Monotonicity Conditions, Monotone Selections, and Equilibria

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March 19, 2009

Abstract

On a partially ordered set a preference relation is given, which depends on an exogenous parameter, the choices of other agents. Various definitions of monotonicity of optimal choices are considered, and restrictions on preferences that ensure monotonicity in one sense or other are obtained. The problem of the existence of monotone selections and fixed points (Nash equilibria) is also studied. *JEL* Classification Numbers: C 72; D 71.

Key words: Best response correspondence; single crossing conditions; quasisupermodular preferences; monotone selection; fixed point

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1 Introduction

Two distinct reasons for the existence of Nash equilibrium were discovered in the 1920s: combinatorial structures of strategy sets, as in games with perfect information, and the concavity of preferences with convex strategy sets, as in mixed extensions of finite games. In recent decades, monotonicity has attracted ever growing attention and become, at least, a successful rival of both predecessors.

McManus (1962, 1964) obtained the existence of Cournot equilibrium in symmetric oligopoly from the "cumulative monotonicity" of the best responses. Topkis (1979) described a class of strategic games where the existence of Nash equilibrium is derivable from Tarski's (1955) fixed point theorem. Novshek (1985) showed that the existence of Cournot equilibrium most often hinges on decreasing best responses even though there was no general fixed point theorem for such mappings at the moment. Bulow et al. (1985) considered the effect of strategy increments on utilities, and coined the terms "strategic substitutes and complements." Impressive advances in the study of the latter property followed (Vives, 1990; Shannon, 1990; Milgrom and Roberts, 1990; Li Calzi and Veinott, 1991). A detailed presentation and a review of relevant economic models can be found, e.g., in Tirole (1988), Fudenberg and Tirole (1991), Topkis (1998).

Milgrom and Shannon (1994) were the first to raise a natural, but very difficult, question: exactly what should be assumed to obtain those results? Their Theorem 4 gave a precise answer concerning comparative statics properties. This paper addresses another version of the same question: what should be assumed about strategy sets and preferences in a game to have increasing best responses (and the existence of a Nash equilibrium to boot)? The very fact that we do not compare optimal choices from different sets makes Milgrom and Shannon's necessity proof irrelevant. A few more differences are worth mentioning.

Their implicit position that the only reasonable way to define an increasing correspondence is with the Veinott order on sublattices is not shared here. We consider quite a number of extensions of an order from a set to its nonempty subsets, including even non-transitive ones, for instance, the "weak Veinott" relation. A particular version of monotonicity for correspondences is worth considering if the existence of a monotone selection can be derived therefrom. Monotone selections are not necessary for the existence of a fixed point of an increasing correspondence, even though Tarski's theorem cannot be invoked otherwise. However, they become practically indispensable, e.g., when the best responses are decreasing and one wants to derive the existence of a Nash equilibrium from the acyclicity of singleton reactions (Kukushkin, 2000, 2003, 2004a, 2006, 2007).

Secondly, we explicitly consider various order structures. A difference between optimization on chains and lattices is shown. In the former case, single crossing conditions (of which we consider four basic versions including two "halves" of Milgrom and Shannon's notion) are necessary and sufficient for monotonicity of the best responses in a single parametric setting. For lattices, characterization results are only obtainable when the same preference relation is put into a wide variety of parametric settings. The quasisupermodularity property is partitioned into four independent constituent parts, each of which is necessary and sufficient (in that sophisticated sense) for a kind (actually, two kinds) of the monotonicity of optima.

We also make two steps beyond lattices. Considering semilattices, a condition on preference orderings is defined which can claim a role similar to that of quasisupermodularity on lattices; unfortunately, its suitability for any interesting application remains unclear. Similar analysis of optimization on general posets, or even directed posets, brings about only discouraging results. Loosely speaking, the common view that lattices constitute the only order structure worth considering in such studies is given some support, but not proven.

Thirdly, we try to go beyond orderings in the description of preferences; the principal interpretation is ε-optimization. The point is that the conventional treatment of relations between properties such as single crossing or quasisupermodularity, on one hand, and the monotonicity of the best responses on the other, heavily relies on the interchangeability of all best responses. If indifference may be intransitive, new techniques are needed. To the best of my knowledge, the previous literature contains no existence result for ε-Nash equilibria in games with strategic complementarities where the existence of the best responses is not guaranteed. Such a result is given by Theorem 8.11 here. The final Theorem 8.12 gives the most general sufficient condition for the existence of a Nash equilibrium because of increasing best responses available at the moment.

The Appendix to Milgrom and Shannon (1994) contains an "almost topology-free" sufficient condition for a utility function on a complete lattice to attain its maximum. Unfortunately, their proof relies on "Theorem A2," ascribed to A.F. Veinott, whereas Example 6.1 from Kukushkin (2007) shows that statement to be plainly wrong. Here we establish the sufficiency of even weaker conditions. However, a plausible conjecture that a quasisupermodular ordering on a complete lattice attains its maximum if it attains a maximum on every complete subchain remains neither proven nor disproved.

Monotonicity considerations also play an important part, e.g., in various incentive and fairness problems (Moulin, 1988), but this topic is not touched on here.

Section 2 reproduces more or less standard definitions, results, and notation. Section 3 starts with a review of various ways to extend an order given on a set to its nonempty subsets. Then a number of sufficient conditions for the existence of a monotone selection are given; the most important are Theorem 3.5 and Theorem 3.10 about weakly ascending, respectively, ascending, correspondences.

Sections 4–6 follow largely the same plan. Considering optimization on, respectively, complete chains, complete lattices, or complete semilattices, we present sufficient conditions for the existence of the best choices (on chains, they are also necessary) first, and then study restrictions on preferences that will ensure the monotonicity of the set of optima in an exogenous factor. Theorems 5.14 and 6.8 provide conditions for the existence of a Nash equilibrium based on the monotonicity.

In Section 7, a similar analysis of monotonicity is attempted on poorer order structures. In the last Section 8, "less rational" preferences are considered.

2 Basic Notions

2.1 Order structures

A *binary relation* on a set A is a Boolean function on $A \times A$; as usual, we write $y \succ x$ when the relation $\succ$ is true on a pair (y, x) and $y \not\succ x$ when it is false. An irreflexive and transitive binary relation is called a *strict order*; a *partial order* is the disjunction of a strict order and the equality relation. A set with a given partial order is called a *poset*; when the order is total, i.e., every two points are comparable, the poset is called a *chain*. An obvious, but important, observation is that, whenever $\succ$ is a strict order, its *reverse*, $y \prec x \rightleftharpoons x \succ y$, is a strict order as well; therefore, every notion and every statement concerning orders admits a *dual* version.

The whole paper is about interrelationships between two orders. One, denoted by symbols like $\succ$, describes preferences of an agent over alternatives from a set A. The other, denoted $>$ (or $\geq$ when appropriate) refers to an internal structure on A, unrelated to preferences. Most attention is paid to specific order structures: chains, lattices, semilattices.

Let A be a poset and $x, y \in A$. A *lower bound* of x and y is $z \in A$ such that $z \leq x$ and $z \leq y$. A poset A is *directed downwards* if a lower bound exists for every pair x and y. A *greatest lower bound* or *meet* of x and y is a lower bound $z \in A$ of theirs such that $z \geq z'$ whenever z' is also a lower bound of x and y. Clearly, the meet is unique if exists at all; we denote it $x \wedge y$. A poset A is a *meet-semilattice* if $x \wedge y$ exists for every pair $x, y \in A$; if $X \subseteq A$ contains $x \wedge y$ for every pair $x, y \in X$, it is called a *meet-subsemilattice* of A.

An *upper bound*, a poset *directed upwards*, a *least upper bound* or *join* of $x, y \in A$, denoted $x \vee y$, a *join-semilattice*, and a *join-subsemilattice* are defined dually. In the following we view the "meet" versions as basic, and use the words "semilattice" and "subsemilattice" accordingly.

A *lattice* is a poset A in which $x \vee y$ and $x \wedge y$ exist for every pair $x, y \in A$. A *sublattice* of A is $X \subseteq A$ which contains both $x \vee y$ and $x \wedge y$ for every pair $x, y \in X$. Given $x, y \in A$, we denote $L(x, y) := \{x, y, x \vee y, x \wedge y\}$, the minimal sublattice of A containing both x and y; clearly, $\# L(x, y) \in \{1, 2, 4\}$. A chain is the simplest example of a lattice, $x \vee y$ and $x \wedge y$ being, respectively, the maximum and minimum of x and y; every subset of a chain is a sublattice.

For every set A, we denote $\mathfrak{B}_A$ the lattice of all subsets of A and $\mathfrak{B}_A^{\emptyset} := \mathfrak{B}_A \setminus \{\emptyset\}$. Let A be a poset and $X \in \mathfrak{B}_A$. A *lower bound* of X is $z \in A$ such that $z \leq x$ for every $x \in X$. A *greatest lower bound* or *meet* of X is a lower bound $z \in A$ of X such that $z \geq z'$ whenever z' is also a lower bound of X. As in the case of two points, the meet is unique if exists at all; we denote it $\bigwedge X$. An *upper bound* and *join* of $X \subseteq A$ are defined dually; we use the notation $\bigvee X$. Note that $\bigwedge \emptyset = \bigvee A$ and $\bigvee \emptyset = \bigwedge A$ by definition. When $X \in \mathfrak{B}_A^{\emptyset}$ is a chain, we use more conventional notation $\inf X := \bigwedge X$ and $\sup X := \bigvee X$.

A lattice is (*relatively*) *complete* if $\bigwedge X$ and $\bigvee X$ exist for all (bounded below or above, respectively) $X \in \mathfrak{B}_A^{\emptyset}$. If a chain is (relatively) complete as a lattice, we call it just a (*relatively*) *complete chain*. A semilattice is (*relatively*) *complete* if $\bigwedge X$ exists for every (bounded below) $X \in \mathfrak{B}_A^{\emptyset}$ and $\sup X$ exists for every (bounded above) chain $X \in \mathfrak{B}_A^{\emptyset}$. If A is a relatively complete lattice, $X \in \mathfrak{B}_A^{\emptyset}$ is a *complete sublattice* of A if $\bigwedge Y$ and $\bigvee Y$ exist and belong to X for all $Y \in \mathfrak{B}_X^{\emptyset}$. A *complete subsemilattice* is defined similarly.

2.2 Preferences

The preference relation $\succ$ is always assumed to be a strict order. Usually, we also add a "rationality requirement"; most often, the relation is an *ordering*, i.e., *negatively transitive* strict order, $z \not\succ y \not\succ x \Rightarrow z \not\succ x$. Every total order is an ordering. In Section 8, the preference relation is a *semiorder* or an *interval order*; the definitions are given there.

Orderings can also be defined in terms of representations in chains: $\succ$ is an ordering if and only if there is a chain $\mathcal{C}$ and a mapping $v \colon A \to \mathcal{C}$ such that $y \succ x \iff v(y) > v(x)$ for all $x, y \in A$. The most usual assumption in game theory is that the preferences of the player are described by a *utility function* $v \colon A \to \mathbb{R}$. Here we work in a purely ordinal framework, so it is natural to replace $\mathbb{R}$ with an arbitrary chain. If $\succ$ is an ordering, then the "non-strict preference" relation $\succeq$ defined by $y \succeq x \rightleftharpoons x \not\succ y$ is reflexive, transitive, and total.

Given $X \in \mathfrak{B}_A^{\emptyset}$, we denote

$$M(X, \succ) := \{x \in X \mid \nexists y \in X \, [y \succ x]\}, \tag{2.1}$$

the set of *maximizers* of $\succ$ on X. The interpretation is that the agent has preferences over the whole A, but may be faced with the necessity to choose from a subset $X \in \mathfrak{B}_A^{\emptyset}$, in which case any point from

$M(X, \succ)$ will do. For such choice to be possible, we need $M(X, \succ) \neq \emptyset$, at least, for "plausible" X. A helpful observation is that every strict order admits a maximizer on every finite nonempty subset of A. Another helpful observation is that $y \succ x$ whenever $\succ$ is an ordering, $x, y \in X$, and $x \notin M(X, \succ) \ni y$.

Of primary importance for us are *parametric optimization problems.* Let A and S be posets, and an ordering on $A \times S$ be represented by a mapping $u\colon A \times S \to \mathcal{C}$, where $\mathcal{C}$ is a chain. Given $X \in \mathfrak{B}_A^{\emptyset}$, the *best response correspondence* $\mathcal{R}^X\colon S \to \mathfrak{B}_X \subseteq \mathfrak{B}_A$ is defined in the usual way:

$$\mathcal{R}^X(s) := \operatorname*{Argmax}_{x \in X} u(x, s) = M(X, \succ^s), \tag{2.2}$$

where $\succ^s$ is the ordering on X represented by the mapping $v_s(\cdot) := u(\cdot, s)$. We are interested in conditions on u ensuring monotonicity, in one sense or another, of $\mathcal{R}^X$. The monotonicity of a single correspondence $\mathcal{R}^X$ may happen just "by accident"; however, when a wide enough class of admissible subsets is taken into account, necessity results become obtainable.

The simplest connection between preferences and order is monotonicity. An ordering $\succ$ on a poset A is (*strictly*) *increasing* if $y \succeq x$ ($y \succ x$) whenever $y > x$; dually, $\succ$ is (*strictly*) *decreasing* if $y \succeq x$ ($y \succ x$) whenever $x > y$. Naturally, we are interested in less straightforward connections.

Whenever $\succ$ is a strict order on a poset A, we consider two auxiliary strict orders:

$$y \mathrel{\underset{>}{\succ}} x \rightleftharpoons [y \succ x \;\&\; y > x]; \tag{2.3a}$$

$$y \mathrel{\underset{<}{\succ}} x \rightleftharpoons [y \succ x \;\&\; y < x]. \tag{2.3b}$$

If $\succ$ is an ordering, we also consider two "weak" versions:

$$y \mathrel{\underset{>}{\succeq}} x \rightleftharpoons [y \succeq x \;\&\; y > x]; \tag{2.3c}$$

$$y \mathrel{\underset{<}{\succeq}} x \rightleftharpoons [y \succeq x \;\&\; y < x]. \tag{2.3d}$$

2.3 Zorn's Lemma and transfinite recursion

When it comes to the "existence" of something, we assume the Axiom of Choice in its strongest form. If purists say that nothing more than the "impossibility of non-existence" is thus proven, so be it. Technically, some well-known corollaries (or equivalent re-formulations) are more convenient to use than the axiom itself.

Theorem A (Zorn's Lemma). *Let $>$ be a strict order on a set X with the property that every chain $Y \in \mathfrak{B}_X^{\emptyset}$ has an upper bound in X. Then $M(X, >) \neq \emptyset$.*

Zorn's Lemma plays an important part in the proof of Theorem 3.5 below. The following useful technical statement is easily derivable from it.

Theorem B (Szpilrajn's Theorem). *On every poset, there exists a strictly increasing total order.*

Zorn's Lemma is helpful in the presentation of various notions of completeness. A poset A is called (*relatively*) *chain-complete* if $\sup X$ and $\inf X$ exist for every (bounded below or above, respectively) chain $X \in \mathfrak{B}_A^{\emptyset}$. If A is a relatively chain-complete poset and $X \in \mathfrak{B}_A^{\emptyset}$, we call X a *chain-complete subset* if $\sup Y$ and $\inf Y$ exist and belong to X for every chain $Y \in \mathfrak{B}_X^{\emptyset}$; if X itself is a chain, we call it a *complete subchain.*

Proposition 2.1. *If A is a chain-complete lattice, then $\bigvee A$ and $\bigwedge A$ exist in A. If A is a chain-complete semilattice, then $\bigwedge A$ exists in A.*

Proof. Given $x \in A$, we denote $X := \{y \in A \mid y \geq x\}$. Clearly, X is chain-complete, hence $M(X, >) \neq \emptyset$ by Zorn' Lemma. By definition of X, we have $y \geq x$ for every $y \in M(X, >)$. Moreover, $M(X, >) \subseteq M(A, >)$. Now if $x, y \in M(A, >)$ and $x \neq y$, then $x \vee y > x$: a contradiction. Therefore, $M(A, >)$ is a singleton, $\{\bigvee A\}$. The existence of $\bigwedge A$ is proven dually. The case of a semilattice is quite similar. □

Proposition 2.2. *Every [relatively] chain-complete (semi)lattice is a [relatively] complete (semi)lattice.*

Proof. Let A be a relatively chain-complete lattice and $X \in \mathfrak{B}_A^{\emptyset}$ be bounded above. We denote $X^- := \{y \in A \mid \forall x \in X\, [y \leq x]\}$. Clearly, X^- is chain-complete and is a sublattice. By Proposition 2.1, there exists $\bigvee X^-$, which is obviously $\bigwedge X$. The existence of $\bigvee X$ for bounded below $X \in \mathfrak{B}_A^{\emptyset}$ is proven dually.

If A is chain-complete, every subset of A is bounded by $\bigvee A$ or $\bigwedge A$, existing by Proposition 2.1. The case of a (relatively) chain-complete semilattice is treated similarly. □

Proposition 2.3. *Let A be a relatively complete (semi)lattice and $X \in \mathfrak{B}_A^{\emptyset}$ be a sub(semi)lattice and a chain-complete subset of A. Then X is a complete sub(semi)lattice of A.*

Proof. By Proposition 2.1, X is a complete lattice. Therefore, $\bigwedge_X Y \in X$ exists for every $Y \in \mathfrak{B}_X^{\emptyset}$; we only have to show that $\bigwedge_X Y = \bigwedge_A Y$. By definition, $\bigwedge_X Y \leq \bigwedge_A Y$. We denote $Y^- := \{x \in X \mid x \geq \bigwedge_A Y \;\&\; \forall y \in Y\, [y \geq x]\}$. Clearly, Y^- is chain-complete and is a sublattice. By Proposition 2.1, there exists $y^* := \bigvee Y^- \in Y^-$. Now we have $y^* \geq \bigwedge_A Y$ by the definition of Y^-; simultaneously, $y^* \leq \bigwedge_X Y$ because $y^* \in X$ and $y^* \leq y$ for all $y \in Y$. Thus, $\bigwedge_X Y \geq \bigwedge_A Y$, hence there is an equality. The equality $\bigvee_X Y = \bigvee_A Y$ is proven dually.

The case of a chain-complete subsemilattice X of a relatively complete semilattice A is treated similarly. □

In the light of Propositions 2.2 and 2.3, there is no point in distinguishing between completeness and chain-completeness; in the following, we use the latter term only in formulations of the results. The set of all complete subchains of a relatively complete poset A is denoted $\mathfrak{C}_A$. Given a relatively complete (semi)lattice A, the set of all complete sub(semi)lattices is denoted $\mathfrak{L}_A$ ($\mathfrak{S}_A$).

Although it is possible to argue that every statement dependent on the Axiom of Choice could be given a proof based on Zorn's Lemma, the practical implementation of the idea could lead to constructions unbearably artificial. A reasonable alternative may be Zermelo's Theorem.

A poset is *well ordered* if every subset contains a least point (then the set obviously must be a chain).

Theorem C (Zermelo's Theorem). *Every set can be well ordered.*

This statement is indispensable in the proof of Theorem 3.10 below. The proofs of Theorems 4.2, 5.3, etc. are based on a more powerful (and heavier) technique.

Zermelo's Theorem implies the existence of an infinite well ordered set Λ with a cardinality greater than that of A. For technical convenience, we assume that $\max \Lambda$ does not exist, hence, for each $\alpha \in \Lambda$, its *successor*, denoted $\alpha + 1$, is uniquely defined as $\min\{\beta \in \Lambda \mid \beta > \alpha\}$. We denote $0 := \min \Lambda$ and $[0, \alpha[:= \{\beta \in \Lambda \mid \beta < \alpha\}$; note that $\alpha \notin [0, \alpha[$.

The principle of *transfinite recursion* allows us to consider a mapping $\lambda\colon \Lambda \to X \in \mathfrak{B}_A^{\emptyset}$ well defined if we have defined $\lambda(0) \in X$ and described how $\lambda(\alpha) \in X$ should be constructed, given $\lambda(\beta) \in X$ for all $\beta \in [0, \alpha[$. Practically, the definition of $\lambda(\alpha+1)$ is usually based on $\lambda(\alpha)$ alone, so *all* $\beta < \alpha$ are only involved when α is a *limit ordinal*, i.e., not the successor to any $\beta \in \Lambda$.

