# Maximizing a preference relation on complete chains and lattices

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**Abstract**

Maximization of a preference relation on a given family of subsets of its domain defines a choice function. Assuming the domain to be a poset or a lattice, and considering subcomplete chains or sublattices as potential feasible sets, we study conditions ensuring the existence of optima, as well as properties of the choice function conducive to monotone comparative statics. Concerning optimization on chains, quite a number of characterization results are obtained; when it comes to lattices, we mostly obtain sufficient conditions. *JEL* Classification Numbers: C 61; D 11.

*Key words*: preference relation; choice function; complete chain; complete lattice; quasisupermodularity

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# 1 Introduction

While the existence of, say, a (pure-strategy) Nash equilibrium is usually viewed as a challenging problem worth attention and effort, the existence of optima is often taken for granted. The assumption that the preferences are upper semicontinuous and the feasible set compact seems almost innocuous. Still, the assumption does not hold, e.g., for (weak) Pareto dominance or lexicographic orderings, even though the existence problem in those cases can be solved. In other words, there is plenty of room where to advance, as well as potential demand for weaker sufficient conditions.

Smith (1974) obtained a simple condition which characterizes orderings (actually, semiorders as well) on a metric space that attain a maximum in every compact subset. Kukushkin (2008b, Theorem 3) characterized interval orders with the same property in a similar way. Arbitrary binary relations on a metric space admitting a maximal element in every compact subset cannot be characterized by any condition of comparable simplicity (Kukushkin, 2008a), but Smith's condition is necessary and sufficient for the "NM-property," i.e., the possibility to dominate every dominated alternative with an undominated one, in every compact subset (Kukushkin, 2008b, Theorem 3). It should be noted that relations between such and similar properties on *finite* subsets have been thoroughly investigated in recent decades (Fishburn, 1973; Sen, 1984; Aizerman and Aleskerov, 1995; Malishevski, 1998), which cannot be said about compact, or convex for that matter, subsets.

Here we study the same questions in the case when the set of alternatives is endowed with an order structure rather than topology. Various conditions for the existence of optima in complete lattices were developed by Veinott (1992). However, he always assumed the chain-completeness of upper contour sets – a natural analog of the upper semicontinuity of preferences – hence could not obtain any *necessary* condition. Here we aim at sufficient conditions as weak as possible, in particular, necessary ones.

We start with a choice function defined by the maximization of a binary relation on all subcomplete chains in an arbitrary poset. A necessary and sufficient condition for the NM-property is obtained (Theorem 3.9); when the preference relation is a semiorder, the same condition (automatically) characterizes the existence of maximizers in all subcomplete chains. When the preference relation is an ordering, a similar, but stronger, condition characterizes the chain-subcompleteness of the set of maximizers (Theorem 3.11).

If certain "countability" assumptions are imposed on the preferences or on the basic poset, those conditions can be re-written in a much simpler form, somewhat resembling Smith's (1974) conditions from the topological context. Moreover, *interval orders* ensuring the existence of optima in all subcomplete chains also admit a simple characterization (Theorems 5.10 and 5.11) under these assumptions, not dissimilarly to Theorem 3 from Kukushkin (2008b). On the other hand, Theorem 5.22 shows that arbitrary binary relations admitting a maximizer in every subcomplete chain cannot be characterized by any condition of comparable simplicity. That result is somewhat similar to Theorem 1 from Kukushkin (2008a) and its proof is based on essentially the same construction (although it is, in a sense, weaker).

Then we switch to a choice function defined by the maximization of a binary relation on all complete sublattices of a given lattice. Most of the results obtained here are restricted to preference *orderings*. We show that some conditions that have already emerged in the study of monotone comparative statics (LiCalzi and Veinott, 1992; Milgrom and Shannon, 1994; Kukushkin, 2013a) are also relevant to the existence of maximizers. In particular, Theorems 4.6 and 4.7 show that either ("upward-looking" or "downward-looking") half of strict quasisupermodularity of an irreflexive and transitive binary relation on a lattice ensures the NM-property in every complete sublattice provided it holds in subcomplete chains.

Two of the four "quarters" (Kukushkin, 2013a) of the quasisupermodularity of an ordering, viz. the "upward-looking" half of meet quasisupermodularity or "downward-looking" half of join quasisupermodularity, ensure equivalence between the existence of optima in every subcomplete chain and in every complete sublattice, provided the whole set of alternatives is a sublattice of a finite product of chains (Theorems 4.8 and 4.9) or a countability assumption holds (Theorems 5.16 and 5.17, borrowed from Kukushkin, 2012). Whether the equivalence holds without any additional assumption remains unknown. An existence result valid for an arbitrary lattice is obtained under an assumption stronger than needed on chains (Theorems 4.10 and 4.11). The other two quarters of quasisupermodularity are insufficient to derive the existence of an optimum in every complete sublattice from that in every subcomplete chain, even if imposed together and even for preferences defined by a numeric function on a complete sublattice of the plane (Example 4.12).

When it comes to *weak* quasisupermodularity, either "half" of the property is sufficient to reduce the problem of the non-emptiness and chain-subcompleteness of the set of optima in every complete sublattice to that in subcomplete chains (Theorems 4.13 and 4.14).

In Section 2, basic definitions and auxiliary results are reproduced. Section 3 starts with three versions of "chain-transitivity" property, and then proceeds to the results about choice from subcomplete chains.

Section 4 contains the results related to choice from complete sublattices. We show that each of the conditions relevant to monotone comparative statics, obtained in Kukushkin (2013a) as a refinement of those from LiCalzi and Veinott (1992) and Milgrom and Shannon (1994), can be used to ensure the existence of optima, as well as other desirable properties of the choice function. A comparison with Veinott's (1992) existence theorems is also provided.

In Section 5, two "countability" assumptions are formulated, one referring to the preferences, the other, to the basic poset. Under those assumptions, the formulations of the previous results become simpler, and stronger results, without analogs in the general case, become possible (Theorems 5.10, 5.11, and 5.16– 5.20). Our impossibility result is also here.

Section 6 is about parametric optimization. A result on the existence of increasing selections, Theorem 6.7, is established, which does not follow from either Veinott (1989) or Kukushkin (2013b). A few concluding remarks are in Section 7.

# 2 Basic notions

## 2.1 Preferences and choice

For every set $A$, we denote $\mathfrak{B}_A$ the set of all nonempty subsets of $A$. Given a binary relation $\succ$ on $A$ ("preference relation") and $X \in \mathfrak{B}_A$, we denote

$$M(X, \succ) := \{x \in X \mid \nexists\, y \in X\, [y \succ x]\}, \tag{2.1}$$

the set of *maximizers* of $\succ$ on $X$. It is often helpful to define the *non-strict preference* relation $\succeq$ by $y \succeq x \rightleftharpoons x \not\succ y$. Then $M(X, \succ) = \{x \in X \mid \forall\, y \in X\, [x \succeq y]\}$.

The mapping $M(\cdot, \succ)\colon \mathfrak{B}_A \to \mathfrak{B}_A$ is called a *choice function*. The most basic property a choice function may, or may not, possess is non-emptiness:

$$M(X, \succ) \neq \emptyset. \tag{2.2}$$

As is well known, (2.2) holds for all $X \in \mathfrak{B}_A$ if and only if $\succ$ is *strongly acyclic*, i.e., there is no sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ in $A$ such that $x^{k+1} \succ x^k$ for all $k$. However, we only require (2.2) to hold for "admissible" subsets, hence strong acyclicity is not indispensable.

We also consider stronger requirements concerning optimal choices. We say that a binary relation $\succ$ has the *NM-property* on a subset $X \in \mathfrak{B}_A$ if

$$\forall x \in X \setminus M(X, \succ)\, \exists y \in M(X, \succ)\, [y \succ x]. \tag{2.3}$$

$\succ$ has the *revealed preference property* on $X$ if

$$\forall x, y \in X\, [x \notin M(X, \succ) \ni y \Rightarrow y \succ x]. \tag{2.4}$$

Clearly, (2.3) implies (2.2). Generally, (2.4) does not imply (2.3); however, the conjunction of (2.4) and (2.2) does imply (2.3).

Many results below need explicit restrictions on a preference relation. A *strict* (partial) *order* is an irreflexive and transitive binary relation. If $\succ$ is a strict order, then $\succeq$ is reflexive and total, but need not be transitive; however, $z \succeq x$ whenever $z \succeq y \succeq x$ and at least one of the two non-strict preferences is actually strict, i.e., either $z \succ y$ or $y \succ x$. A strict order $\succ$ is called an *ordering* if the same condition implies $z \succ x$; this requirement is equivalent to the transitivity of $\succeq$. A strict order $\succ$ is called an *interval order* if $z \succ w$ whenever $z \succ y \succeq x \succ w$. A strict order $\succ$ is called a *semiorder* if $z \succ w$ whenever $z \succeq y \succeq x \succeq w$ and at least two of the three non-strict preferences are actually strict; obviously, any semiorder is an interval order. If $\succ$ is an ordering, then $\succeq$ is reflexive, transitive, and total; if $\succ$ is only a semiorder, $\succeq$ need not be transitive.

The last three classes of preference relations can also be defined in terms of representations in chains; the proofs are not trivial, but are well known.

**Theorem A.** *Let $\succ$ be a binary relation on a set $A$. Then $\succ$ is an ordering if and only if there are a chain $\mathcal{C}$ and a mapping $u\colon A \to \mathcal{C}$ such that*

$$y \succ x \iff u(y) > u(x) \tag{2.5}$$

*for all $x, y \in A$.*

**Theorem B.** *Let $\succ$ be a binary relation on a set $A$. Then $\succ$ is an interval order if and only if there are a chain $\mathcal{C}$ and two mappings $u^+\colon A \to \mathcal{C}$ and $u^-\colon A \to \mathcal{C}$ such that, for all $x, y \in A$:*

$$u^+(x) \geq u^-(x); \tag{2.6a}$$

$$y \succ x \iff u^-(y) > u^+(x). \tag{2.6b}$$

**Theorem C.** *Let $\succ$ be an interval order on a set $A$. Then $\succ$ is a semiorder if and only if there are a representation* (2.6) *and an increasing mapping $\epsilon\colon u^+(A) \to \mathcal{C}$ such that $u^-(x) = \epsilon \circ u^+(x)$ for all $x \in A$.*

**Proposition 2.1.** *A strict order $\succ$ on a set $A$ is an ordering if and only if* (2.4) *holds on every $X \in \mathfrak{B}_A$. If $\succ$ is a semiorder on a set $X$, then $\succ$ has the NM-property* (2.3) *on $X$ if and only if $M(X, \succ) \neq \emptyset$.*

*Proof.* The first statement immediately follows from the definition. Let $\succ$ be a semiorder on a set $X$, $M(X, \succ) \neq \emptyset$, and $x \in X \setminus M(X, \succ)$. We pick $y \in X$ such that $y \succ x$; if $y \in M(X, \succ)$, then we are home. Otherwise, we pick $y' \in X$ such that $y' \succ y$ and $z \in M(X, \succ)$. We have $z \succeq y' \succ y \succ x$, hence $z \succ x$ since $\succ$ is a semiorder. $\square$

## 2.2 Posets and lattices

Henceforth, $A$ is typically a *partially ordered set* (a *poset*) with the order $>$; quite often, a *lattice*. The exact definitions are assumed commonly known. Given $a, b \in A$, we denote $[a, b] := \{x \in A \mid b \geq x \geq a\}$, a *closed order interval.*

Given a strict order $\succ$ on a poset $A$, we consider four auxiliary strict orders:

$$y \mathrel{\substack{\succ \\ >}} x \rightleftharpoons [y \succ x \;\&\; y > x]; \tag{2.7a}$$
$$y \mathrel{\substack{\succ \\ <}} x \rightleftharpoons [y \succ x \;\&\; y < x]; \tag{2.7b}$$
$$y \mathrel{\substack{\succeq \\ >}} x \rightleftharpoons [y \succeq x \;\&\; y > x]; \tag{2.7c}$$
$$y \mathrel{\substack{\succeq \\ <}} x \rightleftharpoons [y \succeq x \;\&\; y < x]. \tag{2.7d}$$

**Theorem D (Zorn's Lemma).** *If a poset $X$ has the property that every chain $Y \in \mathfrak{B}_X$ has an upper bound in $X$, then $M(X, >) \neq \emptyset$.*

A poset $A$ is *chain-complete* if the least upper bound $\sup X$ and the greatest lower bound $\inf X$ exist for every chain $X \in \mathfrak{B}_A$. A poset $A$ is *conditionally chain-complete* if $\sup X$ and $\inf X$ exist for every chain $X \in \mathfrak{B}_A$ which is bounded both above and below. If $A$ is a poset and $X \subseteq A$, we call $X$ *chain-subcomplete* if $\sup Y$ and $\inf Y$ exist in $A$ and belong to $X$ for every chain $Y \in \mathfrak{B}_X$; if $X$ itself is a chain, we call it a *subcomplete chain.* The set of all nonempty subcomplete chains in a poset $A$ is denoted $\mathfrak{C}_A$.

A lattice is *complete* if the *greatest lower bound* or *meet*, $\bigwedge X$, and the *least upper bound* or *join*, $\bigvee X$, exist for every $X \in \mathfrak{B}_A$. A lattice $A$ is *conditionally complete* if meet $\bigwedge X$ exists for all $X \in \mathfrak{B}_A$ that are bounded below, while join $\bigvee X$ exists for all $X \in \mathfrak{B}_A$ that are bounded above. $\mathbb{R}^m$ with the natural partial order is a conditionally complete lattice which is not complete. If $A$ is a lattice, $X \in \mathfrak{B}_A$ is a *complete sublattice* of $A$ if $\bigwedge Y$ and $\bigvee Y$ exist in $A$ and belong to $X$ for all $Y \in \mathfrak{B}_X$. Given a lattice $A$, the set of all (nonempty) complete sublattices is denoted $\mathfrak{L}_A$.

**Theorem E (B. C. Rennie; Veinott, 1989, Lemma 3.1 and Corollaries).** *A lattice $A$ is complete if and only if it is chain-complete as a poset. Then a sublattice of $A$ is a complete sublattice if and only if it is chain-subcomplete.*

The proofs of many theorems here are based on *transfinite recursion*; it seems worthwhile to provide an outline of the method.

A poset is *well ordered* if every subset contains its minimum (hence the poset itself is a chain). When dealing with a well ordered poset $\Lambda$, we usually denote $0 := \min \Lambda$. The *successor* of $\alpha \in \Lambda$, denoted $\alpha + 1$, is uniquely defined as $\min\{\beta \in \Lambda \mid \beta > \alpha\}$ (unless $\alpha = \max \Lambda$, but this does not matter).

**Theorem F (Zermelo).** *Every set can be well ordered.*

The *principle of transfinite recursion* allows us to consider a mapping $\lambda\colon \Lambda \to X$ well defined if we have defined $\lambda(0) \in X$ and described how $\lambda(\alpha) \in X$ should be constructed given $\lambda(\beta) \in X$ for all $\beta < \alpha$. Quite often, the definition of $\lambda(\alpha + 1)$ is based on $\lambda(\alpha)$ alone, so *all* $\beta < \alpha$ are only involved when $\alpha$ is a *limit*, i.e., not the successor to any $\beta \in \Lambda$.

When proving an existence theorem with this technique, we start with a well ordered set $\Lambda$ whose cardinality exceeds that of the set $X$ where we want something to exist. Then we define a mapping $\lambda\colon \Lambda \to X$ in such a way that $\lambda(\alpha') = \lambda(\alpha)$ with $\alpha' > \alpha$ is only possible when $\lambda(\alpha)$ is a point we need. Since the cardinality of $\Lambda$ is greater than that of $X$, the equality must occur at some stage.

# 3 Optimal choice from chains

## 3.1 Chain-transitivity

Given a poset $A$ and $X \in \mathfrak{C}_A$, we denote $X^{\rightarrow} := X \setminus \{\sup X\}$ and $X^{\leftarrow} := X \setminus \{\inf X\}$. A binary relation $\succ$ on a poset $A$ is *weakly chain-transitive* if it is transitive on every chain and satisfies both following conditions:

$$\forall X \in \mathfrak{C}_A \left[ \left( \sup X^{\rightarrow} = \sup X \;\&\; \forall x, y \in X^{\rightarrow} [y > x \Rightarrow y \succ x] \right) \Rightarrow \forall x \in X^{\rightarrow} [\sup X \succeq x] \right]; \quad (3.1a)$$

$$\forall X \in \mathfrak{C}_A \left[ \left( \inf X^{\leftarrow} = \inf X \;\&\; \forall x, y \in X^{\leftarrow} [y < x \Rightarrow y \succ x] \right) \Rightarrow \forall x \in X^{\leftarrow} [\inf X \succeq x] \right]. \quad (3.1b)$$

$\succ$ is *chain-transitive* if it is transitive on every chain and satisfies both following conditions:

$$\forall X \in \mathfrak{C}_A \left[ \left( \sup X^{\rightarrow} = \sup X \;\&\; \forall x, y \in X^{\rightarrow} [y > x \Rightarrow y \succ x] \right) \Rightarrow \forall x \in X^{\rightarrow} [\sup X \succ x] \right]; \quad (3.2a)$$

$$\forall X \in \mathfrak{C}_A \left[ \left( \inf X^{\leftarrow} = \inf X \;\&\; \forall x, y \in X^{\leftarrow} [y < x \Rightarrow y \succ x] \right) \Rightarrow \forall x \in X^{\leftarrow} [\inf X \succ x] \right]. \quad (3.2b)$$

$\succ$ is *strongly chain-transitive* if it is chain-transitive and satisfies both following conditions:

$$\forall X \in \mathfrak{C}_A \left[ \left( \sup X^{\rightarrow} = \sup X \;\&\; \forall x, y \in X^{\rightarrow} [y > x \Rightarrow y \succeq x] \right) \Rightarrow \forall x \in X^{\rightarrow} [\sup X \succeq x] \right]; \quad (3.3a)$$

$$\forall X \in \mathfrak{C}_A \left[ \left( \inf X^{\leftarrow} = \inf X \;\&\; \forall x, y \in X^{\leftarrow} [y < x \Rightarrow y \succeq x] \right) \Rightarrow \forall x \in X^{\leftarrow} [\inf X \succeq x] \right]. \quad (3.3b)$$

If $\succ$ is acyclic on every chain, then either condition (3.2), as well as either condition (3.3), implies the corresponding condition (3.1). If $\succ$ is an ordering on every chain, then it is strongly chain-transitive if and only if $\succeq$ is chain-transitive. Conditions (3.2) and (3.3) are related between themselves only for semiorders.

**Proposition 3.1.** *If $\succ$ is a semiorder on every chain, then either condition* (3.1) *implies the corresponding condition* (3.2).

*Proof.* Let $X \in \mathfrak{C}_A$ satisfy the left hand side of (3.2a); then $X$ satisfies the left hand side of (3.1a). Given $x \in X^{\rightarrow}$, we pick $y, z \in X^{\rightarrow}$ such that $z > y > x$, hence $z \succ y \succ x$. Clearly, $X \cap [z, \sup X]$ satisfies the left hand side of (3.1a) and $\sup X \cap [z, \sup X] = \sup X$; therefore, $\sup X \succeq z$ by (3.1a). Since $\succ$ is a semiorder, we must have $\sup X \succ x$.

The implication (3.1b) $\Rightarrow$ (3.2b) is proven dually. $\square$

**Corollary.** *Let $\succ$ be a semiorder on every chain. Then either condition* (3.3) *implies the corresponding condition* (3.2)*. Besides, $\succ$ is weakly chain-transitive if and only if it is chain-transitive.*

**Example 3.2 (Kukushkin, 2008b, Example 3).** Let $A := [0, 1]$ and $y \succ x \rightleftharpoons 1 > y > x$ for all $y, x \in A$. Then $\succ$ is an interval order satisfying both conditions (3.3) (and (3.2b) for that matter), but not (3.2a), which is violated whenever $X \in \mathfrak{C}_A$ and $\sup X = 1$. Thus, Proposition 3.1 does not hold for interval orders.

Given a binary relation $\succ$ on $A$, we define an *upper contour set* $\mathrm{Up}(x, \succ) := \{y \in X \mid y \succeq x\}$ for any $x \in A$. The assumption that every upper contour set is chain-subcomplete is the most popular analog of the upper semicontinuity of $\succ$ for preferences on a poset.

**Proposition 3.3.** *Let $\succ$ be a binary relation on a poset $A$ such that every upper contour set of $\succ$ is conditionally chain-subcomplete. Then $\succ$ satisfies both conditions* (3.3)*. If $\succ$ is acyclic, then it satisfies both conditions* (3.1)*. If $\succ$ is a semiorder, then it satisfies both conditions* (3.2)*.*

*Proof.* Let $X \in \mathfrak{C}_A$ satisfy the left hand side of (3.3a) and $x \in X^{\rightharpoondown}$. For every $y \in X^{\rightharpoondown}$ such that $y > x$, we have $y \succeq x$, hence $y \in \mathrm{Up}(x, \succ)$. Since $X \cap \mathrm{Up}(x, \succ)$ is chain-subcomplete, we have $\sup X \in \mathrm{Up}(x, \succ)$ as well. The condition (3.1b) is proven dually. All further claims now follow immediately from Proposition 3.1. $\square$

**Remark.** Every upper contour set of $\succ$ in Example 3.2 is chain-subcomplete, i.e., the last claim in Proposition 3.3 does not hold for interval orders.

