i(x_i) \in S_i(x_i)$, $\vartheta_i(x_i) = 1$ otherwise, $\tau_i(x_i) = \langle \sigma_i(x_i),\ \vartheta_i(x_i)\rangle \in \Psi_i^*$ (actually, $\tau_i(x_i)$ is the infimum of $S_i(x_i)$ in $\mathbb{R}^*$) and $\tau_i^+(x_i) = \max\{\tau_i(x_i), x_i\}$. For every $x \in X$, we define:

$$K(x) = \{i \in N |\quad \psi_i(x_{-i}) \geq \tau_i(x_i)\}; \quad \tau^+(x) = \max_{i \notin K(x)} \tau_i^+(x_i);$$

$$M(x) = \{i \in N \setminus K(x) |\quad x_i = \tau^+(x)\}; \quad M^*(x) = \{i \in N \setminus K(x) |\quad \tau_i(x_i) = \tau^+(x)\};$$

$$K^+(x) = \{i \in K(x) |\quad x_i \geq \tau^+(x)\}; \quad K^0(x) = K^+(x) \cup \{i \in N |\quad I(i) \cap K^+(x) \neq \emptyset\}.$$

We also denote:

$$X^{\mathrm{K}} = \{x \in X |\quad K^+(x) \neq \emptyset\}; \quad X^{\mathrm{M}} = \{x \in X |\quad K^+(x) = \emptyset \neq M(x)\};$$

$$X^* = \{x \in X |\quad K^+(x) = \emptyset = M(x)\}.$$

For $y, x \in X$, we define

$$y \succeq' x \iff K^+(x) \subseteq K^+(y) \ \&\ \tau^+(x) \geq \tau^+(y) \ \&\ \forall i \in K^0(x)[y_i \geq x_i], \tag{5.2}$$

denoting $\succ'$ and $\sim'$ the asymmetric and symmetric components of $\succeq'$. Now we set $y \succeq x$ if and only if one of the following conditions holds:

$$y \succ' x; \tag{5.3a}$$

$$x \in X^{\mathrm{K}} \ \&\ y \succeq' x; \tag{5.3b}$$

$$x \in X^{\mathrm{M}} \ \&\ y \sim' x \ \&\ M(x) \supseteq M(y); \tag{5.3c}$$

$$x \in X^* \ \&\ y \in X^* \ \&\ y \sim' x \ \&\ M^*(x) \supseteq M^*(y). \tag{5.3d}$$

Although a complete explanation of the meaning of the constructions can only be found in the proofs to follow, preliminary hints may help a bit. $K(x)$ (at a given profile x) consists of the indices i such that x_i cannot go downwards at the next step; $\tau_i(x_i)$ for $i \notin K(x)$ shows how much "support" would be enough to shift i into K. Therefore, when $i \in K^+(x)$, x_i is large enough to ensure that $I(i) \subseteq K(x)$, so such x_i cannot ever go downwards: once $K^+(x) \neq \emptyset$, we have a subset of N where only upward changes are possible henceforth, and this is enough for a quasipotential. $\tau^+(x)$ could be characterized as "the greatest need for support felt anywhere." At a first glance, it may seem strange that we should put together unsupported levels x_i and their need for support $\tau_i(x_i)$, but this peculiarity is justified by the success of the whole proof; in a similar constructions for additive aggregation as developed in the proof of Theorem 6.1 of Kukushkin (2000), x_i and $\tau_i(x_i)$ are summed up. Since y_i reached at any step is always supported $[y_i \in \rho_i(y_{-i}) \Rightarrow i \in K(y)]$, τ^+ cannot increase; whenever i where $\tau^+(x)$ is attained is touched, "the total need for support" strictly decreases.

Lemma 5.2.1. *$\succeq$ is a preorder.*

Proof. Let us see that $\succeq$ is defined by an arboreous lexicographic construction: $X = C(\alpha^{\min})$ is partitioned into equivalence classes of $\succeq'$; then each class with $x \notin X^{\mathrm{K}}$ is partitioned in accordance with the sets $M(x)$ by (5.3c), so X^* becomes one of the components; finally, (5.3d) defines a preorder on X^*. Now Proposition 3.6 applies. $\square$

Lemma 5.2.2. *For every $x \in X$, $K^0(x) \subseteq K(x)$. For every $i \in N$, $x_i \in X_i$, and $s_i \in \Psi_i$, $s_i \geq \tau_i(x_i) \iff s_i \in S_i(x_i)$.*

Proof. If $j \notin K(x)$ and $i \in K^+(x) \cap I(j)$, then $\psi_j(x_{-j}) \geq x_i \geq \tau^+(x) \geq \tau_j(x_j)$, hence $j \in K(x)$. Let $s_i > \tau_i(x_i)$ [then $s_i > \sigma_i(x_i)$], but $x_i \notin \rho_i(s_i) \ni \xi_i < x_i$. For every $s'_i < s_i$, either $\xi_i \in \rho_i(s'_i)$ or $\xi_i \in \rho_i(s_i) \setminus \rho_i(s'_i)$; in either case, the monotonicity of ρ_i precludes $x_i \in \rho_i(s'_i)$ [there would be $x_i \in \rho_i(s'_i) \setminus \rho_i(s_i)$ otherwise] and ensures $\xi'_i \in \rho_i(s'_i)$ with $\xi'_i \leq \xi_i < x_i$. Therefore, $s'_i \notin S_i(x_i)$, but this contradicts the definition of $\sigma_i(x_i)$. $\square$