When proving an existence theorem with this technique, we construct such a mapping λ with the property that an equality $\lambda(\alpha') = \lambda(\alpha)$ with $\alpha' > \alpha$ is only possible when $\lambda(\alpha)$ is a point we need. Since the cardinality of Λ is greater than that of X, the equality must occur at some stage.

3 Monotonicity

Let S and X be two posets. A mapping $r \colon S \to X$ is *increasing* if $y > x \Rightarrow r(y) \geq r(x)$. As is well known, the property is conducive to the existence of fixed points.

Theorem D (Tarski, 1955). *Let X be a complete lattice and $r \colon X \to X$ be increasing. Then there exists a fixed point of r, i.e., $x \in X$ such that $r(x) = x$. Moreover, the set of fixed points is a complete lattice itself (although not necessarily a complete sublattice of X).*

Theorem E (Abian and Brown, 1961). *Let X be a chain-complete poset, $r \colon X \to X$ be increasing and $r(\bar{x}) \geq \bar{x}$ for an $\bar{x} \in X$. Then there exists a fixed point of r.*

Theorem F (Markowsky, 1976). *Let X be a chain-complete poset containing $\bigwedge X$ and $r \colon X \to X$ be increasing. Then there exists a fixed point of r. Moreover, the set of fixed points is chain-complete and contains a smallest point itself (although it need not be a complete subset of X).*

Remark. Here, those fixed point theorems are used to derive the existence of Nash equilibria in strategic games; no attempt to describe the structure of the set of equilibria, cf. Zhou (1994), is made.

The existence of fixed points of *decreasing* mappings is a much trickier subject (Novshek, 1985; Kukushkin, 1994, 2007; Dubey et al., 2006), which is not touched here at all. It is only worth mentioning that the reversal of the order on S (or, equivalently, on X) transforms an increasing mapping $r \colon S \to X$ into decreasing, and vice versa; therefore, there would be no need for a separate study of monotonicity conditions.

When it comes to *correspondences* $R \colon S \to \mathfrak{B}_X$, it is not quite obvious exactly how their monotonicity "should" be defined. In the next subsection, we consider various ways to extend an order given on A to $\mathfrak{B}_A^{\emptyset}$. Then we study relationships between monotonicity w.r.t. those relations and the existence of monotone selections.

3.1 Extensions of an order to subsets

Let $Y, X \in \mathfrak{B}_A^{\emptyset}$ (the empty subset often behaves in a strange fashion). When A is a lattice, Veinott's order (Topkis, 1978) seems most popular. We define it as a conjunction of the "lower and upper halves," and also define a weak version:

$$Y \geq^{\wedge} X \rightleftharpoons \forall y \in Y \, \forall x \in X \, [y \wedge x \in X]; \tag{3.1a}$$

$$Y \geq^{\vee} X \rightleftharpoons \forall y \in Y \, \forall x \in X \, [y \vee x \in Y]; \tag{3.1b}$$

$$Y \geq^{\mathrm{Vt}} X \rightleftharpoons [Y \geq^{\vee} X \; \& \; Y \geq^{\wedge} X]; \tag{3.1c}$$

$$Y \geq^{\mathrm{wV}} X \rightleftharpoons \forall y \in Y \, \forall x \in X \, [y \vee x \in Y \text{ or } y \wedge x \in X]. \tag{3.1d}$$

The relation $\geq^{\mathrm{Vt}}$ is antisymmetric and transitive on $\mathfrak{B}_A^{\emptyset}$, hence its reflexive closure is a partial order; actually, it is reflexive on sublattices. Neither $\geq^{\mathrm{wV}}$, nor $\geq^{\wedge}$ or $\geq^{\vee}$ need even be transitive although the last two *are* transitive when A is a chain. It is worth noting that $\geq^{\wedge}$ ($\geq^{\vee}$) is defined when A is a (join-)semilattice.

These relations can be defined when A is just a poset:

$$Y \geq^{\mathrm{Inf}} X \rightleftharpoons \forall y \in Y\, \forall x \in X\, \exists x' \in X\, [y \geq x' \;\&\; x \geq x']; \tag{3.1e}$$

$$Y \geq^{\mathrm{inf}} X \rightleftharpoons \forall y \in Y\, \exists x \in X\, [y \geq x]; \tag{3.1f}$$

$$Y \geq^{\mathrm{Sup}} X \rightleftharpoons \forall y \in Y\, \forall x \in X\, \exists y' \in Y\, [y' \geq x \;\&\; y' \geq y]; \tag{3.1g}$$

$$Y \geq^{\mathrm{sup}} X \rightleftharpoons \forall x \in X\, \exists y \in Y\, [y \geq x]; \tag{3.1h}$$

$$Y \geq^{\mathrm{pV}} X \rightleftharpoons [Y \geq^{\mathrm{Inf}} X \;\&\; Y \geq^{\mathrm{Sup}} X]; \tag{3.1i}$$

$$Y \geq^{\mathrm{RS}} X \rightleftharpoons [Y \geq^{\mathrm{inf}} X \;\&\; Y \geq^{\mathrm{sup}} X]. \tag{3.1j}$$

All of them are transitive. $\geq^{\mathrm{inf}}$ and $\geq^{\mathrm{sup}}$ are reflexive; $\geq^{\mathrm{Inf}}$ and $\geq^{\mathrm{Sup}}$ are reflexive on directed (upwards or downwards, respectively) subsets. Obviously, $Y \geq^{\wedge} X \Rightarrow Y \geq^{\mathrm{Inf}} X \Rightarrow Y \geq^{\mathrm{inf}} X$ and $Y \geq^{\vee} X \Rightarrow Y \geq^{\mathrm{Sup}} X \Rightarrow Y \geq^{\mathrm{sup}} X$, hence $Y \geq^{\mathrm{Vt}} X \Rightarrow Y \geq^{\mathrm{pV}} X \Rightarrow Y \geq^{\mathrm{RS}} X$; the converse implications are wrong. If A is a chain, then $\geq^{\mathrm{Inf}}$ is equivalent to $\geq^{\mathrm{inf}}$, while $\geq^{\mathrm{Sup}}$ to $\geq^{\mathrm{sup}}$, and both are orderings; A itself is then a greatest point in $\mathfrak{B}_A^{\emptyset}$ for $\geq^{\mathrm{Sup}}$ and a least for $\geq^{\mathrm{Inf}}$. An analogue of "weak Veinott's" relation can also be defined on an arbitrary poset A:

$$Y \geq^{\mathrm{pwV}} X \rightleftharpoons \forall y \in Y\, \forall x \in X\, \big[\exists x' \in X\, [y \geq x' \;\&\; x \geq x'] \;\text{ or }\; \exists y' \in Y\, [y' \geq x \;\&\; y' \geq y]\big]. \tag{3.1k}$$

Another relation is much stronger:

$$Y \ggg X \rightleftharpoons \forall y \in Y\, \forall x \in X\, [y \geq x]. \tag{3.1l}$$

The relation $\ggg$ is transitive and antisymmetric; it is reflexive on singletons. Clearly, $Y \ggg X$ implies every other relation (3.1) that can be defined on A.

Given posets A and S, a definition of an increasing mapping $R\colon S \to \mathfrak{B}_A^{\emptyset}$ can be based on each relation (3.1) (perhaps assuming the appropriate structure on A). For instance, a correspondence $R\colon S \to \mathfrak{B}_A^{\emptyset}$, where S is a poset and A is a lattice, is *ascending* (Topkis, 1978, 1998) if it increases w.r.t. $\geq^{\mathrm{Vt}}$ while every $R(s)$ is a sublattice of A. Our definition of monotonicity contains a *strict* inequality in the left hand side, so the reflexivity on individual values is not required; not that this is of much importance for anything, but a greater flexibility is allowed. For Roddy and Schröder (2005), the "most natural" notion of an increasing correspondence was that based on the relation $\geq^{\mathrm{RS}}$.

3.2 Monotone selections

Let X and S be posets and $R\colon S \to \mathfrak{B}_X$. Then a *monotone selection* from R is an increasing mapping $r\colon S \to X$ such that $r(s) \in R(s)$ for every $s \in S$. An obvious necessary condition for the existence of a monotone selection is $R(s) \neq \emptyset$ for every $s \in S$.

Proposition 3.1. *Let X and S be posets and $R\colon S \to \mathfrak{B}_X^{\emptyset}$. Then every single-valued selection r from R is increasing if and only if R is increasing w.r.t. $\ggg$.*

A straightforward proof is omitted.

The best-known results on the existence of monotone selections (Topkis, 1998) are applicable to ascending correspondences to complete lattices. Similar results hold under weaker monotonicity conditions.

Proposition 3.2.a. *A correspondence R from a poset S to a relatively chain-complete poset X admits a monotone selection if it is increasing w.r.t. $\geq^{\mathrm{Inf}}$ and every $R(s)$ is a chain-complete subset of X.*

Proof. For every $s \in S$, we denote $R^{-}(s) := \{x \in R(s) \mid \nexists y \in R(s)\,[y < x]\}$. Since $R(s)$ is chain-complete, Zorn's Lemma immediately implies that $R^{-}(s) \neq \emptyset$. Then we pick $r(s) \in R^{-}(s)$ arbitrarily. If $s' > s$, then, by (3.1e), there is $x \in R(s)$ such that $x \leq r(s)$ and $x \leq r(s')$. Since $r(s) \in R^{-}(s)$, we must have $x = r(s)$. □

Corollary. *A correspondence R from a poset S to a relatively complete semilattice X admits a monotone selection if it is increasing w.r.t. $\geq^{\wedge}$ and every $R(s)$ is a chain-complete subset of X.*

Proposition 3.2.b. *A correspondence R from a poset S to a relatively chain-complete poset X admits a monotone selection if it is increasing w.r.t. $\geq^{\mathrm{Sup}}$ and every $R(s)$ is a chain-complete subset of X.*

Example 6.1 from Kukushkin (2007) shows that the chain-completeness of $R(s)$ cannot be dropped in Propositions 3.2 or their corollaries. The replacement of $\geq^{\mathrm{Inf}}$ with $\geq^{\mathrm{inf}}$ also makes Proposition 3.2.a wrong; however, the existence of a fixed point may be provable even in the absence of monotone selections.

Example 3.3. Let $S = X := \{(0,0),(0,1),(1,0),(1,1)\} \subset \mathbb{R}^2$; $R(0,0) := \{(0,1),(1,0)\}$; $R(0,1) := \{(0,1)\}$; $R(1,0) := \{(1,0)\}$; $R(1,1) := \{(1,1)\}$. It is easy to see that R is increasing w.r.t. $\geq^{\mathrm{inf}}$, but there is no monotone selection.

Proposition 3.4. *Let X be a finite poset containing $\bigvee X$ and $R\colon X \to \mathfrak{B}_X^{\emptyset}$ be increasing w.r.t. $\geq^{\mathrm{inf}}$. Then there exists a fixed point of R, i.e., $x \in X$ such that $x \in R(x)$.*

Proof. If $\bigvee X \in R(\bigvee X)$, we have a fixed point. Assuming the contrary, we define $x^0 := \bigvee X$ and pick $x^1 \in R(x^0)$ arbitrarily; clearly, $x^1 < x^0$. Then we recursively define a sequence $x^k \in X$ such that $x^{k+1} \in R(x^k)$ and $x^k \geq x^{k+1}$ for each k. Since X is finite, the sequence must stabilize at some stage; but then we have a fixed point. Given $x^{k+1} < x^k$ and $x^{k+1} \in R(x^k)$, we have $R(x^k) \geq^{\mathrm{inf}} R(x^{k+1})$, hence we can pick $x^{k+2} \in R(x^{k+1})$ such that $x^{k+2} \leq x^{k+1}$. □

When X is a lattice, Corollary to Proposition 3.2.a can be strengthened at the price of a lengthier proof.

Theorem 3.5. *Let X be a relatively complete lattice, S be a poset, $R\colon S \to \mathfrak{B}_X^{\emptyset}$ be increasing w.r.t. $\geq^{\mathrm{wV}}$, and every $R(s)$ ($s \in S$) be a chain-complete subset of X. Then there exists a monotone selection from R.*

Remark. Probably, this result was proven by A.F. Veinott. Milgrom and Shannon (1994) ascribed to him their "Theorem A2," which is actually wrong, see Kukushkin (2007, Example 6.1). The chain-completeness assumption was lacking there, while there was a superfluous assumption that S is a *net*. Proposition 6.2 from Kukushkin (2007) established the existence of a monotone selection under a superfluous assumption that every $R(s)$ is a complete *sublattice*. Most likely, the replacement of the references to "Theorem A2" with those to our Theorem 3.5 makes Milgrom and Shannon's proof of their Theorem A3 valid (actually, stronger existence results can be obtained in a more direct way, see Theorems 5.3 below).

Proof. For every $s \in S$, we denote $R^+(s) := \{x \in R(s) \mid \nexists y \in R(s)\,[y > x]\}$ and $R^-(s) := \{x \in R(s) \mid \nexists y \in R(s)\,[y < x]\}$. Exactly as in the proof of Proposition 3.2.a, Zorn's Lemma immediately implies that $R^-(s) \neq \emptyset \neq R^+(s)$ for every $s \in S$. Moreover, for every $s \in S$ and $x \in R(s)$, there exists $x^* \in R^-(s)$ such that $x^* \leq x$.

Lemma 3.5.1. *If $s' > s$, $y \in R^+(s')$ and $x \in R^-(s)$, then $y \geq x$.*

Proof. Otherwise, we would have $y \vee x > y$ and $y \wedge x < x$; therefore, $y \vee x \notin R(s')$ and $y \wedge x \notin R(s)$, contradicting $R(s') \geq^{\mathrm{wV}} R(s)$. □

We start with an arbitrary mapping $r^0\colon S \to X$ such that $r^0(s) \in R^+(s)$ for every $s \in S$. We define $\mathcal{S}$ as the set of pairs $\langle \bar{S}, r\rangle$ such that $\bar{S} \subseteq S$, $r\colon S \to X$ is a selection from R, and two conditions hold:

$$\forall s \in S \setminus \bar{S}\,[r(s) = r^0(s)];$$

$$\forall s \in \bar{S}\,\forall s' \in S\,[s' > s \Rightarrow r(s') \geq r(s)].$$

Clearly, $\langle \emptyset, r^0\rangle \in \mathcal{S} \neq \emptyset$. We define a partial order on $\mathcal{S}$ by

$$\langle \bar{S}, r\rangle \geq \langle \bar{S}', r'\rangle \rightleftharpoons \bigl[\bar{S} \supseteq \bar{S}' \;\&\; \forall s \in \bar{S}'\,[r(s) \geq r'(s)]\bigr].$$

Applying Zorn's Lemma (Theorem A in Subsection 2.3), we show the existence of a maximizer of the order on $\mathcal{S}$. Given a chain $\{\langle \bar{S}^\alpha, r^\alpha\rangle\}_\alpha \subseteq \mathcal{S}$, we define $\bar{S}^\infty := \bigcup_\alpha \bar{S}^\alpha$; $r^\infty(s) := r^0(s)$ for $s \notin \bar{S}^\infty$; $r^\infty(s) := \sup_{\alpha\colon s \in \bar{S}^\alpha} r^\alpha(s)$ for $s \in \bar{S}^\infty$. Clearly, $\mathcal{S} \ni \langle \bar{S}^\infty, r^\infty\rangle \geq \langle \bar{S}^\alpha, r^\alpha\rangle$ for every α.

Let $\langle \bar{S}, r^*\rangle$ be a maximizer. By definition, r^* is a selection from R; if $\bar{S} = S$, it is increasing and we are home. Supposing $s^* \in S \setminus \bar{S}$, we denote $S^+ := \{s \in S \mid s > s^*\}$, $\bar{S}^- := \{s \in \bar{S} \mid s < s^*\}$, and $x^- := \bigvee_{s \in \bar{S}^-} r^*(s)$. If $\bar{S}^- \neq \emptyset$, then x^- exists because $r^*(s^*) \geq r^*(s)$ for every $s \in \bar{S}^-$ by the definition of $\mathcal{S}$; if $\bar{S}^- = \emptyset$, then x^- is not needed at all in the following (technically, "$x^- := -\infty$"). Now we have $r^*(s) \geq x^-$ for every $s \in S^+ \cup \{s^*\}$; if we replace $R(s)$ for all those s with $\{x \in R(s) \mid x \geq x^-\}$, then both chain-completeness and the monotonicity w.r.t. $\geq^{\mathrm{wV}}$ will obviously survive. In other words, we henceforth assume that $x \geq x^-$ for every $s \in S^+ \cup \{s^*\}$ and $x \in R(s)$.

Now we pick $x^* \in R^-(s^*)$ such that $x^* \leq r^*(s^*)$. We denote $Z^+ := \{s \in S^+ \mid r^*(s) \geq x^*\}$ and $Z^- := S^+ \setminus Z^+$. If $Z^- = \emptyset$, we may define $r^{**}(s) := r^*(s)$ for all $s \neq s^*$ and $r^{**}(s^*) := x^*$; clearly, $\mathcal{S} \ni \langle \bar{S} \cup \{s^*\}, r^{**}\rangle > \langle \bar{S}, r^*\rangle$, contradicting the maximality of the latter. Let $Z^- \neq \emptyset$; then $Z^- \subseteq \bar{S}$ by Lemma 3.5.1 since $r^0(s) \in R^+(s)$ for every $s \in \bar{S}$. We define $r^{**}(s) := r^*(s)$ for $s \in S \setminus (Z^- \cup \{s^*\})$, $r^{**}(s^*) := x^*$, and $r^{**}(s) := r^*(s) \vee x^*$ for $s \in Z^-$. Again, $\mathcal{S} \ni \langle \bar{S} \cup \{s^*\}, r^{**}\rangle > \langle \bar{S}, r^*\rangle$, contradicting the maximality of the latter. □

Corollary. *Let X be a finite lattice, S be a poset, and $R\colon S \to \mathfrak{B}_X^{\emptyset}$ be increasing w.r.t. $\geq^{\mathrm{wV}}$. Then there exists a monotone selection from R.*

Without topological restrictions on values $R(s)$, the existence of a monotone selection can be obtained either for a weakly ascending correspondence from a finite poset, or for an ascending correspondence.

Proposition 3.6. *Let X be a lattice, S be a finite poset, and $R\colon S \to \mathfrak{B}_X^{\emptyset}$ be increasing w.r.t. $\geq^{\mathrm{wV}}$. Then there exists a monotone selection from R.*

Proof. For every $s \in S$ and $x^* \in X$, we denote $R(s; x^*) := \{x \in R(s) \mid x \leq x^*\}$.

Lemma 3.6.1. *Let $s^* \in M(S, >)$. Then there exists $x^* \in R(s^*)$ for which $R(s; x^*) \neq \emptyset$ for every $s \in S^- := \{s \in S \mid s < s^*\}$.*

Proof. For every $x \in R(s^*)$, we denote $Z^+(x) := \{s \in S^- \mid R(s; x^*) \neq \emptyset\}$ and $Z^-(x) := S^- \setminus Z^+(x)$. Then we pick $x^0 \in R(s^*)$ arbitrarily and define a sequence $x^0, x^1, \dots$ by recursion. Let $x^k \in R(s^*)$ have been defined; if $Z^+(x^k) = S^-$, we take x^k as x^* and finish the process. Otherwise, we pick $s \in Z^-(x^k)$ arbitrarily; we have $R(s^*) \geq^{\mathrm{wV}} R(s)$. Picking $x \in R(s)$ arbitrarily, we apply (3.1d) with $y = x^k$. Since $R(s; x^k) = \emptyset$, we have $x \wedge x^k \notin R(s)$, hence $x \vee x^k \in R(s^*)$. Defining $x^{k+1} := x \vee x^k > x^k$, we obtain $Z^+(x^k) \subset Z^+(x^{k+1})$. Since S is finite, the sequence must stabilize at some stage, i.e., reach the situation $Z^+(x^k) = S^-$. $\square$

Now the proposition is proven with straightforward recursion: we pick an undominated $s^* \in S$; define $r(s^*) := x^*$, taking x^* from Lemma 3.6.1; replace $R(s)$ with $R(s; x^*)$ for each $s \in S^-$ – the modified correspondence remains increasing w.r.t. $\geq^{\mathrm{wV}}$; and then apply the same procedure to $S \setminus \{s^*\}$. $\square$

Proposition 3.7. *Let X be a poset, S be a finite chain, and $R\colon S \to \mathfrak{B}_X^{\emptyset}$ be increasing w.r.t. $\geq^{\inf}$. Then there exists a monotone selection from R.*

A straightforward proof (somewhat similar to that of Proposition 3.6) is omitted. On the other hand, the replacement of $\geq^{\mathrm{wV}}$ with $\geq^{\mathrm{pwV}}$ defined by (3.1k) makes Proposition 3.6 wrong.