None of the conditions (3.1), (3.2), or (3.3) implies the chain-subcompleteness of upper contour sets: consider, e.g., a lexicographic order on (a subset of) $\mathbb{R}^m$ (as $\succ$) with the natural partial order (as $>$).

**Proposition 3.4.** *If a binary relation $\succ$ on a poset $A$ has the property that $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$, then $\succ$ is acyclic on every chain and satisfies both conditions* (3.1)*.*

*Proof.* The necessity of acyclicity is quite standard. Let $X \in \mathfrak{C}_A$ satisfy the conditions in the left hand side of (3.1a). Since $X^{\rightharpoondown} \subseteq X \setminus M(X, \succ)$, we must have $M(X, \succ) = \{\sup X\}$, hence the condition in the right hand side of (3.1a) holds as well. The proof of (3.1b) is dual. $\square$

## 3.2 NM-property

**Proposition 3.5.** *If a binary relation $\succ$ on a poset $A$ has the NM-property on every $X \in \mathfrak{C}_A$, then $\succ$ is irreflexive and chain-transitive.*

*Proof.* If $x \succ x$, then $M(\{x\}, \succ) = \emptyset$; if $z \succ y \succ x$, but not $z \succ x$, then there is no NM-property on $\{x, y, z\}$. Let $X \in \mathfrak{C}_A$ satisfy the conditions in the left hand side of (3.2a). Since $X^{\rightharpoondown} \subseteq X \setminus M(X, \succ)$, we must have $M(X, \succ) = \{\sup X\}$. Now the NM-property implies that the condition in the right hand side of (3.2a) holds as well. The proof of (3.2b) is dual. $\square$

**Proposition 3.6.** *Let $\succ$ be a binary relation on a poset $A$; let $\succ$ have the NM-property on every $X \in \mathfrak{C}_A$; let $M(X, \succ) \in \mathfrak{C}_A$ for every $X \in \mathfrak{C}_A$. Then $\succ$ is irreflexive and strongly chain-transitive.*

*Proof.* By Proposition 3.5, $\succ$ is irreflexive and chain-transitive, so we only have to check conditions (3.3). Let $X \in \mathfrak{C}_A$ satisfy the left hand side of (3.3a). Whenever $x, y \in X^{\rightharpoondown}$, the relation $y \succ x$ is only possible when $y > x$. Since $\succ$ has the NM-property on $X$, we have $\sup M(X, \succ) = \sup X$, hence $\sup X \in M(X, \succ)$ and we are home. The proof of (3.3b) is dual. $\square$

Given a binary relation $\succ$ on a poset $A$, we call an $X \in \mathfrak{B}_A$ *$\succ$-subcomplete upwards* if (i) $\sup Y$ exists in $A$ for every chain $Y \in \mathfrak{B}_X$ and (ii) whenever $Y \in \mathfrak{C}_A$ satisfies the left hand side of (3.2a), i.e., $\sup Y^{\to} = \sup Y$ & $\forall x, y \in Y^{\to}\,[y > x \Rightarrow y \succ x]$, and $Y^{\to} \subseteq X$, there holds $\sup Y \in X$ as well. A *$\succ$-subcomplete downwards* subset is defined dually. A chain-subcomplete subset is obviously $\succ$-subcomplete both upwards and downwards for any $\succ$.

**Proposition 3.7.** *Let $\succ$ be a binary relation on a poset $A$ which is irreflexive, transitive on every chain, and satisfies* (3.2a)*; let $X \in \mathfrak{B}_A$ be $\succ$-subcomplete upwards. Then $\overset{\succ}{>}$ has the NM-property on $X$. If $X \in \mathfrak{C}_A$ and $\succ$ satisfies both conditions* (3.2)*, then $M(X, \overset{\succ}{>})$ is $\succ$-subcomplete downwards.*

*Proof.* To prove the NM-property, we pick $x \in X \setminus M(X, \overset{\succ}{>})$ arbitrarily. Let $\Lambda$ be a well ordered set with a cardinality greater than that of $X$. We construct, by (transfinite) recursion, a mapping $\lambda \colon \Lambda \to X$ such that:

$$\forall \alpha, \beta \in \Lambda\, \big[\alpha > \beta \Rightarrow [\lambda(\alpha) = \lambda(\beta) \in M(X, \overset{\succ}{>}) \ \text{ or } \ \lambda(\alpha) \overset{\succ}{>} \lambda(\beta)]\big]; \tag{3.4a}$$

$$\forall \alpha \in \Lambda\, \big[\lambda([0, \alpha]) \in \mathfrak{C}_A\big]. \tag{3.4b}$$

First, we define $\lambda(0) := x$. Let $\lambda(\alpha)$ have been defined. If $\lambda(\alpha) \in M(X, \overset{\succ}{>})$, we define $\lambda(\alpha + 1) := \lambda(\alpha)$; actually, $\lambda(\alpha') = \lambda(\alpha)$ for all $\alpha' > \alpha$ in this case. Otherwise, we pick $\lambda(\alpha + 1) \overset{\succ}{>} \lambda(\alpha)$ arbitrarily. Both requirements (3.4) continue to hold for $\alpha + 1$.

If $\alpha^* \in \Lambda$ is a limit ordinal and $\lambda(\alpha)$ has been defined for all $\alpha < \alpha^*$, we define $\lambda(\alpha^*) := \sup_{\alpha < \alpha^*} \lambda(\alpha)$, ensuring $\lambda([0, \alpha^*]) \in \mathfrak{C}_A$. If $\lambda(\alpha^*) = \lambda(\alpha)$ for some $\alpha < \alpha^*$, then $\lambda(\alpha + 1) = \lambda(\alpha)$, hence $\lambda(\alpha) \in M(X, \overset{\succ}{>})$ and we are home immediately. Otherwise, we have $\lambda(\alpha^*) \in X$ since $X$ is $\succ$-subcomplete upwards, and $\lambda(\alpha^*) \overset{\succ}{>} \lambda(\alpha)$ for every $\alpha < \alpha^*$ by (3.2a).

The final argument is straightforward. An equality $\lambda(\alpha') = \lambda(\alpha)$ with $\alpha' > \alpha$ is only possible when $\lambda(\alpha) \in M(X, \overset{\succ}{>})$. Since the cardinality of $\Lambda$ is greater than that of $X$, the equality must occur at some stage. Since $\lambda(\alpha) \overset{\succ}{>} \lambda(0) = x$, we are home.

To prove the second claim, we assume that $Y \in \mathfrak{C}_A$ satisfies the left hand side of (3.2b) and $Y^{\leftarrow} \subseteq M(X, \overset{\succ}{>})$. If $\inf Y \notin M(X, \overset{\succ}{>})$, there must be $z \in X$ such that $z \overset{\succ}{>} \inf Y$, hence $z > y$ for some $y \in Y \subseteq M(X, \overset{\succ}{>})$. On the other hand, we have $\inf Y \succ y$ by (3.2b), hence $z \succ y$. Thus, $z \overset{\succ}{>} y$, contradicting $y \in M(X, \overset{\succ}{>})$; therefore, $\inf Y \in M(X, \overset{\succ}{>})$ indeed. $\square$

**Proposition 3.8.** *Let $\succ$ be a binary relation on a poset $A$ which is irreflexive, transitive on every chain, and satisfies* (3.2b)*; let $X \in \mathfrak{B}_A$ be $\succ$-subcomplete downwards. Then $\overset{\succ}{<}$ has the NM-property on $X$. If $X \in \mathfrak{C}_A$ and $\succ$ also satisfies* (3.2a)*, then $M(X, \overset{\succ}{<})$ is $\succ$-subcomplete upwards.*

The proof is dual to that of Proposition 3.7.

**Theorem 3.9.** *Let $\succ$ be a binary relation on a poset $A$. Then $\succ$ has the NM-property on every $X \in \mathfrak{C}_A$ if and only if it is irreflexive and chain-transitive.*

*Proof.* The necessity follows from Proposition 3.5. Let $\succ$ satisfy the assumptions in the theorem, $X \in \mathfrak{C}_A$, and $x \in X \setminus M(X, \succ)$. Since $X$ is a chain, we have either $x \notin M(X, \underset{>}{\succ})$ or $x \notin M(X, \underset{<}{\succ})$; without restricting generality, $x \notin M(X, \underset{>}{\succ})$. By Proposition 3.7, there is $y \in M(X, \underset{>}{\succ})$ such that $y \underset{>}{\succ} x$. Applying Proposition 3.8 to $M(X, \underset{>}{\succ})$ and $y$, we obtain $z \in M(M(X, \underset{>}{\succ}), \underset{<}{\succ})$ such that $z \succ x$. Let us show that $z \in M(X, \succ)$.

Suppose the contrary: there is $z' \in X$ such that $z' \succ z$. Since $z \in M(X, \underset{>}{\succ})$, we have $z > z'$. Applying Proposition 3.7 to $X$ and $z'$, we obtain $z'' \in M(X, \underset{>}{\succ})$ such that $z'' \succ z$ (hence $z'' \neq z$). Now $z'' > z$ is incompatible with $z \in M(X, \underset{>}{\succ})$, whereas $z'' < z$ is incompatible with $z \in M(M(X, \underset{>}{\succ}), \underset{<}{\succ})$. ☐

**Corollary.** *A semiorder $\succ$ on a poset $A$ has the property that $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$ if and only if it is chain-transitive.*

**Remark.** The necessity in the corollary does not hold even for interval orders, see Example 3.2 above.

## 3.3 Completeness of the set of optima

**Proposition 3.10.** *Let $\succ$ be a binary relation on a poset $A$; let $\succ$ be a strongly chain-transitive ordering on every chain in $A$. Then $M(X, \succ)$ is chain-subcomplete in $A$ whenever $X \in \mathfrak{B}_A$ is (although $M(X, \succ)$ may be empty).*

*Proof.* Let $X \in \mathfrak{B}_A$ be chain-subcomplete, $Y \subseteq M(X, \succ)$ be a nonempty chain, and $y^* = \sup Y$. Then $y^* \in X$ since $X$ is chain-subcomplete. If $y^* \in Y$, then $y^* \in M(X, \succ)$ immediately. Otherwise, we argue similarly to the proof of Proposition 3.7, taking a well ordered set $\Lambda$ with a cardinality greater than that of $X$, and constructing, by (transfinite) recursion, a mapping $\lambda \colon \Lambda \to X$ such that:

$$\forall \alpha \in \Lambda \left[\lambda(\alpha) \in M(X, \succ) \ \& \ \lambda(\alpha) \leq y^*\right]; \tag{3.5a}$$

$$\forall \alpha \in \Lambda \left[\lambda(\alpha + 1) \in Y\right]; \tag{3.5b}$$

$$\forall \alpha, \beta \in \Lambda \left[\alpha > \beta \Rightarrow \lambda(\alpha) \geq \lambda(\beta)\right]; \tag{3.5c}$$

$$\forall \alpha \in \Lambda \left[\lambda([0, \alpha]) \in \mathfrak{C}_A\right]. \tag{3.5d}$$

First, we pick $\lambda(0) \in Y$ arbitrarily. Let $\lambda(\alpha)$ have been defined. If there are $y \in Y$ such that $y > \lambda(\alpha)$, we pick any one of them as $\lambda(\alpha + 1)$. Otherwise, we set $\lambda(\alpha + 1) := \lambda(\alpha)$; actually, $\lambda(\alpha') = \lambda(\alpha)$ for all $\alpha' > \alpha$ in this case. All requirements (3.5) continue to hold for $\alpha + 1$.

If $\alpha^* \in \Lambda$ is a limit ordinal and $\lambda(\alpha)$ has been defined for all $\alpha < \alpha^*$, we define $\lambda(\alpha^*) := \sup_{\alpha<\alpha^*} \lambda(\alpha)$, ensuring that $\lambda(\alpha^*) \leq y^*$ and $\lambda([0, \alpha^*]) \in \mathfrak{C}_A$. Since $\lambda(\alpha) \in M(X, \succ)$ for all $\alpha < \alpha^*$, there holds the left hand side of (3.3a), hence $\lambda(\alpha^*) \succeq \lambda(\alpha)$ for all $\alpha < \alpha^*$. Since $X$ is subcomplete and $\succ$ is an *ordering* (that assumption is crucial here), $\lambda(\alpha^*) \in M(X, \succ)$. Therefore, all requirements (3.5) continue to hold for a limit $\alpha^*$ as well.

Since the cardinality of $\Lambda$ is greater than that of $X$, we must have $\lambda(\alpha^* + 1) = \lambda(\alpha^*)$ at some stage $\alpha^* \in \Lambda$. If $\alpha^*$ is a successor, i.e., $\alpha^* = \alpha + 1$, then $\lambda(\alpha^*) = y^*$ immediately by (3.5b). Otherwise, we have to show that $\lambda(\alpha^*) \geq y$ for every $y \in Y$. Since $Y$ is a chain, a problem is only possible if $y > \lambda(\alpha + 1)$ for all $\alpha < \alpha^*$; but then $y \geq \lambda(\alpha^*) = \sup_{\alpha<\alpha^*} \lambda(\alpha)$. The case of $y > \lambda(\alpha^*)$ is ruled out by the fact that $\lambda(\alpha^* + 1) = \lambda(\alpha^*)$; therefore, $y = \lambda(\alpha^*)$.

The case of $Y \subseteq M(X, \succ)$ and $y^* = \inf Y$ is treated dually. □

**Theorem 3.11.** *Let $\succ$ be a binary relation on a poset $A$; let $\succ$ be an ordering on every chain in $A$. Then $\succ$ has the property that $M(X, \succ) \in \mathfrak{C}_A$ for every $X \in \mathfrak{C}_A$ if and only if it is strongly chain-transitive.*

*Proof.* The necessity follows from Proposition 3.6; the sufficiency, from Theorem 3.9 and Proposition 3.10. □

The sufficiency part of Theorem 3.11 does not hold even for a semiorder $\succ$.

**Example 3.12.** Let $A := \{0\} \cup [1, 2]$ $(\subset \mathbb{R})$ and $\succ$ be such that $0 \succ 1$ and $y \succeq x$ for all $(x, y) \neq (0, 1)$. Clearly, $\succ$ is a strongly chain-transitive semiorder, but $M(A, \succ) = \{0\} \cup ]1, 2] \notin \mathfrak{C}_A$.

# 4 Optimal choice from (sub)lattices

In this section, we assume $A$ to be a lattice and denote $\mathfrak{L}_A$ the set of all (nonempty) complete sublattices of $A$. Since $\mathfrak{C}_A \subseteq \mathfrak{L}_A$ by Theorem E, the necessity of conditions (3.1) – (3.3), established in Propositions 3.4, 3.5, and 3.6, remains valid if $\mathfrak{C}_A$ is replaced with $\mathfrak{L}_A$. However, the sufficiency statements are not that robust.

**Example 4.1.** Let $A := [0, 1] \times [0, 1]$ with the natural order and $\succ$ on $A$ be represented, in the sense of (2.5), by a function $u\colon A \to \mathbb{R}$ as follows: $u(x_1, x_2) := x_1$ if $x_1 + x_2 = 1$ and $x_2 > 0$, whereas $u(x_1, x_2) := 0$ otherwise. Obviously, $A$ is a complete lattice and $\succ$ is strongly chain-transitive; moreover, all upper contour sets are chain-complete. In accordance with Theorem 3.9, $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$ (which is easy to check by itself). However, $\sup_{x\in A} u(x) = 1$, hence $M(A, \succ) = \emptyset$.

This example is essentially identical to that ascribed by Milgrom and Roberts (1990, pp. 1262–1263) to M. Kandori. The only difference is that their lattice is not even distributive, while here it is the product of two chains.

We consider additional restrictions on the preference relation. Roughly speaking, there are two distinct sources of such restrictions. The first group is considered in Subsections 4.1–4.4; the second, in Subsection 4.5.

## 4.1 Quasisupermodularity conditions

A binary relation $\succ$ on a lattice $A$ is *meet quasisupermodular* ($\wedge$-*QSM*) if

$$\forall x, y \in A\, \big[x \succ y \wedge x \Rightarrow y \vee x \succ y\big]; \tag{4.1a}$$

$\succ$ is *join quasisupermodular* ($\vee$-*QSM*) if

$$\forall x, y \in A\, \big[y \succ y \vee x \Rightarrow y \wedge x \succ x\big]; \tag{4.1b}$$

$\succ$ is *quasisupermodular* (*QSM*) if it satisfies both (4.1a) and (4.1b).

**Remark.** Milgrom and Shannon (1994) called a function $u\colon A \to \mathbb{R}$ quasisupermodular if the ordering $\succ$ represented, in the sense of (2.5), by $u$ satisfies both conditions (4.1). Veinott (1992) used the term "lattice-superextremal mappings" (not necessarily real-valued).

A binary relation $\succ$ on a lattice $A$ is *strictly quasisupermodular* (*SQSM*) if

$$\forall x, y \in A\, \big[[y \vee x > x > y \wedge x \;\&\; x \succeq y \wedge x] \Rightarrow y \vee x \succ y\big]; \tag{4.1c}$$

$\succ$ is *weakly quasisupermodular* (*wQSM*) if

$$\forall x, y \in A\, \big[x \succ y \wedge x \Rightarrow y \vee x \succeq y\big]. \tag{4.1d}$$

It is easy to see that strict quasisupermodularity (4.1c) implies every other condition (4.1). Weak quasisupermodularity (4.1d) is implied by every other condition (4.1).

Even though all our definitions make sense for an arbitrary binary relation on a lattice, the main results of this section, with a single exception, are only applicable to orderings.

Conditions (4.1) play crucial roles in "Type B" (Quah, 2007) monotone comparative statics, i.e., when the choice set is perturbed while the preferences remain fixed (LiCalzi and Veinott, 1992; Milgrom and Shannon, 1994; Shannon, 1995). In "Type A" monotone comparative statics, i.e., when only preferences are perturbed, it makes sense to "partition" each condition (4.1) into two "halves" – "upward-" and "downward-looking" ones (Kukushkin, 2013a).

Using the tautology $(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)$, we may add an equivalent to each condition (4.1). Then we apply a uniform procedure to each of the eight conditions thus obtained, viz. we retain the same left hand side and replace the right hand side with the disjunction of two alternatives. Finally, we put together those conditions where $y \wedge x$ is in the left hand side ("upward-looking" conditions) and those where it is in the right hand side ("downward-looking" conditions).

$$\forall x,y\in A\left[x\succ y\wedge x\Rightarrow[(y\vee x\succ x)\ \text{ or }\ (y\vee x\succ y)]\right];\tag{4.2a}$$
$$\forall x,y\in A\left[x\succeq y\wedge x\Rightarrow[(y\vee x\succeq x)\ \text{ or }\ (y\vee x\succeq y)]\right];\tag{4.2b}$$
$$\forall x,y\in A\left[\,[y\vee x>x>y\wedge x\ \&\ x\succeq y\wedge x]\Rightarrow[(y\vee x\succ x)\ \text{ or }\ (y\vee x\succ y)]\right];\tag{4.2c}$$
$$\forall x,y\in A\left[x\succ y\wedge x\Rightarrow[(y\vee x\succeq x)\ \text{ or }\ (y\vee x\succeq y)]\right].\tag{4.2d}$$

$$\forall x,y\in A\left[y\succ y\vee x\Rightarrow[(y\wedge x\succ x)\ \text{ or }\ (y\wedge x\succ y)]\right];\tag{4.3a}$$
$$\forall x,y\in A\left[y\succeq y\vee x\Rightarrow[(y\wedge x\succeq x)\ \text{ or }\ (y\wedge x\succeq y)]\right];\tag{4.3b}$$
$$\forall x,y\in A\left[\,[y\vee x>x>y\wedge x\ \&\ y\succeq y\vee x]\Rightarrow[(y\wedge x\succ x)\ \text{ or }\ (y\wedge x\succ y)]\right];\tag{4.3c}$$
$$\forall x,y\in A\left[y\succ y\vee x\Rightarrow[(y\wedge x\succeq x)\ \text{ or }\ (y\wedge x\succeq y)]\right].\tag{4.3d}$$

**Remark.** Each of conditions (4.2) and (4.3) holds trivially when $x$ and $y$ are comparable in the basic order.

Conditions (4.2), as well as (4.3), are ordered between themselves in the same way as (4.1): (c) implies all others; (d) is implied by all others. There is no other implication between the conditions, see Kukushkin (2013a, Example 2).

**Remark.** Strictly speaking, conditions (4.1), (4.2) and (4.3) look a bit differently in Kukushkin (2013a); however, the equivalence is easy to check.

**Proposition 4.2.** *Let $\succ$ be a semiorder on a lattice $A$. Then: $\succ$ satisfies condition* (4.1a) *if and only if it satisfies both conditions* (4.2a) *and* (4.3b)*; $\succ$ satisfies condition* (4.1b) *if and only if it satisfies both conditions* (4.2b) *and* (4.3a).

*Proof.* Conditions (4.1) imply conditions (4.2) and (4.3) without any restriction on $\succ$. Let (4.2a) and (4.3b) hold, and let $x\succ y\wedge x$. By (4.2a), either $y\vee x\succ y$ or $y\vee x\succ x$. In the first case, we are home immediately. In the second, we have $y\vee x\succ x\succ y\wedge x$. Since $\succ$ is a semiorder, either $y\vee x\succ y$ again, or $y\succ y\wedge x$, hence $y\vee x\succ y$ yet again by (4.3b).