Lemma 5.2.3. *Let $y \rhd_i x$; then $i \in K(y)$, $i \in K(x)$ implies $y_i > x_i$, and $j \in K(x) \setminus K(y)$ implies $j \neq i$, $i \in I(j)$, $x_i \geq \tau_j(x_j)$, and $i \notin K(x)$.*

Proof. By definition, $y_i \in \rho_i(\psi_i(x_{-i}))$, hence $\psi_i(x_{-i}) \in S_i(y_i)$, hence $\tau_i(y_i) \leq \psi_i(x_{-i})$. If $i \in K(x)$, then $x_i \notin \rho_i(\psi_i(x_{-i}))$ and $\psi_i(x_{-i}) \in S_i(x_i)$ imply $\xi_i > x_i$ for all $\xi_i \in \rho_i(\psi_i(x_{-i}))$, in particular, $y_i > x_i$. Let $j \in K(x) \setminus K(y)$; then $j \neq i$ because $i \in K(y)$, hence $y_j = x_j$, hence $i \in I(j)$ and $x_i = \psi_j(x_{-j}) \geq \tau_j(x_j) > y_i$, hence $i \notin K(x)$. $\square$

Lemma 5.2.4. *Let $y \rhd_i x$; then $y \succeq' x$. If $i \in K^0(x)$, $y \succ' x$; if $x \succeq' y$, $K^0(y) = K^0(x)$.*

Proof. First we show $K^+(x) \subseteq K(y)$. Since $i \in K(y)$ by Lemma 5.2.3, let us suppose $j \neq i$ and $j \in K^+(x) \setminus K(y)$. By Lemma 5.2.3, $i \in I(j)$ and $i \notin K(x)$; however, by Lemma 5.2.2, $i \in K(x)$ because $i \in K^0(x)$.

To show $\tau^+(y) \leq \tau^+(x)$, let us suppose the contrary. Since $i \in K(y)$ by Lemma 5.2.3, this would imply the existence of $j \in K(x) \setminus K(y)$ with either $x_j > \tau^+(x)$ or

$\tau_j(x_j) > \tau^+(x)$. In the first case, $j \in K^+(x)$, hence $j \in K(y)$: a contradiction. In the second case, we must have $i \notin K(x)$ and $x_i \geq \tau_j(x_j)$ by Lemma 5.2.3, but then $x_i > \tau^+(x)$, contradicting $i \notin K(x)$.

Now $j \in K^+(x)$ (for $j \neq i$ or $j = i$) implies $j \in K(y)$ and $y_j \geq x_j \geq \tau^+(x) \geq \tau^+(y)$, i.e., $j \in K^+(y)$. The two other statements follow immediately from Lemmas 5.2.2 and 5.2.3, and from the definition of $\succeq'$. $\square$

Lemma 5.2.5. *Let $x \notin X^{\mathrm{K}}$ and $y \triangleright_i x$; then $y \succeq x$. If $i \in M(x)$, $y \succ x$; if $x \succeq y$, then $y \notin X^{\mathrm{K}}$ and $M(y) = M(x)$; if $x \in X^*$ and $i \in M^*(x)$, $y \succ x$; if $x \in X^*$ and $x \succeq y$, then $y \in X^*$ and $M^*(y) = M^*(x)$.*

Proof. By Lemma 5.2.4, $\tau^+(y) \leq \tau^+(x)$. Let $\tau^+(y) = \tau^+(x)$ and $j \in M(y) \setminus M(x)$; then $j \neq i$ because $i \in K(y)$, hence $x_j = y_j = \tau^+(y) = \tau^+(x)$, so $j \notin M(x)$ implies $j \in K(x)$, but then $j \in K^+(x) = \emptyset$. Let $\tau^+(y) = \tau^+(x)$, $M(y) = M(x) = \emptyset$, and $j \in M^*(y) \setminus M^*(x)$. Again, $j \neq i$, hence $x_j = y_j$, hence $j \in K(x) \setminus K(y)$, hence $x_i \geq \tau_j(x_j) = \tau_j(y_j)$ and $i \notin K(x)$; since $\tau_j(x_j) = \tau^+(x)$, $i \in M(x) = \emptyset$.

All the statements about $y \succ x$ and $x \succeq y$ immediately follow from $i \in K(y)$. $\square$

Lemma 5.2.6. *Let $x^k \to x^\omega$, $i \in N$, $s_i \in \Psi_i^*$, and $\tau_i(x_i^\omega) > s_i$; then there exists $h \in I\!N$ such that $\tau_i(x_i^k) > s_i$ for all $k \geq h$.*