Example 3.8. Let $S = X := \{0, 1, 2\} \subset \mathbb{R}$; $R(0) := \{1\}$; $R(1) := \{0, 2\}$; $R(2) := \{1\}$. It is easy to see that R is increasing w.r.t. $\geq^{\mathrm{pwV}}$, but there is neither monotone selection, nor fixed point.

Proposition 3.9 (Danilov, 2007). *Let X be a lattice, S be a countable poset, and $R\colon S \to \mathfrak{B}_X^{\emptyset}$ be increasing w.r.t. $\geq^{\mathrm{Vt}}$; then there exists a monotone selection from R.*

Theorem 3.10. *Let X be a sublattice of the Cartesian product of a finite number of chains, $X \subseteq \prod_{m \in M} C_m$; let S be a poset and $R\colon S \to \mathfrak{B}_X^{\emptyset}$ be increasing w.r.t. $\geq^{\mathrm{Vt}}$. Then there exists a monotone selection from R.*

Remark. Theorem 6 of Kukushkin (2007) established the existence of a monotone selection under the assumption that X is just a chain. To the best of my knowledge, there is no example of a correspondence increasing w.r.t. $\geq^{\mathrm{Vt}}$, but admitting no monotone selection.

Proof. First, we assume M totally ordered, say, $M = \{0, 1, \dots, \bar{m}\}$; then, invoking Theorem C from Subsection 2.3, we assume each C_m well ordered with an order $\gg_m$ (having nothing to do with the basic order on C_m). Then $X \subseteq \prod_{m \in M} C_m$ is well ordered by the lexicographic combination: Given $y \neq x$, we denote $D(y, x) := \{m \in M \mid y_m \neq x_m\}$, $d := \min D$, and $y \gg x \rightleftharpoons y_d \gg_d x_d$.

We define $r(s) := \min R(s)$ (w.r.t. $\gg$); it exists and is unique. Now r is a selection from R by definition; let us show it is increasing. Let $s' > s$, $y = r(s')$ and $x = r(s)$; since R is increasing w.r.t. $\geq^{\mathrm{Vt}}$, we have $y \wedge x \in R(s)$ and $y \vee x \in R(s')$. The set $D(y, x) := \{m \in M \mid y_m \neq x_m\}$ is partitioned into $D^+ := \{m \in M \mid y_m > x_m\}$ and $D^- := \{m \in M \mid y_m < x_m\}$. If $D^- = \emptyset$, then $y \geq x$ and we are home. Supposing the contrary, we notice that $D^- = D(y, y \vee x) = D(x, y \wedge x)$; let $d := \min D^-$. Since $y \vee x \gg y$ by the definition of r, we have $x_d \gg_d y_d$, but then $x \gg y \wedge x$: a contradiction. $\square$

The replacement of $\geq^{\mathrm{Vt}}$ with the "pseudo-Veinott" order $\geq^{\mathrm{pV}}$ defined by (3.1i) makes Theorem 3.10 wrong even if X is a chain, unless S is a finite chain hence Proposition 3.7 applies. The fact is demonstrated by Example 2.3 from Roddy and Schröder (2005), where $S = X$ and there is no fixed point either; the example is countable, which fact proves that $\geq^{\mathrm{Vt}}$ cannot be replaced with $\geq^{\mathrm{pV}}$ in Proposition 3.9 as well.

4 Optimization on Chains

4.1 Existence

Given a relatively chain-complete poset A and $X \in \mathfrak{C}_A$, we denote $X^{\downarrow} := X \setminus \{\inf X\}$ and $X^{\uparrow} := X \setminus \{\sup X\}$. We consider two rather weak versions of Milgrom and Shannon's (1994) upper semicontinuity on chains, assuming $\succ$ a strict order on A:

$$\forall X \in \mathfrak{C}_A \left[\left(\inf X^{\downarrow} = \inf X \;\&\; \forall x, y \in X^{\downarrow} \left[x > y \Rightarrow y \succ x \right] \right) \Rightarrow \forall x \in X^{\downarrow} \left[\inf X \succ x \right] \right]; \tag{4.1a}$$

$$\forall X \in \mathfrak{C}_A \left[\left(\sup X^{\uparrow} = \sup X \;\&\; \forall x, y \in X^{\uparrow} \left[y > x \Rightarrow y \succ x \right] \right) \Rightarrow \forall x \in X^{\uparrow} \left[\sup X \succ x \right] \right]. \tag{4.1b}$$

Proposition 4.1.a. *Let Y be a chain-complete poset and $\succ$ be a strict order on Y satisfying* (4.1a). *Then for every $x \in Y \setminus M(Y, \underset{<}{\succ})$ (where $\underset{<}{\succ}$ is defined by* (2.3b)*), there is $y \in M(Y, \underset{<}{\succ})$ such that $y \underset{<}{\succ} x$.*

Remark. The statement obviously implies $M(Y, \underset{<}{\succ}) \neq \emptyset$.

Proof. Let Λ be a well ordered set with a cardinality greater than that of Y. We construct, by (transfinite) recursion, a mapping $\lambda \colon \Lambda \to Y$ such that:

$$\forall \alpha, \beta \in \Lambda \left[\alpha > \beta \Rightarrow [\lambda(\alpha) = \lambda(\beta) \;\text{ or }\; \lambda(\alpha) \underset{<}{\succ} \lambda(\beta)] \right]; \tag{4.2a}$$

$$\forall \alpha \in \Lambda \left[\lambda([0, \alpha]) \in \mathfrak{C}_Y \right]. \tag{4.2b}$$

First, we define $\lambda(0) := x$. Let $\lambda(\alpha)$ have been defined. If $\lambda(\alpha) \in M(Y, \underset{<}{\succ})$, we define $\lambda(\alpha + 1) := \lambda(\alpha)$; actually, $\lambda(\alpha') = \lambda(\alpha)$ for all $\alpha' > \alpha$ in this case. Otherwise, we pick $\lambda(\alpha+1) \underset{<}{\succ} \lambda(\alpha)$ arbitrarily. Both requirements (4.2) continue to hold for $\alpha + 1$.

If $\alpha \in \Lambda$ is a limit ordinal and $\lambda(\beta)$ has been defined for all $\beta < \alpha$, we define $\lambda(\alpha) := \inf_{\beta<\alpha} \lambda(\beta)$, ensuring $\lambda([0, \alpha]) \in \mathfrak{C}_Y$. Now (4.1a) implies $\lambda(\alpha) \underset{<}{\succ} \lambda(\beta)$ for every $\beta < \alpha$ unless $\lambda(\alpha) = \lambda(\beta)$.

The final argument is straightforward. An equality $\lambda(\alpha') = \lambda(\alpha)$ with $\alpha' > \alpha$ is only possible when $\lambda(\alpha) \in M(Y, \underset{<}{\succ})$. Since the cardinality of Λ is greater than that of X, the equality must occur at some stage. Since $\lambda(\alpha) \underset{<}{\succ} \lambda(0) = x$, we are home. $\square$

Proposition 4.1.b. *Let Y be a chain-complete poset and $\succ$ be a strict order on Y satisfying* (4.1b). *Then for every $x \in Y \setminus M(Y, \underset{>}{\succ})$ (where $\underset{>}{\succ}$ is defined by* (2.3a)*), there is $y \in M(Y, \underset{>}{\succ})$ such that $y \underset{>}{\succ} x$.*

The proof is dual to that of Proposition 4.1.a.

Theorem 4.2. *Let A be a relatively chain-complete poset and $\succ$ be a strict order on A satisfying both conditions* (4.1). *Then $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$.*

Proof. The basic construction is similar to that from the proof of Proposition 4.1.a. We use both auxiliary strict orders $\underset{>}{\succ}$ and $\underset{<}{\succ}$ defined by (2.3). Let Λ be a well ordered set with a cardinality greater than that of A. Given X, we construct, by (transfinite) recursion, a mapping $\lambda \colon \Lambda \to X$ such that:

$$\forall \alpha \in \Lambda \left[\lambda(\alpha) \in M(X, \underset{<}{\succ}) \right];$$

$$\forall \alpha, \beta \in \Lambda \left[\alpha > \beta \Rightarrow [\lambda(\alpha) = \lambda(\beta) \ \text{ or } \ \lambda(\alpha) \mathrel{\underset{>}{\succ}} \lambda(\beta)]\right];$$
$$\forall \alpha \in \Lambda \left[\lambda([0, \alpha]) \in \mathfrak{C}_X\right].$$

First, we define $\lambda(0) := \min X \in M(X, \mathrel{\underset{<}{\succ}})$. Let $\lambda(\alpha)$ have been defined. If $\lambda(\alpha) \in M(X, \succ)$, we define $\lambda(\alpha + 1) := \lambda(\alpha)$; actually, $\lambda(\alpha') = \lambda(\alpha)$ for all $\alpha' > \alpha$ in this case. Otherwise, we must have $x \in X$ such that $x \mathrel{\underset{>}{\succ}} \lambda(\alpha)$. We denote $Y := \{y \in X \mid y \leq x\} \ni x$. If $x \in M(Y, \mathrel{\underset{<}{\succ}})$, we define $\lambda(\alpha + 1) := x$; otherwise, we apply Proposition 4.1.a, obtaining $\lambda(\alpha + 1) \in M(Y, \mathrel{\underset{<}{\succ}})$ such that $\lambda(\alpha+1) \mathrel{\underset{<}{\succ}} x$. In either case, we have $\lambda(\alpha+1) \succ \lambda(\alpha)$, hence $\lambda(\alpha+1) > \lambda(\alpha)$ because $\lambda(\alpha) \in M(X, \mathrel{\underset{<}{\succ}})$, and $\lambda(\alpha + 1) \in M(X, \mathrel{\underset{<}{\succ}})$ because $y \mathrel{\underset{<}{\succ}} \lambda(\alpha + 1)$ would imply $y \in Y$.

Let $\alpha \in \Lambda$ be a limit ordinal and $\lambda(\beta)$ have been defined for all $\beta < \alpha$. Defining $\lambda(\alpha) := \sup_{\beta<\alpha} \lambda(\beta)$, we ensure $\lambda([0, \alpha]) \in \mathfrak{C}_X$. Now (4.1b) implies $\lambda(\alpha) \mathrel{\underset{>}{\succ}} \lambda(\beta)$ for every $\beta < \alpha$ unless $\lambda(\alpha) = \lambda(\beta)$, so we only have to show that $\lambda(\alpha) \in M(X, \mathrel{\underset{<}{\succ}})$. The existence of $y \in X$ such that $y \mathrel{\underset{<}{\succ}} \lambda(\alpha)$ would imply $y < \lambda(\beta)$ for some $\beta < \alpha$. Since $\lambda(\alpha) \succ \lambda(\beta)$ or $\lambda(\alpha) = \lambda(\beta)$, we have a contradiction with $\lambda(\beta) \in M(X, \mathrel{\underset{<}{\succ}})$.

The final argument is again straightforward. An equality $\lambda(\alpha') = \lambda(\alpha)$ with $\alpha' > \alpha$ is only possible when $\lambda(\alpha) \in M(X, \succ)$. Since the cardinality of Λ is greater than that of X, the equality must occur at some stage. $\square$

Remark. Arguing similarly to the proof of Theorem 1 from Kukushkin (2008), it is easy to show that conditions (4.1) are necessary and sufficient for the choice function $M(\cdot, \succ)$ to be nonempty-valued and path independent on $\mathfrak{C}_A$.

Corollary. *Let $\succ$ be an ordering on a relatively chain-complete poset A. Then $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$ if and only if both conditions* (4.1) *hold.*

Proof. Sufficiency immediately follows from Theorem 4.2. If (4.1a) or (4.1b) is violated for $X \in \mathfrak{C}_A$, then $M(X, \succ) = \emptyset$. $\square$

Remark. Actually, the corollary holds when $\succ$ is a semiorder. Example 3 from Kukushkin (2008) shows that the necessity does not hold even for interval orders.

Assuming $\succ$ an ordering, we introduce two more versions of upper semicontinuity on chains.

$$\forall X \in \mathfrak{C}_A \left[\left(\inf X^{\downarrow} = \inf X \ \& \ \forall x, y \in X^{\downarrow} [x > y \Rightarrow y \succeq x]\right) \Rightarrow \forall x \in X \, [\inf X \succeq x]\right]; \tag{4.3a}$$

$$\forall X \in \mathfrak{C}_A \left[\left(\sup X^{\uparrow} = \sup X \ \& \ \forall x, y \in X^{\uparrow} [y > x \Rightarrow y \succeq x]\right) \Rightarrow \forall x \in X \, [\sup X \succeq x]\right]. \tag{4.3b}$$

It is easy to see that (4.3a)$\Rightarrow$(4.1a) while (4.3b)$\Rightarrow$(4.1b).

Proposition 4.3.a. *Let Y be a chain-complete poset and $\succ$ be an ordering on Y satisfying* (4.3a). *Then for every $x \in Y \setminus M(Y, \mathrel{\underset{<}{\succeq}})$ (where $\mathrel{\underset{<}{\succeq}}$ is defined by* (2.3d)*), there is $y \in M(Y, \mathrel{\underset{<}{\succeq}})$ such that $y \mathrel{\underset{<}{\succeq}} x$.*

Proposition 4.3.b. *Let Y be a chain-complete poset and $\succ$ be an ordering on Y satisfying* (4.3b). *Then for every $x \in Y \setminus M(Y, \mathrel{\underset{>}{\succeq}})$ (where $\mathrel{\underset{>}{\succeq}}$ is defined by* (2.3c)*), there is $y \in M(Y, \mathrel{\underset{>}{\succeq}})$ such that $y \mathrel{\underset{>}{\succeq}} x$.*

Straightforward modifications of the proof of Proposition 4.1.a suffice. The remark after that proposition is relevant here as well.

Theorem 4.4. *Let $\succ$ be an ordering on a relatively chain-complete poset A. Then $M(X, \succ) \in \mathfrak{C}_A$ for every $X \in \mathfrak{C}_A$ if and only if both conditions* (4.3) *hold.*

Proof. Let (4.3a) hold, $X \in \mathfrak{C}_A$ and $Y \in \mathfrak{B}^{\emptyset}_X$; we have to show that $\inf Y \in M(X, \succ)$. We denote $Z := \{x \in X \mid x \geq \inf Y\}$; since $Y \subseteq M(X, \succ)$, we have $Z \cap M(X, \succ) \neq \emptyset$; since $X \in \mathfrak{C}_A$, we have $\inf Y \in Z \in \mathfrak{C}_A$. By Proposition 4.3.a, there exists $y \in M(Z, \overset{\succeq}{<}) \cap M(X, \succ)$. If $y = \inf Y$, we are home; otherwise, $y > \inf Y$, hence there is $z \in Z \cap M(X, \succ)$ such that $z < y$, which contradicts the assumption about y.

Conversely, let (4.3a) be violated for $X \in \mathfrak{C}_A$. If, for every $x \in X$, there is $y \in X$ such that $y \overset{\succ}{<} x$, then we can easily produce $Y \in \mathfrak{C}_A$ violating (4.1a), hence $M(Y, \succ) = \emptyset$. Otherwise, without restricting generality, $x \sim y$ for all $x, y \in X \setminus \{\inf X\}$, hence $M(X, \succ) = X \setminus \{\inf X\} \notin \mathfrak{C}_A$.

Both sufficiency and necessity of (4.3b) are shown in a dual way. □

Remark. Corollary to Theorem 4.2 [Theorem 4.4] remains valid if the condition (4.1b) [(4.3b)] is restricted to well ordered chains, while condition (4.1a) [(4.3a)] to "reversed" well ordered chains. Moreover, in each case it is sufficient only to consider chains of the minimal length in their cardinality. The similarity with Smith's (1974) Theorems 4.1 and 4.2 is manifest.

Proposition 4.5. *Let $\succ$ be an ordering on a relatively chain-complete poset A satisfying both conditions* (4.3)*; let X be a chain-complete subset of A such that $M(X, \succ) \neq \emptyset$. Then $M(X, \succ)$ is a chain-complete subset of X, hence of A as well.*

A straightforward proof is omitted.

Given a relatively chain-complete poset A, a poset S, and a chain $\mathcal{C}$, we call a mapping $u\colon A \times S \to \mathcal{C}$ *regular* if every ordering $\succ^s$ on A represented by $u(\cdot, s)$, $s \in S$, satisfies both conditions (4.1). We call u *strongly regular* if every ordering $\succ^s$ on A represented by $u(\cdot, s)$, $s \in S$, satisfies both conditions (4.3). Theorems 4.2 and 4.4, respectively, immediately imply that $\mathcal{R}^X(s) \neq \emptyset$ [$\mathcal{R}^X(s) \in \mathfrak{C}_A$] for all $s \in S$ whenever u is [strongly] regular, A is relatively chain-complete, and $X \in \mathfrak{C}_A$.

4.2 Monotonicity

It is well known that the monotonicity of best responses hinges on "single crossing" conditions of various kinds (Milgrom and Shannon, 1994). As Savvateev (2007) observed, those conditions are most conveniently presented with the help of a ternary relation on the set of binary relations on a given set: "$\rhd_1$ is closer to $\rhd_0$ than $\rhd_2$ is." In the following, the role of $\rhd_0$ is always played by (the asymmetric component of) the basic order on A, while $\rhd_1$ and $\rhd_2$ are preferences under different exogenous parameters.

Let $\succ$ and $\succ'$ be orderings on a poset A. We consider four conditions:

$$\forall x, y \in A \left[y \overset{\succ}{>} x \Rightarrow y \succ' x\right]; \tag{4.4a}$$

$$\forall x, y \in A \left[y \overset{\succeq}{>} x \Rightarrow y \succeq' x\right]; \tag{4.4b}$$

$$\forall x, y \in A \left[y \overset{\succeq}{>} x \Rightarrow y \succ' x\right]; \tag{4.4c}$$

$$\forall x, y \in A \left[y \overset{\succ}{>} x \Rightarrow y \succeq' x\right]. \tag{4.4d}$$

Each condition defines a binary relation on the set of orderings on A. The first two are preorders. The third is transitive, but generally not reflexive. The last relation need not even be transitive.

Given two posets A and S and a chain $\mathcal{C}$, we say that $u\colon A \times S \to \mathcal{C}$ satisfies the *lower single crossing* condition if (4.4a) is satisfied whenever $s, s' \in S$, $s' > s$, $u(\cdot, s)$ represents $\succ$, and $u(\cdot, s')$ represents $\succ'$. Similarly, u satisfies the *upper* [*strict* or *weak*] *single crossing* condition if (4.4b) [(4.4c) or (4.4d)] is satisfied under the same circumstances. Clearly, u satisfies the *single crossing* condition (Milgrom and Shannon, 1994) if both upper and lower single crossing conditions hold.

For more convenience in further referencing, we consider four "dual" versions of conditions (4.4):

$$\forall x, y \in A \left[y \overset{\succ}{\leq} x \Rightarrow y \succeq' x\right]; \tag{4.5a}$$

$$\forall x, y \in A \left[y \overset{\succ}{<} x \Rightarrow y \succ' x\right]; \tag{4.5b}$$

$$\forall x, y \in A \left[y \overset{\succ}{\leq} x \Rightarrow y \succ' x\right]; \tag{4.5c}$$

$$\forall x, y \in A \left[y \overset{\succ}{<} x \Rightarrow y \succeq' x\right]. \tag{4.5d}$$

It is easily checked that each condition (4.5) is equivalent to the corresponding condition (4.4) after the exchange of the roles of $\succ$ and $\succ'$. Therefore, the (upper, lower, strict, or weak) single crossing conditions could be defined with references to (4.5) as well.

For $\mathcal{C} = \mathbb{R}$, the single crossing condition can be expressed as a single formula with the help of the function $\operatorname{sign}(t)$ on $\mathbb{R}$, which is -1 if $t < 0$, 0 if $t = 0$, and 1 if $t > 0$:

$$[s' > s \;\&\; y > x] \Rightarrow \operatorname{sign}\big(u(y, s') - u(x, s')\big) \geq \operatorname{sign}\big(u(y, s) - u(x, s)\big); \tag{4.6}$$

although subtraction is used in the formulation, the property itself is purely ordinal. Obviously, (4.6) is implied by Topkis's (1979) cardinal *increasing differences* condition:

$$[s' > s \;\&\; y > x] \Rightarrow u(y, s') - u(x, s') \geq u(y, s) - u(x, s), \tag{4.7}$$

while the strict single crossing condition is implied by a version of (4.7) with a strict inequality in the right hand side. As is well known, (4.7) is symmetric w.r.t. both arguments of $u(\cdot, \cdot)$, while (4.6) is not. The weak single crossing condition holds when the differences may decrease, but not "too much." Note that both (4.7) and its strict version are invariant under increasing affine transformations, i.e., relate to an NM utility function u.