The other implication is proven dually. □

**Example 4.3.** Let $A:=\{(0,0),(0,1),(1,0),(1,1)\}\subset\mathbb{R}^2$; we consider an interval order on $A$ defined by two real functions depicted in this matrix (the axes are directed upwards and rightwards):

$$\begin{bmatrix}[0,5] & [4,5]\\ [0,1] & [2,3]\end{bmatrix}.$$

The relation satisfies all conditions (4.2) and (4.3) except for (4.3c), but not (4.1a). Thus, Proposition 4.2 does not hold even for interval orders.

**Proposition 4.4.** *Let $\succ$ be an interval order on a lattice $A$. Then: $\succ$ satisfies condition* (4.1c) *if and only if it satisfies both conditions* (4.2c) *and* (4.3c)*; $\succ$ satisfies condition* (4.1d) *if and only if it satisfies both conditions* (4.2d) *and* (4.3d)*.*

*Proof.* Both claims are proven quite similarly to Proposition 4.2. Let (4.2c) and (4.3c) hold, and let $y \vee x > x > y \wedge x$ and $x \succeq y \wedge x$. By (4.2c), either we are home or $y \succeq y \vee x \succ x$. Applying (4.3c) to the first preference, we obtain $y \wedge x \succ y$. Thus, we have $y \wedge x \succ y \succeq y \vee x \succ x$, hence $y \wedge x \succ x$ since $\succ$ is an interval order.

The other equivalence is proven similarly. □

**Example 4.5.** Let $A := \{(0,0),(0,1),(1,0),(1,1)\} \subset \mathbb{R}^2$; we consider two strict orders on $A$ each defined as a Pareto combination of two orderings represented by these matrices (the axes are directed upwards and rightwards):

$$\mathbf{a.}\begin{bmatrix}\langle 0,2\rangle & \langle 3,1\rangle\\ \langle 1,3\rangle & \langle 2,0\rangle\end{bmatrix} \qquad \mathbf{b.}\begin{bmatrix}\langle 1,3\rangle & \langle 0,2\rangle\\ \langle 2,0\rangle & \langle 3,1\rangle\end{bmatrix}.$$

The relation depicted in matrix "a" satisfies all conditions (4.2) and (4.3), but not (4.1c). The relation in matrix "b" satisfies both (4.2d) and (4.3d), but not (4.1d). Thus, Proposition 4.4 does not hold even for a Pareto combination of two orderings.

If the order on $A$ is reversed, conditions (4.2) and (4.3) transform into each other; naturally, "meet-related" conditions become "join-related" and vice versa.

## 4.2 Implications of strict QSM

**Theorem 4.6.** *Let $A$ be a lattice and $\succ$ be an irreflexive, transitive, and chain-transitive binary relation on $A$, which satisfies* (4.2c)*. Then $\succ$ has the NM-property on every $X \in \mathfrak{L}_A$.*

*Proof.* The basic construction is similar to that in the proof of Proposition 3.7. This time, we use both auxiliary strict orders $\underset{>}{\succ}$ and $\underset{<}{\succ}$ defined by (2.7).

**Lemma 4.6.1.** *If $X$ is a sublattice of $A$, $x \in M(X, \underset{>}{\succ})$ and $X \ni y \succ x$, then either $x > y$ or $y \wedge x \succ y$.*

*Proof of Lemma 4.6.1.* Since $x \in M(X, \underset{>}{\succ})$, inequality $y > x$ is impossible. If $x > y$, we are home. Suppose to the contrary that $y \succeq y \wedge x$ while $y$ and $x$ are incomparable in the basic order. Then we have $y \vee x \succ x$ by (4.2c), which is incompatible with $x \in M(X, \underset{>}{\succ})$. □

Let $X \in \mathfrak{L}$ and $\Lambda$ be a well ordered set with a cardinality greater than that of $X$. Given $x^* \in X \setminus M(X, \succ)$, we construct, by (transfinite) recursion, a mapping $\lambda \colon \Lambda \to X$ such that:

$$\forall \alpha \in \Lambda \setminus \{0\} \left[\lambda(\alpha) \succ x^*\right]; \tag{4.4a}$$

$$\forall\alpha \in \Lambda\,\bigl[\lambda(\alpha) \in M(X, \mathrel{\underset{>}{\succ}})\bigr]; \tag{4.4b}$$

$$\forall\alpha, \beta \in \Lambda\,\bigl[\alpha > \beta \Rightarrow [\lambda(\alpha) = \lambda(\beta) \in M(X, \succ) \ \text{ or } \ \lambda(\alpha) \mathrel{\underset{<}{\succ}} \lambda(\beta)]\bigr]; \tag{4.4c}$$

$$\forall\alpha \in \Lambda\,\bigl[\lambda([0, \alpha]) \in \mathfrak{C}_A\bigr]. \tag{4.4d}$$

If $x^* \in M(X, \mathrel{\underset{>}{\succ}})$, we define $\lambda(0) := x^*$. Otherwise, we apply Proposition 3.7, obtaining $\lambda(0) \in M(X, \mathrel{\underset{>}{\succ}})$ such that $\lambda(0) \mathrel{\underset{>}{\succ}} x^*$.

Let $\lambda(\alpha)$ have been defined. If $\lambda(\alpha) \in M(X, \succ)$, we define $\lambda(\alpha + 1) := \lambda(\alpha)$; actually, $\lambda(\alpha') = \lambda(\alpha)$ for all $\alpha' > \alpha$ in this case. Otherwise, Lemma 4.6.1 implies the existence of $y^* \in X$ such that $y^* \mathrel{\underset{<}{\succ}} \lambda(\alpha)$. We denote $Y := [y^*, \lambda(\alpha)] \cap X$. If $y^* \in M(Y, \mathrel{\underset{>}{\succ}})$, we define $\lambda(\alpha+1) := y^*$; otherwise, we apply Proposition 3.7, obtaining $\lambda(\alpha+1) \in M(Y, \mathrel{\underset{>}{\succ}})$ such that $\lambda(\alpha+1) \mathrel{\underset{>}{\succ}} y^*$. In either case, we have $\lambda(\alpha+1) \succ \lambda(\alpha)$, hence $\lambda(\alpha+1) \mathrel{\underset{<}{\succ}} \lambda(\alpha)$. Let us show $\lambda(\alpha+1) \in M(X, \mathrel{\underset{>}{\succ}})$.

Supposing the contrary, $y \mathrel{\underset{>}{\succ}} \lambda(\alpha + 1)$, we must have $y \notin Y$. Now we apply Lemma 4.6.1 to $x = \lambda(\alpha)$ and $y$, obtaining $\lambda(\alpha) \wedge y \succ y \succ \lambda(\alpha+1)$. Since $\lambda(\alpha) \wedge y \in Y$ and $\lambda(\alpha) \wedge y \mathrel{\underset{>}{\succ}} \lambda(\alpha+1)$, we have a contradiction with the choice of $\lambda(\alpha + 1)$.

Let $\alpha^* \in \Lambda$ be a limit ordinal and $\lambda(\alpha)$ have been defined for all $\alpha < \alpha^*$. Then we define $\lambda(\alpha^*) := \inf_{\alpha<\alpha^*} \lambda(\alpha) \in X$, ensuring that $\lambda([0, \alpha^*]) \in \mathfrak{C}_A$. If $\lambda(\alpha^*) = \lambda(\alpha)$ for some $\alpha < \alpha^*$, then we are home immediately, exactly as in the proof of Proposition 3.7. Otherwise, (3.2b) implies $\lambda(\alpha^*) \mathrel{\underset{<}{\succ}} \lambda(\alpha)$ for every $\alpha < \alpha^*$, so we only have to show that $\lambda(\alpha^*) \in M(X, \mathrel{\underset{>}{\succ}})$.

Assuming that $y^* \mathrel{\underset{>}{\succ}} \lambda(\alpha^*)$ for a $y^* \in X$, we recursively construct a mapping $\mu\colon [0, \alpha^*] \to X$ such that

$$\forall\alpha \in [0, \alpha^*]\,\bigl[\mu(\alpha) \succ \lambda(\alpha^*) \ \&\ \mu(\alpha) \geq \lambda(\alpha^*)\bigr]; \tag{4.5a}$$

$$\forall\alpha \in [0, \alpha^*]\,\bigl[\lambda(\alpha) \geq \mu(\alpha)\bigr]; \tag{4.5b}$$

$$\forall\alpha, \beta \in [0, \alpha^*]\,\bigl[\alpha > \beta \Rightarrow [\mu(\alpha) = \mu(\beta) \ \text{ or } \ \mu(\alpha) \mathrel{\underset{<}{\succ}} \mu(\beta)]\bigr]; \tag{4.5c}$$

$$\forall\alpha \in [0, \alpha^*]\,\bigl[\mu([0, \alpha]) \in \mathfrak{C}_A\bigr]. \tag{4.5d}$$

First, we define $\mu(0) := y^*$; (4.5a) holds for $\alpha = 0$. Having $\mu(\alpha)$ defined, we set $\mu(\alpha + 1) := \mu(\alpha) \wedge \lambda(\alpha + 1)$. Since $\mu(\alpha) \succ \lambda(\alpha + 1)$, we have either $\mu(\alpha + 1) = \mu(\alpha)$ or $\mu(\alpha + 1) \mathrel{\underset{<}{\succ}} \mu(\alpha)$ by Lemma 4.6.1. Thus, all conditions (4.5) continue to hold.

Let $\alpha$ be a limit ordinal and $\mu(\beta)$ have been defined for all $\beta < \alpha$; then we define $\mu(\alpha) := \inf_{\beta<\alpha} \mu(\beta)$, ensuring that $\mu([0, \alpha]) \in \mathfrak{C}_A$. If $\mu(\alpha) = \mu(\beta^*)$ for some $\beta^* < \alpha$, then $\mu(\alpha) = \mu(\beta)$ for all $\beta \in [\beta^*, \alpha]$ and all conditions (4.5) hold for $\alpha$ too. Otherwise, $\mu(\alpha) \mathrel{\underset{<}{\succ}} \mu(\beta)$ for all $\beta < \alpha$ by (3.2b) and conditions (4.5) are again secured.

When it comes to $\alpha = \alpha^*$, conditions (4.5a) and (4.5b) imply that $\mu(\alpha^*) = \lambda(\alpha^*)$, but $\mu(\alpha^*) \succ \lambda(\alpha^*)$, which contradiction proves that $\lambda(\alpha^*) \in M(X, \mathrel{\underset{>}{\succ}})$ indeed.

The final argument is the same. An equality $\lambda(\alpha') = \lambda(\alpha)$ with $\alpha' > \alpha$ is only possible when $\lambda(\alpha) \in M(X, \succ)$. Since the cardinality of $\Lambda$ is greater than that of $X$, the equality must occur at some stage. Note that we have $\lambda(\alpha) \succ x^*$ by (4.4a). $\square$

**Theorem 4.7.** *Let $A$ be a lattice and $\succ$ be an irreflexive, transitive, and chain-transitive binary relation on $A$, which satisfies* (4.3c)*. Then $\succ$ has the NM-property on every $X \in \mathfrak{L}_A$.*

The proof is dual to that of Theorem 4.6.

**Corollary.** *Let $A$ be a lattice and $\succ$ be a semiorder on $A$ satisfying* (4.2c) *or* (4.3c). *Then $M(X,\succ)\neq\emptyset$ for every $X\in\mathfrak{L}_A$ if and only if $M(X,\succ)\neq\emptyset$ for every $X\in\mathfrak{C}_A$.*

**Remark.** The very possibility to replace one condition with another shows that there is no clear prospect for a necessity result, in the style of Corollary to Theorem 3.9 above or Theorem 4.1 of Smith (1974), about (2.2) for every $X\in\mathfrak{L}_A$.

## 4.3 Implications of meet/join QSM

**Theorem 4.8.** *Let $X$ be a complete sublattice of $\prod_{i\in I}\mathcal{C}_i$, where $I$ is a finite set and each $\mathcal{C}_i$ is a chain. Let $\succ$ be a chain-transitive ordering on $X$ satisfying* (4.2a). *Then $M(X,\succ)\neq\emptyset$.*

*Proof.* We argue by induction in $\#I$. When $\#I=1$, $X$ is a chain, hence Theorem 3.9 applies. Let the statement of the theorem hold whenever $\#I\leq m$; we have to prove that the statement holds whenever $\#I\leq m+1$. We start with auxiliary statements and constructions.

**Lemma 4.8.1.** *If $Y$ is a sublattice of $X$, $x\in M(Y,\overset{\succ}{>})$, and $Y\ni y\succ x$, then $y\wedge x\succeq y$.*

*Proof of Lemma 4.8.1.* An assumption that $y\succ y\wedge x$ would imply $y\vee x\succ x$ by (4.2a), which contradicts the maximality of $x$. $\square$

**Lemma 4.8.2.** *If $\bar{x}\in M(X,\overset{\succ}{>})$, then $M([\bigwedge X,\bar{x}],\overset{\succ}{>})\subseteq M(X,\overset{\succ}{>})$.*

*Proof of Lemma 4.8.2.* Let us suppose the contrary: $X\ni y\overset{\succ}{>}x\in M([\bigwedge X,\bar{x}],\overset{\succ}{>})$. Applying Lemma 4.8.1 to $Y:=[\bigwedge X,\bar{x}]$, $\bar{x}$ and $y$, we obtain $\bar{x}\wedge y\succeq y\succ x$. Since $\bar{x}\wedge y\in Y$ and $\bar{x}\wedge y\overset{\succ}{>}x$, we have a contradiction with the assumption that $x\in M([\bigwedge X,\bar{x}],\overset{\succ}{>})$. $\square$

Given $i\in I$ and $x\in X$, we define $X^i(x):=\{z\in X\mid z_i=x_i\}$, where $x_i$ denotes projection of $x$ to $\mathcal{C}_i$. Clearly, $X^i(x)$ is isomorphic to a complete sublattice of $\prod_{j\in I\setminus\{i\}}\mathcal{C}_j$; by the induction hypothesis, $M(X^i(x),\succ)\neq\emptyset$. For $x,y\in X$, we define

$$y\ggcurly x \rightleftharpoons \forall i\in I\,\forall z\in X^i(x)\,[y\succ z].$$

The rest of the proof is organized as *reductio ad absurdum*: henceforth, we assume $M(X,\succ)=\emptyset$.

**Lemma 4.8.3.** *Whenever $x\in M(X,\overset{\succ}{>})$, there is $y\in M(X,\overset{\succ}{>})$ such that $y<x$ and $y\ggcurly x$.*

*Proof of Lemma 4.8.3.* For each $i\in I$, we pick $x^i\in M(X^i(x),\succ)$; our assumption $M(X,\succ)=\emptyset$ implies the existence of $z\in X$ such that $z\succ x^i$ for each $i$, i.e., $z\ggcurly x$. Since $x\in M(X,\overset{\succ}{>})$, we have $z\wedge x\ggcurly x$ by Lemma 4.8.1. By Proposition 3.7, there is $y\in M([\bigwedge X,x],\overset{\succ}{>})$ such that either $y\overset{\succ}{>}z\wedge x$ or $y=z\wedge x$; by Lemma 4.8.2, $y\in M(X,\overset{\succ}{>})$. $\square$

Let $\Lambda$ be a well ordered set with a cardinality greater than that of $X$. Similarly to the proof of Theorem 4.6, we construct, by (transfinite) recursion, a mapping $\lambda\colon \Lambda \to X$ such that:

$$\forall \alpha \in \Lambda \left[\lambda(\alpha) \in M(X, \mathrel{\substack{\succ\\ >}})\right]; \tag{4.6a}$$

$$\forall \alpha, \beta \in \Lambda \left[\alpha > \beta \Rightarrow [\lambda(\alpha) < \lambda(\beta) \;\&\; \lambda(\alpha) \not\prec \lambda(\beta)]\right]; \tag{4.6b}$$

$$\forall \alpha \in \Lambda \left[\lambda([0, \alpha]) \in \mathfrak{C}_X\right]. \tag{4.6c}$$

First, we set $\lambda(0) := \bigvee X \in M(X, \mathrel{\substack{\succ\\ >}})$. All conditions (4.6) for $\alpha = 0$ hold trivially.

Let $\lambda(\alpha) \in M(X, \mathrel{\substack{\succ\\ >}})$ have been defined. We pick a $y$ existing by Lemma 4.8.3, and call it $\lambda(\alpha + 1)$. All conditions (4.6) continue to hold.

Let $\alpha^* \in \Lambda$ be a limit ordinal and $\lambda(\alpha)$ have been defined for all $\alpha < \alpha^*$. Then we define $\lambda(\alpha^*) := \inf_{\alpha<\alpha^*} \lambda(\alpha)$, ensuring that $\lambda([0, \alpha^*]) \in \mathfrak{C}_X$. For every $\alpha < \alpha^*$, we have $\lambda(\alpha^*) \mathrel{\substack{\succ\\ <}} \lambda(\alpha)$ by (3.2b), hence $\lambda(\alpha^*) \not\prec \lambda(\alpha)$ as well. Thus, we only have to show that $\lambda(\alpha^*) \in M(X, \mathrel{\substack{\succ\\ >}})$.

Suppose the contrary: $X \ni y \mathrel{\substack{\succ\\ >}} \lambda(\alpha^*)$. By the definition of $\lambda(\alpha^*)$, there is $\alpha < \alpha^*$ for which $\lambda(\alpha) > y$ does not hold, hence there is $i \in I$ such that $y \wedge \lambda(\alpha) \in X^i(\lambda(\alpha))$. Since $y \succ \lambda(\alpha^*) \not\prec \lambda(\alpha)$, we have $y \succ y \wedge \lambda(\alpha)$, which contradicts Lemma 4.8.1 since $\lambda(\alpha) \in M(X, \mathrel{\substack{\succ\\ >}})$.

Finally, we have $\lambda(\alpha)$ defined for all $\alpha \in \Lambda$ and all of them distinct because of (4.6b). Since the cardinality of $\Lambda$ is greater than that of $X$, we have obtained our final contradiction. $\square$

**Theorem 4.9.** *Let $X$ be a complete sublattice of $\prod_{i \in I} \mathcal{C}_i$, where $I$ is a finite set and each $\mathcal{C}_i$ is a chain. Let $\succ$ be a chain-transitive ordering on $X$ satisfying* (4.3a). *Then $M(X, \succ) \neq \emptyset$.*

The proof is dual to that of Theorem 4.8.

**Corollary.** *Let $A$ be a sublattice of $\prod_{i \in I} \mathcal{C}_i$, where $I$ is a finite set and each $\mathcal{C}_i$ is a chain. Let $\succ$ be an ordering on $A$ satisfying* (4.2a) *or* (4.3a). *Then $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{L}_A$ if and only if $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$.*

**Theorem 4.10.** *Let an ordering $\succ$ on a lattice $A$ satisfy* (3.2b), (3.3a), *and* (4.2a). *Then $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{L}_A$.*

*Proof.* Let $X \in \mathfrak{L}$ and $\Lambda$ be a well ordered set with a cardinality greater than that of $X$. Similarly to the proof of Theorem 4.6, we construct, by (transfinite) recursion, a mapping $\lambda\colon \Lambda \to X$ such that:

$$\forall \alpha \in \Lambda \left[\lambda(\alpha) \in M(X, \mathrel{\substack{\succ\\ >}})\right]; \tag{4.7a}$$

$$\forall \alpha, \beta \in \Lambda \left[\alpha > \beta \Rightarrow [\lambda(\alpha) = \lambda(\beta) \in M(X, \succ) \text{ or } \lambda(\alpha) \mathrel{\substack{\succ\\ <}} \lambda(\beta)]\right]; \tag{4.7b}$$

$$\forall \alpha \in \Lambda \left[\lambda([0, \alpha]) \in \mathfrak{C}_A\right]. \tag{4.7c}$$

First, we set $\lambda(0) := \bigvee X \in M(X, \mathrel{\substack{\succ\\ >}})$. All conditions (4.7) for $\alpha = 0$ hold trivially.

Let $\lambda(\alpha)$ have been defined. If $\lambda(\alpha) \in M(X, \succ)$, we define $\lambda(\alpha + 1) := \lambda(\alpha)$; actually, $\lambda(\alpha') = \lambda(\alpha)$ for all $\alpha' > \alpha$ in this case. Otherwise, Lemma 4.8.1 implies the existence of $y^* \in X$ such that $y^* \mathrel{\underset{<}{\succ}} \lambda(\alpha)$. We denote $Y := [y^*, \lambda(\alpha)] \cap X$. If $y^* \in M(Y, \mathrel{\underset{>}{\succ}})$, we define $\lambda(\alpha+1) := y^*$; otherwise, we apply Proposition 3.7, obtaining $\lambda(\alpha+1) \in M(Y, \mathrel{\underset{>}{\succ}})$ such that $\lambda(\alpha+1) \mathrel{\underset{>}{\succ}} y^*$. In either case, we have $\lambda(\alpha+1) \succ \lambda(\alpha)$, hence $\lambda(\alpha+1) \mathrel{\underset{<}{\succ}} \lambda(\alpha)$. Let us show $\lambda(\alpha+1) \in M(X, \mathrel{\underset{>}{\succ}})$.