Proof. Suppose first that $s_i \in \Psi_i$. By the definition of τ_i, $s_i \notin S_i(x_i^\omega)$, hence $x_i^\omega \notin \rho_i(s_i)$ and there is $\xi_i \in \rho_i(s_i)$ for which $\xi_i < x_i^\omega$. Since $\rho_i(s_i)$ is closed, there exists $h \in I\!N$ such that $\xi_i < x_i^k$ and $x_i^k \notin \rho_i(s_i)$ for all $k \geq h$, hence $s_i \notin S_i(x_i^k)$. Lemma 5.2.2 implies $\tau_i(x_i^k) > s_i$. If $s_i = \langle s_i', 1\rangle$ with $s_i' \in \Psi_i$, there exists $s_i'' \in \Psi_i$ such that $\tau_i(x_i^\omega) > s_i'' > s_i'$, hence $\tau_i(x_i^k) > s_i'' > s_i$ for all k large enough. $\square$

Lemma 5.2.7. *Let $x^k \to x^\omega$ and $x^{k+1} \succ x^k$ for all $k = 0, 1, \ldots$; then $x^\omega \succ x^0$.*

Proof. Obviously, there must be $x^{k+1} \succ' x^k$ for an infinite number of k; without restricting generality, we may assume that the condition holds for all k. Moreover, we may assume that $K(x^k) = K(x^0)$ and $K^+(x^k) = K^+(x^0)$ [hence $K^0(x^k) = K^0(x^0)$] for all k. Since we can start the sequence $\{x^k\}_k$ from $k = 1$, it is sufficient to prove $x^\omega \succeq' x^0$.

As a first step, we notice that $x_i^{k+1} \geq x_i^k$ for all $i \in K^0(x^0)$ and all k implies $x_i^\omega \geq x_i^0$, hence $\psi_i(x_{-i}^\omega) \geq \psi_i(x_{-i}^0)$ for all $i \in K^0(x^0)$. Then we prove that $K^+(x^0) \subseteq K(x^\omega)$; let $i \in K^+(x^0) = K^+(x^k)$ for all k. If $i \notin K(x^\omega)$, then $\tau_i(x_i^\omega) > \psi_i(x_{-i}^\omega)$; by Lemma 5.2.6, $\tau_i(x_i^k) > \psi_i(x_{-i}^\omega) \geq \psi_i(x_{-i}^k)$, hence $i \notin K(x^k)$ for all k large enough, contradicting $i \in K^+(x^k)$.

Now let us show that $\tau^+(x^\omega) \leq \tau^+(x^0)$; let $i \notin K(x^\omega)$ and $\tau_i^+(x_i^\omega) > \tau^+(x^0)$. If $\tau_i^+(x_i^\omega) = x_i^\omega > \tau^+(x^0)$, then $x_i^k > \tau^+(x^0)$ for all k large enough because $x^k \to x^\omega$; if $\tau_i^+(x_i^\omega) = \tau_i(x_i^\omega) > \tau^+(x^0)$, then Lemma 5.2.6 implies $\tau_i(x_i^k) > \tau^+(x^0)$ for all k large enough; in either case, $\tau_i^+(x_i^k) > \tau^+(x^0)$. If $i \notin K(x^k)$, we have $\tau^+(x^k) > \tau^+(x^0)$, contradicting $x^k \succ' x^0$. If $i \in K(x^k)$, then either $\tau^+(x^k) > \tau_i(x^k)$, hence $\tau^+(x^k) > \tau^+(x^0)$ again, or $i \in K^+(x^k)$, hence $i \in K(x^\omega)$, contradicting our assumption.

Finally, if $i \in K^+(x^0) \subseteq K(x^\omega)$, then $x_i^\omega \geq x_i^0 \geq \tau^+(x^0) \geq \tau^+(x^\omega)$, hence $i \in K^+(x^\omega)$. $\square$

Now we define $V(x) = K^0(x)$ for $x \in X^{\mathrm{K}}$, $V(x) = M(x)$ for $x \in X^{\mathrm{M}}$, and $V(x) = M^*(x)$ for $x \in X^*$. Lemmas 5.2.4, 5.2.5, and 5.2.7 show that $\succeq$ is a quasipotential. We only have to show its consistency.

Lemma 5.2.8. *Let $x, y \in X$, $C = V(x)$, $x_C = y_C$, and $\succeq_{-C}$ denote the quasipotential for the reduced system defined by $x_C = y_C$. Then* (5.1) *holds.*

Proof. When the above constructions are developed for the reduced system (fortunately, we only have to consider one reduced system throughout the proof), we add $-C$ somewhere, most often, as a subscript. It is easy to see that $\tau_i(x_i)$ can only diminish when computed for the reduced system; more precisely, $\tau_i^{-C}(x_i) = \min \Psi_i^{-C}$ if $\tau_i(x_i) \leq \max_{j \in I(i) \cap C} x_j$ and $\tau_i^{-C}(x_i) = \tau_i(x_i)$ otherwise. It follows immediately that $i \in K_{-C}(x_{-C}) \iff i \in K(x) \setminus C$.

Suppose that $i \in N \setminus C \setminus K(y)$ and

$$\tau_i^+(y_i) \geq \tau^+(x). \tag{5.4}$$

Then ${}_{-C}\tau_i^+(y_i) = \tau_i^+(y_i)$, so a strict inequality in (5.4) would imply $\tau_{-C}^+(y_{-C}) > \tau_{-C}^+(x_{-C})$, contradicting $y_{-C} \succeq_{-C} x_{-C}$; therefore $\tau^+(x) \geq \tau^+(y)$. If $K^+(x) \neq \emptyset$, then $C = K^0(x) \subseteq K^0(y)$ and $x_j = y_j$ for every $j \in K^0(x)$ imply that $y \succeq x$ and either $y \succ x$ or $V(y) = V(x)$.