Proposition 4.6. *Let A and S be posets, $\mathcal{C}$ be a chain, and u be a mapping $A \times S \to \mathcal{C}$. Then the following statements are equivalent.*

1. *u satisfies the lower single crossing condition.*

2. *There holds $\mathcal{R}^X(s') \geq^{\wedge} \mathcal{R}^X(s)$ whenever $s', s \in S$, $s' > s$, and $X \in \mathfrak{B}_A^{\emptyset}$ is a chain such that $\mathcal{R}^X(s') \neq \emptyset \neq \mathcal{R}^X(s)$.*

3. *There holds $\mathcal{R}^X(s') \geq^{\inf} \mathcal{R}^X(s)$ whenever $s', s \in S$, $s' > s$, and $X \in \mathfrak{B}_A^{\emptyset}$ is a finite chain.*

Proof. Let Statement 1 hold, $s', s \in S$, $s' > s$, and $\mathcal{R}^X(s') \neq \emptyset \neq \mathcal{R}^X(s)$. We have to show $\mathcal{R}^X(s') \geq^{\wedge} \mathcal{R}^X(s)$; let $y \in \mathcal{R}^X(s')$ and $x \in \mathcal{R}^X(s)$. If $y \geq x$, we are home immediately; let $x > y$. If $y \in \mathcal{R}^X(s)$, we are home again. If $y \notin \mathcal{R}^X(s)$, then $u(x, s) > u(y, s)$, hence $u(x, s') > u(y, s')$ by (4.4a), contradicting the assumption $y \in \mathcal{R}^X(s')$.

Let Statement 1 be violated: there are $s', s \in S$ and $x, y \in A$ such that $s' > s$, $y > x$, $u(y, s) > u(x, s)$, but $u(x, s') \geq u(y, s')$. Then we define $X := \{x, y\}$ and immediately obtain $x \in \mathcal{R}^X(s') \setminus \mathcal{R}^X(s)$, hence $\mathcal{R}^X(s') \geq^{\inf} \mathcal{R}^X(s)$ does not hold, i.e., Statement 3 is invalid. $\square$

Proposition 4.7. *Let A and S be posets, $\mathcal{C}$ be a chain, and u be a mapping $A \times S \to \mathcal{C}$. Then the following statements are equivalent.*

1. *u satisfies the upper single crossing condition.*

2. *There holds $\mathcal{R}^X(s') \geq^{\vee} \mathcal{R}^X(s)$ whenever $s', s \in S$, $s' > s$, and $X \in \mathfrak{B}_A^{\emptyset}$ is a chain such that $\mathcal{R}^X(s') \neq \emptyset \neq \mathcal{R}^X(s)$.*

3. *There holds $\mathcal{R}^X(s') \geq^{\sup} \mathcal{R}^X(s)$ whenever $s', s \in S$, $s' > s$, and $X \in \mathfrak{B}_A^{\emptyset}$ is a finite chain.*

The proof is dual to that of Proposition 4.6.

Corollary. *Let A and S be posets, $\mathcal{C}$ be a chain, and u be a mapping $A \times S \to \mathcal{C}$. Then the following statements are equivalent.*

1. *u satisfies the single crossing condition.*

2. *There holds $\mathcal{R}^X(s') \geq^{\vee\mathrm{t}} \mathcal{R}^X(s)$ whenever $s', s \in S$, $s' > s$, and $X \in \mathfrak{B}_A^{\emptyset}$ is a chain such that $\mathcal{R}^X(s') \neq \emptyset \neq \mathcal{R}^X(s)$.*

3. *There hold $\mathcal{R}^X(s') \geq^{\inf} \mathcal{R}^X(s)$ and $\mathcal{R}^X(s') \geq^{\sup} \mathcal{R}^X(s)$ whenever $s', s \in S$, $s' > s$, and $X \in \mathfrak{B}_A^{\emptyset}$ is a finite chain.*

Theorem 4.8. *Let A be a relatively chain-complete poset, S be a poset, $\mathcal{C}$ be a chain, and u be a regular mapping $A \times S \to \mathcal{C}$ satisfying the single crossing condition. Then there exists a monotone selection from $\mathcal{R}^X$ on S for every $X \in \mathfrak{C}_A$.*

Proof. For every $X \in \mathfrak{C}_A$ and $s \in S$, we have $\mathcal{R}^X(s) \neq \emptyset$ by Theorem 4.2 while $\mathcal{R}^X$ is increasing w.r.t. $\geq^{\vee\mathrm{t}}$ by the previous Corollary; therefore, a monotone selection exists by Theorem 3.10. $\square$

Remark. Example 6.1 from Kukushkin (2007) shows that the lower (or upper) single crossing condition is not enough here; however, there is no clear prospect for a necessity result.

Proposition 4.9. *Let A and S be posets, $\mathcal{C}$ be a chain, and u be a mapping $A \times S \to \mathcal{C}$. Then the following statements are equivalent.*

1. *u satisfies the strict single crossing condition.*

2. *There holds $\mathcal{R}^X(s') \ggg \mathcal{R}^X(s)$ whenever $s', s \in S$, $s' > s$, and $X \in \mathfrak{B}_A^{\emptyset}$ is a chain such that $\mathcal{R}^X(s') \neq \emptyset \neq \mathcal{R}^X(s)$.*

3. *There holds $\mathcal{R}^X(s') \ggg \mathcal{R}^X(s)$ whenever $s', s \in S$, $s' > s$, and $X \in \mathfrak{B}_A^{\emptyset}$ is a finite chain.*

Proof. Let the strict single crossing condition hold, $s', s \in S$, $s' > s$, and $\mathcal{R}^X(s') \neq \emptyset \neq \mathcal{R}^X(s)$. We have to show $\mathcal{R}^X(s') \ggg \mathcal{R}^X(s)$; let $y \in \mathcal{R}^X(s')$ and $x \in \mathcal{R}^X(s)$. If $y \geq x$, we are home; let $x > y$. We have $u(x,s) \geq u(y,s)$ since $x \in \mathcal{R}^X(s)$; applying (4.4c), we obtain $u(x,s') > u(y,s')$, which contradicts $y \in \mathcal{R}^X(s')$.

Let the strict single crossing condition be violated: there are $s', s \in S$ and $x, y \in A$ such that $s' > s$, $y > x$, $u(y,s) \geq u(x,s)$, but $u(x,s') \geq u(y,s')$. Then we define $X := \{x, y\}$ and immediately obtain $y \in \mathcal{R}^X(s)$ while $x \in \mathcal{R}^X(s')$, hence $\mathcal{R}^X(s') \ggg \mathcal{R}^X(s)$ does not hold. $\square$

Proposition 4.10. *Let A and S be posets, $\mathcal{C}$ be a chain, and u be a mapping $A \times S \to \mathcal{C}$. Then the following statements are equivalent.*

1. *u satisfies the weak single crossing condition.*

2. *There holds $\mathcal{R}^X(s') \geq^{\mathrm{wV}} \mathcal{R}^X(s)$ whenever $s', s \in S$, $s' > s$, and $X \in \mathfrak{B}_A^{\emptyset}$ is a chain such that $\mathcal{R}^X(s') \neq \emptyset \neq \mathcal{R}^X(s)$.*

3. *There holds $\mathcal{R}^X(s') \geq^{\mathrm{wV}} \mathcal{R}^X(s)$ whenever $s', s \in S$, $s' > s$, and $X \in \mathfrak{B}_A^{\emptyset}$ is a finite chain.*

Proof. Let the weak single crossing condition hold, $s', s \in S$, $s' > s$, and $\mathcal{R}^X(s') \neq \emptyset \neq \mathcal{R}^X(s)$. We have to show $\mathcal{R}^X(s') \geq^{\mathrm{wV}} \mathcal{R}^X(s)$; let $y \in \mathcal{R}^X(s')$ and $x \in \mathcal{R}^X(s)$. If $y \geq x$, we are home; let $x > y$. We have to show that either $y \in \mathcal{R}^X(s)$ or $x \in \mathcal{R}^X(s')$. Since $x \in \mathcal{R}^X(s)$, we have $u(x,s) \geq u(y,s)$. If $u(x,s) = u(y,s)$, we are home; otherwise, we apply (4.4d), obtaining $u(x,s') \geq u(y,s')$, which implies $x \in \mathcal{R}^X(s')$.

Let the weak single crossing condition be violated: there are $s', s \in S$ and $x, y \in A$ such that $s' > s$, $y > x$, $u(y,s) > u(x,s)$, but $u(x,s') > u(y,s')$. Then we define $X := \{x, y\}$ and immediately obtain $\mathcal{R}^X(s) = \{y\}$ while $\mathcal{R}^X(s') = \{x\}$, hence $\mathcal{R}^X(s') \geq^{\mathrm{wV}} \mathcal{R}^X(s)$ does not hold. □

Theorem 4.11. *Let A be a relatively chain-complete poset, S be a poset, $\mathcal{C}$ be a chain, and u be a strongly regular mapping $A \times S \to \mathcal{C}$. Then u satisfies the weak single crossing condition if and only if there exists a monotone selection from $\mathcal{R}^X$ on S for every $X \in \mathfrak{C}_A$.*

Proof. If u satisfies the weak single crossing condition, then $\mathcal{R}^X$ is increasing w.r.t. $\geq^{\mathrm{wV}}$ by Proposition 4.10; $\mathcal{R}^X(s) \in \mathfrak{C}_A$ for every $s \in S$ by Proposition 4.5. Therefore, a monotone selection exists by Theorem 3.5. If (4.4d) is violated, we argue in exactly the same way as in the necessity proof in Proposition 4.10 and see that no monotone selection is possible even on $\{s, s'\} \subseteq S$. □

Naturally, one does not have to be satisfied with maximization on chains, although scalar strategies are met in economics models most often. The *necessity* of single crossing conditions, obviously, holds on any class of admissible subsets that contains all finite chains. The sufficiency is not so robust.

Example 4.12. Let $A := \{(0,0), (0,1), (1,0), (1,1)\} \subset \mathbb{R}^2$, $S := \{0, 1\}$, and a function $u\colon A \times S \to \mathbb{R}$ be defined by the following matrices (the axes are directed upwards and rightwards):

$$\begin{array}{cc} s = 0 & s = 1 \\ \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} & \begin{bmatrix} 5 & 4 \\ 0 & 6 \end{bmatrix} \end{array}$$

Clearly, u satisfies the strict single crossing condition, actually, even the strict version of (4.7). However, $\mathcal{R}(0) = \{(0,1)\}$, while $\mathcal{R}(1) = \{(1,0)\}$. Therefore, even $\mathcal{R}(1) \geq^{\mathrm{pwV}} \mathcal{R}(0)$ does not hold (and neither $\mathcal{R}(1) \geq^{\inf} \mathcal{R}(0)$, nor $\mathcal{R}(1) \geq^{\sup} \mathcal{R}(0)$ for that matter); there is no monotone selection either.

5 Optimization on Lattices

5.1 Existence

A function v on a lattice A is *supermodular* if

$$\forall x, y \in A \left[v(x \vee y) - v(x) \geq v(y) - v(x \wedge y) \right]. \tag{5.1}$$

Milgrom and Shannon (1994) called a function v on a lattice A *quasisupermodular* if

$$\forall x, y \in A \left[\mathrm{sign}\big(v(x \vee y) - v(y)\big) \geq \mathrm{sign}\big(v(x) - v(x \wedge y)\big) \right]. \tag{5.2}$$

Clearly, (5.1) implies (5.2). The former condition is cardinal (refers to an NM utility), while the latter is purely ordinal and can easily be reformulated in terms of binary relations (Savvateev, 2007):

$$\forall x, y \in A\,\big[x \succ y \wedge x \Rightarrow y \vee x \succ y\big]; \tag{5.3a}$$
$$\forall x, y \in A\,\big[y \succ y \vee x \Rightarrow y \wedge x \succ x\big]. \tag{5.3b}$$

We replace conditions (5.3) with a conjunction of four independent conditions:

$$\forall x, y \in A\,\big[x \succ y \wedge x \Rightarrow [(y \vee x \succ y) \ \text{ or } \ (y \vee x \succ x)]\big]; \tag{5.4a}$$
$$\forall x, y \in A\,\big[y \succeq y \vee x \Rightarrow [(y \wedge x \succeq x) \ \text{ or } \ (y \wedge x \succeq y)]\big]; \tag{5.4b}$$
$$\forall x, y \in A\,\big[x \succeq y \wedge x \Rightarrow [(y \vee x \succeq y) \ \text{ or } \ (y \vee x \succeq x)]\big]; \tag{5.4c}$$
$$\forall x, y \in A\,\big[y \succ y \vee x \Rightarrow [(y \wedge x \succ x) \ \text{ or } \ (y \wedge x \succ y)]\big]. \tag{5.4d}$$

Remark. Each condition (5.4) holds trivially when x and y are comparable in the basic order.

Proposition 5.1. *An ordering on a lattice satisfies both conditions* (5.3) *if and only if it satisfies all conditions* (5.4)*.*

Proof. The necessity is obvious. To prove the sufficiency, we suppose the contrary. Let $x \succ y \wedge x$, but $y \succeq y \vee x$; then $y \vee x \succ x$ by (5.4a), hence $y \succ y \wedge x$ by transitivity, which contradicts (5.4b). The proof of the equivalence (5.3b) $\equiv$ [(5.4c) & (5.4d)] is dual. □

Example 5.2. Let $A := \{(0,0),(0,1),(1,0),(1,1)\} \subset \mathbb{R}^2$; we consider four orderings on A represented by these matrices (the axes are directed upwards and rightwards):

$$\textbf{a.}\begin{bmatrix}0 & 0\\ 1 & 2\end{bmatrix} \qquad \textbf{b.}\begin{bmatrix}2 & 2\\ 0 & 1\end{bmatrix} \qquad \textbf{c.}\begin{bmatrix}1 & 0\\ 2 & 2\end{bmatrix} \qquad \textbf{d.}\begin{bmatrix}2 & 1\\ 0 & 0\end{bmatrix}.$$

The ordering represented by the matrix "a" satisfies all conditions (5.4) except (5.4a), and similarly with other matrices.

Theorem 5.3.a. *Let A be a relatively complete lattice and $\succ$ be an ordering on A satisfying* (5.4d)*,* (4.1a)*, and* (4.3b)*. Then $M(X,\succ) \neq \emptyset$ for every $X \in \mathfrak{L}_A$ (where $\mathfrak{L}_A$ denotes the set of all nonempty complete sublattices $X \subseteq A$).*

Proof. The basic construction is similar to that in the proof of Theorem 4.2. We again use both auxiliary strict orders $\underset{>}{\succ}$ and $\underset{<}{\succ}$ defined by (2.3).

Lemma 5.3.1. *If X is a sublattice of A, $x \in M(X, \underset{<}{\succ})$ and $X \ni y \succ x$, then $y \vee x \succeq y\,[\succ x]$.*

Proof. If $y \succ y \vee x$, then condition (5.4d) applies. The second disjunctive term in the right hand side would imply the first one, and the latter is impossible under the condition $x \in M(X, \underset{<}{\succ})$. □

Let Λ be a well ordered set with a cardinality greater than that of A. Given X, we construct, by (transfinite) recursion, a mapping $\lambda\colon \Lambda \to X$ such that:

$$\forall \alpha \in \Lambda\,\big[\lambda(\alpha) \in M(X, \underset{<}{\succ})\big]; \tag{5.5a}$$
$$\forall \alpha, \beta \in \Lambda\,\big[\alpha > \beta \Rightarrow [\lambda(\alpha) = \lambda(\beta) \ \text{ or } \ \lambda(\alpha) \underset{>}{\succ} \lambda(\beta)]\big]; \tag{5.5b}$$

$$\forall\alpha \in \Lambda \left[\lambda([0,\alpha]) \in \mathfrak{C}_X\right]. \tag{5.5c}$$

First, we define $\lambda(0) := \bigwedge X \in M(X, \underset{<}{\succ})$. Let $\lambda(\alpha)$ have been defined. If $\lambda(\alpha) \in M(X, \succ)$, we define $\lambda(\alpha+1) := \lambda(\alpha)$; actually, $\lambda(\alpha') = \lambda(\alpha)$ for all $\alpha' > \alpha$ in this case. Otherwise, Lemma 5.3.1 implies the existence of $x^* \in X$ such that $x^* \underset{>}{\succ} \lambda(\alpha)$. We denote $Y := \{y \in X \mid \lambda(\alpha) \leq y \leq x^*\} \ni x^*$. If $x^* \in M(Y, \underset{<}{\succ})$, we define $\lambda(\alpha+1) := x^*$; otherwise, we apply Proposition 4.1.a, obtaining $\lambda(\alpha+1) \in M(Y, \underset{<}{\succ})$ such that $\lambda(\alpha+1) \underset{<}{\succ} x^*$. In either case, we have $\lambda(\alpha+1) \succ \lambda(\alpha)$, hence $\lambda(\alpha+1) \underset{>}{\succ} \lambda(\alpha)$. Let us show $\lambda(\alpha+1) \in M(X, \underset{<}{\succ})$. Supposing the contrary, $y \underset{<}{\succ} \lambda(\alpha+1)$, we apply Lemma 5.3.1 to $x = \lambda(\alpha)$ and y, obtaining $\lambda(\alpha) \vee y \succeq y \succ \lambda(\alpha+1)$; since $\lambda(\alpha+1) > \lambda(\alpha)$ and $\lambda(\alpha+1) > y$, we have $\lambda(\alpha+1) \geq \lambda(\alpha) \vee y > \lambda(\alpha)$, hence $\lambda(\alpha) \vee y \in Y$ and $\lambda(\alpha) \vee y \underset{<}{\succ} \lambda(\alpha+1)$, which contradicts the choice of $\lambda(\alpha+1)$.

Let $\alpha \in \Lambda$ be a limit ordinal and $\lambda(\beta)$ have been defined for all $\beta < \alpha$. Then we define $\lambda(\alpha) := \sup_{\beta<\alpha} \lambda(\beta)$, ensuring that $\lambda([0,\alpha]) \in \mathfrak{C}_X$. Now (4.1b) implies $\lambda(\alpha) \underset{>}{\succ} \lambda(\beta)$ for every $\beta < \alpha$ unless $\lambda(\alpha) = \lambda(\beta)$, so we only have to show that $\lambda(\alpha) \in M(X, \underset{<}{\succ})$.

If $\lambda(\alpha) = \lambda(\beta)$ for some $\beta < \alpha$, we are home immediately. Otherwise, we suppose the contrary, $y \underset{<}{\succ} \lambda(\alpha)$, and recursively construct a mapping $\mu\colon [0,\alpha] \to X$ such that $\mu(\beta) = \mu(\gamma)$ or $\mu(\beta) \underset{>}{\succeq} \mu(\gamma)$ whenever $\beta > \gamma$. We define $\mu(0) := y$; whenever $\mu(\beta)$ has been defined, $\mu(\beta+1) := \mu(\beta) \vee \lambda(\beta)$; whenever β is a limit ordinal and $\mu(\gamma)$ have been defined for all $\gamma < \beta$, we define $\mu(\beta) := \sup_{\gamma<\beta} \mu(\gamma)$. Both monotonicity requirements are easily checked by Lemma 5.3.1 and (4.3b).

Now we have $\mu(\alpha) \succeq \mu(0) = y \succ \lambda(\alpha)$. On the other hand, $\lambda(\alpha) > y$ and $\lambda(\alpha) > \lambda(\beta)$ for all $\beta < \alpha$; straightforward induction shows that $\lambda(\alpha) \geq \mu(\beta)$ for all $\beta < \alpha$, hence $\lambda(\alpha) \geq \mu(\alpha)$. Besides, we have $\mu(\beta) \geq \lambda(\beta)$ for all $\beta < \alpha$, hence $\mu(\alpha) \geq \lambda(\alpha)$. Therefore, $\lambda(\alpha) = \mu(\alpha)$, hence $\lambda(\alpha) \succ \lambda(\alpha)$. The contradiction proves that $\lambda(\alpha) \in M(X, \underset{<}{\succ})$ indeed.