Supposing the contrary, $y \mathrel{\underset{>}{\succ}} \lambda(\alpha+1)$, we must have $y \notin Y$. Now we apply Lemma 4.8.1 to $x = \lambda(\alpha)$ and $y$, obtaining $\lambda(\alpha) \wedge y \succeq y \succ \lambda(\alpha+1)$. Since $\lambda(\alpha) \wedge y \in Y$ and $\lambda(\alpha) \wedge y \mathrel{\underset{>}{\succ}} \lambda(\alpha+1)$, we have a contradiction with the choice of $\lambda(\alpha+1)$.

Let $\alpha^* \in \Lambda$ be a limit ordinal and $\lambda(\alpha)$ have been defined for all $\alpha < \alpha^*$. Then we define $\lambda(\alpha^*) := \inf_{\alpha<\alpha^*} \lambda(\alpha) \in X$, ensuring that $\lambda([0, \alpha^*]) \in \mathfrak{C}_A$. If $\lambda(\alpha^*) = \lambda(\alpha)$ for some $\alpha < \alpha^*$, then we are home immediately, exactly as in the proof of Proposition 3.7. Otherwise, (3.2b) implies $\lambda(\alpha^*) \mathrel{\underset{<}{\succ}} \lambda(\alpha)$ for every $\alpha < \alpha^*$, so we only have to show that $\lambda(\alpha^*) \in M(X, \mathrel{\underset{>}{\succ}})$.

Assuming that $y^* \mathrel{\underset{>}{\succ}} \lambda(\alpha^*)$ for a $y^* \in X$, we recursively construct a mapping $\mu \colon [0, \alpha^*] \to X$ such that

$$\forall \alpha \in [0, \alpha^*] \left[\mu(\alpha) \succ \lambda(\alpha^*) \;\&\; \mu(\alpha) \geq \lambda(\alpha^*)\right]; \tag{4.8a}$$

$$\forall \alpha \in [0, \alpha^*] \left[\lambda(\alpha) \geq \mu(\alpha)\right]; \tag{4.8b}$$

$$\forall \alpha, \beta \in [0, \alpha^*] \left[\alpha > \beta \Rightarrow [\mu(\alpha) = \mu(\beta) \;\text{ or }\; \mu(\alpha) \mathrel{\underset{<}{\succeq}} \mu(\beta)]\right]; \tag{4.8c}$$

$$\forall \alpha \in [0, \alpha^*] \left[\mu([0, \alpha]) \in \mathfrak{C}_A\right]. \tag{4.8d}$$

First, we define $\mu(0) := y^*$; (4.8a) holds for $\alpha = 0$. Having $\mu(\alpha)$ defined, we set $\mu(\alpha+1) := \mu(\alpha) \wedge \lambda(\alpha+1)$. Since $\mu(\alpha) \succ \lambda(\alpha+1)$, we have either $\mu(\alpha+1) = \mu(\alpha)$ or $\mu(\alpha+1) \mathrel{\underset{<}{\succeq}} \mu(\alpha)$ by Lemma 4.8.1. Thus, all conditions (4.8) continue to hold.

Let $\alpha$ be a limit ordinal and $\mu(\beta)$ have been defined for all $\beta < \alpha$; then we define $\mu(\alpha) := \inf_{\beta<\alpha} \mu(\beta)$, ensuring that $\mu([0, \alpha]) \in \mathfrak{C}_A$. If $\mu(\alpha) = \mu(\beta^*)$ for some $\beta^* < \alpha$, then $\mu(\alpha) = \mu(\beta)$ for all $\beta \in [\beta^*, \alpha]$ and all conditions (4.8) hold for $\alpha$ too. Otherwise, $\mu(\alpha) \mathrel{\underset{<}{\succeq}} \mu(\beta)$ for all $\beta < \alpha$ by (3.2b) and conditions (4.8) are again secured.

When it comes to $\alpha = \alpha^*$, conditions (4.8a) and (4.8b) imply that $\mu(\alpha^*) = \lambda(\alpha^*)$, but $\mu(\alpha^*) \succ \lambda(\alpha^*)$, which contradiction proves that $\lambda(\alpha^*) \in M(X, \mathrel{\underset{>}{\succ}})$ indeed.

The final argument is standard. An equality $\lambda(\alpha') = \lambda(\alpha)$ with $\alpha' > \alpha$ is only possible when $\lambda(\alpha) \in M(X, \succ)$. Since the cardinality of $\Lambda$ is greater than that of $X$, the equality must occur at some stage. $\square$

**Theorem 4.11.** *Let an ordering* $\succ$ *on a lattice* $A$ *satisfy* (3.2a), (3.3b), *and* (4.3a). *Then* $M(X, \succ) \neq \emptyset$ *for every* $X \in \mathfrak{L}_A$.

The proof is dual to that of Theorem 4.10.

The assumptions (4.2a) or (4.3a) in Theorems 4.8–4.11, cannot be replaced with (4.3b) or (4.2b), nor even with their conjunction.

**Example 4.12 (Kukushkin, 2012, Example 3.1).** Let $A := \bigl(\{n/(n+1)\}_{n\in\mathbb{N}} \cup \{1\}\bigr) \times \bigl(\{0\} \cup \{1/(n+1)\}_{n\in\mathbb{N}}\bigr) \subset \mathbb{R}^2$ and $u\colon A \to \mathbb{R}$ be as follows: $u(1, x_2) = u(x_1, 0) := 0$; $u(n_1/(n_1+1), 1/(n_2+1)) := \min\{n_1, n_2\}$. $A$ with the order induced from $\mathbb{R}^2$ is a complete lattice. The function $u$ is *superjoin* in terms of Veinott (1992), i.e., satisfies the condition $u(y \vee x) \vee u(y \wedge x) \geq u(y) \vee u(x)$ for all $y, x \in A$. The ordering $\succ$ represented by $u$ satisfies (3.2); moreover, it satisfies (4.3b) and (4.2b), hence (4.1d) as well. By Theorem 3.9, $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$; however $\sup_{x\in A} u(x) = +\infty$, hence $M(A, \succ) = \emptyset$.

## 4.4 Implications of weak QSM

**Theorem 4.13.** *Let an ordering $\succ$ on a lattice $A$ satisfy* (4.2d). *Then $\succ$ has the properties that $M(X, \succ) \neq \emptyset$ and $M(X, \succ)$ is chain-subcomplete for every $X \in \mathfrak{L}_A$ if and only if it is strongly chain-transitive.*

*Proof.* The necessity immediately follows from Theorem 3.11 and Proposition 3.6.

The sufficiency is proven with a construction rather similar to that in the proofs of Theorems 4.6 and 4.10. Again, we start with a couple of auxiliary statements.

**Lemma 4.13.1.** *If $X$ is a sublattice of $A$, $x \in M(X, \underset{>}{\succeq})$, and $X \ni y \succ x$, then $y \wedge x \succeq y$.*

*Proof of Lemma 4.13.1.* An assumption that $y \succ y \wedge x$ would imply $y \vee x \succeq x$ by (4.2d). Since $x$ is maximal, this is only possible if $y \vee x = x$, hence $y = y \wedge x$. □

We define $X^{\downarrow} := \{x \in X \mid \forall y \in X \, \exists z \in [\bigwedge X, x] \, [z \succeq y]\}$.

**Lemma 4.13.2.** *If $\bar{x} \in X^{\downarrow}$, then $M([\bigwedge X, \bar{x}], \underset{>}{\succeq}) \subseteq X^{\downarrow}$.*

*Proof of Lemma 4.13.2.* Let $x \in M([\bigwedge X, \bar{x}], \underset{>}{\succeq})$ and $y \in X$. If $x \succeq y$, we are immediately home; let $y \succ x$. Since $\bar{x} \in X^{\downarrow}$, there is $z \in [\bigwedge X, \bar{x}]$ such that $z \succeq y$. Applying Lemma 4.13.1 to $[\bigwedge X, \bar{x}]$, $x$, and $z$, we obtain $z \wedge x \succeq z [\succeq y]$. Clearly, $z \wedge x \in [\bigwedge X, x]$. □

Let $\Lambda$ be a well ordered set with a cardinality greater than that of $X$. We construct, by (transfinite) recursion, a mapping $\lambda\colon \Lambda \to X$ such that:

$$\forall \alpha \in \Lambda \left[\lambda(\alpha) \in X^{\downarrow}\right]; \tag{4.9a}$$

$$\forall \alpha \in \Lambda \left[\lambda(\alpha+1) \in M([\bigwedge X, \lambda(\alpha)], \underset{>}{\succeq})\right]; \tag{4.9b}$$

$$\forall \alpha, \beta \in \Lambda \left[\alpha > \beta \Rightarrow [\lambda(\alpha) = \lambda(\beta) \in M(X, \succ) \text{ or } \lambda(\alpha) \underset{<}{\succ} \lambda(\beta)]\right]; \tag{4.9c}$$

$$\forall \alpha \in \Lambda \left[\lambda([0, \alpha]) \in \mathfrak{C}_A\right]. \tag{4.9d}$$

First, we set $\lambda(0) := \bigvee X \in M(X, \underset{>}{\succ})$. All conditions (4.9) for $\alpha = 0$ hold trivially.

Let $\lambda(\alpha)$ have been defined. If $\lambda(\alpha) \in M(X, \succ)$, we define $\lambda(\alpha+1) := \lambda(\alpha)$; actually, $\lambda(\alpha') = \lambda(\alpha)$ for all $\alpha' > \alpha$ in this case. Otherwise, the induction hypothesis (4.9a) implies the existence of $x \in [\bigwedge X, \lambda(\alpha)]$ such that $x \succ \lambda(\alpha)$. Applying Proposition 3.7 to $\succeq$, $[\bigwedge X, \lambda(\alpha)]$ and $x$, we obtain $\lambda(\alpha+1) \in M([\bigwedge X, \lambda(\alpha)], \underset{>}{\succeq})$ such that $\lambda(\alpha+1) \underset{<}{\succ} \lambda(\alpha)$. By Lemma 4.13.2, we have $\lambda(\alpha+1) \in X^{\downarrow}$; other conditions (4.9) also continue to hold for $\alpha+1$.

Let $\alpha^* \in \Lambda$ be a limit ordinal and $\lambda(\alpha)$ have been defined for all $\alpha < \alpha^*$. Then we define $\lambda(\alpha^*) := \inf_{\alpha<\alpha^*} \lambda(\alpha)$, ensuring that $\lambda([0,\alpha^*]) \in \mathfrak{C}_A$. If $\lambda(\alpha^*) = \lambda(\alpha)$ for some $\alpha < \alpha^*$, then we are home immediately, exactly as in the proof of Proposition 3.7. Otherwise, (3.2b) implies $\lambda(\alpha^*) \underset{<}{\succ} \lambda(\alpha)$ for every $\alpha < \alpha^*$, so we only have to show that $\lambda(\alpha^*) \in X^{\downarrow}$.

Let $y^* \succ \lambda(\alpha^*)$, hence $y^* \succ \lambda(\alpha)$ for every $\alpha < \alpha^*$. We recursively construct a mapping $\mu\colon [0,\alpha^*] \to X$ in this way: $\mu(0) := y^*$ $[= \mu(0) \wedge \lambda(0)]$; $\mu(\alpha+1) := \mu(\alpha) \wedge \lambda(\alpha+1)$; $\mu() := \inf_{\beta<\alpha} \mu(\beta)$ whenever $\alpha \le \alpha^*$ is a limit. A straightforward inductive argument, the same as in the proof of Theorem 4.10, shows that $\mu(\alpha) \underset{<}{\succ} \lambda(\alpha)$ for every $\alpha < \alpha^*$ and, whenever $\beta < \alpha \le \alpha^*$, there holds either $\mu(\alpha) = \mu(\beta) \underset{<}{\succeq} y^*$ or $\mu(\alpha) \underset{<}{\succeq} \mu(\beta) \underset{<}{\succeq} y^*$. Therefore, $\lambda(\alpha^*) := \inf_{\alpha<\alpha^*} \lambda(\alpha) \ge \mu(\alpha^*) := \inf_{\alpha<\alpha^*} \mu(\alpha)$ and $\mu(\alpha^*) \succeq y^*$. Thus, $\lambda(\alpha^*) \in X^{\downarrow}$ indeed.

The final argument is standard. An equality $\lambda(\alpha') = \lambda(\alpha)$ with $\alpha' > \alpha$ is only possible when $\lambda(\alpha) \in M(X, \succ)$. Since the cardinality of $\Lambda$ is greater than that of $X$, the equality must occur at some stage. $\square$

**Remark.** In Example 4.12, conditions (3.3) are violated by sequences $\langle\big((k+m)/(k+m+1), 1/(m+1)\big)\rangle_{k\in\mathbb{N}}$ $(m \in \mathbb{N})$ and $\langle\big(m/(m+1), 1/(k+m+1)\big)\rangle_{k\in\mathbb{N}}$ $(m \in \mathbb{N})$, respectively.

**Theorem 4.14.** *Let an ordering $\succ$ on a lattice $A$ satisfy* (4.3d). *Then $\succ$ has the properties that $M(X, \succ) \neq \emptyset$ and $M(X, \succ)$ is chain-subcomplete for every $X \in \mathfrak{L}_A$ if and only if it is strongly chain-transitive.*

The proof is dual to that of Theorem 4.13.

**Theorem 4.15.** *Let $X$ be a complete sublattice of $\prod_{i\in I} \mathcal{C}_i$, where $I$ is a finite set and each $\mathcal{C}_i$ is a chain. Let $\succ$ be an ordering on $X$ satisfying* (3.2b), (3.3a), *and* (4.2d). *Then $M(X, \succ) \neq \emptyset$.*

*Proof.* As in the proof of Theorem 4.8, we argue by induction in $\#I$, employing auxiliary statements and definitions from the proofs of Theorems 4.8 and 4.13. When $\#I = 1$, $X$ is a chain, hence Theorem 3.9 applies. Let the statement of the theorem hold whenever $\#I \le m$; we have to prove that the statement holds whenever $\#I \le m+1$.

Again as in the proof of Theorem 4.8, we assume to the contrary that $M(X, \succ) = \emptyset$.

**Lemma 4.15.1.** *Whenever $x \in X^{\downarrow}$, there is $y \in M([\bigwedge X, x], \underset{>}{\succeq})$ such that $y < x$ and $y \ggcurly x$.*

*Proof of Lemma 4.15.1.* For each $i \in I$, we pick $x^i \in M(X^i(x), \succ)$; our assumption $M(X, \succ) = \emptyset$ implies the existence of $z \in X$ such that $z \succ x^i$ for each $i$, i.e., $z \ggcurly x$. Since $x \in X^{\downarrow}$, we have $z' \in [\bigwedge X, x]$ such that $z' \succeq z$, hence $z' \ggcurly x$. By Proposition 3.7 applied to $\succeq$ and $[\bigwedge X, x]$, there is $y \in M([\bigwedge X, x], \underset{>}{\succeq})$ such that either $y \underset{>}{\succeq} z'$ or $y = z'$; clearly, $y \ggcurly x$. $\square$

Let $\Lambda$ be a well ordered set with a cardinality greater than that of $X$. We construct, by (transfinite) recursion, a mapping $\lambda\colon \Lambda \to X$ such that:

$$\forall \alpha \in \Lambda \left[\lambda(\alpha) \in X^{\downarrow}\right]; \tag{4.10a}$$

$$\forall \alpha \in \Lambda \left[\lambda(\alpha+1) \in M([\bigwedge X, \lambda(\alpha)], \underset{>}{\succeq})\right]; \tag{4.10b}$$

$$\forall \alpha, \beta \in \Lambda \left[\alpha > \beta \Rightarrow [\lambda(\alpha) < \lambda(\beta) \;\&\; \lambda(\alpha) \mathrel{>\!\!\!\succ} \lambda(\beta)]\right]; \tag{4.10c}$$

$$\forall \alpha \in \Lambda \left[\lambda([0, \alpha]) \in \mathfrak{C}_A\right]. \tag{4.10d}$$

First, we set $\lambda(0) := \bigvee X \in M(X, \underset{>}{\succ})$. All conditions (4.10) for $\alpha = 0$ hold trivially.

Let $\lambda(\alpha)$ have been defined. We pick a $y$ existing by Lemma 4.15.1, and call it $\lambda(\alpha+1)$. All conditions (4.10) continue to hold.

Let $\alpha^* \in \Lambda$ be a limit ordinal and $\lambda(\alpha)$ have been defined for all $\alpha < \alpha^*$. Then we define $\lambda(\alpha^*) := \inf_{\alpha<\alpha^*} \lambda(\alpha)$, ensuring that $\lambda([0, \alpha^*]) \in \mathfrak{C}_X$. For every $\alpha < \alpha^*$, we have $\lambda(\alpha^*) \underset{<}{\succ} \lambda(\alpha)$ by (3.2b), hence $\lambda(\alpha^*) \mathrel{>\!\!\!\succ} \lambda(\alpha)$ as well. Thus, we only have to show that $\lambda(\alpha^*) \in X^{\downarrow}$.

Let $y \in X$; if $\lambda(\alpha^*) \geq y$ or $\lambda(\alpha^*) \succeq y$, then we are home with $z := y$ or $z := \lambda(\alpha^*)$ respectively. Assuming that $y \notin [\bigwedge X, \lambda(\alpha^*)]$ and $y \succ \lambda(\alpha^*)$, we must have $\alpha < \alpha^*$ for which $y \in [\bigwedge X, \lambda(\alpha)]$, but $y \notin [\bigwedge X, \lambda(\alpha+1)]$, hence there is $i \in I$ such that $y \wedge \lambda(\alpha+1) \in X^i(\lambda(\alpha+1))$. Since $y \succ \lambda(\alpha^*) \mathrel{>\!\!\!\succ} \lambda(\alpha+1)$, we have $y \succ y \wedge \lambda(\alpha+1)$, which contradicts Lemma 4.13.1 since $\lambda(\alpha+1) \in M([\bigwedge X, \lambda(\alpha)], \underset{>}{\succeq})$.

Now we have $\lambda(\alpha)$ defined for all $\alpha \in \Lambda$ and all of them distinct because of (4.10c). Since the cardinality of $\Lambda$ is greater than that of $X$, we have obtained our final contradiction. $\square$

**Theorem 4.16.** *Let $X$ be a complete sublattice of $\prod_{i\in I} \mathcal{C}_i$, where $I$ is a finite set and each $\mathcal{C}_i$ is a chain. Let $\succ$ be an ordering on $X$ satisfying* (3.2a), (3.3b), *and* (4.3d). *Then $M(X, \succ) \neq \emptyset$.*

The proof is dual to that of Theorem 4.15.

## 4.5 "Algebraic" conditions

Veinott (1992) considered mappings $u$ from a (semi)lattice $A$ to a chain $\mathcal{C}$ satisfying conditions different from those above and obtained three theorems about the existence of *minima*. I modify those conditions and theorems to make them relevant to *maximization*:

$$\forall y, x \in A\, [u(y \vee x) \vee u(y \wedge x) \geq u(y) \wedge u(x)]; \tag{4.11a}$$

$$\forall y, x \in A\, [u(y \vee x) \geq u(y) \wedge u(x)]; \tag{4.11b}$$

$$\forall y, x \in A \left[u(y \vee x) \vee u(y \wedge x) \geq u(y) \vee u(x) \;\text{ or }\; u(y \vee x) \wedge u(y \wedge x) \geq u(y) \wedge u(x)\right]. \tag{4.11c}$$

In Theorems 6.2, 6.12, and 6.41 of Veinott (1992), $u$ satisfied conditions (4.11a), (4.11b), and (4.11c), respectively; in each theorem, every upper contour set was chain-subcomplete; besides, $\mathcal{C}$ in Theorem 6.2 was $\mathbb{R} \cup \{+\infty, -\infty\}$. Then the existence of a maximum was shown.

**Remark.** Conditions (4.11) also ensure certain nice properties of upper contour sets (Veinott, 1989, 1992).

Obviously, if an ordering $\succ$ admits a representation (2.5) with $u$ satisfying any one of conditions (4.11), then *every* representation (2.5) of $\succ$ satisfies the same condition.

**Proposition 4.17.** *An ordering $\succ$ on a lattice $A$ satisfies* (4.11c) *if and only if it satisfies both conditions* (4.2d) *and* (4.3d)*. If $\succ$ satisfies* (4.11c)*, then it satisfies* (4.11a)*. If $\succ$ satisfies* (4.11b)*, then it satisfies* (4.11a) *and* (4.2b)*.*

**Example 4.18.** Let $A := \{(0,0),(0,1),(1,0),(1,1)\} \subset \mathbb{R}^2$; we consider two orderings on $A$ represented by these matrices (the axes are directed upwards and rightwards):

$$\mathbf{a}.\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix} \qquad \mathbf{b}.\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}.$$

The matrix "a" satisfies (4.11b), hence (4.11a) and (4.2b), but none of the conditions (4.2a), (4.3d), or (4.11c). The matrix "b" satisfies all conditions (4.1) hence (4.11c) and (4.11a), but not (4.11b).

In light of Propositions 3.3 and 4.17, Theorems 6.12 (restricted to lattices) and 6.41 of Veinott (1992) follow from Theorem 4.13. Veinott's Theorem 6.2 is discussed in Section 5. We end this section by strengthening Veinott's Theorem 6.12 concerning semilattices; note that (4.11b) is meaningful for a join-semilattice $A$.