If $K^+(x) = \emptyset$, we have $y \succeq' x$ at least and $y \succ x$ if $K^+(y) \neq \emptyset$. Moreover, $\tau_i^+(y_i) < \tau^+(x)$ for all $i \in N \setminus C \setminus K(y)$ implies $y \sim x$ and $V(y) = V(x)$ regardless of whether $V(x) = M(x)$ or $V(x) = M^*(x)$.

If $K^0(x) = \emptyset$ and (5.4) holds as an equality, then $\tau_{-C}^+(x_{-C}) = \tau^+(x) = \tau^+(y)$ because $y_{-C} \succeq_{-C} x_{-C}$, which is only possible if $C = M(x) \neq \emptyset$, hence $M_{-C}(x_{-C}) = \emptyset$. Now $\tau_i^+(y_i) = y_i$ would imply $M_{-C}(y_{-C}) \neq \emptyset$, which is only compatible with $y_{-C} \succeq_{-C} x_{-C}$ if $K_{-C}(y_{-C}) \neq \emptyset$, but then $K^+(y) \neq \emptyset$ because $\tau^+(y) = \tau_{-C}^+(y_{-C})$. Therefore, $\tau_i(y_i) > y_i$ for all i satisfying (5.4) (as an equality) and we have $M(y) = M(x)$, hence $y \sim x$ and $V(y) = V(x)$. $\square$

A reference to Theorem 5.4 of Kukushkin (2000) completes the proof. $\square$

5.3 Decreasing Reactions

Theorem 8. *Let $\mathcal{S}$ be a system of reactions such that each X_i ($i \in N$) is a compact subset of $I\!R$ and every $R_i(x_{-i})$ ($i \in N$, $x_{-i} \in X_{-i}$) is closed in X_i; let, for each $i \in N$, a subset $I(i) \subseteq N$ be given such that $j \in I(i) \iff i \in I(j)$; let every reaction R_i be decreasing w.r.t. the ordering on X_{-i} defined by the function $\psi_i(x_{-i}) = \max_{j \in I(i)} x_j$ and the partial order on $2^{X_i} \setminus \{\emptyset\}$ defined in Section 4. Then $\mathcal{S}$ is Ω-acyclic.*

Remark. As in Theorem 7, the case of $I(i) = \emptyset$ for some $i \in N$ is dismissed as trivial.

Proof. For each $i \in N$, we define $\Psi_i = \psi_i(X_{-i})$; clearly, there exists a decreasing mapping $\rho_i : \Psi_i \to 2^{X_i} \setminus \{\emptyset\}$ such that $R_i(x_{-i}) = \rho_i(\psi_i(x_{-i}))$ for every $x_{-i} \in X_{-i}$. For $x_i \in X_i$, we denote $S_i^-(x_i) = \{s_i \in \Psi_i |\ x_i \in \rho_i(s_i) \text{ or } \forall \xi_i \in \rho_i(s_i)\, [\xi_i < x_i]\}$, $S_i^+(x_i) = \{s_i \in \Psi_i |\ x_i \in \rho_i(s_i) \text{ or } \forall \xi_i \in \rho_i(s_i)\, [\xi_i > x_i]\}$, $\sigma_i(x_i) = \inf S_i^-(x_i)$, $\vartheta_i(x_i) = 0$ if $\sigma_i(x_i) \in S_i^-(x_i)$, $\vartheta_i(x_i) = 1$ otherwise, and $\tau_i(x_i) = \langle \sigma_i(x_i),\ \vartheta_i(x_i) \rangle \in I\!R^*$ (again, $\tau_i(x_i)$ is the infimum of $S_i^-(x_i)$ in $I\!R^*$). For every $x \in X$, we define:

$$K(x) = \{i \in N |\quad \psi_i(x_{-i}) \in S_i^+(x_i)\}; \quad L(x) = \{i \in N |\quad \psi_i(x_{-i}) \geq \tau_i(x_i)\};$$

$$\lambda(x) = \max_{i \notin K(x)} x_i; \quad \nu(x) = \max_{i \notin L(x)} \tau_i(x_i);$$

$$\tau^+(x) = \max\{\nu(x), \lambda(x)\};$$

$$M(x) = \{i \in N \setminus K(x) |\quad x_i = \tau^+(x)\}; \quad M^*(x) = \{i \in N \setminus L(x) |\quad \tau_i(x_i) = \tau^+(x)\};$$

$$K^+(x) = \{i \in K(x) |\quad x_i \geq \tau^+(x)\}; \quad L^0(x) = \{i \in N |\quad I(i) \cap K^+(x) \neq \emptyset\}.$$

