Maximizing an interval order on compact subsets of its domain

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January 10, 2008

Abstract

Maximal elements of a binary relation on compact subsets of a metric space define a choice function. An infinite extension of transitivity is necessary and sufficient for such a choice function to be nonempty-valued and path independent (or satisfy the outcast axiom). An infinite extension of acyclicity is necessary and sufficient for the choice function to have nonempty values provided the underlying relation is an interval order. *JEL classification*: D 71.

Keywords: Maximal element, Path independence, Outcast axiom, Interval order, Semiorder

1 Introduction

The notion of a choice function plays a central role in the decision theory (Fishburn, 1973; Sen, 1984; Aizerman and Aleskerov, 1995; Malishevski, 1998; Aleskerov et al., 2007), the most important being the case of a choice function defined by a binary relation. Traditionally, attention was focused on choice functions on finite sets; connections between properties of such a choice function and properties of the underlying binary relation have been studied in detail. When the number of conceivable alternatives is large enough, a

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model with an infinite set of alternatives becomes appealing, but then many familiar results may prove irrelevant and intuitions based on the finite case misleading.

Let a binary relation be given on a metric space. There is a considerable literature developing sufficient conditions for the nonemptiness of the choice function on compact subsets (Gillies, 1959; Smith, 1974; Bergstrom, 1975; Mukherji, 1977; Walker, 1977; Kiruta et al., 1980; Danilov and Sotskov, 1985; Campbell and Walker, 1990). In accordance with the needs of those building and studying mathematical models in concrete fields, two objectives are usually kept in mind. First, one tries to obtain a wider applicable, i.e., *weaker*, condition; second, a *simpler* one.

How difficult it is to check whether a general condition applies to a concrete model depends, to a large extent, on the latter; accordingly, the notion of the simplicity of a condition admits no ultimate formal expression. Nonetheless, comparisons are sometimes possible; for instance, it is universally accepted that conditions such as transitivity or acyclicity are as simple as one could wish. Comparisons in generality, on the contrary, can be based on pure logic. In particular, a sufficient condition cannot be weakened if it is also necessary.

It is no surprise that the two objectives are not easy to reconcile. The only example of achieving both in the above literature is due to Smith (1974), who found that a nice condition is necessary and sufficient for an *ordering* (weak order) to admit a maximum on every compact subset of its domain. Actually, Kukushkin (2007) proved that no "simple" (in an exact sense) condition could characterize arbitrary binary relations with this property.

Here we bypass the impossibility theorem by assuming (or demanding) a certain degree of rationality behind the binary relation (or the choice function it defines). Theorem 1 below shows that an infinite extension of transitivity is necessary and sufficient for the choice function generated by a binary relation to be nonempty-valued and path independent on all compact subsets. The fact that the transitivity of the underlying relation is equivalent to path independence (and to the outcast axiom) on all *finite* subsets of the domain is well known. Theorem 3 shows that an infinite extension of acyclicity is necessary and sufficient for the choice function to be nonempty-valued on all compact subsets provided the underlying relation is an interval order. As is well known, a finite analogue is valid without the restriction to interval orders. Both conditions are equivalent (and equivalent to Smith's condition) for semiorders (Theorem 4).

The term "interval order" refers to representations in chains (see, e.g., Fishburn, 1985). Most popular are numeric representations, which need not exist generally. Propositions 6, 7, and 8 provide very simple conditions on the representations that ensure the nonemptiness or path independence of the choice function. Characterization results are actually available (Kukushkin, 2006, Sections 5 and 6), but the mathematical apparatus involved appears too sophisticated.