**Theorem 4.19.** *Let $A$ be a join-semilattice and $\succ$ be an ordering on $A$ satisfying* (3.2b), (3.3a), *and* (4.11b)*. Then $M(X,\succ) \neq \emptyset$ for every $X \in \mathfrak{S}_A$ (where $\mathfrak{S}_A$ denotes the set of all nonempty complete subsemilattices $X \subseteq A$).*

*Proof.* The basic construction is similar to that from the proof of Theorem 4.10. We again use auxiliary strict orders $\mathrel{\underset{<}{\succ}}$ and $\mathrel{\underset{>}{\succeq}}$ defined by (2.7).

**Lemma 4.19.1.** *If $X$ is a subsemilattice of $A$, $x \in M(X, \mathrel{\underset{>}{\succeq}})$ and $X \ni y \succ x$, then $y \mathrel{\underset{<}{\succ}} x$.*

*Proof of Lemma 4.19.1.* If $y \not< x$, then $y \vee x > x$, hence $x \succ y \vee x$ because $x \in M(X, \mathrel{\underset{>}{\succeq}})$. Now (4.11b) applies, hence $x \succ y \vee x \succeq y$, which contradicts the condition of the lemma. $\square$

Let $\Lambda$ be a well ordered set with a cardinality greater than that of $A$. Given $X$, we construct, by (transfinite) recursion, a mapping $\lambda\colon \Lambda \to X$ such that:

$$\forall \alpha \in \Lambda \left[\lambda(\alpha) \in M(X, \mathrel{\underset{>}{\succeq}})\right]; \tag{4.12a}$$

$$\forall \alpha, \beta \in \Lambda \left[\alpha > \beta \Rightarrow [\lambda(\alpha) = \lambda(\beta) \in M(X,\succ) \ \text{ or } \ \lambda(\alpha) \mathrel{\underset{<}{\succ}} \lambda(\beta)]\right]; \tag{4.12b}$$

$$\forall \alpha \in \Lambda \left[\lambda([0,\alpha]) \in \mathfrak{C}_X\right]. \tag{4.12c}$$

First, we define $\lambda(0) := \bigvee X \in M(X, \underset{>}{\succeq})$. Let $\lambda(\alpha)$ have been defined. If $\lambda(\alpha) \in M(X, \succ)$, we define $\lambda(\alpha+1) := \lambda(\alpha)$; actually, $\lambda(\alpha') = \lambda(\alpha)$ for all $\alpha' > \alpha$ in this case. Otherwise, Lemma 4.19.1 implies the existence of $x^* \in X$ such that $x^* \underset{<}{\succ} \lambda(\alpha)$. Applying Proposition 3.7 to $\succeq$, $[\bigwedge X, \lambda(\alpha)]$ and $x^*$, we obtain $\lambda(\alpha+1) \in M([\bigwedge X, \lambda(\alpha)], \underset{>}{\succeq})$ such that $\lambda(\alpha+1) \underset{<}{\succ} \lambda(\alpha)$. Let us show $\lambda(\alpha+1) \in M(X, \underset{>}{\succeq})$. Supposing the contrary, $y \underset{>}{\succeq} \lambda(\alpha+1)$, we apply Lemma 4.19.1 to $x = \lambda(\alpha)$ and $y$, obtaining $y < \lambda(\alpha)$, which is incompatible with the choice of $\lambda(\alpha+1)$.

Let $\alpha^* \in \Lambda$ be a limit ordinal and $\lambda(\alpha)$ have been defined for all $\alpha < \alpha^*$. Then we define $\lambda(\alpha^*) := \sup_{\alpha<\alpha^*} \lambda(\alpha)$, ensuring that $\lambda([0, \alpha^*]) \in \mathfrak{C}_X$. Now (3.2b) implies $\lambda(\alpha^*) \underset{<}{\succ} \lambda(\alpha)$ for every $\alpha < \alpha^*$ unless $\lambda(\alpha^*) = \lambda(\alpha)$, so we only have to show that $\lambda(\alpha^*) \in M(X, \underset{>}{\succeq})$.

If $\lambda(\alpha^*) = \lambda(\alpha)$ for some $\alpha < \alpha^*$, we are home immediately. Otherwise, $\lambda(\alpha^*) \underset{<}{\succ} \lambda(\alpha)$ for every $\alpha < \alpha^*$, hence $y \underset{>}{\succeq} \lambda(\alpha^*)$ would imply $y \succ \lambda(\alpha)$, hence $y < \lambda(\alpha)$ by Lemma 4.19.1. Since $\alpha$ is arbitrary, we would have $y \le \lambda(\alpha^*)$, contradicting the assumption about $y$.

The final argument is again standard. □

**Remark.** The preference relation in Example 4.12 satisfies (4.11b) as well as both (3.2), but not (3.3a). Therefore, the replacement of (3.3a) in Theorem 4.19 with (3.2a) would make it wrong.

**Proposition 4.20.** *Let $A$ be a join-semilattice and $\succ$ be an ordering on $A$ satisfying* (4.11b). *Then $M(X, \succ) \in \mathfrak{S}_A$ for every $X \in \mathfrak{S}_A$ if and only if $\succ$ satisfies* (3.3).

*Proof.* The necessity follows from Proposition 3.6; the sufficiency, from Theorem 4.19 and Proposition 3.10. □

# 5 Countability assumptions

## 5.1 Basic definitions

We call a chain $\mathcal{C}$ *regular*, just for want of a better term, if every $V \in \mathfrak{B}_{\mathcal{C}}$ contains a countable *cofinal* subset, i.e., a countable subset $W \subseteq V$ such that for every $v \in V$ there is $w \in W$ such that $w \ge v$. Clearly, then every $V \in \mathfrak{B}_{\mathcal{C}}$ contains either a maximum or a *maximizing sequence*, i.e., $\langle v^k \rangle_{k\in\mathbb{N}}$ such that: (i) $v^k \in V$ and $v^{k+1} > v^k$ for all $k$; (ii) for every $v \in V \setminus M(V, >)$, there exists $k \in \mathbb{N}$ such that $v^k > v$. An obvious example of a regular chain is $\mathbb{R}$; a somewhat less obvious example is $\mathbb{R}^m$ with a lexicographic order.

We call a strict order $\succ$ on a set $A$ *regular* if every chain w.r.t. $\succ$ in $A$ is regular. Given a strict order $\succ$ on $A$ and $X \in \mathfrak{B}_A$, we call a sequence $\langle x^k \rangle_{k\in\mathbb{N}}$ in $X$ *optimizing* if (i) $x^{k+1} \succ x^k$ for all $k$; (ii) for every $x \in X \setminus M(X, \succ)$, there is $k \in \mathbb{N}$ such that $x^k \succ x$.

**Proposition 5.1.** *Let $\succ$ be an interval order on $A$. Then $\succ$ is regular if and only if for every $X \in \mathfrak{B}_A$, either $M(X, \succ) \neq \emptyset$, or there exists an optimizing sequence in $X$.*

*Proof.* The sufficiency is straightforward. Let $\succ$ be a regular interval order on $A$, let $X \in \mathfrak{B}_A$, and let $M(X,\succ) = \emptyset$. We denote $\mathfrak{P}$ the set of all chains w.r.t. $\succ$ in $X$ with set inclusion as a partial order. Zorn's Lemma (Theorem D) shows that $M(\mathfrak{P},\supset) \neq \emptyset$; let $C \in M(\mathfrak{P},\supset)$. Since $M(X,\succ) = \emptyset$, $M(C,\succ) = \emptyset$ as well. Since $\succ$ is regular, there is a sequence $\langle x^k \rangle_{k\in\mathbb{N}}$ such that $x^k \in C$ and $x^{k+1} \succ x^k$ for each $k \in \mathbb{N}$, and for every $x \in C$ there is $k \in \mathbb{N}$ for which $x^k \succ x$. Let us show that $\langle x^k \rangle_{k\in\mathbb{N}}$ is an optimizing sequence in $X$.

Let $y \in X$. Since $M(X,\succ) = \emptyset$, there is $z \in X$ such that $z \succ y$. An assumption that $z \succ x^k$ for each $k \in \mathbb{N}$ would imply that $z \succ x$ for every $x \in C$, hence $C \cup \{z\} \in \mathfrak{P}$, contradicting the maximality of $C$. Thus, $x^k \succeq z$ for some $k \in \mathbb{N}$. Now we have $x^{k+1} \succ x^k \succeq z \succ y$; since $\succ$ is an interval order, it follows that $x^{k+1} \succ y$. Since $y$ was arbitrary, we are home. $\square$

**Proposition 5.2.** *If an interval order $\succ$ on $A$ admits a representation* (2.6) *with a regular chain $\mathcal{C}$, then it is regular itself.*

*Proof.* Let $B \subseteq A$ be a chain w.r.t. $\succ$ in $A$. If $M(B,\succ) \neq \emptyset$, we are home immediately. Otherwise, $u^-(B) \subseteq \mathcal{C}$ also does not contain its maximum, hence it contains a maximizing sequence $\langle v^k \rangle_{k\in\mathbb{N}}$. Now we recursively define a sequence $\langle x^k \rangle_{k\in\mathbb{N}}$ in $B$ such that $u^-(x^k) = v^{h(k)}$ for each $k \in \mathbb{N}$ and $h\colon \mathbb{N} \to \mathbb{N}$ is strictly increasing. First, we pick $x^0$ such that $u^-(x^0) = v^0$ and set $h(0) := 0$. Given $x^k$ for which $u^-(x^k) = v^{h(k)}$, we pick $x \in B$ such that $x \succ x^k$, hence $u^-(x) > u^-(x^k) = v^{h(k)}$. Since $u^-(x) \in u^-(B)$, we have $v^m > u^-(x)$ for all $m \in \mathbb{N}$ large enough. Now we pick $h(k+1) \in \mathbb{N}$ such that $v^{h(k+1)} > u^-(x)$, hence $h(k+1) > h(k)$, and pick $x^{k+1} \in B$ for which $u^-(x^{k+1}) = v^{h(k+1)}$. Clearly, $\langle x^k \rangle_{k\in\mathbb{N}}$ is an optimizing sequence in $B$. $\square$

**Proposition 5.3.** *Every regular ordering $\succ$ on $A$ admits a representation* (2.6) *with a regular chain $\mathcal{C}$.*

*Proof.* Starting with an arbitrary representation $u\colon A \to \mathcal{C}$, we set $Z := u(A)$; then $u\colon A \to Z$ is still a representation of $\succ$ in the sense of (2.5). Let $V \in \mathfrak{B}_Z$; we define $X := u^{-1}(V)$. If $M(X,\succ) \neq \emptyset$, then $u(M(X,\succ))$ is a singleton cofinal subset of $V$. Otherwise, there is an optimizing sequence $\langle x^k \rangle_{k\in\mathbb{N}}$ in $X$. Denoting $w^k := u(x^k)$, we immediately see that $W := \{w^k\}_{k\in\mathbb{N}}$ is a maximizing sequence in $V$. $\square$

**Example 5.4.** Let $\Lambda$ be a well-ordered uncountable set. We define $\Omega := \{a \in \Lambda \mid \{x \in \Lambda \mid x < a\}$ is countable$\}$; it is easily seen that $\Omega$ is uncountable and conditionally complete, but $\max\Omega$ does not exist. Then we set $A := \{0,1\} \times \Omega$ with a lexicographic order

$$(\vartheta',a') > (\vartheta,a) \rightleftharpoons \bigl[\vartheta' > \vartheta \ \text{ or } \ (\vartheta' = \vartheta \ \& \ a' > a)\bigr].$$

Finally, we define a preference relation on $A$:

$$(\vartheta',a') \succ (\vartheta,a) \rightleftharpoons \bigl[\vartheta' > \vartheta \ \& \ a' > a]\bigr].$$

It is easily checked that $\succ$ is a regular semiorder; every chain w.r.t. $\succ$ in $A$ contains two points at most. On the other hand, whenever $a' > a$, we have $(0,a') \succeq (1,a') \succ (0,a)$, hence

$u^+(0,a') > u^+(0,a)$ for every representation (2.6) of $\succ$. It follows immediately that the chain $u^+(0,\Omega)$ in $\mathcal{C}$ does not have a maximum. Moreover, for every increasing sequence $\langle v^k\rangle_{k\in\mathbb{N}}$ in $u^+(0,\Omega)$, we have $\sup_k v^k \in u^+(0,\Omega)$, i.e., the sequence cannot be maximizing. Thus, chain $\mathcal{C}$ is not regular.

We see that Proposition 5.3 could not be extended even to semiorders.

We call a binary relation $\succ$ on a poset $A$ *weakly mono-$\omega$-transitive* if it is transitive on every chain and satisfies both following conditions:

$$\big[\forall k \in \mathbb{N}\,[x^{k+1} \succ x^k \;\&\; x^{k+1} > x^k] \;\&\; x^\omega = \sup\{x^k\}_k\big] \Rightarrow x^\omega \succeq x^0; \tag{5.1a}$$

$$\big[\forall k \in \mathbb{N}\,[x^{k+1} \succ x^k \;\&\; x^{k+1} < x^k] \;\&\; x^\omega = \inf\{x^k\}_k\big] \Rightarrow x^\omega \succeq x^0. \tag{5.1b}$$

We call $\succ$ *mono-$\omega$-transitive* if it is transitive on every chain and satisfies both following conditions:

$$\big[\forall k \in \mathbb{N}\,[x^{k+1} \succ x^k \;\&\; x^{k+1} > x^k] \;\&\; x^\omega = \sup\{x^k\}_k\big] \Rightarrow x^\omega \succ x^0; \tag{5.2a}$$

$$\big[\forall k \in \mathbb{N}\,[x^{k+1} \succ x^k \;\&\; x^{k+1} < x^k] \;\&\; x^\omega = \inf\{x^k\}_k\big] \Rightarrow x^\omega \succ x^0. \tag{5.2b}$$

We call an ordering $\succ$ *strongly mono-$\omega$-transitive* if it is mono-$\omega$-transitive and satisfies both following conditions:

$$\big[\forall k \in \mathbb{N}\,[x^{k+1} \succeq x^k \;\&\; x^{k+1} > x^k] \;\&\; x^\omega = \sup\{x^k\}_k\big] \Rightarrow x^\omega \succeq x^0; \tag{5.3a}$$

$$\big[\forall k \in \mathbb{N}\,[x^{k+1} \succeq x^k \;\&\; x^{k+1} < x^k] \;\&\; x^\omega = \inf\{x^k\}_k\big] \Rightarrow x^\omega \succeq x^0. \tag{5.3b}$$

**Remark.** There is an obvious similarity with the (topological) notion of "$\omega$-transitivity" (Gillies, 1959; Smith, 1974; Kukushkin, 2008b).

Clearly, each condition (3.1), (3.2) or (3.3) implies the corresponding condition (5.1), (5.2) or (5.3). The converse is generally wrong.

**Proposition 5.5.** *If $\succ$ is a regular strict order on a poset A, then each condition* (5.1) *or* (5.2) *is equivalent to the corresponding condition* (3.1) *or* (3.2)*.*

*Proof.* Implications in one direction only deserve attention. Let $X \in \mathfrak{C}_A$ satisfy the left hand side of (3.1a) or (3.2a), i.e., $\sup X^{\rightharpoondown} = \sup X$ and $y > x \Rightarrow y \succ x$ for all $x, y \in X^{\rightharpoondown}$. The first condition implies that $M(X^{\rightharpoondown}, >) = \emptyset$, hence $M(X^{\rightharpoondown}, \succ) = \emptyset$ too. Since $\succ$ is regular, there exists an optimizing sequence $\langle x^k\rangle_{k\in\mathbb{N}}$ in $X^{\rightharpoondown}$; by the monotonicity assumption, we have $x^{k+1} > x^k$ for each $k \in \mathbb{N}$ and $\sup_{k\geq m} x^k = \sup X^{\rightharpoondown} = \sup X$ for any $m \in \mathbb{N}$. Therefore, $\sup X \succeq x^m$ for any $m \in \mathbb{N}$ by (5.1a) or $\sup X \succ x^m$ for any $m \in \mathbb{N}$ by (5.2a). For any $x \in X^{\rightharpoondown}$, there is $m \in \mathbb{N}$ such that $x^m \succ x$, hence $\sup X \succeq x$ in the first case, or $\sup X \succ x$ in the second, by transitivity.

The proofs of corresponding implications from (5.1b) or (5.2b) are similar. $\square$

We call a poset $A$ *regular* if every chain $\mathcal{C} \in \mathfrak{B}_A$ is regular in both the basic order and its reverse. An example is the set of uniformly Lipshitz continuous functions on a compact metric space with the point-wise order. The Cartesian product of a countable number of regular posets with the product order is also regular; ditto for every subset of a regular poset.

**Proposition 5.6.** *If $\succ$ is a strict order on a regular poset $A$, then each condition* (5.1), (5.2) *or* (5.3) *is equivalent to the corresponding condition* (3.1), (3.2) *or* (3.3).

*Proof.* If $X \in \mathfrak{C}_A$ and $\sup X^{\rightharpoondown} = \sup X$, then $M(X^{\rightharpoondown}, >) = \emptyset$. Therefore, there exists a maximizing sequence $\langle x^k \rangle_{k\in\mathbb{N}}$ in $X^{\rightharpoondown}$. For any $m \in \mathbb{N}$, we have $\sup_{k\geq m} x^k = \sup X^{\rightharpoondown} = \sup X$. Therefore, we have $\sup X \succeq x^m$ if $X$ satisfies the left hand side of (3.1a) or (3.3a) and $\succ$ satisfies (5.1) or (5.3), or $\sup X \succ x^m$ if $X$ satisfies the left hand side of (3.2a) and $\succ$ satisfies (5.2). The end of the proof is virtually identical with that of Proposition 5.5. $\square$

## 5.2 Chains

Combining Theorem 3.9 and Propositions 5.5 and 5.6, we immediately obtain two following theorems. What is much more interesting, regularity assumptions allow us to obtain characterization results unavailable in the general case.

**Theorem 5.7.** *A regular strict order $\succ$ on a poset $A$ has the NM-property on every $X \in \mathfrak{C}_A$ if and only if it is mono-$\omega$-transitive.*

**Corollary.** *A regular semiorder $\succ$ on a poset $A$ has the property that $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$ if and only if $\succ$ is mono-$\omega$-transitive.*

**Theorem 5.8.** *Let $A$ be a regular poset and $\succ$ be a binary relation on $A$. Then $\succ$ has the NM-property on every $X \in \mathfrak{C}_A$ if and only if it is irreflexive and mono-$\omega$-transitive.*

**Corollary.** *A semiorder $\succ$ on a regular poset $A$ has the property that $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$ if and only if $\succ$ is mono-$\omega$-transitive.*

**Remark.** There is an obvious similarity with Theorem 4.1 of Smith (1974).

**Example 5.9 (Kukushkin, 2012, Example 6.1).** Let well-ordered uncountable sets $\Lambda$ and $\Omega$ be as in Example 5.4; without restricting generality, $\Omega \subset \Lambda$. We define $A := \Omega \cup \{\sup \Omega\}$ (clearly, $\sup \Omega \notin \Omega$). It is easy to see that $A$ is a complete chain.

Then we define a preference ordering (actually, a linear order) on $A$:

$$y \succ x \rightleftharpoons \big[y \in \Omega \;\&\; [y > x \;\text{ or }\; x = \sup \Omega]\big].$$

Condition (5.2b) holds vacuously; (5.2a), because $\sup\{x^k\}_k \in \Omega$ whenever $\{x^k\}_k \subseteq \Omega$. However, $M(A, \succ) = \emptyset$. We see that Theorem 5.7 does not hold without the regularity of preferences, whereas Theorem 5.8 does not hold without the regularity of $A$.

**Theorem 5.10.** *Let $A$ be a poset and $\succ$ be a regular interval order on every chain in $A$. Then $\succ$ has the property that $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$ if and only if it is weakly mono-$\omega$-transitive.*

*Proof.* The necessity follows from Proposition 3.4. Let $\succ$ be a regular interval order on $X \in \mathfrak{C}_A$ satisfying (5.1). If $M(X, \succ) \neq \emptyset$, we are home immediately. Otherwise, let $\langle x^k \rangle_{k \in \mathbb{N}}$ be an optimizing sequence in $X$; it exists by Proposition 5.1. Without restricting generality, we may assume $x^{k+1} > x^k$ for each $k \in \mathbb{N}$, hence $\sup_k x^k \succeq x^k$ by (5.1a), hence $\sup_k x^k \in M(X, \succ)$. $\square$

**Theorem 5.11.** *Let $A$ be a regular poset and $\succ$ be an interval order on every chain in $A$. Then $\succ$ has the property that $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$ if and only if it is weakly mono-$\omega$-transitive.*

*Proof.* The necessity again follows from Proposition 3.4. The sufficiency proof is organized somewhat similarly to that of Theorem 3.9 and is based on analogs of Propositions 3.7 and 3.8. Let $X \in \mathfrak{C}_A$.