We also denote:

$$X^{\mathrm{K}} = \{x \in X |\quad K^+(x) \neq \emptyset\}; \quad X^{\mathrm{M}} = \{x \in X |\quad K^+(x) = \emptyset \neq M(x)\};$$

$$X^* = \{x \in X |\quad K^+(x) = \emptyset = M(x)\}.$$

For $y, x \in X$, we define

$$y \succeq' x \iff K^+(x) \subseteq K^+(y) \ \&\ \tau^+(x) \geq \tau^+(y) \ \&$$
$$\forall i \in K^+(x)[y_i \geq x_i] \ \&\ \forall i \in L^0(x)[y_i \leq x_i], \tag{5.5}$$

denoting $\succ'$ and $\sim'$ the asymmetric and symmetric components of $\succeq'$. Now we set $y \succeq x$ if and only if one of the following conditions holds:

$$y \succ' x; \tag{5.6a}$$

$$x \in X^{\mathrm{K}} \ \&\ y \succeq' x; \tag{5.6b}$$

$$x \in X^{\mathrm{M}} \ \&\ y \sim' x \ \&\ M(x) \supseteq M(y); \tag{5.6c}$$

$$x \in X^* \ \&\ y \in X^* \ \&\ y \stackrel{\prime}{\sim} x \ \&\ M^*(x) \supseteq M^*(y). \tag{5.6d}$$

The constructions are rather similar to those of Theorem 7. $K^+(x)$ consists of the indices that have already decided always to go upwards; $L^0(x)$, downwards. If $K^+(x) \cup L^0(x) \subset N$, then $\tau^+(x)$ measures, in a peculiar way, "the degree of undecidedness" in possible further movements.

Lemma 5.3.1. *$\succeq$ is a preorder.*

Proof. Indeed, $\succeq$ is defined by an arboreous lexicographic construction exactly in the same way as in Lemma 5.2.1. $\square$

Lemma 5.3.2. *For every $x \in X$, $L^0(x) \subseteq L(x)$. For every $i \in N$, $x_i \in X_i$, and $s_i, s'_i \in \Psi_i$, $s_i \geq \tau_i(x_i) \iff s_i \in S_i^-(x_i)$ and $[s_i > s'_i \ \&\ s_i \in S_i^+(x_i)] \Rightarrow s'_i \in S_i^+(x_i)$.*

Proof. If $j \notin L(x)$ and $i \in K^+(x) \cap I(j)$, then $\psi_j(x_{-j}) \geq x_i \geq \tau^+(x) \geq \tau_j(x_j)$, hence $j \in L(x)$. Let $s_i > \tau_i(x_i)$ (then $s_i > \sigma_i(x_i)$), but $x_i \notin \rho_i(s_i) \ni \xi_i > x_i$. For every $s'_i < s_i$, either $\xi_i \in \rho_i(s'_i)$ or $\xi_i \in \rho_i(s_i) \setminus \rho_i(s'_i)$; in either case, the monotonicity of ρ_i precludes $x_i \in \rho_i(s'_i)$ (there would be $x_i \in \rho_i(s'_i) \setminus \rho_i(s_i)$ otherwise) and ensures $\xi'_i \in \rho_i(s'_i)$ with $\xi'_i \geq \xi_i > x_i$. Therefore, $s'_i \notin S_i^-(x_i)$, but this contradicts the definition of $\sigma_i(x_i)$. The last statement follows from the monotonicity of ρ_i in a similar way. $\square$

Lemma 5.3.3. *Let $y \triangleright_i x$. Then $i \in K(y) \cap L(y)$; $i \in K(x)$ implies $y_i > x_i$; $i \in L(x)$ implies $y_i < x_i$; $j \in L(x) \setminus L(y)$ implies $j \neq i$, $i \in I(j)$, $x_i \geq \tau_j(x_j)$, and $i \notin K(x)$.*

Proof. By definition, $y_i \in \rho_i(\psi_i(x_{-i}))$, hence $\psi_i(x_{-i}) \in S_i^-(y_i) \cap S_i^+(y_i)$, hence $\tau_i(y_i) \leq \psi_i(x_{-i}) = \psi_i(y_{-i})$. If $i \in K(x)$, then $x_i \notin \rho_i(\psi_i(x_{-i}))$ and $\psi_i(x_{-i}) \in S_i^+(x_i)$ imply $\xi_i > x_i$ for all $\xi_i \in \rho_i(\psi_i(x_{-i}))$, in particular, $y_i > x_i$. For $i \in L(x)$, a similar argument works. Let $j \in L(x) \setminus L(y)$; then $j \neq i$ because $i \in L(y)$, hence $y_j = x_j$, hence $i \in I(j)$ and $x_i = \psi_j(x_{-j}) \geq \tau_j(x_j) > y_i$, hence $i \notin K(x)$. $\square$

Lemma 5.3.4. *Let $y \triangleright_i x$; then $y \succeq' x$. If $i \in K^+(x) \cup L^0(x)$, then $y \succ' x$; if $x \succeq' y$, then $K^+(y) = K^+(x)$ and $L^0(y) = L^0(x)$.*

Proof. First we show $K^+(x) \subseteq K(y)$. Since $i \in K(y)$ by Lemma 5.3.3, let us suppose $j \neq i$ and $j \in K^+(x)$. If $i \notin I(j)$, then $y_j = x_j$ and $\psi_j(y_{-j}) = \psi_j(x_{-j})$, hence $j \in K(y)$ as well. If $i \in I(j)$, then $i \in L^0(x)$, hence $i \in L(x)$ by Lemma 5.3.2, hence $y_i < x_i$, hence $\psi_j(y_{-j}) \leq \psi_j(x_{-j})$, hence $\psi_j(y_{-j}) \in S_j^+(x_j)$ by Lemma 5.3.2, hence $j \in K(y)$.

