Maximizing a preference relation on complete chains and lattices

Nikolai S. Kukushkin

Russian Academy of Sciences,
Dorodnicyn Computing Center
ququns@inbox.ru
http://www.ccas.ru/mmes/perstaff/Kukushkin.html

MSC2010 Classifications: 91A10; 49J27
JEL Classifications: C 61; D 11

Key words:
preference relation; choice function; complete chain; complete lattice; quasisupermodularity

Introduction

Need existence results be boring?

Weierstrass Theorem

Theorem.
(…Bergstrom, 1973; Bondareva, 1973; Walker, 1977; …)
An acyclic binary relation with open lower contour sets admits a maximizer in every compact subset of its domain.

Unapplicable to Pareto dominance, lexicography, …

Smith (1974)

Lattice Programming

Veinott, A.F., Jr. (1934–2012)

Milgrom & Roberts (1990); Milgrom & Shannon (1994): weaker theorems with bizarre proofs.

Plan of Presentation

Preferences and choice

Optimization on compact subsets

Optimization on chains

- Countability assumptions

Optimization on lattices

- QSM conditions
- Veinott's conditions

An impossibility result

Preferences and choice

A: *alternatives*;
$\succ$: *preference relation* on A;
$\mathfrak{B}_A$: nonempty subsets of A.

For $X \in \mathfrak{B}_A$,

$$M(X,\succ) := \{x \in X \mid \nexists y \in X\,[y \succ x]\}. \tag{1}$$

non-strict preference relation on A:
$y \succeq x \rightleftharpoons x \not\succ y$;
$M(X,\succ) = \{x \in X \mid \forall y \in X\,[x \succeq y]\}$.

A *choice function*

$$M(\cdot, \succ)\colon \mathfrak{B}_A \to 2^A.$$

Basic properties

$(X \in \mathfrak{B}_A)$

$$M(X, \succ) \neq \emptyset. \tag{2}$$

The *NM-property*:

$$\forall x \in X \setminus M(X, \succ)\, \exists y \in M(X, \succ)\, [y \succ x]. \tag{3}$$

(3) $\Rightarrow$ (2).

Easy cases:

$M(X,\succ) \neq \emptyset$ for all finite $X \in \mathfrak{B}_A$
if and only $\succ$ is acyclic.

$\succ$ has the NM-property on every finite $X \in \mathfrak{B}_A$
if and only if it is irreflexive and transitive.

$M(X,\succ) \neq \emptyset$ for all $X \in \mathfrak{B}_A$
if and only if $\succ$ is *strictly acyclic*,
i.e., admits no infinite improvement path.

$\succ$ has the NM-property on every $X \in \mathfrak{B}_A$
if and only if it is strictly acyclic and transitive.

Path Independence; Outcast Axiom.

“Rationality” restrictions

An *ordering*:

$\mathcal{C}$ is a chain; $u\colon A \to \mathcal{C}$

$$y \succ x \iff u(y) > u(x). \tag{4}$$

An *interval order*:

$\mathcal{C}$ is a chain; $u^+, u^-\colon A \to \mathcal{C}$

$$u^+(x) \geq u^-(x); \tag{5a}$$

$$y \succ x \iff u^-(y) > u^+(x). \tag{5b}$$

A *semiorder*:

a representation (5) with $u^-(x) = \lambda \circ u^+(x)$

[$\lambda\colon u^+(A) \to \mathcal{C}$ is increasing].

(3) $\equiv$ (2).

Topological case

A: a metric space;

$\mathfrak{C}_A$: nonempty compact subsets of A.

ω-*transitivity*:

$$\left[\forall k \in \mathbb{N}\,[x^{k+1} \succ x^k]\ \&\ x^\omega = \lim_{k\to\infty} x^k\right] \Rightarrow x^\omega \succ x^0. \qquad (6)$$

Theorem (Smith, 1974).

Let $\succ$ be an ordering [semiorder] on a metric space A.
Then $M(X,\succ) \neq \emptyset$ for all $X \in \mathfrak{C}_A$
if and only if $\succ$ is ω-transitive.