**Lemma 5.11.1.** *Whenever $x \in X \setminus M(X, \mathrel{\substack{\succ \\ >}})$, there is $y \in M(X, \mathrel{\substack{\succ \\ >}})$ for which either $y \mathrel{\substack{\succ \\ >}} x$ or there is an infinite sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ in $X$ such that $y = \sup_k x^k$ and $x^{k+1} \mathrel{\substack{\succ \\ >}} x^k \mathrel{\substack{\succ \\ >}} x$ for each $k \in \mathbb{N}$.*

*Proof of Lemma 5.11.1.* We set $Y := \{y \in X \mid y \mathrel{\substack{\succ \\ >}} x\}$; since $x \notin M(X, \mathrel{\substack{\succ \\ >}})$, $Y \neq \emptyset$. Similarly to the proof of Proposition 5.1, we denote $\mathfrak{P}$ the set of all chains w.r.t. $\mathrel{\substack{\succ \\ >}}$ in $Y$ with set inclusion as a partial order. Zorn's Lemma (Theorem D) shows that $M(\mathfrak{P}, \supset) \neq \emptyset$; let $C \in M(\mathfrak{P}, \supset)$. Since $A$ is regular, there exists either $\max C$ or a maximizing sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ in $C$. Accordingly, we set $y := \max C$ in the first case or $y := \sup_k x^k$ in the second; we only have to show that $y \in M(X, \mathrel{\substack{\succ \\ >}})$. Supposing the contrary, $X \ni z \mathrel{\substack{\succ \\ >}} y$, we see that $z \in Y$: in the first case, it follows just from the transitivity of $\succ$; in the second, from $z \succ y \succeq x^{k+1} \succ x^k \succ x$, since $\succ$ is an interval order. Now we have $C \subset C \cup \{z\} \in \mathfrak{P}$, contradicting the maximality of $C$. $\square$

The rest of the proof goes dually to the proof of the lemma. We have $M(X, \mathrel{\substack{\succ \\ >}}) \neq \emptyset$ by Lemma 5.11.1. Denoting $\mathfrak{P}$ the set of all chains w.r.t. $\mathrel{\substack{\succ \\ <}}$ in $M(X, \mathrel{\substack{\succ \\ >}})$, we pick $C \in M(\mathfrak{P}, \supset)$ (Zorn's Lemma again). Since $A$ is regular, there exists either $\min C$ or a minimizing sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ in $C$. Again, we set $y := \min C$ in the first case or $y := \inf_k x^k$ in the second.

In the first case, we have $y \in M(X, \mathrel{\substack{\succ \\ >}})$ immediately; let us show the same in the second case. Supposing the contrary, $X \ni z \mathrel{\substack{\succ \\ >}} y$, we have $z > x^k$ for some $k \in \mathbb{N}$ and $z \succ y \succeq x^{k+1} \succ x^k$, hence $z \succ x^k$ as well. Thus $z \mathrel{\substack{\succ \\ >}} x^k$, which contradicts the fact that $x^k \in C \subseteq M(X, \mathrel{\substack{\succ \\ >}})$.

Finally, let us show that $y \in M(X, \mathrel{\substack{\succ \\ <}})$. Let $X \ni z \mathrel{\substack{\succ \\ <}} y$. Applying Lemma 5.11.1 to $z$, we obtain the existence of $z^* \in M(X, \mathrel{\substack{\succ \\ >}})$ such that $z^* \succ y$. Since $y \in M(X, \mathrel{\substack{\succ \\ >}})$, we must have $z^* \mathrel{\substack{\succ \\ <}} y$; besides, $z^* \mathrel{\substack{\succ \\ <}} x^k$ for each $k$ in the second case. Thus, $C \subset C \cup \{z^*\} \in \mathfrak{P}$, again contradicting the maximality of $C$. $\square$

**Example 5.12.** Let $\Omega$ and $A$ be the same as in Example 5.9. Let $\Omega_1 \subset \Omega$ consist of successors, i.e., $\Omega_1 := \{a \in \Omega \mid \exists b \in \Omega\, [a = b+1]\}$. We define a preference relation on $A$:

$$y \succ x \rightleftharpoons \big[\, [y \in \Omega \;\&\; x = \sup\Omega] \;\text{ or }\; [y \in \Omega_1 \;\&\; y > x] \,\big].$$

Whenever $y \succ x \succeq b \succ a$, we have $y, b \in \Omega$, hence $x \in \Omega$ too, hence $y \in \Omega_1$ and $y > x$. If $a = \sup\Omega$, then $y \succ a$ immediately. Otherwise, we have $b \in \Omega_1$, $b > a$, and $x \geq b$; therefore, $y > a$, hence $y \succ a$ again. Thus, $\succ$ is an interval order. Condition (3.1b) holds vacuously; (3.1a), because the left hand side condition can only hold when $X = \{x^k\}_k \subseteq \Omega_1$ and $x^{k+1} > x^k$ for each $k \in \mathbb{N}$, while $\sup\{x^k\}_k \in \Omega \setminus \Omega_1$ in this case. However, $M(A, \succ) = \emptyset$.

We see that Theorem 5.10 does not hold without the regularity of preferences, whereas Theorem 5.11 does not hold without the regularity of $A$.

**Example 5.13.** Let $A := [-1, 1] \subset \mathbb{R}$. We define a preference relation on $A$:

$$y \succ x \rightleftharpoons \big[\, [y \in\, ]-1,0[\; \&\; x \in\, ]-1,0[\cup\{1\} \;\&\; y < x] \;\text{ or }\; [y \in\, ]0,1[\; \&\; x \in \{-1\}\cup]0,1[\; \&\; y > x] \,\big].$$

Clearly, $A$ is a regular chain while $\succ$ is a regular strict order. Conditions (5.1) are also easy to check. Meanwhile, $M(A, \succ) = \emptyset$.

Thus, the assumption in Theorems 5.10 and 5.11 that $\succ$ is an interval order is essential.

**Theorem 5.14.** *An ordering $\succ$ on a chain-complete, regular poset $A$ has the property that $M(X, \succ) \in \mathfrak{C}_A$ for every $X \in \mathfrak{C}_A$ if and only if $\succ$ is strongly mono-$\omega$-transitive.*

Immediately follows from Theorem 3.11 and Proposition 5.6

**Example 5.15.** We define $A$ and $\Omega$ in the same way as in Example 5.9, and consider an ordering $\succ$ on $A$ represented by this utility function $u\colon A \to \mathbb{R}$, $u(x) := 1$ for $x \in \Omega$ and $u(x) := 0$ otherwise. Clearly, $\succ$ is regular and strongly mono-$\omega$-transitive; however, $M(A, \succ) = \Omega$, which is not chain-complete. Thus, Theorem 5.14 does not hold without the regularity of $A$ even for a regular ordering.

## 5.3 Lattices

Under appropriate regularity conditions, Theorems 4.6, 4.7, 4.8, 4.10, 4.11, 4.13, 4.14, and 4.19 can be reformulated with chain-transitivity replaced with mono-$\omega$-transitivity. What is much more interesting, sometimes stronger results can be obtained for regular preferences.

**Theorem 5.16 (Kukushkin, 2012, Theorem 3.2).** *Let $A$ be a lattice and $\succ$ be a regular ordering on $A$ satisfying* (4.2a). *Then $\succ$ has the property that $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{L}_A$ if and only if it is mono-$\omega$-transitive.*

**Theorem 5.17 (Kukushkin, 2012, Theorem 3.3).** *Let $A$ be a lattice and $\succ$ be a regular ordering on $A$ satisfying* (4.3a). *Then $\succ$ has the property that $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{L}_A$ if and only if it is mono-$\omega$-transitive.*

**Corollary.** *A $\wedge$-QSM function, as well as a $\vee$-QSM function, on a complete lattice attains its maximum if it attains a maximum on every subcomplete chain.*

**Remark.** Example 4.12 presents a regular ordering on a regular lattice; therefore, conditions (4.2a) or (4.3a) cannot be replaced even with the conjunction of (4.3b) and (4.2b).

**Theorem 5.18.** *Let $A$ be a lattice and $\succ$ be a regular ordering on $A$ satisfying* (5.2b), (3.3a), *and* (4.2d). *Then $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{L}_A$.*

*Proof.* Exactly as in the proof of Theorem 4.13, we define $X^{\downarrow} := \{x \in X \mid \forall y \in X\, \exists z \in [\bigwedge X, x]\, [z \succeq y]\}$. Lemma 4.13.2 remains valid.

Let $X \in \mathfrak{L}_A$; if $M(X, \succ) \neq \emptyset$, then we are already home. Supposing the contrary, we must have an optimizing sequence $\langle y^h \rangle_{h \in \mathbb{N}}$ in $X$. We recursively define a sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ such that

$$\forall k \in \mathbb{N}\, \big[x^k \in X^{\downarrow}\big]; \tag{5.4a}$$

$$\forall k \in \mathbb{N}\, \big[x^{k+1} \in M([\textstyle\bigwedge X, x^k], \underset{>}{\succeq})\big]; \tag{5.4b}$$

$$\forall k \in \mathbb{N}\, \big[x^{k+1} \succ y^k\big]; \tag{5.4c}$$

$$\forall k, h \in \mathbb{N}\, \big[k > h \Rightarrow x^k \underset{<}{\succeq} x^h\big]. \tag{5.4d}$$

First, $x^0 := \bigvee X$.

Let $x^k$ have been defined. Since $M(X, \succ) = \emptyset$, we may pick $h \in \mathbb{N}$ such that $y^h \succ x^k$ and $h > k$, hence $y^h \succ y^k$. The induction hypothesis (5.4a) implies the existence of $x \in [\bigwedge X, x^k]$ such that $x \succeq y^h$, hence $x \underset{<}{\succeq} x^k$. Applying Proposition 3.7 to $\succeq$, $[\bigwedge X, x^k]$ and $x$, we obtain $x^{k+1} \in M([\bigwedge X, x^k], \underset{>}{\succeq})$ such that $x^{k+1} \underset{<}{\succeq} x^k$ and $x^{k+1} \succ y^k$. By Lemma 4.13.2, we have $x^{k+1} \in X^{\downarrow}$; condition (5.4d) also continues to hold for $x^{k+1}$.

Finally, we set $x^{\infty} := \inf_k x^k$. By (5.2b), we have $x^{\infty} \succ x^k$ for each $k \in \mathbb{N}$. Since $X$ is subcomplete, we have $x^{\infty} \in X$. Since $x^{k+1} \succ y^k$ for each $k$, we have $x^{\infty} \in M(X, \succ)$, which contradicts our assumption $M(X, \succ) = \emptyset$. $\square$

**Theorem 5.19.** *Let $A$ be a lattice and $\succ$ be a regular ordering on $A$ satisfying* (5.2a), (3.3b), *and* (4.3d). *Then $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{L}_A$.*

The proof is dual to that of Theorem 5.18.

**Theorem 5.20.** *Let $A$ be a lattice and $\succ$ be a regular ordering on $A$ satisfying* (4.11a) *and such that every upper contour set is conditionally chain-subcomplete. Then $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{L}_A$.*

**Remark.** Technically, this result is a bit stronger than Veinott's Theorem 6.2: we do not assume that $\succ$ can be represented by a real-valued function. However, the proof is essentially the same. It is reproduced here because Veinott's lectures have never been published.

*Proof.* As usual, we start with an auxiliary statement.

**Lemma 5.20.1.** *If $X$ is a sublattice of $A$, $x \in M(X, \succneqq)$, and $X \ni y \succ x$, then $y \wedge x \succeq x$.*

*Proof of Lemma 5.20.1.* An assumption that $x \succ y \wedge x$ would imply $y \vee x \succeq x$ by (4.11a). Since $x$ is maximal, this is only possible if $y \vee x = x$, hence $y = y \wedge x$: a contradiction. $\square$

Let $X \in \mathfrak{L}_A$; if $M(X, \succ) \neq \emptyset$, then we are already home. Supposing the contrary, we must have an optimizing sequence $\langle y^k \rangle_{k \in \mathbb{N}}$ in $X$. Taking into account Propositions 3.7 and 3.3, we may assume $y^k \in M(X, \succneqq)$ for each $k \in \mathbb{N}$.

Now we recursively define a "double sequence" $\langle x^{k,h} \rangle_{k,h \in \mathbb{N}}$: $x^{k,0} := y^k$; $x^{k,h+1} := y^k \wedge x^{k+1,h}$.

**Lemma 5.20.2.** *There hold these conditions:*

$$\forall k, h \in \mathbb{N} \left[ x^{k,h} \geq x^{k,h+1} \right]; \tag{5.5a}$$

$$\forall k, h \in \mathbb{N} \left[ x^{k,h} \succeq y^k \right]; \tag{5.5b}$$

$$\forall k', k, h \in \mathbb{N} \left[ k' > k \Rightarrow x^{k',h} \geq x^{k,k'-k+h} \right]. \tag{5.5c}$$

*Proof of Lemma 5.20.2.* In each case, we argue by induction. For (5.5a), it is induction in $h$ (for all $k$ simultaneously) based on the recursive definition. For (5.5b), it is again induction in $h$ based on both the recursive definition and Lemma 5.20.1 since $y^k \in M(X, \succneqq)$. To prove (5.5c), we rewrite it as $x^{k+d,h-d} \geq x^{k,h}$ and argue by induction in $d$. $\square$

For each $k \in \mathbb{N}$, we set $z^k := \inf_h x^{k,h}$. By (5.5b), (5.5a), and chain-subcompleteness, we have $z^k \in \mathrm{Up}(y^k)$ for each $k$. By (5.5c), $z^{k+1} \geq z^k$ for each $k$. Then we set $z^\infty := \sup_k z^k$. Whenever $k' > k$, we have $\mathrm{Up}(y^{k'}) \subset \mathrm{Up}(y^k)$; therefore, $z^\infty \in \mathrm{Up}(y^k)$ for all $k$, hence $z^\infty \in M(X, \succ)$. $\square$

**Example 5.21.** Let $A := \big(\{n/(n+1)\}_{n \in \mathbb{N}} \cup \{1\}\big) \times \big(\{0\} \cup \{1/(n+1)\}_{n \in \mathbb{N}}\big) \subset \mathbb{R}^2$, exactly as in Example 4.12; $A$ with the order induced from $\mathbb{R}^2$ is a regular, complete lattice. We define $u\colon A \to \mathbb{R}$ by $u(1, x_2) = u(x_1, 0) := 0$ and $u(n_1/(n_1+1), 1/(n_2+1)) := U(n_1, n_2)$, where $U(k,k) := k$ while $U(k+h, k) = U(k, k+h) := k + 1/(h+1)$ $(h > 0)$. It is easily checked that $u$ satisfies (4.11a). The ordering $\succ$ represented by $u$ is regular and strongly mono-$\omega$-transitive, hence $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$ by Theorem 3.9. On the other hand, $\sup_{x \in A} u(x) = +\infty$, hence $M(A, \succ) = \emptyset$.

Thus, the chain-subcompleteness of upper contour sets in Theorem 5.20 cannot be replaced even with (3.3), unlike in every other result here.

## 5.4 An impossibility result

We start with an adaptation of the basic concepts from Kukushkin (2008a) to our current situation. Recall that $\mathbb{N} = \{0, 1, \dots\}$ is the chain of natural numbers starting from zero. An *abstract configuration* $C$ consists of $\operatorname{Dom} C \subseteq \mathbb{N}$; $C_=, C_{\neq}, C_>, C_{\geq}, C_{\triangleright}, C_{\ntriangleright} \subseteq \operatorname{Dom} C \times \operatorname{Dom} C$; $C_{\wedge}, C_{\not\wedge}, C_{\vee}, C_{\not\vee} \subseteq \operatorname{Dom} C \times \operatorname{Dom} C \times \operatorname{Dom} C$; and $C_{\bigwedge}, C_{\not\bigwedge}, C_{\bigvee}, C_{\not\bigvee} \subseteq (\operatorname{Dom} C)^{\mathbb{N}}$, where $(\operatorname{Dom} C)^{\mathbb{N}}$ means the set of mappings $\mathbb{N} \to \operatorname{Dom} C$, i.e., sequences in $\operatorname{Dom} C$.

Let $\succ$ be a binary relation on a poset $A$ and $C$ be an abstract configuration. A *realization* of $C$ in $A$ for $\succ$ is a mapping $\mu : \operatorname{Dom} C \to A$ such that: $\mu(k') = \mu(k)$ whenever $(k', k) \in C_=$; $\mu(k') \neq \mu(k)$ whenever $(k', k) \in C_{\neq}$; $\mu(k') > \mu(k)$ whenever $(k', k) \in C_>$; $\mu(k') \geq \mu(k)$ whenever $(k', k) \in C_{\geq}$; $\mu(k') \succ \mu(k)$ whenever $(k', k) \in C_{\triangleright}$; $\mu(k') \not\succ \mu(k)$ whenever $(k', k) \in C_{\ntriangleright}$; $\mu(k'') = \mu(k') \wedge \mu(k)$ whenever $(k'', k', k) \in C_{\wedge}$; $\mu(k'') \neq \mu(k') \wedge \mu(k)$ whenever $(k'', k', k) \in C_{\not\wedge}$; $\mu(k'') = \mu(k') \vee \mu(k)$ whenever $(k'', k', k) \in C_{\vee}$; $\mu(k'') \neq \mu(k') \vee \mu(k)$ whenever $(k'', k', k) \in C_{\not\vee}$; $\mu(\nu(0)) = \bigwedge\{\mu(\nu(k))\}_{k>0}$ whenever $\nu \in C_{\bigwedge}$; $\mu(\nu(0)) \neq \bigwedge\{\mu(\nu(k))\}_{k>0}$ whenever $\nu \in C_{\not\bigwedge}$; $\mu(\nu(0)) = \bigvee\{\mu(\nu(k))\}_{k>0}$ whenever $\nu \in C_{\bigvee}$; $\mu(\nu(0)) \neq \bigvee\{\mu(\nu(k))\}_{k>0}$ whenever $\nu \in C_{\not\bigvee}$.

Many natural properties of binary relations, including all those considered in this section, can be expressed as the impossibility to realize a certain configuration (or every configuration from a certain set). For example, the irreflexivity of $\succ$ is equivalent to the impossibility to realize a configuration with $\operatorname{Dom} C = \{0\}$, $C_{\triangleright} = \{(0,0)\}$, and other sets empty; transitivity, with $\operatorname{Dom} C = \{0, 1, 2\}$, $C_{\triangleright} = \{(1,0), (2,1)\}$ and $C_{\ntriangleright} = \{(2,0)\}$. If we only want transitivity on every chain, we can prohibit the realization of several similar configurations, indicating all possible order relations between $\mu(0)$, $\mu(1)$, and $\mu(2)$. To define acyclicity, we have to prohibit the realization of each of a countable set of configurations parameterized with $m \in \mathbb{N}$: $\operatorname{Dom} C^{(m)} = \{0, \dots, m+1\}$, $C_{\triangleright}^{(m)} = \{(1,0), (2,1), \dots, (m+1, m)\}$ and $C_=^{(m)} = \{(0, m+1)\}$. Condition (5.2a), e.g., is equivalent to the impossibility to realize this configuration: $\operatorname{Dom} C = \mathbb{N}$, $C_> = C_{\triangleright} = \{(k+1, k)\}_{k=1,2,\dots}$, $C_{\ntriangleright} = \{(0,1)\}$, and $C_{\bigvee} = \{\nu^0\}$, where $\nu^0(k) = k$.

Obviously, if a configuration admits no realization in a poset/lattice $A$, it also admits no realization in any subcomplete subset/sublattice $A' \subset A$. It seems natural, therefore, to use such prohibitions when trying to characterize properties of binary relations which are inherited in this sense (like the existence of a maximizer in every subcomplete chain).

**Theorem 5.22.** *There exists no set $\mathcal{N}$ of abstract configurations such that a binary relation $\succ$ on a subset $A$ of a finite-dimensional Euclidean space (with its natural partial order) admits a maximizer in every nonempty subcomplete chain in $A$ if and only if no configuration $C \in \mathcal{N}$ admits a realization in $A$ for $\succ$.*

*Proof.* Supposing to the contrary that such a collection of configurations $\mathcal{N}$ exists, we set $A := [0,1] \subset \mathbb{R}$ and consider a mapping $\varphi \colon A \to \mathbb{C}$ defined by $\varphi(x) := \mathrm{e}^{2\pi \mathrm{i} x}$ (where $\mathrm{i} = \sqrt{-1}$, i.e., $\varphi(A)$ is a circle embedded into the plane of complex numbers). Then we define a binary relation on $A$ by $y \succ x \rightleftharpoons \varphi(y) = \mathrm{e}^{\mathrm{i}} \cdot \varphi(x)$. Since $A$ is complete itself and there is no maximizer for $\succ$ on $A$, there must be $C \in \mathcal{N}$ admitting a realization $\mu$ in $A$.