To show $\tau^+(y) \leq \tau^+(x)$, let us suppose the contrary. Let $j \notin K(y)$ and $y_j \geq \tau^+(x)$; then $j \neq i$, hence $x_j = y_j$, hence $j \in K(x)$ would imply $j \in K^+(x)$ contradicting the findings of the previous paragraph, whereas $j \notin K(x)$ is only possible if $y_j = x_j = \tau^+(x)$, i.e., $j \in M(x)$. Let $j \in L(x) \setminus L(y)$ and $\tau_j(y_j) > \tau^+(x)$; by Lemma 5.3.3, $j \neq i$, $x_i \geq \tau_j(y_j)$, and $i \notin K(x)$; but then $\lambda(x) \geq x_i > \tau^+(x)$: a contradiction.

If $j \in K^+(x) \subseteq K(y)$, then $y_j \geq x_j \geq \tau^+(x) \geq \tau^+(y)$, i.e., $j \in K^+(y)$. The two other statements follow immediately from Lemmas 5.3.2 and 5.3.3, and from the definition of $\succeq'$. □

Lemma 5.3.5. *Let $x \notin X^{\mathrm{K}}$ and $y \rhd_i x$; then $y \succeq x$. If $i \in M(x)$, $y \succ x$; if $x \succeq y$, then $y \notin X^{\mathrm{K}}$ and $M(y) = M(x)$; if $x \in X^*$ and $i \in M^*(x)$, then $y \succ x$; if $x \in X^*$ and $x \succeq y$, then $y \in X^*$ and $M^*(y) = M^*(x)$.*

Proof. By Lemma 5.3.4, $\tau^+(y) \leq \tau^+(x)$; moreover, if $\tau^+(y) = \tau^+(x)$, then $M(y) \subseteq M(x)$. Let $\tau^+(y) = \tau^+(x)$, $M(y) = M(x) = \emptyset$, and $j \in M^*(y) \setminus M^*(x)$; then $j \neq i$, hence $x_j = y_j$, hence $j \in L(x) \setminus L(y)$, hence $x_i \geq \tau_j(x_j)$ and $i \notin K(x)$ by Lemma 5.3.3; since $\tau_j(x_j) = \tau^+(x)$, $i \in M(x) = \emptyset$.

All the statements about $y \succ x$ and $x \succeq y$ immediately follow from $i \in K(y) \cap L(y)$. □

Lemma 5.3.6. *Let $x^k \to x^\omega$, $i \in N$, $s_i \in \Psi_i^*$, and $\tau_i(x_i^\omega) > s_i$; then there exists $h \in I\!N$ such that $\tau_i(x_i^k) > s_i$ for all $k \geq h$.*

The proof is virtually the same as that of Lemma 5.2.6.

Lemma 5.3.7. *Let $x^k \to x^\omega$ and $x^{k+1} \succ' x^k$ for all $k = 0, 1, \ldots$; then $x_i^\omega \geq x_i^k$ and $\psi_i(x_{-i}^\omega) \leq \psi_i(x_{-i}^k)$ for all $i \in K^+(x^0)$ and all k; $x_i^\omega \leq x_i^k$ and $\psi_i(x_{-i}^\omega) \geq \psi_i(x_{-i}^k)$ for all $i \in L^0(x^0)$ and all k; and $K^+(x^0) \subseteq K(x^\omega)$.*

Proof. Both inequalities on x_i^ω immediately follow from $x_i^{k+1} \geq x_i^k$ for $i \in K^+(x^k)$ and $x_i^{k+1} \leq x_i^k$ for $i \in L^0(x^k)$. If $i \in K^+(x^0) \subseteq K^+(x^k)$, then $I(i) \subseteq L^0(x^k)$ by definition, hence $\psi_i(x_{-i}^{k+1}) \leq \psi_i(x_{-i}^k)$ for all k. If $j \in L^0(x^0) \subseteq L^0(x^k)$, $i \in I(j)$, and $x_i^k = \psi_j(x_{-j}^k)$, then, obviously, $i \in K^+(x^k)$, hence $\psi_j(x_{-j}^{k+1}) \geq x_i^{k+1} \geq x_i^k = \psi_j(x_{-j}^k)$ for all k.

Finally, let $i \in K^+(x^0)$, but $i \notin K(x^\omega)$; then $\psi_i(x_{-i}^\omega) \notin S_i^+(x_i^\omega)$, hence $x_i^\omega \notin \rho_i(\psi_i(x_{-i}^\omega)) \ni \xi_i < x_i^\omega$. We define $\xi_i^\omega = \max \rho_i(s_i) \cap [\xi_i, x_i^\omega] < x_i^\omega$ [$\rho_i(s_i)$ is closed]. By the previous argument, $\psi_i(x_{-i}^\omega) \leq \psi_i(x_{-i}^k)$ for all k; because $x^k \to x^\omega$, $\xi_i^\omega < x_i^k$ for all k large enough; by the monotonicity of ρ_i, $x_i^k \notin \rho_i(\psi_i(x_{-i}^k)) \ni \xi_i^k < x_i^k$. Thus, $\psi_i(x_{-i}^k) \notin S_i^+(x_i^k)$, hence $i \notin K(x^k)$ for all k large enough, contradicting our assumption. □

Lemma 5.3.8. *Let $x^k \to x^\omega$ and $x^{k+1} \succ x^k$ for all $k = 0, 1, \ldots$; then $x^\omega \succ x^0$.*

Proof. Obviously, there must be $x^{k+1} \succ' x^k$ for an infinite number of k; without restricting generality, we may assume that the condition holds for all k. Moreover, we may assume that $K(x^k) = K(x^0)$, $L(x^k) = L(x^0)$, and $K^+(x^k) = K^+(x^0)$ [hence $L^0(x^k) = L^0(x^0)$] for all k. Since we can start the sequence $\{x^k\}_k$ from $k = 1$, it is sufficient to prove $x^\omega \succeq' x^0$.

