On the Existence of Optima in Complete Chains and Lattices*

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Abstract

A necessary and sufficient condition for a preference ordering defined on a chain-complete poset to attain its maximum in every subcomplete chain is obtained. A meet-superextremal, or join-superextremal, function on a complete lattice attains its maximum in every subcomplete sublattice if and only if it attains a maximum in every subcomplete chain.

Key words Lattice Optimization · Complete Chain · Complete Lattice · Superextremal function

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1 Introduction

The familiar observation that an upper semicontinuous function attains its maximum in every compact set resolves the question of the existence of optimal choices in many situations, but not in all. First, sometimes preferences *have* to be described by discontinuous relations, e.g., lexicographic combinations. Second, there may be no "natural" topology, hence the question of whether the preferences are (semi)continuous may become intolerably vague.

This paper studies conditions for the existence of optimal choices that do not refer to any topological notion. Instead, we assume an order structure on the set of alternatives, following in this respect the approach of Veinott [1]. The main difference is that we obtain *necessary* and sufficient conditions, while none of the existence results from [1] makes any claim to necessity.

We formulate a property of an ordering on a chain-complete poset, "mono-ω-transitivity," which is a natural analog of the topological notion of "ω-transitivity" [2, 3] in an order context. This property of a preference ordering is shown to be necessary and sufficient for the existence of optima in every subcomplete chain. Then we show that if an ordering on a complete lattice satisfies either of

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two “combinatorial” conditions that have already emerged in the studies of monotone comparative statics [4], then it attains its maximum in every subcomplete sublattice if and only if it attains a maximum in every subcomplete chain. Finally, we show that none of the conditions considered in [1] ensures this equivalence: an ordering on a complete lattice may satisfy Veinott’s combinatorial conditions and attain its maximum in every subcomplete chain, but still fail to attain its maximum in the whole lattice.

2 Preliminaries

For every set A, we denote $\mathfrak{B}_A$ the set of all nonempty subsets of A. Given a binary relation $\succ$ on A and $X \in \mathfrak{B}_A$, we denote

$$M(X, \succ) := \{x \in X \mid \nexists y \in X\, [y \succ x]\}, \tag{1}$$

the set of *maximizers* of $\succ$ in X. The interpretation is that an agent has (strict) preferences described by relation $\succ$ over the whole A, but may be faced with the necessity to choose from a subset $X \in \mathfrak{B}_A$, in which case any point from $M(X, \succ)$ will do. For such choice to be possible, we need $M(X, \succ) \neq \emptyset$, at least, for “plausible” X.

Typically, A is a *partially ordered set* (a *poset*) with the order $>$; most often, a *lattice*. The exact definitions are assumed commonly known.

Theorem A (Zorn’s Lemma). *If a poset X has the property that every chain $Y \in \mathfrak{B}_X$ has an upper bound in X, then $M(X, >) \neq \emptyset$.*

A poset A is *chain-complete* iff $\sup X$ and $\inf X$ exist for every chain $X \in \mathfrak{B}_A$. If A is a chain-complete poset and $X \in \mathfrak{B}_A$, we call X *chain-subcomplete* iff $\sup Y$ and $\inf Y$ belong to X for every chain $Y \in \mathfrak{B}_X$; if X itself is a chain, we call it a *subcomplete chain.* The set of all subcomplete chains in a chain-complete poset A is denoted $\mathfrak{C}_A$.

A lattice is *complete* iff the *greatest lower bound* or *meet*, $\bigwedge X$, and the *least upper bound* or *join*, $\bigvee X$, exist for every $X \in \mathfrak{B}_A$. If A is a complete lattice, $X \in \mathfrak{B}_A$ is a *subcomplete sublattice* of A iff $\bigwedge Y$ and $\bigvee Y$ belong to X for all $Y \in \mathfrak{B}_X$. Given a complete lattice A, the set of all subcomplete sublattices is denoted $\mathfrak{L}_A$.