Theorem (Kukushkin, 2008).

Let $\succ$ be a binary relation on a metric space A.
Then $\succ$ has the NM-property on every $X \in \mathfrak{C}_A$
if and only if $\succ$ is irreflexive and ω-transitive.

Topological case (continued)

ω-*acyclicity* (Mukherji, 1977):

$$\Big[\forall k \in \mathbb{N}\,[x^{k+1} \succ x^k]\;\&\; x^\omega = \lim_{k\to\infty} x^k\Big] \Rightarrow x^\omega \neq x^0\ [x^\omega \succeq x^0]. \quad (7)$$

Theorem (Kukushkin, 2008).
Let $\succ$ be an interval order on a metric space A.
Then $M(X,\succ) \neq \emptyset$ for all $X \in \mathfrak{C}_A$
if and only if $\succ$ is ω-acyclic.

Example (Kukushkin, 2008).
Let $A = [0,1]$ and $y \succ x \rightleftharpoons 1 > y > x$ for all $y, x \in A$.
$\succ$ is an ω-acyclic, but not ω-transitive, interval order.
$M(X,\succ) \neq \emptyset$ for all $X \in \mathfrak{C}_A$,
but there is no NM-property on A itself [$M(A,\succ) = \{1\}$].

Topological case (continued further)

Theorem (Smith, 1974).
Let $\succ$ be an ordering on a metric space A.
Then $M(X,\succ) \in \mathfrak{C}_A$ for all $X \in \mathfrak{C}_A$
if and only if $\succeq$ is ω-transitive.

$$\left[\forall k,h \in \mathbb{N}\,[x^k \succeq x^h]\ \&\ x^\omega = \lim_{k\to\infty} x^k\right] \Rightarrow x^\omega \succeq x^0. \qquad (8)$$

Theorem (Kukushkin, 2008).
Let $\succ$ be an interval order on a metric space A.
Then $M(X,\succ) \in \mathfrak{C}_A$ for all $X \in \mathfrak{C}_A$
if and only if $\succeq$ satisfies (8).

Henceforth, A is a *poset*

$X \in \mathfrak{B}_A$ is (*chain-*)*subcomplete*
if $\sup Y$ and $\inf Y$ exist in A and belong to X
for every chain $Y \in \mathfrak{B}_X$.

"Upper semicontinuity" in posets:
All *upper contour sets*, $\{y \in X \mid y \succeq x\}$,
are subcomplete.

Admissible subsets

$\mathfrak{C}_A$: nonempty subcomplete chains in A;
$\mathfrak{L}_A$: nonempty subcomplete sublattices in A.

Optimization on chains

$\succ$ is *chain-transitive* if
it is transitive on every chain and:

$$\forall X \in \mathfrak{C}_A \Big[\big(\sup X^{\rightarrow} = \sup X \;\&\; \forall x, y \in X^{\rightarrow} [y > x \Rightarrow y \succ x] \big) \\ \Rightarrow \forall x \in X^{\rightarrow} \big[\sup X \succ x \big] \Big]; \quad (9a)$$

$$\forall X \in \mathfrak{C}_A \Big[\big(\inf X^{\leftarrow} = \inf X \;\&\; \forall x, y \in X^{\leftarrow} [y < x \Rightarrow y \succ x] \big) \\ \Rightarrow \forall x \in X^{\leftarrow} \big[\inf X \succ x \big] \Big]. \quad (9b)$$

$$[\, X^{\rightarrow} := X \setminus \{\sup X\};\; X^{\leftarrow} := X \setminus \{\inf X\} \,].$$

Theorem.
Let $\succ$ be a binary relation on a poset A.
Then $\succ$ has the NM-property on every $X \in \mathfrak{C}_A$
if and only if $\succ$ is irreflexive and chain-transitive.

Not necessary for mere existence
even if $\succ$ is an interval order
(the same example as above).

Theorem.