We pick $x^* \in A \setminus [\mu(\mathbb{N}) \cup \{0,1\}]$, define $B := [0, x^*[ \cup ]x^*, 1] \subset A$, and consider the restriction of $\succ$ to $B$. Clearly, the same $\mu$ is a realization of $C$ in $B$; therefore, there must exist $X \in \mathfrak{C}_B$ such that $M(X, \succ) = \emptyset$. Picking $x^0 \in X$, we notice that there is a unique $x^1 \in A$ such that $x^1 \succ x^0$; since $M(X, \succ) = \emptyset$, we must have $x^1 \in X$ too. Repeating this observation, we obtain an infinite sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ such that and $x^k \in X$ and $x^{k+1} \succ x^k$ for each $k \in \mathbb{N}$. The last relation meaning that $\varphi(x^{k+1}) = \mathrm{e}^{\mathrm{i}} \cdot \varphi(x^k)$, the sequence $\langle \varphi(x^k) \rangle_{k \in \mathbb{N}}$ is dense in $\varphi(A)$ by the Jacobi theorem (see, e.g., Billingsley, 1965), hence the sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ is dense in $A$, hence the set $N^- := \{k \in \mathbb{N} \mid x^k \in [0, x^*[\}$ is not empty. Defining $x^+ := \sup\{x^k \mid k \in N^-\}$, we have $x^+ < x^*$ since $X$ is subcomplete; on the other hand, there must be $k \in \mathbb{N}$ such that $x^k \in ]x^+, x^*[$ since $\langle x^k \rangle_{k \in \mathbb{N}}$ is dense in $A$. Thus, $k \in N^-$ whereas $x^k > x^+$, which contradiction proves the theorem. $\square$

This proof is based on the same construction as the proof of Theorem 1 in Kukushkin (2008a). However, the statement is much weaker here: we did not consider "disjunctive forms." Whether Theorem 5.22 would survive such an extension of the class of admissible conditions remains unclear. What is certain is that this construction is insufficient to prove the stronger statement: there is a continuum of abstract configurations such that every one of them can be realized in $A$, whereas for every $x^* \in ]0, 1[$, there is a configuration in the list that cannot be realized in $[0, x^*[ \cup ]x^*, 1]$.

**Remark.** Since $B \subset A$ in the proof, every configuration realizable in $B$ can be realized in $A$. Therefore, the same proof would work if "positive" requirements, i.e., that every configuration from a certain list must have a realization, were also allowed, as in Kukushkin (2008a).

# 6 Parametric preferences

## 6.1 Single crossing

We consider a *parametric family* $\langle \overset{s}{\succ} \rangle_{s \in S}$ of binary relations on $X$. To simplify notations, we define the *best response correspondence*:

$$\mathcal{R}(s) := M(X, \overset{s}{\succ}).$$

Let $X$ and $S$ be posets. A *monotone selection* from $\mathcal{R}$ is an increasing mapping $r \colon S \to X$ such that $r(s) \in \mathcal{R}(s)$ for every $s \in S$; "increasing" here means that $r(s'') \geq r(s')$ whenever $s', s'' \in S$ and $s'' \geq s'$.

Although monotone comparative statics is most often expressed in terms of comparison between $\mathcal{R}(s)$ and $\mathcal{R}(s')$ whenever $s$ and $s'$ are comparable in the order on $S$, monotone selections from the best response correspondence sometimes become the main object of interest (Edlin and Shannon, 1998; Strulovici and Weber, 2010).

The *single crossing* condition (on a parametric family $\langle \succ^s \rangle_{s \in S}$) is the conjunction of these two:

$$\forall x, y \in X\, \forall s, s' \in S\, \big[[s' > s\ \&\ y \succ^s x\ \&\ y > x] \Rightarrow y \succ^{s'} x\big]; \tag{6.1a}$$

$$\forall x, y \in X\, \forall s, s' \in S\, \big[[s' > s\ \&\ y \succ^{s'} x\ \&\ y < x] \Rightarrow y \succ^s x\big]. \tag{6.1b}$$

This definition is equivalent to Milgrom and Shannon's (1994) if every $\succ^s$ is an ordering represented by a numeric function.

When relations $\succ^s$ are orderings, (6.1) is sufficient for $\mathcal{R}$ to be *ascending* (Veinott, 1989) provided $X$ is a chain. Moreover, it is also necessary if we want the ascendence to hold, say, for all finite subsets of $X$. The sufficiency perishes if we allow a broader class of preferences.

**Example 6.1.** Let $X := \{0, 1, 2, 3\}$, $S := \{0, 1\}$ (both with natural orders), and relations $\succ^s$ be defined by: $0 \succ^0 1$ and $3 \succ^1 1$. Clearly, $R(0) = \{0, 2, 3\}$ while $R(1) = \{0, 1, 3\}$. Conditions (6.1) hold vacuously. There are plenty of monotone selections. On the other hand, $\mathcal{R}$ is not even *weakly ascending* (Veinott, 1989): $2 \in R(0) \setminus R(1)$ whereas $1 \in R(1) \setminus R(0)$. Both relations $\succ^0$ and $\succ^1$ are semiorders (though not orderings, naturally).

For a family of preference relations defined by $\varepsilon$-improvements

$$y \succ^s x \rightleftharpoons u(y, s) > u(x, s) + \varepsilon$$

($\varepsilon > 0$), both conditions (6.1) hold if $u(x, s)$ satisfies Topkis's (1979) *increasing differences* condition:

$$\forall x, y \in X\, \forall s, s' \in S\, \big[[s' \geq s\ \&\ y \geq x] \Rightarrow u(y, s') - u(x, s') \geq u(y, s) - u(x, s)\big]. \tag{6.2}$$

When $X$ and $S$ are chains, the condition is equivalent to the *supermodularity* of $u$ (as a function on the lattice $X \times S$).

**Remark.** It is not difficult to see that the semiorders in Example 6.1 cannot be generated by the construction just described. Otherwise, $\mathcal{R}$ would have been *weakly* ascending.

We say that a binary relation $\succ$ has the *strong NM-property* on a set $X$ if

$$\forall \{x^0, \ldots, x^m\} \subseteq X \setminus M(X, \succ)\, \exists y \in M(X, \succ)\, \forall k \in \{0, \ldots, m\}\, [y \succ x^k]. \tag{6.3}$$

Clearly, (6.3) implies (2.3); besides, (6.3) is implied by the conjunction of (2.4) and (2.2).

The strong NM-property is connected with "being an interval order" in the same way that the "revealed preference" property (2.4) is connected with "being an ordering."

**Proposition 6.2.** *Let $\succ$ be a binary relation on a set $A$. Then $\succ$ has the strong NM-property on every finite subset $X \in \mathfrak{B}_A$ if and only if it is an interval order.*

*Proof.* Let $\succ$ be an interval order and $\{x^0, \ldots, x^m\} \subseteq X \setminus M(X, \succ)$. The NM-property on every finite subset is obvious, hence there are $\{y^0, \ldots, y^m\} \subseteq M(X, \succ)$ such that $y^k \succ x^k$ for each $k = 0, \ldots, m$. Invoking Theorem B, we obtain a representation (2.6). Now let $u^-(y^h) \geq u^-(y^k)$ for all $k$; obviously, $y^h \succ x^k$ for each $k$.

Conversely, if $z \succ y \succeq x \succ w$ and $\succ$ has the strong NM-property on $X := \{w, x, y, z\}$, then both $y$ and $w$ do not belong to $M(X, \succ)$, hence $z \in M(X, \succ)$ and $z \succ w$. □

**Proposition 6.3.** *Let $\succ$ be a binary relation on a poset $A$. Then $\succ$ has the strong NM-property on every $X \in \mathfrak{C}_A$ if and only if it is chain-transitive and is an interval order on every chain.*

*Proof.* The sufficiency is proven with a reference to Theorem 3.9 combined with the same argument as in the proof of Proposition 6.2. The necessity of chain-transitivity immediately follows from Theorem 3.9; the necessity for $\succ$ to be an interval order is proven in the same way as in Proposition 6.2. □

For the main result of this section, we need a strengthened version of chain-transitivity:

$$\forall X \in \mathfrak{B}_A \Big[ \big( X \text{ is a chain } \& \ \exists \sup X \ \& \ \sup X^{\rightharpoondown} = \sup X \ \& \\ \forall x, y \in X^{\rightharpoondown} [y > x \Rightarrow y \succ x] \big) \Rightarrow \forall x \in X^{\rightharpoondown} [\sup X \succ x] \Big]; \quad (6.4a)$$

$$\forall X \in \mathfrak{B}_A \Big[ \big( X \text{ is a chain } \& \ \exists \inf X \ \& \ \inf X^{\leftharpoondown} = \inf X \ \& \\ \forall x, y \in X^{\leftharpoondown} [y < x \Rightarrow y \succ x] \big) \Rightarrow \forall x \in X^{\leftharpoondown} [\inf X \succ x] \Big]. \quad (6.4b)$$

Obviously, each condition (6.4) implies the corresponding condition (3.2), hence (5.2) as well. Example 6.4 shows that the converse is generally wrong. However, it becomes valid under a regularity condition.

**Example 6.4.** Let $\Omega$, $A$, and $\Omega_1$ be the same as in Example 5.12. We set $\Omega_\infty := \Omega \setminus \Omega_1$ and define a preference ordering (actually, a linear order) on $A$:

$$y \succ x \rightleftharpoons \big[ [y \neq \sup \Omega = x] \ \text{ or } \ [y \in \Omega_\infty \ \& \ x \in \Omega_1] \ \text{ or } \\ [y \in \Omega_1 \ \& \ x \in \Omega_1 \ \& \ y > x] \ \text{ or } \ [y \in \Omega_\infty \ \& \ x \in \Omega_\infty \ \& \ y < x] \big].$$

Condition (3.2b), as well as (6.4b) for that matter, hold vacuously; (3.2a), because the left hand side condition can only hold when $X$ either is a finite subset of $\Omega_1$, or contains just one point $x \in \Omega_\infty$, which is $\max X$. On the other hand, (6.4a) is violated for $X = \Omega_1$, since $\sup \Omega_1 = \sup \Omega$.

**Proposition 6.5.** *If $\succ$ is a regular strict order on a poset $A$, then each condition* (5.2) *is equivalent to the corresponding condition* (6.4)*. The same equivalences hold if $A$ is regular.*

The proof is essentially the same as in Propositions 5.5 and 5.6.

## 6.2 Monotone selections

Let $S$ be a poset. We call a subset $S' \subseteq S$ *interval-like* if $s \in S'$ whenever $s' < s < s''$ and $s', s'' \in S'$. The intersection of any number of interval-like subsets is interval-like too.

**Proposition 6.6.** *Let a parametric family* $\langle \succcurlyeq^s \rangle_{s \in S}$ *of binary relations on a chain* $X$ *satisfy both conditions* (6.1). *Let every* $\succcurlyeq^s$ *have the NM-property on* $X$. *Then the set* $\{s \in S \mid x \in \mathcal{R}(s)\}$, *for every* $x \in X$, *is interval-like.*

*Proof.* Suppose the contrary: $s' < s < s''$ and $x \in \mathcal{R}(s') \cap \mathcal{R}(s'')$, but $x \notin \mathcal{R}(s)$. By (2.3), we can pick $x^* \in \mathcal{R}(s)$ such that $x^* \succ^s x$. If $x^* > x$, we have $x^* \succ^{s''} x$ by (6.1a), contradicting the assumed $x \in \mathcal{R}(s'')$. If $x^* < x$, we have $x^* \succ^{s'} x$ by (6.1b) with the same contradiction. $\square$

Let $S$ be a chain, $S' \subseteq S$ be interval-like, and $s \in S \setminus S'$; then either $s > s'$ for all $s' \in S'$, or $s' > s$ for all $s' \in S'$. We write $s > S'$ in the first case, and $s < S'$ in the second.

**Theorem 6.7.** *Let* $X$ *and* $S$ *be chains, and* $X$ *be complete. Let a parametric family* $\langle \succcurlyeq^s \rangle_{s \in S}$ *of transitive binary relations on* $X$ *satisfy single crossing conditions* (6.1). *Let every* $\succcurlyeq^s$ *satisfy both conditions* (6.4) *and have the strong NM-property on* $X$. *Let* $s^0 \in S$ *and* $x^0 \in \mathcal{R}(s^0)$. *Then there exists a monotone selection* $r$ *from* $\mathcal{R}$ *on* $S$ *such that* $r(s^0) = x^0$.

*Proof.* Let $\Lambda$ be a well ordered set of a cardinality greater than that of $S$. By transfinite recursion, we construct a chain of subsets $\Sigma(\alpha) \subseteq S$ ($\alpha \in \Lambda$) such that $\Sigma(\beta) \subseteq \Sigma(\alpha)$ whenever $\beta < \alpha$, with an equality only possible when $\Sigma(\beta) = S$; we also construct increasing mappings ("partial monotone selections") $r_\alpha \colon \Sigma(\alpha+1) \to X$ such that $r_\alpha(s) \in \mathcal{R}(s)$ for every $s \in \Sigma(\alpha+1)$ and $r_\alpha|_{\Sigma(\beta+1)} = r_\beta$ whenever $\beta < \alpha$. Since the cardinality of $\Lambda$ is greater than that of $S$, there must be $\bar\alpha \in \Lambda$ such that $\Sigma(\bar\alpha) = \Sigma(\bar\alpha + 1) = S$ hence $r_{\bar\alpha} \colon S \to X$ is a monotone selection from $\mathcal{R}$.

First, we set $\Sigma(0) := \emptyset$. The recursive definition of $\Sigma(\alpha) \subseteq S$ for $\alpha > 0$ uses a number of auxiliary constructions recursively defined whenever $\Sigma(\alpha) \subset S$, namely $\sigma(\alpha) \in S$, $S(\alpha) \subseteq S$, $\xi(\alpha) \in X$, and $\vartheta(\alpha) \in \{-1, 0, 1\}$ such that:

$$\sigma(\alpha) \in S(\alpha); \tag{6.5a}$$

$$S(\alpha) \text{ is interval-like}; \tag{6.5b}$$

$$\forall s \in S(\alpha)\, \big[\xi(\alpha) \in \mathcal{R}(s)\big]; \tag{6.5c}$$

$$\forall \beta < \alpha\, \big[S(\alpha) \cap S(\beta) = \emptyset\big]; \tag{6.5d}$$

$$\forall s \in S\, \big[[\xi(\alpha) \in \mathcal{R}(s) \;\&\; s < S(\alpha)] \Rightarrow \exists \beta < \alpha\, [s \in S(\beta) \;\text{ or }\; s < \sigma(\beta) < \sigma(\alpha)]\big]; \tag{6.5e}$$

$$\forall s \in S\, \big[[\xi(\alpha) \in \mathcal{R}(s) \;\&\; s > S(\alpha)] \Rightarrow \exists \beta < \alpha\, [s \in S(\beta) \;\text{ or }\; s > \sigma(\beta) > \sigma(\alpha)]\big]; \tag{6.5f}$$

$$\forall \beta < \alpha\, \big[[\sigma(\alpha) < \sigma(\beta) \Rightarrow \xi(\alpha) < \xi(\beta)] \;\&\; [\sigma(\alpha) > \sigma(\beta) \Rightarrow \xi(\alpha) > \xi(\beta)]\big]; \tag{6.5g}$$

$\vartheta(\alpha) \le 0 \Rightarrow$
$$\forall \beta < \alpha \left[ \xi(\alpha) \succcurlyeq^{\sigma(\alpha)} \xi(\beta) \;\text{ or }\; \sigma(\beta) > \sigma(\alpha) \;\text{ or }\; \exists \gamma < \beta \left( \sigma(\gamma) \in [\sigma(\beta), \sigma(\alpha)] \right) \right]; \quad (6.5h)$$

$\vartheta(\alpha) \ge 0 \Rightarrow$
$$\forall \beta < \alpha \left[ \xi(\alpha) \succcurlyeq^{\sigma(\alpha)} \xi(\beta) \;\text{ or }\; \sigma(\beta) < \sigma(\alpha) \;\text{ or }\; \exists \gamma < \beta \left( \sigma(\gamma) \in [\sigma(\alpha), \sigma(\beta)] \right) \right]; \quad (6.5i)$$

$$\vartheta(\alpha) = -1 \Rightarrow \forall s < S(\alpha) \, \exists \beta < \alpha \, [\vartheta(\beta) \le 0 \;\&\; s < \sigma(\beta) < \sigma(\alpha)]; \quad (6.5j)$$

$$\vartheta(\alpha) = 1 \Rightarrow \forall s > S(\alpha) \, \exists \beta < \alpha \, [\vartheta(\beta) \ge 0 \;\&\; s > \sigma(\beta) > \sigma(\alpha)]. \quad (6.5k)$$

To start with, we set $\sigma(0) := s^0 \in S$, $\xi(0) := x^0 \in \mathcal{R}(\sigma(0))$, $\vartheta(0) := 0$, and $S(0) := \{s \in S \mid \xi(0) \in \mathcal{R}(s)\}$. Now (6.5a), (6.5c), (6.5e) and (6.5f) for $\alpha = 0$ immediately follow from the definitions; (6.5b), from Proposition 6.6; (6.5d), (6.5g), (6.5h), (6.5i), (6.5j), and (6.5k) hold vacuously.

Let $\alpha \in \Lambda \setminus \{0\}$, and let $\sigma(\beta) \in S$, $S(\beta) \subseteq S$, $\xi(\beta) \in X$, and $\vartheta(\beta)$ satisfying (6.5) have been defined for all $\beta < \alpha$. First of all, we define $\Sigma(\alpha) := \bigcup_{\beta<\alpha} S(\beta)$. For every $s \in \Sigma(\alpha)$, there is a unique, by (6.5d), $\varkappa(s) \in \Lambda$ such that $\varkappa(s) < \alpha$ and $s \in S(\varkappa(s))$. By (6.5c), $r := \xi \circ \varkappa$ is a selection from $\mathcal{R}$ on $\Sigma(\alpha)$. The conditions (6.5b) and (6.5g) imply that $r$ is increasing. If $\Sigma(\alpha) = S$, then we already have a monotone selection, so we effectively finish the process, setting $S(\alpha) := \emptyset$, hence $S(\beta) = \emptyset$ and $\Sigma(\beta) = S$ for all $\beta > \alpha$; there is no need to define $\sigma(\alpha)$, $\xi(\alpha)$, and $\vartheta(\alpha)$ in this case.

Otherwise, we pick $s^* \in S \setminus \Sigma(\alpha)$ arbitrarily and define $\Lambda^- := \{\beta \in \Lambda \mid \beta < \alpha \;\&\; \sigma(\beta) < s^*\}$ and $\Lambda^+ := \{\beta \in \Lambda \mid \beta < \alpha \;\&\; \sigma(\beta) > s^*\}$. Since $\alpha > 0$, both $\Lambda^-$ and $\Lambda^+$ cannot be empty; if one of them is empty, everything related to it in the following should be just ignored. We also define $I := \{s \in S \setminus \Sigma(\alpha) \mid \forall \beta' \in \Lambda^- \, \forall \beta'' \in \Lambda^+ \, [\sigma(\beta') < s < \sigma(\beta'')]\} \, [\ni s^*]$; (6.5a) and (6.5b) ensure that $I$ is an interval.

Supposing $\Lambda^- \neq \emptyset$, we define $x^- := \sup\{\xi(\beta)\}_{\beta\in\Lambda^-}$ (its existence is ensured by the completeness of $X$), $\Lambda^\uparrow := \{\beta \in \Lambda^- \mid \forall \gamma \in \Lambda^- \, [\gamma \ge \beta \;\text{ or }\; \sigma(\gamma) < \sigma(\beta)]\}$, and $\Xi^\uparrow := \{\xi(\beta)\}_{\beta\in\Lambda^\uparrow}$.

**Lemma 6.7.1.** *$\vartheta(\beta) \le 0$ whenever $\beta \in \Lambda^\uparrow$.*

Immediately follows from condition (6.5k) for $\beta$ and $s^*$.

**Lemma 6.7.2.** *For every $\gamma \in \Lambda^-$, there is $\beta \in \Lambda^\uparrow$ such that $\sigma(\beta) \ge \sigma(\gamma)$.*

*Proof of Lemma 6.7.2.* We define $\mathrm{B} := \{\gamma' \in \Lambda^- \mid \gamma' < \gamma \;\&\; \sigma(\gamma') > \sigma(\gamma)\}$. If $\mathrm{B} = \emptyset$, then $\gamma \in \Lambda^\uparrow$. Otherwise, $\min \mathrm{B} \in \Lambda^\uparrow$. □

**Lemma 6.7.3.** *$x^- = \sup \Xi^\uparrow$.*

Immediately follows from Lemma 6.7.2 and (6.5g).

**Lemma 6.7.4.** *For every $s \in I$, there holds $x^- \succcurlyeq^s \xi(\beta)$ for every $\beta \in \Lambda^\uparrow$, except $\beta = \max \Lambda^\uparrow$ if it exists (then $x^- = \xi(\max \Lambda^\uparrow)$).*

*Proof of Lemma 6.7.4.* Let $\beta, \beta' \in \Lambda^{\uparrow}$ and $\beta' > \beta$; then $\sigma(\beta') > \sigma(\beta)$ by definition and $\xi(\beta') > \xi(\beta)$ by (6.5g). Lemma 6.7.1 and (6.5h) for $\beta'$ imply $\xi(\beta') \succcurlyeq^{\sigma(\beta')} \xi(\beta)$ because the third disjunctive term in (6.5h) is incompatible with $\beta \in \Lambda^{\uparrow}$. Therefore, $\xi(\beta') \succcurlyeq^{s} \xi(\beta)$ by (6.1a). We see that condition (6.4a) applies to $\Xi^{\uparrow}$ and $\succcurlyeq^{s}$, hence $x^{-} \succcurlyeq^{s} \xi(\beta)$. $\square$

Supposing $\Lambda^{+} \neq \emptyset$, we define $x^{+} := \inf\{\xi(\beta)\}_{\beta\in\Lambda^{+}}$, $\Lambda^{\downarrow} := \{\beta \in \Lambda^{+} \mid \vartheta(\beta) \geq 0 \;\&\; \forall\gamma \in \Lambda^{+}\, [\gamma \geq \beta \;\text{ or }\; \sigma(\gamma) > \sigma(\beta)]\}$, and $\Xi^{\downarrow} := \xi(\Lambda^{\downarrow})$.