Theorem B ([5], Lemma 3.1 and Corollaries). *A lattice A is complete if and only if it is chain-complete as a poset. Then a sublattice of A is subcomplete if and only if it is chain-subcomplete.*

The preference relation $\succ$ is always assumed to be an *ordering*, i.e., irreflexive, transitive, and *negatively transitive*, $z \not\succ y \not\succ x \Rightarrow z \not\succ x$. Then the “non-strict preference” relation $\succeq$ defined by $y \succeq x \rightleftharpoons x \not\succ y$ is reflexive, transitive, and total. Orderings can also be defined in terms of representations in chains: $\succ$ is an ordering if and only if there is a chain $\mathcal{C}$ and a mapping $u\colon A \to \mathcal{C}$ such that

$$y \succ x \iff u(y) > u(x) \tag{2}$$

for all $x, y \in A$. Then $M(X, \succ) = \operatorname{Argmax}_{x \in X} u(x)$ for every $X \in \mathfrak{B}_A$.

The most usual assumption in, say, game theory is that the preferences of an agent are described by a *utility function* $u\colon A \to \mathbb{R}$. In a purely ordinal framework, it is natural to replace $\mathbb{R}$ with an arbitrary chain. Here, we take an intermediate position, considering orderings satisfying an additional restriction.

Given an ordering $\succ$ on A and $X \in \mathfrak{B}_A$, we call an infinite sequence $\langle x^k \rangle_{k\in\mathbb{N}}$ in X *optimizing* iff (i) $x^{k+1} \succ x^k$ for all k; (ii) for every $x \in X \setminus M(X, \succ)$, there is $k \in \mathbb{N}$ such that $x^k \succ x$. We call $\succ$ *regular*, just for want of a better term, iff, for every $X \in \mathfrak{B}_A$, either $M(X, \succ) \neq \emptyset$, or there is an optimizing sequence in X.

Every ordering admitting a representation (2) with $\mathcal{C} = \mathbb{R}$ is regular. Another example, not so obvious, but still straightforward, emerges when $\mathcal{C} = \mathbb{R}^m$ with a lexicographic order. An example of an ordering that is *not* regular in this sense is given in Section 6.4.

3 Theorems

We call an ordering $\succ$ on a chain-complete poset A *mono-ω-transitive* iff both following conditions hold:

$$\forall k \in \mathbb{N}\, [x^{k+1} \succ x^k \;\&\; x^{k+1} > x^k] \Rightarrow \sup\{x^k\}_k \succ x^0; \tag{3a}$$

$$\forall k \in \mathbb{N}\, [x^{k+1} \succ x^k \;\&\; x^{k+1} < x^k] \Rightarrow \inf\{x^k\}_k \succ x^0. \tag{3b}$$

An ordering is mono-ω-transitive if every upper contour set, $\{y \in A \mid y \succeq x\}$ $(x \in A)$, is chain-subcomplete. The converse is wrong: consider, e.g., a lexicographic order (as $\succ$) on a compact subset of $\mathbb{R}^m$ with the natural partial order (as $>$).

Theorem 3.1. *A regular ordering $\succ$ on a chain-complete poset A has the property that $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$ if and only if $\succ$ is mono-ω-transitive.*

Proof. The sufficiency proof is deferred to after Theorem 3.2. To prove necessity, let a sequence $\langle x^k \rangle_{k\in\mathbb{N}}$ satisfy the conditions in the left hand side of (3a). We set $X := \{x^k\}_{k\in\mathbb{N}} \cup \{\sup_k x^k\}$; clearly, $X \in \mathfrak{C}_A$, hence $M(X, \succ) \neq \emptyset$, hence $M(X, \succ) = \{\sup_k x^k\}$, hence $\sup_k x^k \succ x^0$. The proof of (3b) is obtained by reversing the order on A. □

Remark 3.1. Clearly, the necessity proof does not need the regularity assumption.

Following [1, 6], we call an ordering $\succ$ on a lattice A *meet-superextremal* iff

$$\forall x, y \in A\, \big[x \succ y \wedge x \Rightarrow y \vee x \succ y\big] \tag{4a}$$

and *join-superextremal* iff

$$\forall x, y \in A\, \big[y \succ y \vee x \Rightarrow y \wedge x \succ x\big]. \tag{4b}$$

An ordering satisfying both conditions (4) is called *lattice-superextremal.*

Remark 3.2. Milgrom and Shannon [8] called a real-valued function *quasisupermodular* iff it generates an ordering satisfying both conditions (4).