The final argument is standard. An equality $\lambda(\alpha') = \lambda(\alpha)$ with $\alpha' > \alpha$ is only possible when $\lambda(\alpha) \in M(X, \succ)$. Since the cardinality of Λ is greater than that of X, the equality must occur at some stage. □

Remark. If X is a complete chain, all conditions (5.4) become vacuous when restricted to $x, y \in X$, while (4.3b) can be replaced with (4.1b) (Theorem 4.2 above). If X is not a chain, (4.1b) is not sufficient for the above proof to remain valid, but no counterexample to the statement itself is known.

Theorem 5.3.b. *Let A be a relatively complete lattice and $\succ$ be an ordering on A satisfying* (5.4a), (4.1b)*, and* (4.3a)*. Then $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{L}_A$.*

The proof is dual to that of Theorem 5.3.a.

Theorem 5.3.c. *Let A be a relatively complete lattice and $\succ$ be an ordering on A satisfying* (5.4c) *and both conditions* (4.3)*. Then $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{L}_A$.*

Proof. The general scheme of the proof is the same as in Theorem 5.3.a. We replace (5.5a) with $\lambda(\alpha) \in M(X, \underset{<}{\succeq})$, and make an appropriate modification of Lemma 5.3.1. The reference to Proposition 4.1.a is replaced with that to Proposition 4.3.a; (4.3a) is needed here rather than (4.1a); (4.3b) is needed for the same reasons. □

Theorem 5.3.d. *Let A be a relatively complete lattice and $\succ$ be an ordering on A satisfying* (5.4b) *and both conditions* (4.3)*. Then $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{L}_A$.*

The proof is dual to that of Theorem 5.3.c.

Example 5.4. Let $X := [\{n/(n+1)\}_{n\in\mathbb{N}} \cup \{1\}] \times [\{0\} \cup \{1/(n+1)\}_{n\in\mathbb{N}}] \subset \mathbb{R}^2$ and $v\colon X \to \mathbb{R}$ be as follows: $v(1, x_2) = v(x_1, 0) := 0$; $v(n_1/(n_1+1), 1/(n_2+1)) := \min\{n_1, n_2\}$. Viewing X as a complete sublattice of $\mathbb{R}^2$, it is easy to check that the ordering represented by v satisfies (5.4c), (5.4b), and both (4.1); conditions (4.3b) and (4.3a) are violated by chains $\{n_1/(n_1+1)\}_{n_1\geq n_2} \times \{1/(n_2+1)\}$ and $\{n_1/(n_1+1)\} \times \{1/(n_2+1)\}_{n_2\geq n_1}$, respectively. Obviously, $\sup_{x\in X} v(x) = +\infty$, hence there is no maximizer.

Remark. Conditions (5.3) are meaningful for an arbitrary binary relation $\succ$; conditions (5.4) can easily be modified to the same effect. It remains unclear whether an analogue of Theorems 5.3 could be obtained for an interval order or, at least, a semiorder.

Proposition 5.5. *Let X be a lattice and $\succ$ be an ordering on X such that $M(X, \succ) \neq \emptyset$. Then $M(X, \succ)$ is a subsemilattice of X if $\succ$ satisfies* (5.4a) *or* (5.4b)*; $M(X, \succ)$ is a join-subsemilattice of X if $\succ$ satisfies* (5.4c) *or* (5.4d)*.*

A straightforward proof is omitted.

Theorem 5.6. *Let A be a relatively complete lattice and $\succ$ be an ordering on A satisfying* [(5.4a) *or* (5.4b)]*,* [(5.4c) *or* (5.4d)]*, and both conditions* (4.3)*. Then $M(X, \succ) \in \mathfrak{L}_A$ for every $X \in \mathfrak{L}_A$.*

Proof. Given $X \in \mathfrak{L}_A$, we have $M(X, \succ) \neq \emptyset$ by an appropriate version of Theorem 5.3. By Proposition 5.5, $M(X, \succ)$ is a sublattice of X, hence of A as well. By Proposition 4.5, it is chain-complete. Now $M(X, \succ) \in \mathfrak{L}_A$ by Proposition 2.2. □

Remark. When A is not a chain, the very possibility to replace one condition with another shows that there is no clear prospect for a necessity result in the style of Corollary to Theorem 4.2 or Theorem 4.4.

5.2 Monotonicity

To obtain characterization results for preferences ensuring monotonicity of $\mathcal{R}^X$ for sublattices $X \in \mathfrak{B}_A^{\emptyset}$, we have to consider variations of the problem itself. The sufficiency parts of Propositions 4.6–4.10 can be interpreted as the monotonicity of the correspondence $M(X, \cdot)$ w.r.t. relations (4.4) on the set of orderings when X is a chain. Here, each condition (5.4) is shown to be necessary and sufficient for a kind of such monotonicity on sublattices.

Proposition 5.7. *Let A be a lattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ satisfies* (5.4a)*.*
2. *There holds $M(X, \succ') \geq^{\wedge} M(X, \succ)$ whenever X is a sublattice of A and $\succ'$ is an ordering on X such that $M(X, \succ') \neq \emptyset \neq M(X, \succ)$ and* (4.4a) *holds on X.*
3. *There holds $M(X, \succ') \geq^{\mathrm{wV}} M(X, \succ)$ whenever X is a sublattice of A and $\succ'$ is an ordering on X such that $M(X, \succ') \neq \emptyset \neq M(X, \succ)$ and* (4.4d) *holds on X.*
4. *There holds $M(X, \succ') \geq^{\mathrm{Inf}} M(X, \succ)$ whenever X is a finite sublattice of A and $\succ'$ is an ordering on A such that* (4.4a) *and* (4.4b) *hold on A.*
5. *There holds $M(X, \succ') \geq^{\mathrm{pwV}} M(X, \succ)$ whenever X is a finite sublattice of A and $\succ'$ is an ordering on A such that* (4.4a) *holds on A.*

Proof. The implications Statement 2 $\Rightarrow$ Statement 4 and Statement 3 $\Rightarrow$ Statement 5 are obvious.

Statement 1 $\Rightarrow$ Statement 2. Let $M(X,\succ) \neq \emptyset \neq M(X,\succ')$, and (5.4a) and (4.4a) hold. We have to show that $y \wedge x \in M(X,\succ)$ whenever $y \in M(X,\succ')$ and $x \in M(X,\succ)$. Supposing the contrary, we have $x \succ y \wedge x$, hence $y \vee x \succ y$ by (5.4a) and the optimality of x. Therefore, $y \vee x \succ' y$ by (4.4a), contradicting the optimality of y.

Statement 1 $\Rightarrow$ Statement 3. Let $M(X,\succ) \neq \emptyset \neq M(X,\succ')$, and (5.4a) and (4.4d) hold. We have to show $M(X,\succ') \geq^{\mathrm{wV}} M(X,\succ)$; let $y \in M(X,\succ')$ and $x \in M(X,\succ)$. If $y \wedge x \in M(X,\succ)$, we are home; otherwise, $x \succ y \wedge x$, hence $y \vee x \succ y$ by (5.4a) and the optimality of x. Therefore, $y \vee x \succeq' y$ by (4.4d), hence $y \vee x \in M(X,\succ')$.

Statement 4 $\Rightarrow$ Statement 1. Let (5.4a) be violated: there are $x, y \in A$ such that $x \succ y \wedge x$, but $y \succeq y \vee x$ and $x \succeq y \vee x$. Without restricting generality, $x \succeq y$. We define $X := L(x,y)$, so $y \wedge x \notin M(X,\succ) \ni x$, and $Y := \{z \in A \mid z \geq y\}$; our assumption implies $x \notin Y$. Then we define an ordering $\succ'$ on A: it coincides with $\succ$ on $A \setminus Y$ and on Y, whereas $z' \succ' z$ whenever $z \notin Y \ni z'$. Both (4.4a) and (4.4b) are obvious: Whenever $z' \gneq z$ and $z \in Y$, we have $z' \in Y$ as well. Meanwhile, $y \in M(X,\succ')$, hence $M(X,\succ') \geq^{\mathrm{Inf}} M(X,\succ)$ does not hold, i.e., Statement 4 is invalid.

Statement 5 $\Rightarrow$ Statement 1. Let (5.4a) be violated. We pick x and y as in the previous paragraph, define $X := L(x,y)$, so $y \wedge x \notin M(X,\succ) \ni x$ again, and then define $\succ'$ in the same manner, but with $Y := \{z \in A \mid z \gneq y\} \cup \{y\}$. Clearly, $M(X,\succ') = \{y\}$, hence $M(X,\succ') \geq^{\mathrm{pwV}} M(X,\succ)$ does not hold. Since $\succ'$ and $\succ$ satisfy (4.4a), Statement 5 is invalid. $\square$

Proposition 5.8. *Let A be a lattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ satisfies* (5.4b)*.*
2. *There holds $M(X,\succ) \geq^{\wedge} M(X,\succ')$ whenever X is a sublattice of A and $\succ'$ is an ordering on X such that $M(X,\succ') \neq \emptyset \neq M(X,\succ)$ and* (4.5a) *holds on X.*
3. *There holds $M(X,\succ) \ggg M(X,\succ')$ whenever X is a sublattice of A and $\succ'$ is an ordering on X such that $M(X,\succ') \neq \emptyset \neq M(X,\succ)$ and* (4.5c) *holds on X.*
4. *There holds $M(X,\succ) \geq^{\mathrm{inf}} M(X,\succ')$ whenever X is a finite sublattice of A and $\succ'$ is an ordering on A such that* (4.5c) *holds on A.*

Proof. Let (5.4b) hold, $x \in M(X,\succ')$ and $y \in M(X,\succ)$. Let us show that $y \wedge x \in M(X,\succ')$ if (4.5a) holds. Supposing the contrary, we have $x \succ' y \wedge x$, hence $x \succ y \wedge x$ by (4.5a). Since $y \succeq x$, (5.4b) implies that $y \vee x \succ y$, which contradicts the optimality of y. Let us show that $y \geq x$ if (4.5c) holds. Supposing the contrary, we have $x > y \wedge x$; since $y \succeq y \vee x$ and $y \succeq x$, we have $y \wedge x \succeq x$ by (5.4b). Therefore, $y \wedge x \succ' x$ by (4.5c), which contradicts the optimality of x.

Let (5.4b) be violated: there are $x, y \in A$ such that $x \succ y \wedge x$ and $y \succ y \wedge x$, but $y \succeq y \vee x$. Without restricting generality, $y \succeq x$; we define $X := L(x,y)$, so $y \in M(X,\succ)$. Then we define an ordering $\succ'$ on A in the same manner as in the proof of Proposition 5.7, but with $Y := \{z \in A \mid z \leq x\}$. On every equivalence class E of $\succeq'$, we pick a strictly increasing total order $\gg_E$, existing by the Szpilrajn theorem (Theorem B in Subsection 2.3). Then we define $\succ''$ as a lexicography: $z' \succ'' z$ if $z' \succ' z$, or if they belong to the same equivalence class E and $z \gg_E z'$. Clearly, $\succ''$ is a total order, and both $z' > z$ and $z' \succeq'' z$ can only hold together when $z' \succ z$; therefore, (4.5c) holds for $\succ''$ and $\succ$. Meanwhile, $M(X,\succ'') = \{x\}$, hence $M(X,\succ) \geq^{\mathrm{inf}} M(X,\succ'')$ does not hold. $\square$

Proposition 5.9. *Let A be a lattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ satisfies* (5.4c)*.*

2. *There holds $M(X, \succ') \geq^{\vee} M(X, \succ)$ whenever X is a sublattice of A and $\succ'$ is an ordering on X such that $M(X, \succ') \neq \emptyset \neq M(X, \succ)$ and* (4.4b) *holds on X.*

3. *There holds $M(X, \succ') \ggg M(X, \succ)$ whenever X is a sublattice of A and $\succ'$ is an ordering on X such that $M(X, \succ') \neq \emptyset \neq M(X, \succ)$ and* (4.4c) *holds on X.*

4. *There holds $M(X, \succ') \geq^{\sup} M(X, \succ)$ whenever X is a finite sublattice of A and $\succ'$ is an ordering on A such that* (4.4c) *holds on A.*

Proof. Let (5.4c) hold, $y \in M(X, \succ')$ and $x \in M(X, \succ)$. We have $x \succeq y \wedge x$, hence $y \vee x \succeq y$ by (5.4c) and the optimality of x. Now (4.4b) implies $y \vee x \succeq' y$, hence $y \vee x \in M(X, \succ')$. If (4.4c) holds, we have either $y \geq x$ or $y \vee x \succ' y$, but the latter would contradict the optimality of y.

Let (5.4c) be violated: there are $x, y \in A$ such that $y \succ y \vee x$ and $x \succ y \vee x$, but $x \succeq y \wedge x$. Without restricting generality, $x \succeq y$. We define $X := L(x, y)$, so $x \in M(X, \succ)$, and then define an ordering $\succ'$ exactly as in the proof of Proposition 5.7, with $Y := \{z \in A \mid z \geq y\}$. On every equivalence class E of $\succeq'$, we pick a strictly increasing total order $\gg_E$, existing by the Szpilrajn theorem. Then we define $\succ''$ as a lexicography: $z' \succ'' z$ if $z' \succ' z$, or if they belong to the same equivalence class E and $z' \gg_E z$. Clearly, $\succ''$ is a total order, and $z' \succ'' z$ whenever $z' \gneqq z$; therefore, (4.4c) holds for $\succ''$ and $\succ$. Meanwhile, $M(X, \succ'') = \{y\}$, hence $M(X, \succ'') \geq^{\sup} M(X, \succ)$ does not hold. $\square$

Proposition 5.10. *Let A be a lattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ satisfies* (5.4d)*.*

2. *There holds $M(X, \succ) \geq^{\vee} M(X, \succ')$ whenever X is a sublattice of A and $\succ'$ is an ordering on X such that $M(X, \succ') \neq \emptyset \neq M(X, \succ)$ and* (4.5b) *holds on X.*

3. *There holds $M(X, \succ) \geq^{\mathrm{wV}} M(X, \succ')$ whenever X is a sublattice of A and $\succ'$ is an ordering on X such that $M(X, \succ') \neq \emptyset \neq M(X, \succ)$ and* (4.5d) *holds on X.*

4. *There holds $M(X, \succ) \geq^{\mathrm{Sup}} M(X, \succ')$ whenever X is a finite sublattice of A and $\succ'$ is an ordering on A such that* (4.5a) *and* (4.5b) *hold on A.*

5. *There holds $M(X, \succ) \geq^{\mathrm{pwV}} M(X, \succ')$ whenever X is a finite sublattice of A and $\succ'$ is an ordering on A such that* (4.5a) *holds on A.*

Proof. Let (5.4d) hold, $x \in M(X, \succ')$ and $y \in M(X, \succ)$. If $y \vee x \in M(X, \succ)$, this is sufficient for both Statements 2 and 3. Supposing the contrary, we have $y \succ y \vee x$, hence $y \wedge x \succ x$ by (5.4d) and the optimality of y. If (4.5b) holds, we have $y \wedge x \succ' x$, contradicting the optimality of x. If (4.5d) holds, we have $y \wedge x \succeq' x$, hence $y \wedge x \in M(X, \succ')$.

Let (5.4d) be violated: there are $x, y \in A$ such that $y \succ y \vee x$, but $x \succeq y \wedge x$ and $y \succeq y \wedge x$. Without restricting generality, $y \succeq x$. We define $X := L(x, y)$, so $y \vee x \notin M(X, \succ) \ni y$. Then we define an ordering $\succ'$ on A in exactly the same manner as in the proof of Proposition 5.8. Both (4.5a) and (4.5b) are obvious, but $x \in M(X, \succ')$, hence $M(X, \succ) \geq^{\mathrm{Sup}} M(X, \succ')$ does not hold. If we define $\succ'$ in the same manner, but with $Y := \{z \in A \mid z < x \;\&\; z \succ x\} \cup \{x\}$, we have $M(X, \succ') = \{x\}$, hence $M(X, \succ) \geq^{\mathrm{pwV}} M(X, \succ')$ does not hold whereas $\succ'$ and $\succ$ satisfy (4.5a). $\square$

Proposition 5.11. *Let A be a lattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *Both conditions* (5.4a) *and* (5.4c) *hold.*
2. *There holds $M(X,\succ') \geq^{\mathrm{Vt}} M(X,\succ)$ whenever X is a sublattice of A and $\succ'$ is an ordering on X such that $M(X,\succ') \neq \emptyset \neq M(X,\succ)$ and* (4.4a) *and* (4.4b) *hold on X.*
3. *There holds $M(X,\succ') \geq^{\mathrm{Vt}} M(X,\succ)$ whenever X is a finite sublattice of A and $\succ'$ is an ordering on A such that* (4.4a) *and* (4.4b) *hold on A.*

The equivalence immediately follows from Propositions 5.7 and 5.9.

Proposition 5.12. *Let A be a lattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *Both conditions* (5.4b) *and* (5.4d) *hold.*
2. *There holds $M(X,\succ) \geq^{\mathrm{Vt}} M(X,\succ')$ whenever X is a sublattice of A and $\succ'$ is an ordering on X such that $M(X,\succ') \neq \emptyset \neq M(X,\succ)$ and* (4.5a) *and* (4.5b) *hold on X.*
3. *There holds $M(X,\succ) \geq^{\mathrm{Vt}} M(X,\succ')$ whenever X is a finite sublattice of A and $\succ'$ is an ordering on A such that* (4.5a) *and* (4.5b) *hold on A.*

The equivalence immediately follows from Propositions 5.8 and 5.10.

Theorem 5.13. *Let A be a lattice and $\succ$ be an ordering on A. Then each of the two following lists of requirements is equivalent to the quasisupermodularity of $\succ$.*

1. (a) *There holds $M(X,\succ') \geq^{\mathrm{Vt}} M(X,\succ)$ whenever X is a sublattice of A and $\succ'$ is an ordering on X such that $M(X,\succ') \neq \emptyset \neq M(X,\succ)$ and* (4.4a) *and* (4.4b) *hold on X;*

   (b) *there holds $M(X,\succ) \geq^{\mathrm{Vt}} M(X,\succ')$ whenever X is a sublattice of A and $\succ'$ is an ordering on X such that $M(X,\succ') \neq \emptyset \neq M(X,\succ)$ and* (4.5a) *and* (4.5b) *hold on X.*

2. (a) *There holds $M(X,\succ') \ggg M(X,\succ)$ whenever X is a sublattice of A and $\succ'$ is an ordering on X such that $M(X,\succ') \neq \emptyset \neq M(X,\succ)$ and* (4.4c) *holds on X;*

   (b) *there holds $M(X,\succ) \ggg M(X,\succ')$ whenever X is a sublattice of A and $\succ'$ is an ordering on X such that $M(X,\succ') \neq \emptyset \neq M(X,\succ)$ and* (4.5c) *holds on X;*

   (c) *there holds $M(X,\succ') \geq^{\mathrm{wV}} M(X,\succ)$ whenever X is a sublattice of A and $\succ'$ is an ordering on X such that $M(X,\succ') \neq \emptyset \neq M(X,\succ)$ and* (4.4d) *holds on X;*

   (d) *there holds $M(X,\succ) \geq^{\mathrm{wV}} M(X,\succ')$ whenever X is a sublattice of A and $\succ'$ is an ordering on X such that $M(X,\succ') \neq \emptyset \neq M(X,\succ)$ and* (4.5d) *holds on X.*

Remark. In contrast to Theorem 4 of Milgrom and Shannon (1994), here we obtained a characterization result without comparing optimization problems with different choice sets. An ordering on a lattice is quasisupermodular if and only if it cannot destroy the monotonicity of best responses in any parametric optimization problem satisfying (strong or weak) single crossing conditions.

Proof. The first equivalence immediately follows from Propositions 5.11 and 5.12; the second, from Propositions 5.7–5.10. □

5.3 Implications for strategic games

A *strategic game* (*with ordinal preferences*) is defined by a finite set of players N, and strategy sets X_i and preferences on $X_N := \prod_{i\in N} X_i$ for all $i \in N$. We assume each player's preference relation to be an ordering represented by a mapping $u_i \colon X_i \times X_{-i} \to \mathcal{C}_i$, where $X_{-i} := \prod_{j\neq i} X_j$ and $\mathcal{C}_i$ is a chain; "less rational" preferences are only considered in Section 8. Defining the *best response correspondence* in the usual way,

$$\mathcal{R}_i(x_{-i}) := \operatorname*{Argmax}_{x_i\in X_i} u_i(x_i, x_{-i}), \tag{5.6}$$

a *Nash equilibrium* is $x_N \in X_N$ such that $x_i \in \mathcal{R}_i(x_{-i})$ for each $i \in N$.