There is a strand in economics and decision literature where the nonemptiness of chosen sets is addressed in a radically different manner: the idea of simplicity is dropped altogether while necessity is given a peculiar turn. For instance, Theorem 1 of Alcantud (2006) provides a list of requirements on a choice function which is necessary and sufficient for the nonemptiness of its value on a *single* set; quite a few similar results are referred to in that paper. An unpleasant feature of such conditions is that they are not "inherited" (Walker, 1977), i.e., their validity on a set does not imply the same on any of its subsets. Actually, this fact is closely related to their necessity in that sense: the final paragraph of Walker (1977) contains an argument to this effect, ascribed to P. Fishburn. The author of a concrete model trying to derive, from such an abstract theorem, the nonemptiness of chosen sets in the model would immediately find that it is not enough to check the conditions on the whole domain (as is the case with, e.g., transitivity or acyclicity). Instead, a family of independent conditions parameterized with potential feasible sets would have to be checked; typically, a continuum of different requirements, each of which is complicated enough. No practical way to do that was ever suggested in that literature; apparently, no such application was intended.

The next section contains basic definitions. Section 3 establishes equivalence between "ω-transitivity" and path independence (as well as the outcast axiom); Section 4, between "ω-acyclicity" and the existence of maximal elements of interval orders. Section 5 contains sufficient conditions for both properties in terms of representations in a chain. Section 6 briefly summarizes the main findings of the paper and discusses possible extensions.

2 Basic Notions

A *binary relation* on a set A is a Boolean function on $A \times A$; as usual, we write $y \succ x$ when the relation $\succ$ is true on a pair (y, x) and $y \not\succ x$ when it is false. An *interval order* is an irreflexive relation $\succ$ such that

$$[y \succ x \;\&\; a \succ b] \Rightarrow [y \succ b \;\text{ or }\; a \succ x]; \tag{1}$$

every interval order is transitive. An interval order is called a *semiorder* if

$$z \succ y \succ x \Rightarrow \forall a \in A\, [z \succ a \;\text{ or }\; a \succ x]. \tag{2}$$

A relation $\succ$ is called a *strict ordering* if it is asymmetric, i.e., $y \succ x \Rightarrow x \not\succ y$, and *negatively transitive*, i.e., $z \not\succ y \not\succ x \Rightarrow z \not\succ x$; every strict ordering is a semiorder.

We consider binary relations on a set A and denote $\mathfrak{B}$ the lattice of all subsets of A. Given $X \in \mathfrak{B}$, a point $x \in X$ is a *maximal element* of $\succ$ on X if $y \not\succ x$ for any $y \in X$. The set of all maximal elements of $\succ$ on X is denoted $M_\succ(X)$. A relation $\succ$ has the *NM property* on $X \in \mathfrak{B}$ if for every $x \in X \setminus M_\succ(X)$ there is $y \in M_\succ(X)$ such that $y \succ x$. The property means that $M_\succ(X)$ is a von Neumann–Morgenstern solution on X.

Whenever a binary relation $\succ$ on A is given, $M_\succ(\cdot)$ defines a mapping $\mathfrak{B} \to \mathfrak{B}$ with the property $M_\succ(X) \subseteq X$ for every $X \in \mathfrak{B}$, i.e., a *choice function*. The simplest desirable property of a choice function is

$$M_\succ(X) \neq \emptyset \tag{3}$$

(for $X \neq \emptyset$, naturally). We also consider two rather weak rationality requirements.

A choice function $M_\succ$ satisfies the *path independence axiom* (*PI*) if

$$X = X' \cup X'' \Rightarrow M_\succ(X) = M_\succ(M_\succ(X') \cup X'') \tag{4}$$

for all $X, X', X'' \in \mathfrak{B}$; it satisfies the *outcast axiom* (*O*) if

$$M_\succ(X) \subseteq X' \subseteq X \Rightarrow M_\succ(X') = M_\succ(X) \tag{5}$$

for all $X, X' \in \mathfrak{B}$. Sometimes PI and O are called Plott's and Nash's properties, respectively. As is well known, a choice function defined by a binary relation satisfies PI if and only if it satisfies O. For general choice functions, PI is equivalent to the conjunction of O and the *heredity axiom* (Chernoff's property), $X' \subseteq X \Rightarrow M_{\succ}(X) \cap X' \subseteq M_{\succ}(X')$; however, the latter is obviously satisfied without any restrictions on $\succ$.