Now let us show that $\tau^+(x^\omega) \le \tau^+(x^0)$. Let $i \notin K(x^\omega)$ and $x_i^\omega > \tau^+(x^0)$; then $x_i^k > \tau^+(x^0)$ for all k large enough because $x^k \to x^\omega$. Now $i \notin K(x^k)$ is incompatible with $\tau^+(x^k) \le \tau^+(x^0)$ following from $x^k \not\succ' x^0$, whereas $i \in K(x^k)$ would imply $i \in K^+(x^k)$, hence $i \in K(x^\omega)$ by Lemma 5.3.7.

Let $i \notin L(x^\omega)$ and $\tau_i(x_i^\omega) > \tau^+(x^0)$; then Lemma 5.3.6 implies $\tau_i(x_i^k) > \tau^+(x^0)$ for all k large enough. If $i \notin L(x^k)$, or $i \in L(x^k)$ and $\tau^+(x^k) \ge \tau_i(x_i^k)$ for some k, we have $\tau^+(x^k) > \tau^+(x^0)$ contradicting $x^k \not\succ' x^0$. Let $i \in L(x^k)$ and $\tau^+(x^k) < \tau_i(x_i^k)$ for all k; then $\tau^+(x^k) < \tau_i(x_i^k) \le \psi_i(x_{-i}^k)$, hence there is $j \in I(i) \cap K^+(x^k)$, hence $i \in L^0(x^k)$. Now $i \notin L(x^\omega)$ means that $\tau_i(x_i^\omega) > \psi_i(x_{-i}^\omega)$; by Lemmas 5.3.6 and 5.3.7, $\tau_i(x_i^k) > \psi_i(x_{-i}^\omega) \ge \psi_i(x_{-i}^k)$ for all k large enough, hence $i \notin L(x^k)$: a contradiction.

Finally, if $i \in K^+(x^0)$, then $i \in K(x^\omega)$ by Lemma 5.3.7; besides, $x_i^\omega \ge x_i^0 \ge \tau^+(x^0) \ge \tau^+(x^\omega)$, hence $i \in K^+(x^\omega)$. $\square$

Now we define $V(x) = K^+(x) \cup L^0(x)$ for $x \in X^{\mathrm{K}}$, $V(x) = M(x)$ for $x \in X^{\mathrm{M}}$, and $V(x) = M^*(x)$ for $x \in X^*$. Lemmas 5.3.4, 5.3.5, and 5.3.8 show that $\succeq$ is a quasipotential. We only have to show its consistency.

Lemma 5.3.9. *Let $x, y \in X$, $C = V(x)$, $x_C = y_C$, and $\succeq_{-C}$ denote the quasipotential for the reduced system defined by $x_C = y_C$. Then* (5.1) *holds.*

Proof. When the above constructions are developed for the only reduced system we have to consider throughout the proof, we add $-C$ somewhere, most often, as a subscript. It is easy to see that $\tau_i^{-C}(x_i) \le \tau_i(x_i)$; more precisely, $\tau_i^{-C}(x_i) = \min \Psi_i^{-C}$ if $\tau_i(x_i) \le \max_{j \in I(i) \cap C} x_j$ and $\tau_i^{-C}(x_i) = \tau_i(x_i)$ otherwise. It follows immediately that $i \in L_{-C}(x_{-C}) \iff i \in L(x) \setminus C$; quite similarly, $i \in K_{-C}(x_{-C}) \iff i \in K(x) \setminus C$.

If $i \in N \setminus C \setminus L(y)$ and $\tau_i(y_i) \ge \tau^+(x)$, then $\tau_i^{-C}(y_i) = \tau_i(y_i)$. Therefore, $\tau^+(x) < \tau^+(y)$ would imply $\tau_{-C}^+(y_{-C}) > \tau_{-C}^+(x_{-C})$, contradicting $y_{-C} \succeq_{-C} x_{-C}$.

If $K^+(x) \ne \emptyset$, then $C = [K^+(x) \cup L^0(x)] \subseteq [K^+(y) \cup L^0(y)]$ and $x_j = y_j$ for every $j \in K^+(x) \cup L^0(x)$ imply that $y \succeq x$ and either $y \succ x$ or $V(y) = V(x)$. If $K^+(x) = \emptyset$, we have $y \succeq' x$ at least; moreover, $\tau_{-C}^+(y_{-C}) < \tau^+(x)$ implies that either $y \succ x$ [if $K^+(y) \ne \emptyset$] or $y \sim x$ and $V(y) = V(x)$ regardless of whether $V(x) = M(x)$ or $V(x) = M^*(x)$.