Let $\succ$ be an ordering on a poset A.
Then $M(X,\succ) \in \mathfrak{C}_A$ for all $X \in \mathfrak{C}_A$
if and only if $\succeq$ is chain-transitive.

Example.

$A := \{0\} \cup [1,2]\, (\subset \mathbb{R})$;
$0 \succ 1$ and $y \succeq x$ for all $(x,y) \neq (0,1)$.
$\succ$ is a semiorder; $\succeq$ satisfies (9).
However, $M(A,\succ) = \{0\} \cup]1,2] \notin \mathfrak{C}_A$.

!!!

$$\forall X \in \mathfrak{C}_A \left[\left(\sup X^{\rightarrow} = \sup X \;\&\; \forall x, y \in X^{\rightarrow} [y > x \Rightarrow y \succ x] \right) \right.$$
$$\left. \Rightarrow \forall x \in X^{\rightarrow} [\sup X \succeq x] \right]; \quad (10a)$$

$$\forall X \in \mathfrak{C}_A \left[\left(\inf X^{\leftarrow} = \inf X \;\&\; \forall x, y \in X^{\leftarrow} [y < x \Rightarrow y \succ x] \right) \right.$$
$$\left. \Rightarrow \forall x \in X^{\leftarrow} [\inf X \succeq x] \right]. \quad (10b)$$

(Follows from the subcompleteness of upper contour sets.)

Proposition.

If a binary relation $\succ$ on a poset A has the property that $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$, then $\succ$ is acyclic on every chain and satisfies both conditions (10).

Insufficient for existence even if $\succ$ is an interval order.

!!!

Countability assumptions

A "*regular*" poset:
every chain contains
a countable *cofinal* and *coinitial* subset.

A "*regular*" interval order $\succ$:
for every $X \in \mathfrak{B}_A$, either $M(X,\succ) \neq \emptyset$,
or there exists an *optimizing sequence* in X, i.e.,
(i) $\forall k\,[x^{k+1} \succ x^k]$;
(ii) $\forall x \in X \exists k\,[x^k \succ x]$.

mono-ω-transitivity:

$$\Big[\forall k \in \mathbb{N}\,[x^{k+1} \succ x^k \;\&\; x^{k+1} > x^k] \;\&\; x^\omega = \sup\{x^k\}_k\Big] \Rightarrow x^\omega \succ x^0; \quad (11a)$$

$$\Big[\forall k \in \mathbb{N}\,[x^{k+1} \succ x^k \;\&\; x^{k+1} < x^k] \;\&\; x^\omega = \inf\{x^k\}_k\Big] \Rightarrow x^\omega \succ x^0. \quad (11b)$$

Theorem (∼Kukushkin, 2012).
Let $\succ$ be a regular semiorder on a poset A.
Then $M(X,\succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$
if and only if it is mono-ω-transitive.

Theorem.
Let $\succ$ be a binary relation on a regular poset A.
Then $\succ$ has the NM-property on every $X \in \mathfrak{C}_A$
if and only if it is irreflexive and mono-ω-transitive.

weak mono-ω-transitivity:

$$\left[\forall k \in \mathbb{N}\,[x^{k+1} \succ x^k \;\&\; x^{k+1} > x^k]\;\&\; x^\omega = \sup\{x^k\}_k\right] \Rightarrow x^\omega \succeq x^0; \quad (12a)$$

$$\left[\forall k \in \mathbb{N}\,[x^{k+1} \succ x^k \;\&\; x^{k+1} < x^k]\;\&\; x^\omega = \inf\{x^k\}_k\right] \Rightarrow x^\omega \succeq x^0. \quad (12b)$$

Theorem.
Let $\succ$ be a regular interval order on a poset A.
Then $M(X,\succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$
if and only if it is weakly mono-ω-transitive.

Theorem.
Let $\succ$ be an interval order on a regular poset A.
Then $M(X,\succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$
if and only if it is weakly mono-ω-transitive.

!!!