Conditions (4) play crucial roles in the study of monotone comparative statics of optima when a preference ordering $\succ$ is fixed while a sublattice of feasible alternatives X is varied [6]. (Actually, two more conditions of a similar structure were considered in [6], but we do not need them here.) When the preference ordering is varied, those roles are taken [4] by conditions (5) and (6):

$$\forall x, y \in A\, \big[(x \succ y \wedge x \ \text{ or } \ y \succ y \wedge x) \Rightarrow (y \vee x \succ x \ \text{ or } \ y \vee x \succ y)\big]; \tag{5a}$$
$$\forall x, y \in A\, \big[(x \succ y \vee x \ \text{ or } \ y \succ y \vee x) \Rightarrow (y \wedge x \succ x \ \text{ or } \ y \wedge x \succ y)\big]; \tag{5b}$$

$$\forall x, y \in A\, \big[(x \succ y \wedge x \,\&\, y \succ y \wedge x) \Rightarrow (y \vee x \succ x \,\&\, y \vee x \succ y)\big]; \tag{6a}$$
$$\forall x, y \in A\, \big[(x \succ y \vee x \,\&\, y \succ y \vee x) \Rightarrow (y \wedge x \succ x \,\&\, y \wedge x \succ y)\big]. \tag{6b}$$

Proposition 3.1. *An ordering $\succ$ on a lattice A satisfies any one of the conditions* (4) *if and only if it satisfies both corresponding conditions* (5) *and* (6)*.*

Proof. The implications $\big[(4a) \Rightarrow [(5a)\ \&\ (6a)]\big]$ and $\big[(4b) \Rightarrow [(5b)\ \&\ (6b)]\big]$ are straightforward. Let (5a) and (6a) hold and $x \succ y \wedge x$. By (5a), we have either $y \vee x \succ x$ or $y \vee x \succ y$; in the latter case, we are home immediately. If $y \succeq y \vee x$, then $y \succ x$, hence $y \vee x \succ y$ by (6a), contradicting the assumption. The other implication is proved by reversing the order on A. $\square$

Remark 3.3. Essentially, our Proposition 3.1 is equivalent to Proposition 11 from [4], but conditions (4) – (6) look a bit differently there.

All four conditions (5) and (6) are mutually independent: any three of them may hold without the fourth [4, Example 2]. Each pair (5) and (6) consists of mutually dual conditions: they turn into one another when the order on A is reversed. When x and y are comparable in the basic order, each of the conditions (4) – (6) holds trivially.

Theorem 3.2. *Let A be a complete lattice and $\succ$ be a regular ordering on A satisfying* (5a)*. Then $\succ$ has the property that $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{L}_A$ if and only if it is mono-ω-transitive.*

The proof is deferred to Section 4.

Proof of sufficiency in Theorem 3.1. Let $X \in \mathfrak{C}_A$. The restriction of $\succ$ to X satisfies (5a), hence the sufficiency part of Theorem 3.2 applies. $\square$

Corollary 3.1. *A meet-superextremal function on a complete lattice attains its maximum if it attains a maximum in every subcomplete chain.*

Theorem 3.3. *Let A be a complete lattice and $\succ$ be a regular ordering on A satisfying* (5b)*. Then $\succ$ has the property that $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{L}_A$ if and only if it is mono-ω-transitive.*

The proof is obtained by reversing the order on A in the proof of Theorem 3.2.

Corollary 3.2. *A join-superextremal function on a complete lattice attains its maximum if it attains a maximum in every subcomplete chain.*

Conditions (5a) or (5b) cannot be replaced with (6a) or (6b), nor even with their conjunction.

Example 3.1. Let $A := \big(\{n/(n+1)\}_{n\in\mathbb{N}} \cup \{1\}\big) \times \big(\{0\} \cup \{1/(n+1)\}_{n\in\mathbb{N}}\big) \subset \mathbb{R}^2$ and $u\colon A \to \mathbb{R}$ be as follows: $u(1, x_2) = u(x_1, 0) := 0$; $u(n_1/(n_1+1), 1/(n_2+1)) := \min\{n_1, n_2\}$. A with the order induced from $\mathbb{R}^2$ is a complete lattice. The ordering $\succ$ represented, in the sense of (2), by u is regular because u maps to reals and mono-ω-transitive because the left hand side conditions in either (3a) or (3b) never hold. By Theorem 3.1, $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$. On the other hand, $\sup_{x\in A} u(x) = +\infty$, hence $M(A, \succ) = \emptyset$.