Given a poset S, a *monotonic pseudopartition* of S consists of two subsets $S^-, S^+ \subseteq S$ such that $S^- \cap S^+ = \emptyset$ and $\forall s', s \in S \left[s' > s \Rightarrow [s \in S^- \text{ or } s' \in S^+]\right]$. Clearly, any two points outside $S^- \cup S^+$ must be incomparable.

Given a strategic game Γ and $i \in N$, we consider these four requirements.

$$X_i \text{ is a complete chain; } u_i \text{ is regular and satisfies the single crossing condition.} \tag{5.7a}$$

$$X_i \text{ is a complete chain; } u_i \text{ is strongly regular and satisfies the weak single crossing condition.} \tag{5.7b}$$

$$\begin{gathered} X_i \text{ is a complete sublattice of the Cartesian product of a finite number of complete chains;}\\ u_i \text{ satisfies the single crossing condition;}\\ \text{there is a monotonic pseudopartition } \langle S^-, S^+\rangle \text{ of } X_{-i} \text{ such that}\\ \text{the ordering } \succ \text{ on } X_i \text{ represented by } v(\cdot) := u(\cdot, x_{-i})\\ \text{satisfies (4.1b), (4.3a), (5.4a), and (5.4c) for } x_{-i} \in S^-,\\ \text{(4.1a), (4.3b), (5.4b), and (5.4d) for } x_{-i} \in S^+,\\ \text{and all conditions of one of Theorems 5.3 for } x_{-i} \in X_{-i} \setminus (S^- \cup S^+). \end{gathered} \tag{5.7c}$$

$$\begin{gathered} X_i \text{ is a complete lattice;}\\ u_i \text{ is strongly regular and satisfies the weak single crossing condition;}\\ \text{there is a monotonic pseudopartition } \langle S^-, S^+\rangle \text{ of } X_{-i} \text{ such that}\\ \text{the ordering } \succ \text{ on } X_i \text{ represented by } v(\cdot) := u(\cdot, x_{-i})\\ \text{satisfies (5.4a) for } x_{-i} \in S^-, \text{ (5.4d) for } x_{-i} \in S^+,\\ \text{and one of conditions (5.4) for } x_{-i} \in X_{-i} \setminus (S^- \cup S^+). \end{gathered} \tag{5.7d}$$

Remark. It is easy to see that (5.7a) $\Rightarrow$ (5.7c), while (5.7b) $\Rightarrow$ (5.7d). However, it seems more convenient to have those simpler versions written down explicitly.

Theorem 5.14. *Let Γ be a strategic game such that, for each $i \in N$, at least one of the conditions (5.7) holds. Then Γ possesses a Nash equilibrium.*

Proof. Let us show the existence a monotone selection r_i from the best response correspondence $\mathcal{R}_i$ for each $i \in N$. If (5.7a) or (5.7b) holds, we apply either Theorem 4.8 or Theorem 4.11.

Let (5.7c) hold. We have $\mathcal{R}_i(x_{-i}) \neq \emptyset$ for every $x_{-i} \in X_{-i}$ by one of Theorems 5.3. The correspondence $\mathcal{R}_i$ is increasing w.r.t. $\geq^{\mathrm{Vt}}$ by Propositions 5.11 or 5.12. Therefore, r_i exists by Theorem 3.10.

Let (5.7d) hold. Then $\mathcal{R}_i(x_{-i})$ is nonempty by one of Theorems 5.3 and complete by Proposition 4.5. The correspondence $\mathcal{R}_i$ is increasing w.r.t. $\geq^{\mathrm{wV}}$ by Propositions 5.7 or 5.10. Therefore, r_i exists by Theorem 3.5.

Now Tarski's fixed point theorem (Theorem D in Section 3) can be applied to the Cartesian product of the monotone selections. $\square$

Remark. Agliardi (2000) called a function v on a lattice A *pseudosupermodular* if the ordering represented by v satisfies (5.4a) and (5.4c). Lemma 3.1 from Kukushkin et al. (2005) showed that the best response correspondence is increasing if the utility function is pseudosupermodular and satisfies the single crossing condition. Obviously, (5.7) provide weaker sufficient conditions for increasing best responses. It might be appropriate to call an ordering *pseudosupermodular upwards* [*downwards*] if it satisfies (5.4a) and (5.4c) [(5.4b) and (5.4d)].

Proposition 3.6 can be applied to strategic games where all strategy sets but one are finite; Proposition 3.9, to games where all strategy sets but one are countable. However, both conditions seem too exotic to dwell on them. An extension of Theorem 3.10 to "infinite-dimensional" lattices would be of much greater interest. Unfortunately, there is no idea so far about how this could be done.

6 Optimization on Semilattices

6.1 Existence

Assuming $\succ$ an ordering on a semilattice A, we consider three conditions somewhat reminiscent of quasisupermodularity.

$$\forall x, y \in A \, \big[[y \not\geq x \;\&\; x \not\geq y] \Rightarrow y \wedge x \succ x\big]; \tag{6.1a}$$

$$\forall x, y \in A \, \big[[y \not\geq x \;\&\; x \not\geq y] \Rightarrow y \wedge x \succeq x\big]; \tag{6.1b}$$

$$\forall x, y \in A \, \big[x \succ y \wedge x \Rightarrow y \wedge x \succeq y\big]. \tag{6.1c}$$

Obviously, (6.1a) $\Rightarrow$ (6.1b) $\Rightarrow$ (6.1c); the first two conditions are very close to the (anti)monotonicity of preferences in the basic order (though do not coincide with it). The third is not that stringent; a term like *semiquasisupermodularity* might be appropriate.

Proposition 6.1. *Let A be a semilattice, $\succ$ be an ordering on A satisfying* (6.1c)*, and $\succ'$ an ordering on A satisfying* (4.5a)*. Then $\succ'$ satisfies* (6.1c)*.*

Proof. Let $x \succ' y \wedge x$. Then $x \succ y \wedge x$ by (4.5a) [or rather by (4.4a) with the reversed roles of $\succ$ and $\succ'$], hence $y \wedge x \succeq y$ by (6.1c), hence $y \wedge x \stackrel{\succeq}{<} y$, hence $y \wedge x \succeq' y$ by (4.5a). We see that (6.1c) holds for $\succ'$ as well. $\square$

Remark. There seems to be no analog of Proposition 6.1 for conditions (5.4) on lattices, or conditions (4.1) or (4.3) for that matter.

Theorem 6.2. *Let A be a relatively complete semilattice and $\succ$ be an ordering on A satisfying* (6.1c)*,* (4.1b)*, and* (4.3a)*. Then $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{S}_A$ (where $\mathfrak{S}_A$ denotes the set of all nonempty complete subsemilattices $X \subseteq A$).*

Proof. The basic construction is similar to that from the proof of Theorem 5.3.a. We again use auxiliary strict orders $\stackrel{\succeq}{>}$ and $\stackrel{\succeq}{<}$ defined by (2.3).

Lemma 6.2.1. *If X is a subsemilattice of A, $x \in M(X, \underset{<}{\succeq})$ and $X \ni y \succ x$, then $y \underset{>}{\succ} x$.*

Proof. If $y \not> x$, then $y \wedge x < x$, hence $x \succ y \wedge x$ because $x \in M(X, \underset{<}{\succeq})$. Now (6.1c) applies, hence $x \succ y \wedge x \succeq y$, which contradicts the condition of the lemma. □

Let Λ be a well ordered set with a cardinality greater than that of A. Given X, we construct, by (transfinite) recursion, a mapping $\lambda \colon \Lambda \to X$ such that:

$$\forall \alpha \in \Lambda \left[\lambda(\alpha) \in M(X, \underset{<}{\succeq}) \right]; \tag{6.2a}$$

$$\forall \alpha, \beta \in \Lambda \left[\alpha > \beta \Rightarrow [\lambda(\alpha) = \lambda(\beta) \ \text{ or } \ \lambda(\alpha) \underset{>}{\succ} \lambda(\beta)] \right]; \tag{6.2b}$$

$$\forall \alpha \in \Lambda \left[\lambda([0, \alpha]) \in \mathfrak{C}_X \right]. \tag{6.2c}$$

First, we define $\lambda(0) := \bigwedge X \in M(X, \underset{<}{\succeq})$. Let $\lambda(\alpha)$ have been defined. If $\lambda(\alpha) \in M(X, \succ)$, we define $\lambda(\alpha + 1) := \lambda(\alpha)$; actually, $\lambda(\alpha') = \lambda(\alpha)$ for all $\alpha' > \alpha$ in this case. Otherwise, Lemma 6.2.1 implies the existence of $x^* \in X$ such that $x^* \underset{>}{\succ} \lambda(\alpha)$. We denote $Y := \{y \in X \mid \lambda(\alpha) \leq y \leq x^*\} \ni x^*$. If $x^* \in M(Y, \underset{<}{\succeq})$, we define $\lambda(\alpha+1) := x^*$; otherwise, we apply Proposition 4.3.a, obtaining $\lambda(\alpha+1) \in M(Y, \underset{<}{\succeq})$ such that $\lambda(\alpha + 1) \underset{<}{\succeq} x^*$. In either case, we have $\lambda(\alpha + 1) \succ \lambda(\alpha)$, hence $\lambda(\alpha + 1) \underset{>}{\succ} \lambda(\alpha)$. Let us show $\lambda(\alpha + 1) \in M(X, \underset{<}{\succeq})$. Supposing the contrary, $y \underset{<}{\succeq} \lambda(\alpha + 1)$, we apply Lemma 6.2.1 to $x = \lambda(\alpha)$ and y, obtaining $y > \lambda(\alpha)$, hence $y \in Y$, which is incompatible with $\lambda(\alpha + 1) \in M(Y, \underset{<}{\succeq})$.

Let $\alpha \in \Lambda$ be a limit ordinal and $\lambda(\beta)$ have been defined for all $\beta < \alpha$. Then we define $\lambda(\alpha) := \sup_{\beta<\alpha} \lambda(\beta)$, ensuring that $\lambda([0, \alpha]) \in \mathfrak{C}_X$. Now (4.1b) implies $\lambda(\alpha) \underset{>}{\succ} \lambda(\beta)$ for every $\beta < \alpha$ unless $\lambda(\alpha) = \lambda(\beta)$, so we only have to show that $\lambda(\alpha) \in M(X, \underset{<}{\succeq})$.

If $\lambda(\alpha) = \lambda(\beta)$ for some $\beta < \alpha$, we are home immediately. Otherwise, $\lambda(\alpha) \underset{>}{\succ} \lambda(\beta)$ for every $\beta < \alpha$, hence $y \underset{<}{\succeq} \lambda(\alpha)$ would imply $y \succ \lambda(\beta)$, hence $y > \lambda(\beta)$ by Lemma 6.2.1. Since β is arbitrary, we would have $y \geq \lambda(\alpha)$, contradicting the assumption about y.

The final argument is again standard. □

Remark. The preference relation in Example 5.4 above satisfies (6.1c) as well as both (4.1), but not (4.3a). Therefore, the replacement of (4.3a) in Theorem 6.2 with (4.1a) would make it wrong.

Proposition 6.3. *Let A be a relatively complete semilattice and $\succ$ be an ordering on A satisfying* (6.1c) *and* (4.3). *Then $M(X, \succ) \in \mathfrak{S}_A$ for every $X \in \mathfrak{S}_A$.*

The proof is essentially the same as in Theorem 5.6.

Proposition 6.4. *Let A be a relatively complete semilattice and $\succ$ be an ordering on A. Then $M(X, \succ') \in \mathfrak{S}_A$ for every $X \in \mathfrak{S}_A$ and every ordering $\succ'$ on A satisfying* (4.3) *and* (4.5a) *if and only if $\succ$ satisfies* (6.1c).

Proof. If (6.1c) and (4.5a) hold, then $\succ'$ satisfies (6.1c) by Proposition 6.1. If $\succ'$ satisfies (4.3) as well, then Proposition 6.3 applies.

Conversely, let (6.1c) be violated: there are $x, y \in A$ such that $x \succ y \wedge x$ and $y \succ y \wedge x$ (hence x and y are incomparable in the basic order); without restricting generality, $x \succeq y$. We denote $X := \{x, y, y \wedge x\} \in \mathfrak{S}_A$, $Y := \{z \in A \mid y \geq z \;\&\; z \succeq y\}$. Then we define $\succ'$: it coincides with $\succ$ on $A \setminus Y^*$ and on Y; whenever $z \notin Y \ni z'$, $z \succ' z'$ never holds while $z' \succ' z \rightleftharpoons [z' \succ y \ \text{ or } \ x \succ z]$. It is easily checked that $\succ'$ is an ordering too. Now (4.5a) is obvious, while $M(X, \succ') = \{x, y\} \notin \mathfrak{S}_A$. □

6.2 Monotonicity

Proposition 6.5. *Let A be a semilattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ satisfies* (6.1c)*.*
2. *There holds $M(X, \succ) \geq^{\wedge} M(X, \succ')$ whenever X is a subsemilattice of A and $\succ'$ is an ordering on X such that $M(X, \succ) \neq \emptyset \neq M(X, \succ')$ and* (4.5a) *holds on X.*
3. *There holds $M(X, \succ) \ggg M(X, \succ')$ whenever X is a subsemilattice of A and $\succ'$ is an ordering on X such that $M(X, \succ) \neq \emptyset \neq M(X, \succ')$ and* (4.5c) *holds on X.*
4. *There holds $M(X, \succ) \geq^{\text{inf}} M(X, \succ')$ whenever X is a finite subsemilattice of A and $\succ'$ is an ordering on X such that* (4.5c) *holds on A.*

Proof. Let (6.1c) hold, $y \in M(X, \succ)$ and $x \in M(X, \succ')$. If (4.5a) holds, we have to show $y \wedge x \in M(X, \succ')$. Supposing the contrary, $x \succ' y \wedge x$, we have $x \succ y \wedge x$ by (4.5a), hence $y \wedge x \succeq y$ by (6.1c), hence $x \succ y$, contradicting the optimality of y. Similarly, if (4.5c) holds, we have to show $y \geq x$. Otherwise, we would have $y \wedge x < x$; since $x \succeq' y \wedge x$, we have $x \succ y \wedge x$ by (4.5c), hence $y \wedge x \succeq y$ by (6.1c) with the same contradiction.

Let (6.1c) be violated: there are $x, y \in A$ such that $y \succ y \wedge x$ and $x \succ y \wedge x$. Without restricting generality, $y \succeq x$. Then we define $X := \{x, y, y \wedge x\}$, so $y \in M(X, \succ)$. Defining an ordering $\succ''$ on A in exactly the same manner as in the proof of Proposition 5.8, we see that (4.5c) holds for $\succ''$ and $\succ$. Meanwhile, $M(X, \succ'') = \{x\}$, hence $M(X, \succ) \geq^{\text{inf}} M(X, \succ'')$ does not hold. □

Remark. When A is a lattice, (6.1c) obviously implies (5.4b). In this sense, Proposition 6.5 is in perfect accord with Proposition 5.8.

Proposition 6.6. *Let A be a semilattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ satisfies* (6.1b)*.*
2. *There holds $M(X, \succ') \geq^{\wedge} M(X, \succ)$ whenever X is a subsemilattice of A and $\succ'$ is an ordering on X such that $M(X, \succ') \neq \emptyset \neq M(X, \succ)$ and* (4.4a) *holds on X.*
3. *There holds $M(X, \succ') \geq^{\text{Inf}} M(X, \succ)$ whenever X is a finite subsemilattice of A and $\succ'$ is an ordering on A such that* (4.4c) *hold on A.*

Proof. Let (6.1b) and (4.4a) hold, $y \in M(X, \succ')$ and $x \in M(X, \succ)$. If x and y are incomparable in the basic order, we have $y \wedge x \succeq x$ by (6.1b), hence $y \wedge x \in M(X, \succ)$. If they are comparable and $y \geq x$ or $y \sim x$, we are home too. Finally, if $x \mathrel{\substack{\succ \\ >}} y$, we have $x \succ' y$ by (4.4a), which contradicts the optimality of y.

Let (6.1b) be violated: there are $x, y \in A$ such that x and y are incomparable in the basic order, but $x \succ y \wedge x$. Without restricting generality, $x \succeq y$. We define $X := \{x, y, y \wedge x\}$, so $y \wedge x \notin M(X, \succ) \ni x$. Defining an ordering $\succ''$ on A in exactly the same manner as in the proof of Proposition 5.9, we see that (4.4c) holds for $\succ''$ and $\succ$. Meanwhile, $M(X, \succ'') = \{y\}$, hence $M(X, \succ'') \geq^{\text{Inf}} M(X, \succ)$ does not hold. □

Proposition 6.7. *Let A be a semilattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ satisfies* (6.1a).
2. *There holds $M(X, \succ') \ggg M(X, \succ)$ whenever X is a subsemilattice of A and $\succ'$ is an ordering on X such that $M(X, \succ') \neq \emptyset \neq M(X, \succ)$ and* (4.4c) *holds on X.*
3. *There holds $M(X, \succ') \ggg M(X, \succ)$ whenever X is a finite subsemilattice of A and $\succ'$ is an ordering on A such that* (4.4c) *holds on A.*

Proof. Let (6.1a) and (4.4c) hold, $y \in M(X, \succ')$ and $x \in M(X, \succ)$. If x and y are incomparable in the basic order, we have $y \wedge x \succ x$ by (6.1a), which contradicts the optimality of x. If $x > y$, we have $x \underset{>}{\succeq} y$, hence $x \succ' y$ by (4.4c), which contradicts the optimality of y.

Let (6.1a) be violated: there are $x, y \in A$ such that x and y are incomparable in the basic order, but $x \succeq y \wedge x$. Without restricting generality, $x \succeq y$. We define $X := \{x, y, y \wedge x\}$, so $x \in M(X, \succ)$. Defining an ordering $\succ''$ on A in exactly the same manner as in the proof of Proposition 5.9 (or 6.6), we see that (4.4c) holds for $\succ''$ and $\succ$. Meanwhile, $M(X, \succ'') = \{y\}$, hence $M(X, \succ'') \ggg M(X, \succ)$ does not hold. $\square$

6.3 Implications for strategic games

Theorem 6.8. *Theorem 5.14 remains valid if one more condition is added to the list* (5.7)*:*

$$\begin{aligned} &X_i \text{ is a complete semilattice;} \\ &u_i \text{ is strongly regular and satisfies the lower single crossing condition;} \\ &\text{the ordering } \succ \text{ on } X_i \text{ represented by } v(\cdot) := u(\cdot, x_{-i}) \\ &\text{satisfies (6.1c) for every } x_{-i} \in X_{-i}. \end{aligned} \tag{6.3}$$

Proof. Let (6.3) hold for an $i \in N$. By Proposition 6.3, we have $\mathcal{R}_i(x_{-i}) \in \mathfrak{S}_{X_i}$ for every $x_{-i} \in X_{-i}$; by Proposition 6.5, the correspondence $\mathcal{R}_i$ is increasing w.r.t. $\geq^{\wedge}$. Therefore, there exists a monotone selection r_i from $\mathcal{R}_i$ by Corollary to Propositions 3.2.a.

Now X_N is a complete semilattice, hence a Nash equilibrium exists by Abian and Brown's fixed point theorem (Theorem E in Section 3) applied to the Cartesian product of the monotone selections. $\square$

Obviously, (6.1c) holds when the preferences are decreasing, in which case $\bigwedge X_i$ is among the best responses to any x_{-i} and Theorem 6.8 becomes trivial. In the rest of the subsection, we describe a class of preferences satisfying (6.3) in a not-so-trivial way.