Optimization on lattices

$\mathfrak{C}_A \subseteq \mathfrak{L}_A$, hence the necessity results remain valid.

Example.

($\sim$Milgrom and Roberts (1990) [M. Kandori])

$A := [0,1] \times [0,1]$ with the natural order;

$u \colon A \to \mathbb{R}$ as follows:

$$u(x_1, x_2) := \begin{cases} x_1, & x_1 + x_2 = 1 \;\&\; x_2 > 0; \\ 0, & \text{otherwise.} \end{cases}$$

A is a complete lattice;

all upper contour sets are chain-complete.

$M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$.

However, $\sup_{x \in A} u(x) = 1$, hence $M(A, \succ) = \emptyset$.

meet quasisupermodularity ($\wedge$-*QSM*):

$$\forall x, y \in A \left[x \succ y \wedge x \Rightarrow y \vee x \succ y\right]; \tag{13a}$$

join quasisupermodularity ($\vee$-*QSM*):

$$\forall x, y \in A \left[y \succ y \vee x \Rightarrow y \wedge x \succ x\right]; \tag{13b}$$

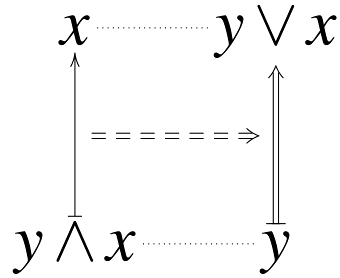
quasisupermodularity (*QSM*): both (13a) and (13b).

strict quasisupermodularity (*SQSM*):

$$\forall x, y \in A \left[[y \vee x > x > y \wedge x \;\&\; x \succeq y \wedge x] \Rightarrow y \vee x \succ y\right]; \tag{13c}$$

weak quasisupermodularity (*wQSM*):

$$\forall x, y \in A \left[x \succ y \wedge x \Rightarrow y \vee x \succeq y\right]. \tag{13d}$$



LiCalzi and Veinott (1992)
Milgrom and Shannon (1994)

$$\frac{\partial^2 u}{\partial x_i \partial x_j} \geq 0 \, [> 0]$$

$$
\begin{array}{ccccc}
 & & \vee\text{-QSM} & \Longrightarrow & \text{wQSM} \\
 & & \Uparrow & & \Uparrow \\
 & & \text{QSM} & \Longrightarrow & \wedge\text{-QSM} \\
 & \nearrow & & & \\
\text{SQSM} & & & &
\end{array}
$$

$$\text{QSM} \equiv [\, \vee\text{-QSM} \bigwedge \wedge\text{-QSM} \,];$$

$$\text{wQSM} \Leftarrow [\, \vee\text{-QSM} \bigvee \wedge\text{-QSM} \,].$$

"Upward-looking halves":

$$\forall y,x \in A\,[u(y) \vee u(x) > u(y \wedge x) \Rightarrow u(y \vee x) > u(y) \wedge u(x)];$$

$$\forall y,x \in A\,[u(y) \vee u(x) \geq u(y \wedge x) \Rightarrow u(y \vee x) \geq u(y) \wedge u(x)];$$

$$(\forall) y,x \in A\,\big[\,[u(y) \vee u(x) \geq u(y \wedge x) \Rightarrow u(y \vee x) > u(y) \wedge u(x)];$$

$$\forall y,x \in A\,[u(y) \vee u(x) > u(y \wedge x) \Rightarrow u(y \vee x) \geq u(y) \wedge u(x)].$$