It is easily checked that u satisfies the condition $u(y \vee x) \wedge u(y \wedge x) \geq u(y) \wedge u(x)$, hence both (6a) and (6b). Naturally, neither (5a) nor (5b) are satisfied. Thus, from the viewpoint of the existence of optima, (5a) is a "greater half" of (4a) than (6a); ditto for (5b), (6b), and (4b).

4 Proof of Theorem 3.2

The necessity follows from that in Theorem 3.1 because $\mathfrak{C}_A \subseteq \mathfrak{L}_A$ by Theorem B above. We start the sufficiency proof with the definition of two auxiliary orders (irreflexive and transitive relations):

$$y \mathrel{\overset{\succ}{>}} x \rightleftharpoons [y \succ x \;\&\; y > x];$$
$$y \mathrel{\overset{\succ}{<}} x \rightleftharpoons [y \succ x \;\&\; y < x].$$

Claim 4.1. *If X is a sublattice of A, $x \in M(X, \overset{\succ}{>})$, and $X \ni y \succ x$, then $y \wedge x \succeq y$.*

Proof. If $y \succ y \wedge x$, then (5a) would imply that $y \vee x \succ x$, which contradicts the assumption $x \in M(X, \overset{\succ}{>})$ since X is a sublattice. □

Claim 4.2. *If $x \in X \in \mathfrak{L}_A$, then either $x \in M(X, \overset{\succ}{>})$ or there is $y \in M(X, \overset{\succ}{>})$ such that $y \mathrel{\overset{\succ}{>}} x$.*

Proof. We define $X^* := \{y \in X \mid y \mathrel{\overset{\succ}{>}} x\}$. It is sufficient to show that $M(X^*, \overset{\succ}{>}) \neq \emptyset$ if $X^* \neq \emptyset$; we do that applying Zorn's Lemma (Theorem A above). Let $Y \subseteq X^*$ be a chain w.r.t. $\overset{\succ}{>}$; i.e., Y is a chain such that $y' \succ y$ whenever $y', y \in Y$ and $y' > y$. If $\bar{y} \in M(Y, \overset{\succ}{>}) \neq \emptyset$, then $\bar{y}$ is an upper bound of Y. If $M(Y, \overset{\succ}{>}) = \emptyset$, there exists an optimizing sequence $\langle y^k \rangle_{k\in\mathbb{N}}$ in Y since $\succ$ is regular. We define $y^\infty := \sup_k y^k$; $y^\infty \in X$ because the latter is subcomplete. For every $y \in Y$, there is $m \in \mathbb{N}$ such that $y^m \mathrel{\overset{\succ}{>}} y$; since $y^\infty = \sup_{k\geq m} y^k$, we have $y^\infty \mathrel{\overset{\succ}{>}} y^m$ by (3a), hence $y^\infty \mathrel{\overset{\succ}{>}} y \mathrel{\overset{\succ}{>}} x$. Therefore, $y^\infty \in X^*$ and is an upper bound of Y in X^*. □

Let $X \in \mathfrak{L}_A$ and $\langle y^h \rangle_{h\in\mathbb{N}}$ be an optimizing sequence in X. (If there is no such sequence, then $M(X, \succ) \neq \emptyset$, and we are already home.) We recursively define a sequence $\langle x^k \rangle_{k\in\mathbb{N}}$ such that $x^k \in M(X, \overset{\succ}{>})$, $x^{k+1} \mathrel{\overset{\succ}{<}} x^k$, and $x^{k+1} \succ y^{k+1}$ for all k. First, $x^0 := \bigvee X$.

Having $x^k \in M(X, \overset{\succ}{>})$ defined, we first check whether $x^k \in M(X, \succ)$; if the answer is "yes," we are home again. Otherwise, we pick $h \in \mathbb{N}$ such that $y^h \succ x^k$ and $h > k+1$, hence $y^h \succ y^{k+1}$.