Let an agent allocate a single resource (money or time, effort, etc.) among m "needs," each need k receiving $x_k \geq 0$. There is a function $v_k(x_k, s)$, dependent also on an exogenous parameter s (the choices of other agents); we assume each v_k ($k \in \{1, \ldots m\}$) increasing in x_k. Denoting x_0 the amount of the resource that is left undistributed, we assume an increasing function $v_0(x_0)$; it reflects "personal consumption" and is independent of any externality. The set of strategies is $X := \{(x_0, x_1, \ldots x_m) \in \mathbb{R}^{m+1} \mid \sum_{k=0}^{m} x_k = K\}$. An order on X is given by $y \geq x \rightleftharpoons \forall k \in \{1, \ldots, m\}\, [y_k \geq x_k]$; clearly, X is a complete semilattice. The overall preferences of the agent on X (depending on the parameter s) are represented by the function

$$u(x, s) := \min\{v_0(x_0), \min_{1 \leq k \leq m} v_k(x_k, s)\}. \tag{6.4}$$

Proposition 6.9. *The ordering on X represented by* (6.4) *with v_0 increasing in x_0 and each v_k increasing in x_k satisfies* (6.1c) *for every $s \in S$.*

Proof. Since s is fixed throughout, we drop it from the arguments of each v_k to simplify notations. Given $x, y \in X$, we denote $M^+ := \{k \in \{1, \dots m\} \mid x_k > y_k\}$ and $M^- := \{k \in \{1, \dots m\} \mid x_k < y_k\}$. If x and y are comparable in the basic order, then (6.1c) obviously holds; therefore, we may assume that $M^+ \neq \emptyset \neq M^-$ and $x > z := y \wedge x$ and $y > z$, hence $v_0(x_0) \leq v_0(z_0) \geq v_0(y_0)$. Let $x \succ z$; then $\mathrm{Argmin}_{0 \leq k \leq m} v_k(z_k) \subseteq M^+$. Since $y_k = z_k$ for each $k \in M^+$, we have $\mathrm{Argmin}_{0 \leq k \leq m} v_k(y_k) \subseteq M^+ \cup \{0\}$, hence $z \succeq y$. □

Proposition 6.10. *If S is a poset and there are increasing functions $\bar{v} \colon S \to \mathbb{R}$ and $\bar{v}_k \colon \mathbb{R}_+ \to \mathbb{R}$ such that $v_k(x_k, s) = \bar{v}_k(x_k) - \bar{v}(s)$ for all k, x_k, and s, then $u(x, s)$ defined by* (6.4) *satisfies the lower single crossing condition.*

Proof. Let $y > x$, $s' > s$, and $u(y, s) > u(x, s)$. Since $x_0 > y_0$, we have $0 \notin \mathrm{Argmin}_{0 \leq k \leq m} v_k(x_k, s)$, i.e., $v_0(x_0) \geq v_0(y_0) > v_k(x_k, s)$ for a $k \in \{1, \dots, m\}$. Therefore, $v_0(x_0) \geq v_0(y_0) > v_k(x_k, s')$, hence $\mathrm{Argmin}_{0 \leq k \leq m} v_k(x_k, s') = \mathrm{Argmin}_{0 \leq k \leq m} v_k(x_k, s)$ and $u(y, s') > u(x, s')$. □

The minimum aggregation in a utility function, i.e., the "absolute complementarity" of components, is not met in economic models very often; however, it is not exceptionally rare either. Galbraith (1958, Chapter XVIII) effectively viewed the rule as most natural in the evaluation of tradeoffs between public and private consumption ("social balance"). The model of Germeier and Vatel' (1974) employed utility functions like (6.4) with an additional assumption that, roughly speaking, all players have the same functions v_k. Later research (Kukushkin et al., 1985; Kukushkin, 2004b) showed that this form of utilities is sufficient by itself, without any monotonicity assumptions, for quite a number of nice properties of the game. Here, on the contrary, we see that the existence of a Nash equilibrium, at least, can be derived from (6.4) without the "common intermediate objectives" assumption. It is also worthwhile to note that increasing, or decreasing, best responses to the minimum of the partners' choices (when strategies are scalar) ensure the acyclicity of Cournot tatonnement (Kukushkin, 2003, Theorems 7 and 8)

7 Optimization on Poorer Order Structures

Proposition 7.1. *Let A be a poset and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *There holds $x \sim y$ whenever x and y are incomparable in the basic order.*
2. *There holds $M(X, \succ') \geq^{\inf} M(X, \succ)$ whenever $X \in \mathfrak{B}_A^{\emptyset}$ and $\succ'$ is an ordering on X such that $M(X, \succ) \neq \emptyset \neq M(X, \succ')$ and* (4.4a) *holds on X.*
3. *There holds $M(X, \succ') \geq^{\inf} M(X, \succ)$ whenever X is a finite subset of A and $\succ'$ is an ordering on A such that* (4.4c) *holds on A.*

Proof. Let Statement 1 and (4.4a) hold, and $y \in M(X, \succ')$; we need $x \in M(X, \succ)$ such that $y \geq x$. If $y \in M(X, \succ)$, we are home; otherwise, there is $x \in M(X, \succ)$ such that $x \succ y$. By Statement 1, x and y are comparable in the basic order; if $y > x$, we are home again. If $x > y$, then (4.4a) implies $x \succ' y$, contradicting the optimality of y.

Let Statement 1 not hold: there are incomparable x and y such that $x \succ y$. Denoting $X := \{x, y\}$, we have $M(X, \succ) = \{x\}$. Then we define an ordering $\succ''$ in exactly the same manner as in the proof of Proposition 5.9. Now we see that (4.4c) holds for $\succ''$ and $\succ$, while $M(X, \succ'') = \{y\}$, hence $M(X, \succ'') \geq^{\mathrm{inf}} M(X, \succ)$ does not hold. ☐

Proposition 7.2. *Let A be a poset and $\succ$ be an ordering on A such that $M(X, \succ) \geq^{\mathrm{inf}} M(X, \succ')$ whenever X is a finite subset of A and $\succ'$ is an ordering on A such that* (4.5c) *holds. Then A is a chain.*

Remark. The converse statement immediately follows from Proposition 4.9.

Proof. Indeed, if there are incomparable $x, y \in A$, we may, without restricting generality, assume $y \succeq x$. Denoting $X := \{x, y\}$, we have $y \in M(X, \succ)$. Then we define an ordering $\succ''$ on A in exactly the same way as in the proof of Proposition 5.8, obtaining $M(X, \succ'') = \{x\}$, hence $M(X, \succ) \geq^{\mathrm{inf}} M(X, \succ'')$ does not hold while (4.5c) holds for $\succ''$ and $\succ$. ☐

Dual statements concerning $\geq^{\mathrm{sup}}$ are not worth writing explicit formulations.

Corollary. *A poset A is a chain if there is an ordering $\succ$ on A such that $M(X, \succ') \geq^{\mathrm{RS}} M(X, \succ)$ whenever X is a finite subset of A and $\succ'$ is an ordering on A such that* (4.4c) *holds; or if there is an ordering $\succ$ on A such that $M(X, \succ) \geq^{\mathrm{RS}} M(X, \succ')$ whenever X is a finite subset of A and $\succ'$ is an ordering on A such that* (4.5c) *holds.*

In other words, monotonicity w.r.t. $\geq^{\mathrm{RS}}$ does not look promising from the viewpoint of parametric optimization on general posets. In the light of Proposition 3.2, one could wonder whether monotonicity w.r.t. $\geq^{\mathrm{Inf}}$ or $\geq^{\mathrm{Sup}}$ on directed subsets could lead to something interesting. Unfortunately, we have to impose restrictions even closer to the (anti)monotonicity of preferences than (6.1) even though we do not have to restrict ourselves to chains.

A triple $x, y, z \in A$ is called a *downward triad* if $x > z$ and $y > z$ while x and y are incomparable. An ordering $\succ$ on A is *almost* (*strictly*) *decreasing* if $z \succeq x$ ($z \succ x$) whenever there is $y \in A$ such that (x, y, z) is a downward triad. An *upward triad* and an *almost* (*strictly*) *increasing* ordering on A are defined dually.

Proposition 7.3. *Let A be a poset and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ is almost decreasing.*
2. *There holds $M(X, \succ') \geq^{\mathrm{Inf}} M(X, \succ)$ whenever X is a downwards directed subset of A and $\succ'$ is an ordering on X such that $M(X, \succ) \neq \emptyset \neq M(X, \succ')$ and* (4.4a) *holds on X.*
3. *There holds $M(X, \succ) \ggg M(X, \succ')$ whenever X is a downwards directed subset of A and $\succ'$ is an ordering on X such that $M(X, \succ) \neq \emptyset \neq M(X, \succ')$ and* (4.5c) *holds on X.*
4. *There holds $M(X, \succ') \geq^{\mathrm{inf}} M(X, \succ)$ whenever X is a finite downwards directed subset of A and $\succ'$ is an ordering on A such that* (4.4c) *holds on A.*
5. *There holds $M(X, \succ) \geq^{\mathrm{inf}} M(X, \succ')$ whenever X is a finite downwards directed subset of A and $\succ'$ is an ordering on A such that* (4.5c) *holds on A.*

Proof. Let $\succ$ be almost decreasing, $y \in M(X, \succ')$, and $x \in M(X, \succ)$. If x and y are comparable in the basic order, then both Statements 2 and 3 immediately follow from the sufficiency parts of Propositions 4.6 and 4.9, respectively. Otherwise, let $z \in X$ be a lower bound of x and y in X. Since $\succ$ is almost decreasing, we have $z \succeq x$ and $z \succeq y$. The first relation implies that $z \in M(X, \succ)$, which is exactly what is needed for Statement 2. If (4.5c) holds, we have $z \succ' y$, contradicting the optimality of y.

Let $\succ$ not be almost decreasing: there is a downward triad (x, y, z) such that $x \succ z$. Without restricting generality, $x \succeq y$. Then we define $X := \{x, y, z\}$, so $z \notin M(X, \succ) \ni x$. Defining an ordering $\succ''$ on A in exactly the same manner as in the proof of Proposition 5.9, we see that (4.4c) holds for $\succ''$ and $\succ$ whereas $M(X, \succ'') = \{y\}$, hence $M(X, \succ'') \succeq^{\mathrm{Inf}} M(X, \succ)$ does not hold. Defining an ordering $\succ''$ on A in exactly the same manner as in the proof of Proposition 5.8, but with $Y := \{z \in A \mid z \leq y\}$, we see that (4.5c) holds for $\succ''$ and $\succ$. Meanwhile, $M(X, \succ'') = \{y\}$, hence $M(X, \succ) \succeq^{\mathrm{inf}} M(X, \succ'')$ does not hold. $\square$

Proposition 7.4. *Let A be a poset and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *There holds $z \succeq y$ whenever (x, y, z) is a downward triad and $x \succ z$.*
2. *There holds $M(X, \succ) \succeq^{\mathrm{Inf}} M(X, \succ')$ whenever X is a downwards directed subset of A and $\succ'$ is an ordering on X such that $M(X, \succ) \neq \emptyset \neq M(X, \succ')$ and* (4.5a) *holds on X.*
3. *There holds $M(X, \succ) \succeq^{\mathrm{inf}} M(X, \succ')$ whenever X is a finite downwards directed subset of A and $\succ'$ is an ordering on A such that* (4.5c) *holds on A.*

Remark. Statement 1 looks very similar to condition (6.1c), but is much stronger. It need not hold even for preferences (6.4).

Proof. Let Statement 1 and (4.5a) hold, $y \in M(X, \succ)$, and $x \in M(X, \succ')$. If x and y are comparable in the basic order, then again the sufficiency part of Proposition 4.6 suffices. Otherwise, let $z \in X$ be a lower bound of x and y in X; if $z \in M(X, \succ')$, we are home. Otherwise, we have $x \succ' z$, hence $x \succ z$ by (4.5a), hence $z \succeq y$ by Statement 1, hence $x \succ y$, contradicting the optimality of y.

Let Statement 1 not hold: there is a downward triad (x, y, z) such that $y \succ z$ and $x \succ z$. Without restricting generality, $y \succeq x$. Then we define $X := \{x, y, z\}$, so $y \in M(X, \succ)$. Defining an ordering $\succ''$ on A in exactly the same manner as in the proof of Proposition 5.8, we see that (4.5c) holds for $\succ''$ and $\succ$, while $M(X, \succ'') = \{x\}$, hence $M(X, \succ) \succeq^{\mathrm{inf}} M(X, \succ'')$ does not hold. $\square$

Proposition 7.5. *Let A be a poset and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ is almost strictly decreasing.*
2. *There holds $M(X, \succ') \ggg M(X, \succ)$ whenever X is a downwards directed subset of A and $\succ'$ is an ordering on X such that $M(X, \succ) \neq \emptyset \neq M(X, \succ')$ and* (4.4c) *holds on X.*
3. *There holds $M(X, \succ') \succeq^{\mathrm{sup}} M(X, \succ)$ whenever X is a finite downwards directed subset of A and $\succ'$ is an ordering on A such that* (4.4c) *holds on A.*

Proof. Let $\succ$ be almost strictly decreasing, $y \in M(X, \succ')$, and $x \in M(X, \succ)$. If x and y are comparable in the basic order, then again the sufficiency part of Proposition 4.9 suffices. Otherwise, let $z \in X$ be a lower bound of x and y in X. Since $\succ$ is almost strictly decreasing, we have $z \succ x$, which contradicts the optimality of x.

Let $\succ$ not be almost strictly decreasing: there is a downward triad (x, y, z) such that $x \succeq z$. Without restricting generality, $x \succeq y$. Then we define $X := \{x, y, z\}$, so $x \in M(X, \succ)$. Defining an ordering $\succ''$ on A in exactly the same manner as in the proof of Proposition 5.9, we see that (4.4c) holds for $\succ''$ and $\succ$. Meanwhile, $M(X, \succ'') = \{y\}$, hence $M(X, \succ'') \geq^{\text{sup}} M(X, \succ)$ does not hold. $\square$

8 ε-Optimization

A binary relation $\succ$ is *strongly acyclic* if there exists no *infinite improvement path*, i.e., no sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ such that $x^{k+1} \succ x^k$ for all k. It is well known, and easy to check anyway, that a binary relation $\succ$ on A is strongly acyclic if and only if $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{B}_A^{\emptyset}$. A strongly acyclic strict order always satisfies both (4.1) by default.

Proposition 8.1. *Let $\succ$ be a strongly acyclic strict order on a set X and $X \ni x \notin M(X, \succ)$. Then there is $y \in M(X, \succ)$ such that $y \succ x$.*

A straightforward proof is omitted.

An ordering represented by a mapping $v \colon A \to \mathcal{C}$, where $\mathcal{C}$ is a chain, is strongly acyclic if and only if $v(A) \subseteq \mathcal{C}$ is well ordered in the reversed order. With the exception of a finite $v(A)$, there is something exotic in such orderings. However, the property may be quite natural for "less rational" preferences. Let $v \colon A \to \mathbb{R}$ be bounded above and $\varepsilon > 0$; let the preference relation be

$$y \succ x \rightleftharpoons v(y) > v(x) + \varepsilon. \tag{8.1}$$

It is easily seen that $\succ$ is strongly acyclic. Maximizers for $\succ$ are exactly ε-maxima of v.

Let us reproduce standard definitions, see, e.g., Fishburn (1985). An *interval order* is a strict order $\succ$ such that

$$\forall x, y, a, b \in A \left[[y \succ x \ \& \ a \succ b] \Rightarrow [y \succ b \ \text{ or } \ a \succ x] \right]. \tag{8.2}$$

A *semiorder* is an interval order such that

$$\forall x, y, z \in A \left[z \succ y \succ x \Rightarrow \forall a \in A \left[z \succ a \ \text{ or } \ a \succ x \right] \right]. \tag{8.3}$$

Every ordering is a semiorder. It is also easily checked that $\succ$ defined by (8.1) satisfies both (8.2) and (8.3), i.e., is a semiorder.

Proposition 8.2. *Let $\succ$ be a semiorder on a set X and $X \ni x \notin M(X, \succ) \neq \emptyset$. Then there is $y \in M(X, \succ)$ such that $y \succ x$.*

A straightforward proof is omitted. Note that Proposition 8.2 would be wrong for an interval order (see, e.g., Example 3 from Kukushkin, 2008).

Like orderings, interval orders and semiorders can also be defined in terms of representations in chains. An *interval representation* of a binary relation $\succ$ on a set A consists of a chain $\mathcal{C}$ and two mappings $v^+, v^- \colon A \to \mathcal{C}$ such that:

$$\forall x \in A \left[v^+(x) \geq v^-(x) \right]; \tag{8.4a}$$

$$\forall x, y \in A \left[y \succ x \iff v^-(y) > v^+(x) \right]. \tag{8.4b}$$

A *semiorder representation* of a binary relation $\succ$ on a set A consists of a chain $\mathcal{C}$ and two mappings $v\colon A \to \mathcal{C}$ and $\delta\colon v(A) \to \mathcal{C}$ such that:

$$\forall x, y \in A\,[v(y) > v(x) \Rightarrow \delta \circ v(y) \geq \delta \circ v(x)]; \tag{8.5a}$$

$$\forall x \in A\,[v(x) \geq \delta \circ v(x)]; \tag{8.5b}$$

$$\forall x, y \in A\,[y \succ x \iff \delta \circ v(y) > v(x)]. \tag{8.5c}$$

Clearly, if $\langle v, \delta\rangle$ is a semiorder representation of $\succ$, then $\langle v, \delta \circ v\rangle$ is an interval representation of $\succ$.

Proposition 8.3. *Let $\succ$ be a binary relation on a set A. Then $\succ$ is an interval order if and only if it admits an interval representation.*

Proposition 8.4. *Let $\succ$ be a binary relation on a set A. Then $\succ$ is a semiorder if and only if it admits a semiorder representation.*

The proofs are not quite trivial, but both statements are well known.

A *strong semiorder representation* of a binary relation $\succ$ on a set A consists of a complete chain $\mathcal{C}$ and two mappings $v\colon A \to \mathcal{C}$ and $\delta\colon \mathcal{C} \to \mathcal{C}$ such that, for all $x, y \in A$ and $w', w \in \mathcal{C}$:

$$\forall w', w \in \mathcal{C}\,[w' > w \Rightarrow \delta(w') \geq \delta(w)]; \tag{8.6a}$$

$$\forall w \in \mathcal{C}\,[w \geq \delta(w)]; \tag{8.6b}$$

$$\forall w \in \mathcal{C}\,[w = \delta(w) \Rightarrow w = \min \mathcal{C}]; \tag{8.6c}$$

$$\forall x, y \in A\,[y \succ x \iff \delta \circ v(y) > v(x)]. \tag{8.6d}$$

Proposition 8.5. *Let $\succ$ be a binary relation on a set A. Then $\succ$ is a strongly acyclic semiorder if and only if it admits a strong semiorder representation.*

Proof. Let (8.6) hold; by the sufficiency part of Proposition 8.4, $\succ$ is a semiorder. Suppose there is an infinite improvement path $x^0, x^1, \ldots$; then $v(x^{k+1}) > v(x^k)$ for all k. We denote $w^+ := \sup_k v(x^k)$ and $w^- := \sup_k \delta \circ v(x^k)$; by (8.6b) and (8.6c), $w^+ > \delta(w^+) \geq w^-$. Therefore, there is k such that $v(x^k) > w^- \geq \delta \circ v(x^{k+1})$, but this contradicts the supposed dominance $x^{k+1} \succ x^k$.

Let $\succ$ be a strongly acyclic semiorder. Applying the necessity part of Proposition 8.4, we obtain a semiorder representation. Then we move from (8.5) to (8.6) in two steps, the first being the standard Dedekind construction.

We define $\mathcal{C}^* := \{V \subseteq \mathcal{C} \mid \forall w', w \in \mathcal{C}\,\big[[w \in V \;\&\; w > w'] \Rightarrow w' \in V\big]\}$. $\mathcal{C}^*$ with the set inclusion is a complete chain, $\sup C$ for $C \subseteq \mathcal{C}^*$ being the set union of C. Then we define $v^*\colon A \to \mathcal{C}^*$ by $v^*(x) := \{w \in \mathcal{C} \mid v(x) \geq w\}$ and $\delta^*\colon v^*(A) \to \mathcal{C}^*$ by $\delta^* \circ v^*(x) := \{w \in \mathcal{C} \mid \delta \circ v(x) > w\}$. It is easy to see that v^* and δ^* define a semiorder representation of $\succ$ with a strict inequality in (8.5b).

As the next step, denoting $C := v^*(A) \cup \delta^* \circ v^*(A) \subseteq \mathcal{C}^*$, we define $\mathcal{C}^{**}$ as the intersection of all complete subchains of $\mathcal{C}^*$ containing C, i.e., $\mathcal{C}^{**}$ consists of the least upper bounds of all nonempty subsets of C. For every $w \in \mathcal{C}^{**}$, we define $\Phi(w) := \{w' \in v^*(A) \mid w' \leq w\} \subseteq C$, $\delta^{**}(w) := \min \mathcal{C}^{**}$ if $\Phi(w) = \emptyset$, and $\delta^{**}(w) := \sup_{w' \in \Phi(w)} \delta^*(w') \in \mathcal{C}^{**}$ otherwise.