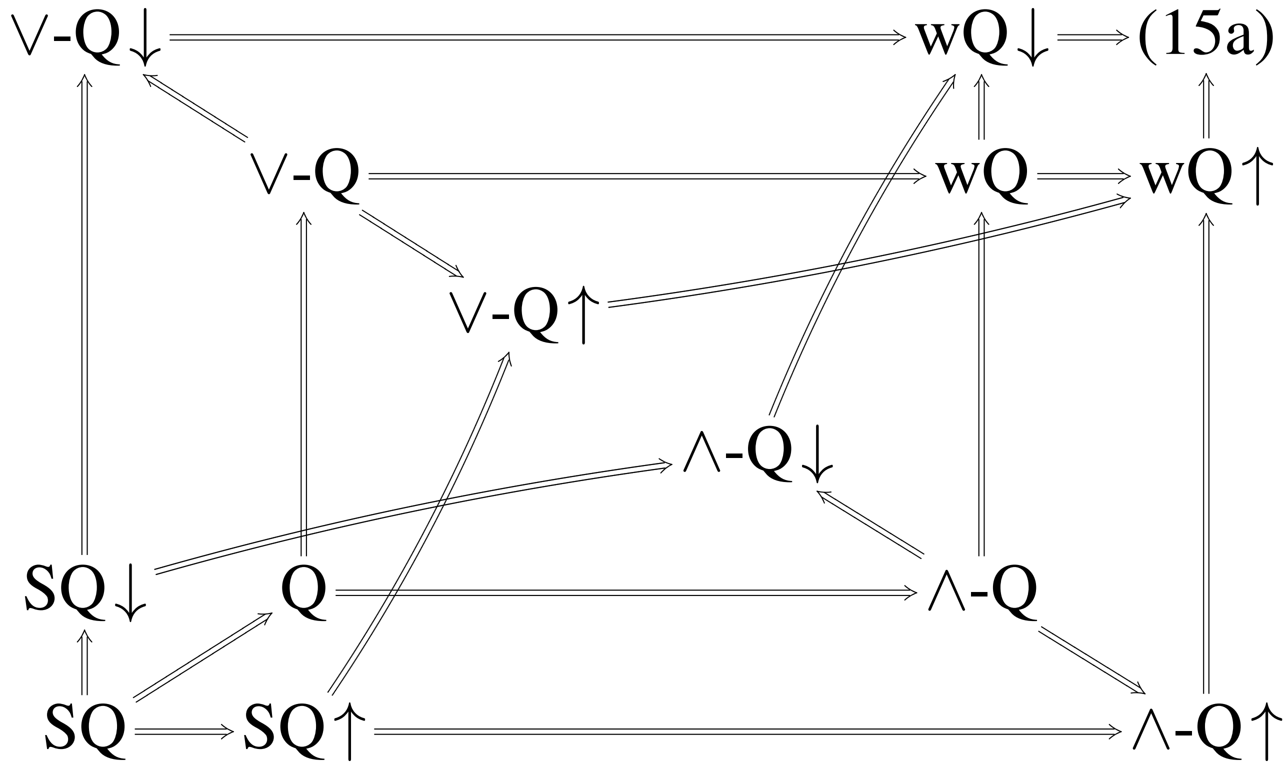
"Downward-looking halves":

$$\forall y,x \in A\,[u(y) \vee u(x) > u(y \vee x) \Rightarrow u(y \wedge x) > u(y) \wedge u(x)];$$

$$\forall y,x \in A\,[u(y) \vee u(x) \geq u(y \vee x) \Rightarrow u(y \wedge x) \geq u(y) \wedge u(x)];$$

$$(\forall) y,x \in A\,\big[\,[u(y) \vee u(x) \geq u(y \vee x) \Rightarrow u(y \wedge x) > u(y) \wedge u(x)];$$

$$\forall y,x \in A\,[u(y) \vee u(x) > u(y \vee x) \Rightarrow u(y \wedge x) \geq u(y) \wedge u(x)].$$



$$\textbf{a}.\begin{bmatrix}3 & 2\\ 0 & 1\end{bmatrix} \qquad \textbf{b}.\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}.$$

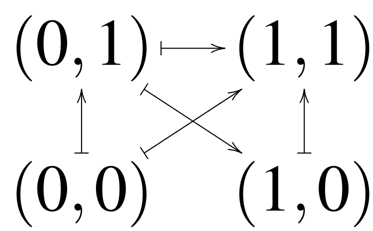
a: SQSM↑, but not even wQSM↓;
b: SQSM↓, but not even wQSM↑.