Denoting $x^* := y^h \wedge x^k$, we have $x^* \succeq y^h \succ x^k$ by Claim 4.1, hence $x^* \underset{<}{\succ} x^k$. Now we define $Y := \{x \in X \mid x^* \leq x \leq x^k\}$ and, applying Claim 4.2, obtain $x^{k+1} \in M(Y, \underset{>}{\succ})$ such that $x^{k+1} \succeq x^*$.

Let us show that $x^{k+1} \in M(X, \underset{>}{\succ})$. Otherwise, there is $y \in X$ such that $y \underset{>}{\succ} x^{k+1}$, hence $y \succ x^k$. We define $y^* := y \wedge x^k$ and apply Claim 4.1, obtaining $y^* \succeq y \succ x^{k+1}$. Besides, $x^k \geq y^* \geq x^{k+1} \geq x^*$, hence $y^* \in Y$ and $y^* \underset{>}{\succ} x^{k+1}$, which contradicts $x^{k+1} \in M(Y, \underset{>}{\succ})$. Thus, $x^{k+1} \in M(X, \underset{>}{\succ})$ indeed.

Finally, we set $x^\infty := \inf_k x^k$. By (3b), we have $x^\infty \succ x^k$ for each $k \in \mathbb{N}$. Since X is subcomplete, we have $x^\infty \in X$. Since $x^{k+1} \succ y^{k+1}$ for each k, we have $x^\infty \in M(X, \succ)$.

5 Comparison with Veinott's Results

Veinott [1] obtained three independent theorems on the existence of optima: Theorems 6.2, 6.12, and 6.41. First of all, he considered *minimization*; to make comparisons with this paper possible, we "transform" his assumptions and results to the case of *maximization*. Sometimes, the corresponding statement is already in [1]; sometimes, it is easy to formulate by duality.

In each theorem of [1], an assumption is made (and used in the proof) that every upper contour set is chain-subcomplete. As was already noted, this assumption implies mono-ω-transitivity of $\succ$, but is not implied by it. Lexicographic preferences provide an example of a situation where our theorems show the existence of optima while Veinott's theorems are inapplicable.

Besides the completeness assumption, each theorem imposes a "combinatorial" condition. Following [1], we call a mapping u from a lattice A to a chain *dual quasilattice mapping*, *meet supermorphism*, or *superextremal mapping* iff, respectively,

$$\forall y, x \in A\, [u(y \vee x) \vee u(y \wedge x) \geq u(y) \wedge u(x)], \tag{7a}$$

$$\forall y, x \in A\, [u(y \wedge x) \geq u(y) \wedge u(x)], \tag{7b}$$

$$\forall y, x \in A\, \big[u(y \vee x) \vee u(y \wedge x) \geq u(y) \vee u(x) \ \text{ or } \ u(y \vee x) \wedge u(y \wedge x) \geq u(y) \wedge u(x)\big]. \tag{7c}$$

It is easy to see that either of (5a) and (5b) implies (7c), which, in turn, implies (7a). However, even (4a) and (4b) together do not imply (7b). The latter implies (7a) [and (6a)], but neither of conditions (5), nor (7c).

Example 5.1. Let $A := \{(0,0), (0,1), (1,0), (1,1)\} \subset \mathbb{R}^2$; we consider two orderings on A represented by these matrices (the axes are directed upwards and rightwards):

$$\mathbf{a.} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{b.} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

The matrix "a" satisfies (7b), hence (7a) and (6a), but none of the conditions (5), (6b), or (7c). The matrix "b" satisfies all conditions (4) hence (7c) and (7a), but not (7b).

The function u in Example 3.1 satisfies both (7b) and (7c) for the same reason as (6a) and (6b). Thus, none of Veinott's conditions (7) allows us to reduce the problem of the existence of optima in subcomplete sublattices to that in subcomplete chains.