Let us assume $\tau^+(y) = \tau^+(x)$ and $K^+(y) = \emptyset$. Let $x \in X^{\mathrm{M}}$, i.e., $C = M(x) \ne \emptyset$. If $y_i = \tau^+(x)$ for $i \in N \setminus C \setminus K(y)$, then $M_{-C}(y_{-C}) \ne \emptyset = M_{-C}(x_{-C})$, which is incompatible with $y_{-C} \succeq_{-C} x_{-C}$; therefore, $M(y) = M(x)$ and $y \sim x$. Finally, let $x \in X^*$, i.e., $\emptyset \ne C = M^*(x) \subseteq M^*(y)$; then $\tau_{-C}^+(x_{-C}) < \tau^+(x)$. Now if $y \in X^M$ or $M^*(x) \subset M^*(y)$, then $\tau_{-C}^+(x_{-C}) < \tau_{-C}^+(y_{-C})$, which is incompatible with $y_{-C} \succeq_{-C} x_{-C}$. $\square$

A reference to Theorem 5.4 of Kukushkin (2000) completes the proof. $\square$

5.4 Discussion

If we redefine $\psi_i(x_{-i})$ as $\min_{j \in I(i)} x_j$, retaining all other assumptions, Theorems 7 and 8 remain valid: to prove this, it is sufficient to replace each X_i with $-X_i$.

Let N be the set of players in a strategic game, $X_i \subseteq I\!\!R$ $(i \in N)$ be the set of strategies, and let the utility functions be of the form $u_i(x) = U_i(x_i, \max_{j \in I(j)} x_j)$ for all $i \in N$ [or $u_i(x) = U_i(x_i, \min_{j \in I(j)} x_j)$ for all $i \in N$]. If we assume that each X_i is compact and each U_i is upper semicontinuous in the first argument, the best reply correspondences $R_i(\cdot)$ will have nonempty and closed values for each $x_{-i} \in X_{-i}$. If each function $U_i(x_i, s_i)$ satisfies the ordinal strategic complements (single crossing, Milgrom and Shannon, 1994) condition:

$$\operatorname{sign}(U_i(x_i + \delta, s_i + \Delta) - U_i(x_i, s_i + \Delta)) \geq \operatorname{sign}(U_i(x_i + \delta, s_i) - U_i(x_i, s_i)) \qquad (5.7)$$

for each $x_i \in X_i$, $s_i \in \Psi_i$, and $\delta, \Delta \geq 0$, then the correspondences are increasing in the appropriate sense. If each function $U_i(x_i, s_i)$ satisfies the ordinal strategic substitutes condition:

$$\operatorname{sign}(U_i(x_i + \delta, s_i - \Delta) - U_i(x_i, s_i - \Delta)) \geq \operatorname{sign}(U_i(x_i + \delta, s_i) - U_i(x_i, s_i)) \qquad (5.8)$$

for each $x_i \in X_i$, $s_i \in \Psi_i$, and $\delta, \Delta \geq 0$, the correspondences are decreasing in the appropriate sense. Thus, the conditions of Theorems 7 and 8 have a reasonable interpretation.

Games with the maximum (minimum) aggregation are considered in the literature now and then (Bliss and Nalebuff, 1984; Hirshleifer, 1983), although they cannot match in importance, for instance, games with additive aggregation. The acyclicity of systems of increasing (decreasing) singleton reactions with additive aggregation was established in Kukushkin (2000). Multiple reactions were considered in Kukushkin (2001), but only under the finiteness assumption; the case of decreasing reactions could only be handled under a stronger monotonicity condition, and the validity of the result when monotonicity is interpreted exactly as here remains an open question. Taking into account that the maximum aggregate is a coarsening of the leximax one, and the latter is separable exactly like the sum, Theorems 7 and 8 can be regarded as bridgeheads into the area.

The restriction of the maximum to the "neighbouring" indices $I(i)$ destroys the separability (and even the "partial separability" of Segal and Sobel, 2002); one cannot help wondering whether $\max_{j \in I(i)}$ could be replaced with $\sum_{j \in I(i)}$. A positive answer for the case of singleton reactions on finite sets can be easily obtained with the technique developed by Dubey et al. (2002) for different purposes. Unfortunately, the technique resists application to a more general situation.

The closedness of each $R_i(x_{-i})$ is much weaker than a closed graph of each correspondence $R_i(\cdot)$. However, the assumption is not innocuous: it precludes the application of the theorems to games with compact X_i, bounded (but not necessarily upper semicontinuous) utilities, and cardinal strategic complements (substitutes) if each $R_i(x_{-i})$ is the set of ε-optimal responses. Without the assumption, Lemmas 5.2.6 and 5.2.7 (as well as 5.3.6, 5.3.7, and 5.3.8) are just wrong; however, no counterexample to either theorem itself is known.

Finally, instead of $X_i \subseteq \mathbb{R}$, we could assume that $\psi_i(x_{-i}) = \max_{j \in I(i)} \varphi_j(x_j)$, where each $\varphi_j : X_j \to \mathbb{R}$ ($j \in N$) is continuous, and that each R_i is monotonic w.r.t. the orderings defined by ψ_i and φ_i. The scheme of the proofs would remain principally the same, but become even more cumbersome.

6 References

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