Theorem.

Let A be a lattice and
$\succ$ be an irreflexive, transitive, and
chain-transitive binary relation on A
which satisfies SQSM$\uparrow$ or SQSM$\downarrow$.
Then $\succ$ has the NM-property on every $X \in \mathfrak{L}_A$.

Example.

Let $A := \{(0,0),(0,1),(1,0),(1,1)\} \subset \mathbb{R}^2$;
let a binary relation $\succ$ on A be as follows:



There is the NM-property on every sublattice
(including A itself),
but no transitivity since $(0,0) \not\succeq (1,0)$.

Theorem.
Let X be a complete sublattice of $\prod_{i \in I} C_i$,
where I is a finite set and each C_i is a chain.
Let $\succ$ be a chain-transitive ordering on X
satisfying $\wedge$-QSM$\uparrow$ or $\vee$-QSM$\downarrow$.
Then $M(X, \succ) \neq \emptyset$.

Theorem (Kukushkin, 2012).
Let $\succ$ be a regular chain-transitive ordering
on a complete lattice X.
Let $\succ$ satisfy $\wedge$-QSM$\uparrow$ or $\vee$-QSM$\downarrow$.
Then $M(X, \succ) \neq \emptyset$.

$$\forall X \in \mathfrak{L}_A \left[M(X, \succ) \neq \emptyset \right] \equiv \forall X \in \mathfrak{C}_A \left[M(X, \succ) \neq \emptyset \right]$$

under the assumptions of either theorem.

Example (Kukushkin, 2012).

$A := \big(\{n/(n+1)\}_{n\in\mathbb{N}} \cup \{1\}\big) \times \big(\{0\} \cup \{1/(n+1)\}_{n\in\mathbb{N}}\big) \subset \mathbb{R}^2$;

$u\colon A \to \mathbb{R}$ as follows:

$u(1,x_2) = u(x_1,0) := 0$;

$u(n_1/(n_1+1), 1/(n_2+1)) := \min\{n_1,n_2\}$.

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & 1 & \dots & 0 \\ 0 & 1 & 2 & 2 & 2 & \dots & 0 \\ 0 & 1 & 2 & 3 & 3 & \dots & 0 \\ 0 & 1 & 2 & 3 & 4 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{matrix}$$

Both $\wedge$-QSM$\downarrow$ and $\vee$-QSM$\uparrow$ are satisfied, hence wQSM as well.

$M(X,\succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$;

however, $\sup_{x\in A} u(x) = +\infty$, hence $M(A,\succ) = \emptyset$.

$\succ$ is *strongly chain-transitive* if
$\succeq$ is chain-transitive:

$$\forall X \in \mathfrak{C}_A \left[\left(\sup X^{\rightarrow} = \sup X \;\&\; \forall x, y \in X^{\rightarrow} [y > x \Rightarrow y \succeq x] \right) \right.$$
$$\left. \Rightarrow \forall x \in X^{\rightarrow} [\sup X \succeq x] \right]; \quad (14a)$$

$$\forall X \in \mathfrak{C}_A \left[\left(\inf X^{\leftarrow} = \inf X \;\&\; \forall x, y \in X^{\leftarrow} [y < x \Rightarrow y \succeq x] \right) \right.$$
$$\left. \Rightarrow \forall x \in X^{\leftarrow} [\inf X \succeq x] \right]. \quad (14b)$$

Theorem.
If an ordering $\succ$ on a complete lattice X satisfies (9b), (14a), and $\wedge$-QSM$\uparrow$, then $M(X, \succ) \neq \emptyset$.

Theorem.
If an ordering $\succ$ on a complete lattice X satisfies (14b), (9a), and $\vee$-QSM$\downarrow$, then $M(X, \succ) \neq \emptyset$.

Theorem.
Let an ordering $\succ$ on a lattice A satisfy wQSM$\uparrow$ or wQSM$\downarrow$.
Then $\succ$ has the properties that $M(X,\succ) \neq \emptyset$ and
$M(X,\succ)$ is chain-subcomplete for every $X \in \mathfrak{L}_A$
if and only if it is strongly chain-transitive.