6 Concluding Remarks

6.1. We may call a lattice A *conditionally complete* iff meet $\bigwedge X$ exists for all $X \in \mathfrak{B}_A$ that are bounded below, while join $\bigvee X$ exists for all $X \in \mathfrak{B}_A$ that are bounded above. $\mathbb{R}^m$ with the natural order is a conditionally complete lattice which is not complete. If A is a conditionally complete lattice, $X \in \mathfrak{B}_A$ is a *subcomplete sublattice* of A iff $\bigwedge Y$ and $\bigvee Y$ exist in A and belong to X for all $Y \in \mathfrak{B}_X$. All our theorems admit straightforward generalizations to conditionally complete lattices or "conditionally chain-complete posets."

6.2. There is no clear prospect for a necessary and sufficient condition on $\succ$ ensuring $M(X, \succ) \neq \emptyset$ for all $X \in \mathfrak{L}_A$, in the style of Theorem 3.1: When A is finite, every ordering attains a maximum in every nonempty subset, while conditions like (5), (6), or (7) remain biting.

6.3. An anonymous referee raised a question of whether mono-ω-transitivity can be described as (semi)continuity in an appropriate topology. Very technically speaking, the answer is positive. We may call a subset U of A open iff, whenever an infinite sequence $\langle x^k \rangle_k$ satisfies the left hand side conditions of either (3a) or (3b), and $x^k \in A \setminus U$ for every k, there holds $\sup_k x^k \in A \setminus U$, or, respectively, $\inf_k x^k \in A \setminus U$, too. Then an ordering $\succ$ is mono-ω-transitive if and only if it is upper semicontinuous in this topology. The problem with this approach is that our topology explicitly depends on $\succ$: if we want to switch attention to another ordering, we have to switch to another topology. Thus, a negative answer may be more appropriate.

6.4. An ordering that is not regular in our sense could hardly be relevant to any decision problem. Still, one may wonder whether the same results could be obtained without the assumption. The necessity part of Theorem 3.1, as well as of Theorems 3.2 and 3.3, obviously remains valid, but the sufficiency perishes.

Example 6.1. Let A' be a well-ordered uncountable set. We define $A^* := \{a \in A' \mid \{x \in A' \mid x < a\}$ is countable$\}$ and $A := A^* \cup \{\sup A^*\}$. It is easy to see that A is a complete chain and $\sup A^* \notin A^*$.

Then we define a preference ordering (actually, a linear order) on A:

$$y \succ x \rightleftharpoons \big[y \in A^* \ \&\ [y > x \ \text{ or } \ x = \sup A^*]\big].$$

Condition (3b) holds vacuously; (3a), because $\sup\{x^k\}_k \in A^*$ whenever $\{x^k\}_k \subseteq A^*$. However, $M(A, \succ) = \emptyset$. We see that Theorem 3.1 does not hold without the regularity of preferences.

On the other hand, it remains an open question whether the meet superextremal function in Corollary 3.1 can be replaced with an ordering satisfying (5a) (or even both conditions (4) for that matter).

6.5. If we replace the strict preference $\succ$ in both conditions (3a) and (3b) with $\succeq$, we obtain a requirement that is stronger than mono-ω-transitivity, but still weaker than the chain-subcompleteness of upper contour sets. Combining it with conditions (6) or (7), we may obtain further sufficient conditions for the existence of optima in complete lattices, independent of both Veinott's results and Theorems 3.2 and 3.3 here. There are some preliminary results and counterexamples in the area, but the whole picture is far from clear.

6.6. Veinott [1] formulates conditions (7), as well as a number of similar ones, without the assumption that u maps to a chain. Moreover, he proves a few statements concerning such more general maps, although his Theorems 6.2, 6.12, and 6.41 are restricted to “scalar” u. It would be interesting to obtain the existence of maximizers of more general preferences along those lines, but there is no clear prospect for that at the moment.

6.7. Similarly to [3, Theorem 4], Theorem 3.1 remains valid if $\succ$ is a semiorder (e.g., ε-improvement). In principle, the sufficiency part can be extended beyond semiorders, cf. [3, Theorem 1], but the notion of an optimizing sequence has to be modified considerably. The necessity does not hold even for interval orders [3, Example 3]. As to Theorems 3.2 and 3.3, it is unclear whether they remain valid, say, for semiorders. On the other hand, the impossibility result of [9] does not apply here, hence even broader characterization theorems may yet emerge.

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