Let us show that the mappings $v^*\colon A \to \mathcal{C}^{**}$ and $\delta^{**}\colon \mathcal{C}^{**} \to \mathcal{C}^{**}$ satisfy conditions (8.6). Monotonicity (8.6a) immediately follows from the definition of δ^{**}; (8.6b), from the same definition and (8.5b). Proper representation condition (8.6d) follows from (8.5c) because δ^{**} coincides with δ^* on $v^*(A)$: if $x \in A$, then $v^*(x) = \max \Phi(v^*(x))$, hence $\delta^{**} \circ v^*(x) = \delta^* \circ v^*(x)$.

Finally, let us check (8.6c). When $w = v^*(x)$, a strict inequality follows from the equality $\delta^{**}(w) = \delta^*(w)$ and the strict inequality in (8.5b). If $\Phi(w) = \emptyset$, then $\delta^{**}(w) = \min \mathcal{C}^{**}$, hence (8.6c) holds trivially. Let $w = \sup \Phi(w) > \min \mathcal{C}^{**}$ and $w \notin v^*(A)$; if $\delta^{**}(w) = w$, we pick $x^0 \in (v^*)^{-1}\Phi(w)$ arbitrarily, and then define an infinite sequence of $x^k \in (v^*)^{-1}\Phi(w)$ inductively, in the following way. Since $v^*(x^k) \in \Phi(w)$ and $\delta^{**}(w) = w$, there must be $x^{k+1} \in A$ such that $v^*(x^{k+1}) \in \Phi(w)$ and $\delta^{**} \circ v^*(x^{k+1}) > v^*(x^k)$. Now we have $x^{k+1} \succ x^k$ for all k, i.e., a contradiction with the strong acyclicity of $\succ$. Finally, if $w > w^* := \sup \Phi(w)$, then $\delta^{**}(w) = \delta^{**}(w^*) < w^* < w$. $\square$

A strong semiorder representation of $\succ$ defined by (8.1) is obtained if we define $\mathcal{C} := [-\infty, \bar{w}]$, where $\bar{w}$ is an upper bound for v, and $\delta(w) := w - \varepsilon$.

Remark. The first paragraph in the necessity proof of Proposition 8.5 shows that Proposition 8.4 remains valid if we modify conditions (8.5) adding the completeness of $\mathcal{C}$ and a strict inequality in (8.5b). Then the only characteristic feature of *strongly acyclic* semiorders will be the fact that δ can be defined on the whole $\mathcal{C}$.

Proposition 8.6. *Let $\succ$ be a binary relation on a set A. Then $\succ$ is a strongly acyclic interval order if and only if it admits an interval representation $v^+, v^- \colon A \to \mathcal{C}$ and there is a strong semiorder representation $v \colon A \to \mathcal{C}$ and $\delta \colon \mathcal{C} \to \mathcal{C}$ such that $v^+(x) = v(x)$ and $v^-(x) \leq \delta \circ v(x)$ for all $x \in A$.*

The sufficiency proof is the same as in Proposition 8.5; the necessity proof is only slightly different.

Interval (or semiorder) representations also help in studying monotonicity. To achieve uniformity with the previous sections, we say that an interval order $\succ$ is *upper semi-represented* on A by an ordering $\ggcurly$ if there is an interval representation $\langle v^+, v^- \rangle$ of $\succ$ such that v^+ represents $\ggcurly$. It is worth noting that every ordering is upper semi-represented by itself (and only itself).

Proposition 8.7. *Let $\succ$ be an interval order on a set X, upper semi-represented by an ordering $\ggcurly$; let $X \ni x \notin M(X, \succ) \ni y$. Then $y \ggcurly x$.*

A straightforward proof is omitted. Note that $y \succ x$ cannot be asserted unless $\succ$ itself is an ordering.

Proposition 8.8. *Let A be a lattice and $\ggcurly$ be an ordering on A. Then the following statements are equivalent.*

1. *$\ggcurly$ satisfies* (5.3a).

2. *There holds $M(X, \succ') \geq^{\mathrm{wV}} M(X, \succ)$ whenever X is a sublattice of A, and $\succ$ and $\succ'$ are interval orders on X for which $M(X, \succ') \neq \emptyset \neq M(X, \succ)$ and there is an ordering $\ggcurly'$ on X such that $\succ$ and $\succ'$ are upper semi-represented by $\ggcurly$ and $\ggcurly'$, respectively, and* (4.4d) *holds on X for $\ggcurly'$ and $\ggcurly$.*

3. *There holds $M(X, \succ') \geq^{\mathrm{pwV}} M(X, \succ)$ whenever X is a finite sublattice of A, and an ordering $\ggcurly'$ and semiorders $\succ$ and $\succ'$ on A are such that* (4.4a) *holds on A for $\ggcurly'$ and $\ggcurly$, while $\succ$ and $\succ'$ are upper semi-represented by $\ggcurly$ and $\ggcurly'$, respectively.*

Proof. Let $y \in M(X, \succ')$ and $x \in M(X, \succ)$. If $y \wedge x \in M(X, \succ)$, we are home; otherwise, $x \ggcurly y \wedge x$, by Proposition 8.7, hence $y \vee x \ggcurly y$ by (5.3a). Therefore, $y \vee x \succeq' y$ by (4.4d), hence $y \vee x \in M(X, \succ')$ again by Proposition 8.7.

Let (5.3a) not hold: there are $x, y \in A$ such that $x \ggcurly y \wedge x$, but $y \not\succeq y \vee x$. We define $X := L(x, y)$, and an ordering $\succ'$ exactly as in the proof of implication [Statement 5 $\Rightarrow$ Statement 1] in Proposition 5.7. Exactly as in that proof, we have $M(X, \succ') = \{y\}$ while (4.4a) holds for $\succ'$ and $\ggcurly$. In simpler words, we pick $\succ'$ as $\ggcurly'$. Let $\mathcal{C}$ be a chain and $v \colon A \to \mathcal{C}$ represent $\ggcurly$. We define $\delta \colon v(A) \to \mathcal{C}$ by $\delta \circ v(z) = v(x)$ if $v(z) \geq v(x)$ and $\delta \circ v(z) = v(z)$ otherwise; let $\succ$ be the semiorder represented by $\langle v, \delta \rangle$. Obviously, $y \wedge x \notin M(X, \succ) \ni x$. Therefore, $M(X, \succ') \geq^{\mathrm{pwV}} M(X, \succ)$ does not hold and we are home. $\square$

Remark. When compared to Proposition 5.7, Proposition 8.8 states that (5.4b) must be added to (5.4a) if we want to retain the monotonicity w.r.t. $\geq^{\mathrm{wV}}$ for "ε-optimization" of the ordering.

Corollary. *Let X be a lattice, S be a poset, $u \colon X \times S \to \mathbb{R}$ be supermodular and bounded above in the first argument, and satisfy the increasing differences condition* (4.7)*; let $\mathcal{R}(s)$ for every $s \in S$ consist of ε-maxima of $u(\cdot, s)$ ($\varepsilon > 0$) on X. Then $\mathcal{R} \colon S \to X$ is increasing w.r.t. $\geq^{\mathrm{wV}}$.*

Proposition 8.9. *Let A be a lattice and $\ggcurly$ be an ordering on A. Then the following statements are equivalent.*

1. *$\ggcurly$ satisfies* (5.3b).
2. *There holds $M(X, \succ') \geq^{\mathrm{wV}} M(X, \succ)$ whenever X is a sublattice of A, and $\succ$ and $\succ'$ are interval orders on X for which $M(X, \succ') \neq \emptyset \neq M(X, \succ)$ and there is an ordering $\ggcurly'$ on X such that $\succ$ and $\succ'$ are upper semi-represented by $\ggcurly$ and $\ggcurly'$, respectively, and* (4.5d) *holds on X for $\ggcurly'$ and $\ggcurly$.*
3. *There holds $M(X, \succ') \geq^{\mathrm{pwV}} M(X, \succ)$ whenever X is a finite sublattice of A, and an ordering $\ggcurly'$ and semiorders $\succ$ and $\succ'$ on A are such that* (4.5a) *holds on A for $\ggcurly'$ and $\ggcurly$, while $\succ$ and $\succ'$ are upper semi-represented by $\ggcurly$ and $\ggcurly'$, respectively.*

The proof is dual to that of Proposition 8.8.

As usual, we finish the section with implications for strategic games. Were "Theorem A2" of Milgrom and Shannon (1994) true, Corollary to Proposition 8.8 would give us a nice existence result for ε-Nash equilibria. Probably, this accounts for the absence of interest to ε-equilibria in the literature on games with strategic complementarities. Theorem 3.5 is a poor substitute for "Theorem A2" in this respect because ε-optimization is most natural when there is no topological restriction on preferences, but then there is no ground to expect the completeness of the set of ε-best responses. Although none of the known counterexamples to "Theorem A2" can be interpreted as an ε-best response correspondence of this kind, there is no result on the existence of monotone selections in this situation either. It should be noted that even the replacement of (4.7) in Corollary to Proposition 8.8 with its strict analogue would not give us monotonicity w.r.t. $\geq^{\mathrm{Vt}}$.

Example 8.10. Let $X := \{0, 1, 2\}$, $S := \{0, 1\}$, $u \colon X \times S \to \mathbb{R}$ be represented by the matrix (the axes are directed upwards and rightwards)

$$\begin{matrix} 1 & 3 \\ 0 & 1 \\ 3 & 0 \end{matrix}$$

and $\varepsilon := 2$. Since X is a chain, $u(\cdot, s)$ satisfies (5.1) trivially for either s; the strict analogue of (4.7) is obvious. However, $1 \notin \mathcal{R}(0) \ni 2$ and $1 \in \mathcal{R}(1)$, so even $\mathcal{R}(1) \geq^{\wedge} \mathcal{R}(0)$ does not hold.

Once the preferences of a player i are described by a strict order $\succ_i$ on X_N rather than an ordering, the notion of the best response correspondence (5.6) becomes inadequate. For every $x_{-i} \in X_{-i}$, we denote $\succ_i^{x_{-i}}$ the "restriction" of $\succ_i$ to X_i given x_{-i}, i.e.,

$$y_i \succ_i^{x_{-i}} x_i \rightleftharpoons (y_i, x_{-i}) \succ_i (x_i, x_{-i}).$$

The "best" (rather, undominated) response correspondence is now defined as

$$\mathcal{R}_i(x_{-i}) := M(X_i, \succ_i^{x_{-i}}), \tag{8.7}$$

while a *Nash equilibrium* is still $x_N \in X_N$ such that $x_i \in \mathcal{R}_i(x_{-i})$ for each $i \in N$.

For the reader's convenience, the principal way to obtain the existence of an ε-Nash equilibrium is demonstrated in Theorem 8.11 below. Then Theorem 8.12 gives the most general existence result available at the moment. In both cases, we have to assume that strategy sets are chains; both proofs bypass the problem of monotone selections.

Theorem 8.11. *Let Γ be a strategic game where each player i's strategy set X_i is a complete chain and preference relation is a strict order on X_N such that: (i) every relation $\succ_i^{x_{-i}}$ is strongly acyclic; (ii) whenever $x'_{-i} > x_{-i}$, the relations $\succ_i^{x'_{-i}}$ and $\succ_i^{x_{-i}}$ satisfy* (4.4a) *and* (4.4b). *Then Γ possesses a Nash equilibrium.*

Proof. The key role is played by the following recursive definition of a sequence $x_N^k \in X_N$ ($k \in \mathbb{N}$) such that $x_N^{k+1} \geq x_N^k$ and $x_i^{k+1} \in \mathcal{R}_i(x_{-i}^k)$ for all $k \in \mathbb{N}$ and $i \in N$. By the latter condition, x_N^k is a Nash equilibrium if $x_N^{k+1} = x_N^k$. On the other hand, the sequence must stabilize at some stage because of the strong acyclicity assumption.

We define $x_i^0 := \min X_i$ for each $i \in N$. Given x_N^k, we, for each $i \in N$ independently, check whether $x_i^k \in \mathcal{R}_i(x_{-i}^k)$ holds. If it does, we define $x_i^{k+1} := x_i^k$; otherwise, we pick $x_i^{k+1} \in \mathcal{R}_i(x_{-i}^k)$ such that $x_i^{k+1} \succ_i^{x_{-i}^k} x_i^k$ (it exists by Proposition 8.1). Supposing $x_i^{k+1} < x_i^k$, hence $k > 0$, we obtain $x_i^{k+1} \succ_i^{x_{-i}^{k-1}} x_i^k$ by (4.4b), contradicting the induction hypothesis $x_i^k \in \mathcal{R}_i(x_{-i}^{k-1})$. Therefore, $x_i^{k+1} > x_i^k$, hence $x_N^{k+1} \geq x_N^k$.

Supposing that $x_N^{k+1} > x_N^k$ for all $k \in \mathbb{N}$, we denote $x_i^\omega := \sup_k x_i^k$ for each $i \in N$; the completeness of X_i is essential here. Whenever $x_i^{k+1} \neq x_i^k$, we have $x_i^{k+1} \succ_i^{x_{-i}^k} x_i^k$ and $x_i^{k+1} > x_i^k$ as was shown in the previous paragraph; since $x_{-i}^\omega \geq x_{-i}^k$, we have $x_i^{k+1} \succ_i^{x_{-i}^\omega} x_i^k$ by (4.4a). Since N is finite, there must be $i \in N$ such that $x_i^{k+1} > x_i^k$ for an infinite number of k. Clearly, the elimination of repetitions in the sequence $\{x_i^k\}_k$ makes it an infinite improvement path for the relation $\succ_i^{x_{-i}^\omega}$, which contradicts the supposed strong acyclicity. $\square$

Corollary. *Let Γ be a strategic game where each player i's strategy set X_i is a complete chain and utility function $u_i \colon X_N \to \mathbb{R}$ is bounded above in own strategy and satisfies the increasing differences condition* (4.7). *Then Γ possesses an ε-Nash equilibrium for every $\varepsilon > 0$.*

Remark. The proof of Theorem 8.11 is invalid for multi-dimensional strategy sets because x_i^{k+1} extracted from Proposition 8.1 may be incomparable with x_i^k: arguments like Lemma 5.3.1 are only valid for orderings. No counterexample to such an extension of the theorem is known, however.

Theorem 8.12. *Theorem 5.14 remains valid if two more conditions are added to the list* (5.7) *and* (6.3)*:*

X_i is a complete chain;

the relation $\succ_i$ is a strict order satisfying both conditions
(i) and (ii) from Theorem 8.11. (8.8a)

X_i is a complete chain;

the relation $\succ_i$ satisfies condition (ii) from Theorem 8.11;

every relation $\succ_i^{x_{-i}}$ is a semiorder satisfying both conditions (4.1). (8.8b)

Proof. We combine the basic ideas from the proofs of Theorems 5.3, 5.14, and 8.11. Let N^0 denote the set of $i \in N$ for which either one of (5.7) or (6.3) holds; let $N^\varepsilon := N \setminus N^0$. For each $i \in N^0$, we fix a monotone selection r_i from the best response correspondence $\mathcal{R}_i$.

Let Λ be a well ordered set with a cardinality greater than that of X_N. By (transfinite) recursion, we construct a mapping $\lambda_N \colon \Lambda \to X_N$ such that:

$$\forall i \in N\, \forall \alpha \in \Lambda \left[\lambda_i(\alpha+1) \in \mathcal{R}_i(\lambda_{-i}(\alpha))\right]; \tag{8.9a}$$

$$\forall i \in N^0\, \forall \alpha \in \Lambda \left[\lambda_i(\alpha+1) = r_i(\lambda_{-i}(\alpha))\right]; \tag{8.9b}$$

$$\forall \alpha, \beta \in \Lambda \left[\alpha > \beta \Rightarrow [\lambda_N(\alpha) \geq \lambda_N(\beta)]\right]; \tag{8.9c}$$

$$\forall i \in N^\varepsilon\, \forall \alpha, \beta \in \Lambda \left[\alpha > \beta \Rightarrow [\lambda_i(\alpha) = \lambda_i(\beta) \text{ or } \lambda_i(\alpha) \succ_i^{\lambda_{-i}(\alpha)} \lambda_i(\beta)]\right]; \tag{8.9d}$$

$$\forall i \in N\, \forall \alpha \in \Lambda \left[\lambda_i([0, \alpha]) \in \mathfrak{C}_{X_i}\right]. \tag{8.9e}$$

First, we define $\lambda_i(0) := \min X_i$ for each $i \in N$. Let $\lambda_N(\alpha)$ have been defined. For $i \in N^0$, we define $\lambda_i(\alpha+1) := r_i(\lambda_{-i}(\alpha))$, thus ensuring (8.9a) and (8.9b). For $i \in N^\varepsilon$, we define $\lambda_i(\alpha+1) := \lambda_i(\alpha)$ if $\lambda_i(\alpha) \in \mathcal{R}_i(\lambda_{-i}(\alpha))$, ensuring (8.9a) again as well as the continuation of (8.9d). Otherwise, we pick $\lambda_i(\alpha+1) \in \mathcal{R}_i(\lambda_{-i}(\alpha))$ such that $\lambda_i(\alpha+1) \succ_i^{\lambda_{-i}(\alpha)} \lambda_i(\alpha))$ [it exists by Proposition 8.1 under (8.8a), or by Proposition 8.2 under (8.8b)], thus ensuring (8.9d) for $\lambda_i(\alpha+1)$ and $\lambda_i(\alpha)$. Since at most one point is added to $\lambda_i([0, \alpha])$, (8.9e) continues to hold. The check of (8.9c), as well as (8.9d) for $\beta < \alpha$, is postponed till after the definition of $\lambda_i(\alpha)$ for limit ordinals.

Let α be a limit ordinal, and $\lambda_N(\beta)$ have been defined for all $\beta < \alpha$. Then we define $\lambda_i(\alpha) := \sup_{\beta<\alpha} \lambda_i(\beta)$ for each $i \in N$, ensuring (8.9c) and (8.9e). By (4.4a), (8.9c) and (8.9d), we have $\lambda_i(\beta') \succ_i^{\lambda_{-i}(\alpha)} \lambda_i(\beta)$ whenever $\beta', \beta < \alpha$ and $\lambda_i(\beta') > \lambda_i(\beta)$. If $\lambda_i(\alpha) = \lambda_i(\beta)$ for some $\beta < \alpha$, which holds, in particular, under (8.8a), then (8.9d) is valid; otherwise, the chain $\lambda_i([0, \alpha])$ satisfies the "left-hand-side" condition in (4.1b), hence $\lambda_i(\alpha) \succ_i^{\lambda_{-i}(\alpha)} \lambda_i(\beta)$ for all $\beta < \alpha$, i.e., (8.9d) holds again.

Now let us return to a "successor step." If α itself is a successor ordinal, then condition (4.4b), exactly as in the proof of Theorem 8.11, ensures that $\lambda_i(\alpha+1) > \lambda_i(\alpha)$, hence (8.9c) continues to hold. If α is a limit ordinal, the assumption $\lambda_i(\alpha+1) < \lambda_i(\alpha)$ would imply $\lambda_i(\alpha+1) < \lambda_i(\beta)$ for some $\beta < \alpha$, hence $\lambda_i(\alpha+1) < \lambda_i(\beta+1)$, and a contradiction with the optimality of $\lambda_i(\beta+1)$ obtained in exactly the same way. In either case, (8.9d) for $\lambda_i(\alpha+1)$ and $\lambda_i(\beta)$ with $\beta < \alpha$ holds by (4.4a).

The final argument is again standard. We must have $\lambda_N(\alpha) = \lambda_N(\beta)$ for some $\beta < \alpha$. Then we have $\lambda_N(\beta+1) = \lambda_N(\beta)$ by (8.9c); therefore, $\lambda_N(\beta)$ is a Nash equilibrium by (8.9a). □

9 Acknowledgments

Financial support from a Presidential Grant for the State Support of the Leading Scientific Schools (NSh-2982.2008.1), the Russian Foundation for Basic Research (project 08-07-00158), the Russian Foundation for Humanities (project 08-02-00347), and the Spanish Ministry of Education and Science (project SEJ 2007-67135) is acknowledged. I have benefitted from a visit to U.S. universities; I thank Hervé Moulin and Francisco Marhuenda, whose effort and generosity made the trip possible. I also thank Vladimir Danilov and Alexei Savvateev for very fruitful discussions.

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