**Lemma 6.7.5.** $\vartheta(\beta) \geq 0$ *whenever* $\beta \in \Lambda^{\downarrow}$.

**Lemma 6.7.6.** *For every* $\gamma \in \Lambda^{+}$*, there is* $\beta \in \Lambda^{\downarrow}$ *such that* $\sigma(\beta) \leq \sigma(\gamma)$.

**Lemma 6.7.7.** $x^{+} = \inf \Xi^{\downarrow}$.

**Lemma 6.7.8.** *For every* $s \in I$*, there holds* $x^{+} \succcurlyeq^{s} \xi(\beta)$ *for every* $\beta \in \Lambda^{\downarrow}$*, except* $\beta = \max\Lambda^{\downarrow}$ *if it exists* (*then* $x^{+} = \xi(\max\Lambda^{\downarrow})$ ).

The proofs are dual to those of Lemmas 6.7.1–6.7.4.

**Lemma 6.7.9.** $x^{-} \leq x^{+}$ *(if both are defined).*

*Proof of Lemma 6.7.9.* Whenever $\beta \in \Lambda^{+}$ and $\gamma \in \Lambda^{-}$, we have $\xi(\beta) \geq \xi(\gamma)$ by (6.5g) for $\max\{\beta, \gamma\} < \alpha$. Therefore, $x^{-} = \sup \Xi^{\uparrow} \leq \inf \Xi^{\downarrow} = x^{+}$. $\square$

**Lemma 6.7.10.** *Let* $s \in I$ *and* $y \in X$*. If* $y \succcurlyeq^{s} x^{-}$*, then* $y > x^{-}$*. If* $y \succcurlyeq^{s} x^{+}$*, then* $y < x^{+}$*.*

*Proof of Lemma 6.7.10.* Let $y < x^{-}$; by Lemma 6.7.3, there is $\beta \in \Lambda^{\uparrow}$ such that $y < \xi(\beta)$. If $y \succcurlyeq^{s} x^{-}$, then, by Lemma 6.7.4, $y \succcurlyeq^{s} \xi(\beta)$, hence $y \succcurlyeq^{\sigma(\beta)} \xi(\beta)$ by (6.1b), which contradicts (6.5a) and (6.5c) for $\beta$. The case of $y > x^{+}$ is treated dually. $\square$

Now we consider two alternatives.

**A.** Let there exist $s \in I$ such that neither $x^{-}$, nor $x^{+}$ belong to $\mathcal{R}(s)$. Then we pick one of them as $\sigma(\alpha)$, set $\vartheta(\alpha) := 0$, and, invoking (6.3), obtain $\xi(\alpha) \in \mathcal{R}(\sigma(\alpha))$ such that $\xi(\alpha) \succcurlyeq^{\sigma(\alpha)} x^{-}$ and $\xi(\alpha) \succcurlyeq^{\sigma(\alpha)} x^{+}$. Finally, we set $S(\alpha) := \{s \in I \mid \xi(\alpha) \in \mathcal{R}(s)\} \ni \sigma(\alpha)$.

**B.** Otherwise, we set $\sigma(\alpha) := s^{*}$ and consider two alternatives again. If $x^{-} \in \mathcal{R}(s^{*})$, then we set $\vartheta(\alpha) := -1$, $\xi(\alpha) := x^{-}$, and $S(\alpha) := \{s \in I \mid x^{-} \in \mathcal{R}(s)\} \ni \sigma(\alpha)$. If $x^{-} \notin \mathcal{R}(s^{*})$, then $x^{+} \in \mathcal{R}(s^{*})$ because the alternative A does not hold; we set $\vartheta(\alpha) := 1$, $\xi(\alpha) := x^{+}$, and $S(\alpha) := \{s \in I \mid x^{+} \in \mathcal{R}(s)\} \ni \sigma(\alpha)$.

Let us check conditions (6.5). First, (6.5a), (6.5c), and (6.5d) immediately follow from the definitions; (6.5b), from Proposition 6.6.

If $s \in S$ satisfies the conditions in the left hand side of (6.5e), then $s \notin I$, hence there is $\beta \in \Lambda^{-}$ such that $s < \sigma(\beta)$; obviously, the right hand side of (6.5e) holds with that $\beta$. Condition (6.5f) is checked dually.

If the alternative A holds, we obtain $x^{-} \leq \xi(\alpha) \leq x^{+}$ from Lemma 6.7.10; otherwise, the inequalities follow just from the definition. Therefore, (6.5g) holds whenever $\xi(\beta) < x^{-}$

or $\xi(\beta) > x^+$. Let $\beta < \alpha$ and $x^- \le \xi(\beta) \le x^+$. If $\beta \in \Lambda^-$, we have $\xi(\beta) = x^-$ and $\sigma(\beta) = \max\{\sigma(\gamma)\}_{\gamma\in\Lambda^-}$, hence $x^- \notin \mathcal{R}(s^*)$ by (6.5f) for $\beta$ and $s^*$, hence $\xi(\alpha) > x^-$. The case of $\beta \in \Lambda^+$ is treated dually.

To check (6.5h), let us assume $\vartheta(\alpha) \le 0$, hence $\xi(\alpha) \succ^{\sigma(\alpha)} x^-$ or $\xi(\alpha) = x^-$. In the latter case, the existence of $\beta^* \in \Lambda^\uparrow$ such that $\xi(\beta^*) = x^-$ would imply a contradiction with (6.5f) for $\beta^*$ and $\sigma(\alpha)$ exactly as in the previous paragraph. Therefore, $\xi(\alpha) \succ^{\sigma(\alpha)} \xi(\beta)$ for every $\beta \in \Lambda^\uparrow$ by Lemma 6.7.4 and (6.1a). Finally, the set $\Lambda^- \setminus \Lambda^\uparrow$ consists of $\beta \in \Lambda^-$ for which there exists a $\gamma < \beta$ as in the last disjunctive term in (6.5h). Condition (6.5i) is checked dually.

Let us check (6.5j). If $\vartheta(\alpha) = -1$, then $\xi(\alpha) = x^- \in \mathcal{R}(\sigma(\alpha))$. If $s \in I \setminus S(\alpha)$, then $x^- \notin \mathcal{R}(s)$. By (2.3), there is $y \in \mathcal{R}(s)$ such that $y \succ^s x^-$; by Lemma 6.7.10, $y > x^-$. If $s < \sigma(\alpha)$ then $y \succ^{\sigma(\alpha)} x^-$ by (6.1a), which is incompatible with $x^- \in \mathcal{R}(\sigma(\alpha))$. Thus, $s < S(\alpha)$ is only possible if $s < I$. Then there is $\gamma \in \Lambda^-$ such that $s < \sigma(\gamma)$; Lemma 6.7.2 implies the existence of $\beta \in \Lambda^\uparrow$ such that $\sigma(\beta) \ge \sigma(\gamma)$; Lemma 6.7.1 implies that $\vartheta(\beta) \le 0$. Condition (6.5k) is checked dually.

The theorem is proven. □

**Proposition 6.8.** *Let $X$ and $S$ be chains. Let a parametric family $\langle\succ^s\rangle_{s\in S}$ of binary relations on $X$ satisfy* (6.1a)*. Let every $\succ^s$ $(s \in S)$ have the NM-property on $X$. If either $X$ or $S$ is finite, then there exists a monotone selection from $\mathcal{R}$ on $S$.*

*Proof.* Let $S$ be finite. We start with $s^+ := \max S$ and pick $r(s^+) \in \mathcal{R}(s^+)$ arbitrarily. Then we move along $S$ downwards, denoting $s+1$ the point in $S$ immediately above $s$. If $r(s+1) \in \mathcal{R}(s)$, we set $r(s) := r(s+1)$; otherwise, we invoke (2.3) and pick $r(s) \in \mathcal{R}(s)$ such that $r(s) \succ^s r(s+1)$. The inequality $r(s) > r(s+1)$ would, by (6.1a), imply $r(s) \succ^{s+1} r(s+1)$, contradicting the induction hypothesis; therefore, $r(s) \le r(s+1)$ for all $s \in S$.

Now let $X$ be finite. For every $s \in S$, we set $r(s) := \min \mathcal{R}(s)$. The inequalities $s' > s$ and $r(s) > r(s')$ would imply $r(s') \notin \mathcal{R}(s)$, hence $\mathcal{R}(s) \ni y \succ^s r(s')$ by (2.3). By the definition of $r(s)$, we have $y \ge r(s) > r(s')$, hence $y \succ^{s'} r(s')$ by (6.1a), contradicting the definition of $r(s')$. □

**Remark.** By duality, (6.1a) can be replaced with (6.1b).

**Proposition 6.9.** *Let $X$ and $S$ be chains. Let a parametric family $\langle\succ^s\rangle_{s\in S}$ of binary relations on $X$ satisfy both conditions* (6.1)*. Let every $\succ^s$ $(s \in S)$ have the NM-property on $X$. Let $s^0 \in S$ and $x^0 \in \mathcal{R}(s^0)$. If either $X$ or $S$ is finite, then there exists a monotone selection $r$ from $\mathcal{R}$ on $S$ such that $r(s^0) = x^0$.*

*Proof.* If $S$ is finite, then we apply Proposition 6.8 to $\{s \in S \mid s \le s^0\}$ and its dual to $\{s \in S \mid s \ge s^0\}$. If $X$ is finite, we set $r(s) := \min \mathcal{R}(s)$ for $s < s^0$ and $r(s) := \max \mathcal{R}(s)$ for $s > s^0$. $r$ is increasing for the same (or dual) reason as in the proof of Proposition 6.8. □

## 6.3 "Counterexamples"

The single crossing conditions (6.1) alone, without any restriction on preferences, do not imply the monotonicity of the best responses even for finite chains $X$ and $S$ (Example 6.10 below). Without the finiteness of either $X$ or $S$, there may be no monotone selection from $\mathcal{R}$ if either "half" of (6.1) does not hold, even when preferences are defined by a utility function (Example 6.11); or if the strong NM-property in Theorem 6.7 is replaced with just NM-property (Example 6.12). Finally, the existence of a monotone selection with a given value at a given point cannot be asserted in Proposition 6.8, even if all $\succ^s$ are orderings (Example 6.13).

**Example 6.10.** Let $X := \{0, 1, 2, 3, 4\}$, $S := \{0, 1\}$ (both with natural orders), and relations $\succ^s$ be defined by: $2 \succ^0 4 \succ^0 0 \succ^0 1 \succ^0 3$; $1 \succ^1 3 \succ^1 2 \succ^1 4 \succ^1 0$. Clearly, $R(0) = \{2\}$ while $R(1) = \{1\}$, so there is no monotone selection. On the other hand, conditions (6.1) are easy to check: (6.1b) is nontrivial only for $4 \succ^0 0$; (6.1a), only for $1 \succ^1 3$ and $2 \succ^1 4$.

**Example 6.11.** Let $X := S := [0, 1] \times [0, 1]$ with the natural order and every relation $\succ^s$ be represented, in the sense of (2.5), by a function

$$u^s(x) := \begin{cases} 0, & \text{if } x = 0, \\ 2, & \text{if } x > 0 = s, \\ 2 - sx, & \text{if } x \geq s > 0, \\ (2 - s^2)x/s, & \text{if } x \leq s > 0. \end{cases}$$

The function is continuous on $X \times S$, except at $(0, 0)$. Every $\succ^s$ is a chain-transitive ordering. When $s > 0$, $u^s$ is strictly increasing in $x$ while $x < s$ and strictly decreasing while $x > s$; therefore, (6.1a) holds everywhere.

The best responses are $\mathcal{R}(0) = ]0, 1]$ and $\mathcal{R}(s) = \{s\}$ for $s > 0$. Clearly, there is no increasing selection from $\mathcal{R}$: monotonicity is bound to be violated for $0 < s < r(0)$. Of all the assumptions of Theorem 6.7, only (6.1b) is violated.

**Remark.** If every $\succ^s$ were a *strongly* chain-transitive ordering, the existence of a monotone selection could be derived from the existing literature, even without (6.1b), in a rather roundabout way; $S$ could then be an arbitrary poset. If, besides, $X$ were a complete *lattice*, it would be sufficient to add (4.2d).

**Example 6.12.** Let $X := [-2, 2]$, $S := [-1, 1]$ (both with natural orders), and relations $\succ^s$ be defined by

$$y \succ^s x \rightleftharpoons [u_1(y, s) > u_1(x, s) \ \&\ u_2(y, s) > u_2(x, s)], \tag{6.6}$$

where $u\colon X \times S \to \mathbb{R}^2$ is this: $u(1, s) := \langle 5, 2 \rangle$ and $u(-1, s) := \langle 2, 5 \rangle$ for all $s \in S$; $u(2, s) := u(-2, s) := u(x, s) := \langle 0, 0 \rangle$ for all $x \in ] - 1, 1[$ and $s \in S$; whenever $x \in ]1, 2[$ and $s \geq 0$,

$$u_1(x, s) := \begin{cases} x + s - 1, & \text{if } x + s \leq 2, \\ x + s + 4, & \text{if } x + s > 2, \end{cases}$$

while $u_2(x,s) := 6 - x - s$; whenever $x \in \,]-2,-1[$ and $s \geq 0$, $u(x,s) := \langle x+6, -1-x \rangle$; finally, $u_i(x,s)$ for all $s < 0$, $i = 1,2$, and $x \in \,]-2,-1[\,\cup\,]1,2[$ is such that the equality

$$u_i(x,s) = u_{3-i}(-x,-s) \tag{6.7}$$

holds for all $s \in S$, $i = 1,2$, and $x \in X$.

The very form of (6.6) ensures that every $\succ^s$ is irreflexive and transitive. Whenever $x \in \{-2\} \cup ]-1,1[ \cup \{2\}$ and $y \in \,]-2,-1] \cup [1,2[$, $y \succ^s x$ for every $s \in S$. Whenever $x, y \in \,]-2,-1[$ or $x, y \in \,]1,2[$, $y \succ^s x$ does not hold for any $s \in S$. Let $s \geq 0$; if $-2 < x < -1$, then $u_1(x) < 5$ and $u_2(x) \leq 1$, hence $1 \succ^s x$; if $1 < x \leq 2 - s < 2$, then $u_1(x) \leq 1$ and $u_2(x) < 5$, hence $-1 \succ^s x$; if $2 - s < y < 2$, then $u_1(y) > 6$ and $u_2(y) > 3$, hence $y \succ^s 1$. "Dually," by (6.7), $y \succ^s -1 \succ^s x$ whenever $s < 0$, $-2 < y < -2 - s$, and $1 < x < 2$; $1 \succ^s x$ whenever $s < 0$ and $-2 - s \leq x < -1$. Thus, we see that every relation $\succ^s$ is strongly acyclic: no more than three consecutive improvements can be made from any starting point (e.g., $2 - s/2 \succ^s 1 \succ^s -1.5 \succ^s -2$ when $s > 0$). Single crossing conditions (6.1) are also easy to check.

Suppose there is a monotone selection $r$ from $R$. If $r(s) > -1$ for some $s > 0$, then $2 > r(s) > 2 - s$; defining $s' := 2 - r(s) > 0$, we have $s' < s$, hence $r(s') \leq r(s)$, hence $r(s') < 2 - s'$, hence $r(s') \in R(s')$ is only possible if $r(s') = -1$. Therefore, $r(s) = -1$ for some $s > 0$; dually, $r(s) = 1$ for some $s < 0$. We have a contradiction, i.e., there is no monotone selection: Theorem 6.7 cannot be extended to strongly acyclic and transitive preference relations.

**Remark.** Although every $\succ^s$ in Example 6.12 is strongly acyclic, the orderings defined by functions $u_1(\cdot,s)$ and $u_2(\cdot,s)$ are *not*. Moreover, if, given arbitrary chains $X$ and $S$, we considered a preference relation

$$y \succ^s x \rightleftharpoons [u_1(y,s) > u_1(x,s) + \varepsilon \;\&\; u_2(y,s) > u_2(x,s) + \varepsilon]$$

with $\varepsilon > 0$ and both $u_i$ satisfying (6.2) and bounded above in $x$ for every $s \in S$, then $\mathcal{R}(s)$ would contain the set of $\varepsilon$-optima of $u_1(x,s) + u_2(x,s)$, which admits a monotone selection by Theorem 6.7. Thus, the example cannot claim to present typical problems with parametric multi-criterial $\varepsilon$-optimization.

**Example 6.13.** Let $X := \{0,1\}$, $S := \{0,1\}$ (both with natural orders), and relations $\succ^s$ be defined by $0 \succ^1 1$; condition (6.1a) holds vacuously while (6.1b) does not. We have $\mathcal{R}(0) = \{0,1\}$ and $\mathcal{R}(1) = \{0\}$, so there is no increasing selection with $r(0) = 1$.

# 7 Concluding remarks

**7.1.** The fact that *all* characterization results here are related to optimization on *chains* may not be very surprising: chains are indeed a much simpler object than lattices. It may be worth noting, nonetheless, that all similarity with the choice from finite, or even compact, subsets is lost when we go from chains to sublattices. For instance, transitivity on every sublattice means just transitivity, and it is *not* necessary for the NM-property on every sublattice.

**Example 7.1.** Let $A := \{(0,0),(0,1),(1,0),(1,1)\} \subset \mathbb{R}^2$; let a binary relation $\succ$ on $A$ be as follows: $(1,1) \succ (1,0) \succ (0,1) \succ (0,0)$, $(1,1) \succ (0,1)$, and $(1,1) \succ (0,0)$. The NM-property on every sublattice (including $A$ itself) is obvious, but there is no transitivity since $(0,0) \succeq (1,0)$.

**7.2.** Regularity of preferences and regularity of the poset have the same implications in Propositions 5.5 and 5.6, as well as in Theorems 5.10 and 5.11. When it comes to optimization on *lattices*, however, it is only the former that gives us something unavailable in the general case (Theorems 5.16, 5.17, 5.18, 5.19, and 5.20). On the other hand, Theorem 5.14 only accepts the regularity of the *poset* (see Example 5.15).

**7.3.** There is a temptation to go beyond Example 5.12 and claim that interval orders admitting a maximizer in every subcomplete chain cannot be characterized in the style of Corollary to Theorem 3.9 unless some sort of regularity is present. However, there is no clear prospect for an appropriate definition of an "uncountable configuration."

**7.4.** Theorem 2 of Kukushkin (2008a) showed the impossibility to characterize *transitive* relations admitting a maximizer in every compact subset. No analog, however weak, has been established in the order context so far.

**7.5.** The proof of Theorem 6.7 would become invalid if conditions (6.4) were replaced with chain-transitivity (3.2); it remains unclear, however, what would happen to the theorem itself in this case. Example 6.4 only shows that the former conditions are indeed stronger than the latter.

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# 9 References


Aizerman, M.A., Aleskerov, F.T., 1995. Theory of Choice. North-Holland, Amsterdam.

Billingsley, P., 1965. Ergodic Theory and Information. J. Wiley & Sons, N.Y.–London–Sydney.

Edlin, A.S., and C. Shannon, 1998. Strict monotonicity in comparative statics. Journal of Economic Theory 81, 201–219.

Fishburn, P.C., 1973. The Theory of Social Choice. Princeton University Press, Princeton.

Gillies, D.B., 1959. Solutions to general non-zero-sum games. In: Tucker, A.W., and R.D. Luce (Eds.) Contributions to the Theory of Games 4 (Annals of Mathematical Studies 40). Princeton University Press, 47–85.

Kukushkin, N.S., 2008a. On the existence of maximal elements: An impossibility theorem. Discrete Applied Mathematics 156, 1988–1993.

Kukushkin, N.S., 2008b. Maximizing an interval order on compact subsets of its domain. Mathematical Social Sciences 56, 195–206.

Kukushkin, N.S., 2012. On the existence of optima in complete chains and lattices. Journal of Optimization Theory and Applications 154, 759–767.

Kukushkin, N.S., 2013a. Monotone comparative statics: changes in preferences versus changes in the feasible set. Economic Theory 52, 1039–1060.

Kukushkin, N.S., 2013b. Increasing selections from increasing multifunctions. Order 30, 541–555.

LiCalzi, M., and A.F. Veinott, Jr., 1992. Subextremal functions and lattice programming. Unpublished manuscript. Available at http://ideas.repec.org/p/wpa/wuwpge/0509001.html

Malishevski, A.V., 1998. Qualitative Models in the Theory of Complex Systems. Nauka-Fizmatlit, Moscow (in Russian and English).

Milgrom, P., and J. Roberts, 1990. Rationalizability, learning, and equilibrium in games with strategic complementarities. Econometrica 58, 1255–1277.

Milgrom, P., and C. Shannon, 1994. Monotone comparative statics. Econometrica 62, 157–180.

Quah, J., 2007. The comparative statics of constrained optimization problems. Econometrica 75, 401–431.

Sen, A.K., 1984. Collective Choice and Social Welfare. North-Holland, Amsterdam.

Shannon, C., 1995. Weak and strong monotone comparative statics. Economic Theory 5, 209–227.

Smith, T.E., 1974. On the existence of most-preferred alternatives. International Economic Review 15, 184–194.

Strulovici, B., and T.A. Weber, 2010. Generalized monotonicity analysis. Economic Theory 43, 377–406.

Veinott, A.F., Jr., 1989. Lattice Programming. Unpublished lectures.

Veinott, A.F., Jr., 1992. Lattice Programming. Qualitative Optimization and Equilibria. Seminar notes.

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