Theorem.
Let X be a complete sublattice of $\prod_{i \in I} C_i$,
where I is a finite set and each C_i is a chain.
Let $\succ$ be an ordering on X satisfying
(9b), (14a), and wQSM$\uparrow$.
Then $M(X,\succ) \neq \emptyset$.

Theorem.
Let X be a complete lattice.
Let $\succ$ be a regular ordering on X satisfying
(11b), (14a), and wQSM$\uparrow$.
Then $M(X,\succ) \neq \emptyset$.

Theorem.

Let X be a complete sublattice of $\prod_{i \in I} C_i$,
where I is a finite set and each C_i is a chain.
Let $\succ$ be an ordering on X satisfying
(14b), (9a), and wQSM$\downarrow$.
Then $M(X, \succ) \neq \emptyset$.

Theorem.

Let X be a complete lattice.
Let $\succ$ be a regular ordering on X satisfying
(14b), (11a), and wQSM$\downarrow$.
Then $M(X, \succ) \neq \emptyset$.

Veinott's conditions

dual quasilattice mapping:

$$\forall y,x \in A\,[u(y \vee x) \vee u(y \wedge x) \geq u(y) \wedge u(x)]; \tag{15a}$$

meet supermorphism:

$$\forall y,x \in A\,[u(y \vee x) \geq u(y) \wedge u(x)]; \tag{15b}$$

superextremal mapping:

$$\forall y,x \in A\,\big[u(y \vee x) \vee u(y \wedge x) \geq u(y) \vee u(x) \text{ or } u(y \vee x) \wedge u(y \wedge x) \geq u(y) \wedge u(x)\big]. \tag{15c}$$

In Theorems 6.2, 6.12, and 6.41 of Veinott (1992), u satisfied conditions (15a), (15b), and (15c), respectively; in each theorem, every upper contour set was subcomplete; besides, $\mathcal{C}$ in Theorem 6.2 was $\mathbb{R} \cup \{+\infty, -\infty\}$. Then the existence of a maximum was shown.

Proposition.

wQSM $\equiv$ (15c) $\Rightarrow$ (15a).

(15b) $\Rightarrow$ [(15a) & $\vee$-QSM$\uparrow$].

Example.

Let $A := \{(0,0),(0,1),(1,0),(1,1)\} \subset \mathbb{R}^2$;

$$\mathbf{a}.\ \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix} \qquad \mathbf{b}.\ \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}.$$

The matrix "a" satisfies (15b), hence (15a) and $\vee$-QSM$\uparrow$, but none of the conditions $\wedge$-QSM$\uparrow$, wQSM$\downarrow$, or (15c). The matrix "b" satisfies even SQSM, hence (15c) and (15a), but not (15b).

Theorem.

Let X be a complete join-semilattice and $\succ$ be an ordering on X satisfying (9b), (14a), and (15b). Then $M(X, \succ) \neq \emptyset$.

Theorem (∼Veinott, 1992).

Let X be a complete lattice and $\succ$ be a regular ordering on X satisfying (15a) and such that every upper contour set is subcomplete. Then $M(X, \succ) \neq \emptyset$.

Example.

$A := \left(\{n/(n+1)\}_{n\in\mathbb{N}} \cup \{1\}\right) \times \left(\{0\} \cup \{1/(n+1)\}_{n\in\mathbb{N}}\right) \subset \mathbb{R}^2$;

$u\colon A \to \mathbb{R}$ as follows:

$u(1,x_2) = u(x_1,0) := 0$;

$u(n_1/(n_1+1), 1/(n_2+1)) := U(n_1,n_2)$,

where $U(k,k) := k$ while

$U(k+h,k) = U(k,k+h) := k + 1/(h+1)$ $(h > 0)$.

$$\begin{array}{ccccccc}
0 & 1/2 & 1/3 & 1/4 & 1/5 & \dots & 0 \\
1/2 & 1 & 3/2 & 4/3 & 5/4 & \dots & 0 \\
1/3 & 3/2 & 2 & 5/2 & 7/3 & \dots & 0 \\
1/4 & 4/3 & 5/2 & 3 & 7/2 & \dots & 0 \\
1/5 & 5/4 & 7/3 & 7/2 & 4 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \dots & 0
\end{array}$$

(15a) is satisfied.

The ordering is regular and strongly mono-ω-transitive,

hence $M(X,\succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$.

However, $\sup_{x\in A} u(x) = +\infty$, hence $M(A,\succ) = \emptyset$.

Impossibility results

Theorem (Kukushkin, 2008).
There is no “simple” condition such that
a transitive binary relation $\succ$ on a subset A of $\mathbb{R}^n$
has the property that
$M(X,\succ) \neq \emptyset$ for every compact $X \in \mathfrak{B}_A$
if and only if $\succ$ satisfies the condition.

Convex (or convex and compact) subsets as admissible sets
are even worse than that.

An *abstract configuration* C:

$\operatorname{Dom} C \subseteq \mathbb{N}$;

$C_=, C_{\neq}, C_>, C_{\not>}, C_{\triangleright}, C_{\not\triangleright} \subseteq \operatorname{Dom} C \times \operatorname{Dom} C$;

$C_\wedge, C_{\not\wedge}, C_\vee, C_{\not\vee} \subseteq \operatorname{Dom} C \times \operatorname{Dom} C \times \operatorname{Dom} C$;

$C_\bigwedge, C_{\not\bigwedge}, C_\bigvee, C_{\not\bigvee} \subseteq (\operatorname{Dom} C)^{\mathbb{N}}$.

A *realization* of C in A for $\succ$ is

a mapping $\mu : \operatorname{Dom} C \to A$ such that:

$\mu(k') = \mu(k)$ for $(k', k) \in C_=$; $\mu(k') \neq \mu(k)$ for $(k', k) \in C_{\neq}$;

$\mu(k') > \mu(k)$ for $(k', k) \in C_>$; $\mu(k') \not> \mu(k)$ for $(k', k) \in C_{\not>}$;

$\mu(k') \succ \mu(k)$ for $(k', k) \in C_{\triangleright}$; $\mu(k') \not\succ \mu(k)$ for $(k', k) \in C_{\not\triangleright}$;

$\mu(k'') = \mu(k') \wedge \mu(k)$ for $(k'', k', k) \in C_\wedge$;

$\mu(k'') \neq \mu(k') \wedge \mu(k)$ for $(k'', k', k) \in C_{\not\wedge}$;

$\mu(k'') = \mu(k') \vee \mu(k)$ for $(k'', k', k) \in C_\vee$;

$\mu(k'') \neq \mu(k') \vee \mu(k)$ for $(k'', k', k) \in C_{\not\vee}$;

$\mu(\nu(0)) = \bigwedge\{\mu(\nu(k))\}_{k>0}$ for $\nu \in C_\bigwedge$;

$\mu(\nu(0)) \neq \bigwedge\{\mu(\nu(k))\}_{k>0}$ for $\nu \in C_{\not\bigwedge}$;

$\mu(\nu(0)) = \bigvee\{\mu(\nu(k))\}_{k>0}$ for $\nu \in C_\bigvee$;

$\mu(\nu(0)) \neq \bigvee\{\mu(\nu(k))\}_{k>0}$ for $\nu \in C_{\not\bigvee}$.

Theorem.

There exists no set $\mathcal{N}$ of abstract configurations such that
a binary relation $\succ$ on a subset A of $\mathbb{R}^n$
has the property that $M(X,\succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$
if and only if
no configuration $C \in \mathcal{N}$ admits a realization in A for $\succ$.

Remark.

In the topological context,
a broader class of conditions was shown to be insufficient
(disjunctions were allowed too).
Moreover, an *a priori* restriction to transitive relations
would not change the result.

That’s all for now