It is important to recognize that the standard equivalence PI $\equiv$ O is only relevant when *all* $X, X', X'' \in \mathfrak{B}$ are admissible. The assumption looks natural enough if A is finite, but not otherwise. In economics literature, for instance, attention is usually restricted to either compact, or convex and compact subsets (or even to budget sets). There are also technical reasons for restrictions: one could hardly imagine a nicer binary relation than the order $>$ on the real line, but the choice function $M_{>}$ does not satisfy even (5) when $\sup X \notin X \supset X' \ni \sup X'$.

Proposition 1. *Let $\mathfrak{B}' \subseteq \mathfrak{B}$ and $\succ$ be a binary relation on A. If $\succ$ has the NM property on all $X \in \mathfrak{B}'$, then $M_{\succ}$ satisfies* (3) *and* (4) *for all $X \in \mathfrak{B}'$ and $X', X'' \in \mathfrak{B}$. The latter property is equivalent to* (3) *and* (5) *for all $X \in \mathfrak{B}'$ and $X' \in \mathfrak{B}$, and implies* (3) *and* (4) *for all $X, X', X'' \in \mathfrak{B}'$. The latter property implies* (3) *and* (5) *for all $X, X' \in \mathfrak{B}'$.*

Routine proofs are left to the reader.

Proposition 2. *Let $A = \mathbb{R}$ and $\mathfrak{B}' \subset \mathfrak{B}$ consist of bounded closed intervals $[a, b]$ such that $b \geq a + 1$. There are binary relations $\succ_1, \succ_2, \succ_3$ on $\mathbb{R}$ such that: $M_{\succ_1}$ satisfies* (3) *and* (4) *for all $X \in \mathfrak{B}'$ and $X', X'' \in \mathfrak{B}$, but $\succ_1$ does not have the NM property on any $X \in \mathfrak{B}'$; $M_{\succ_2}$ satisfies* (3) *and* (4) *for all $X, X', X'' \in \mathfrak{B}'$, but there is no $X \in \mathfrak{B}'$ such that* (5) *would hold for all $X' \in \mathfrak{B}$; $M_{\succ_3}$ satisfies* (3) *and* (5) *for all $X, X' \in \mathfrak{B}'$, but there are $X, X', X'' \in \mathfrak{B}'$ for which* (4) *is violated.*

Proof. Denoting $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ the sets of all integer and rational numbers, respectively, we define:

$$y \succ_1 x \rightleftharpoons \big[y = x \in \mathbb{R} \setminus \mathbb{Z}\big];$$
$$y \succ_2 x \rightleftharpoons \big[\,[y \in \mathbb{Z} \;\&\; x \in \mathbb{Q} \setminus \mathbb{Z}] \;\text{ or }\; [y \in \mathbb{Q} \setminus \mathbb{Z} \;\&\; x \in \mathbb{R} \setminus \mathbb{Q}]\,\big];$$
$$y \succ_3 x \rightleftharpoons \big[y > x \geq 0 \;\text{ or }\; 0 \geq y > x\big].$$

Straightforward checks are left to the reader. In the case of $\succ_3$, we can take, e.g., $X = X' = [-2, 1]$ and $X'' = [-2, -1]$. $\square$

It is easy to see that the first two implications in Proposition 1 can be reversed if and only if $\mathfrak{B}'$ contains all singleton subsets of its members. The last implication seems to admit no such simple criterion; moreover, it is irreversible for quite natural classes of admissible sets: e.g., $\succ_3$ from Proposition 2 retains its unpleasant properties when $\mathfrak{B}'$ consists of *all* bounded and closed intervals in $\mathbb{R}$.

Throughout the paper, we consider binary relations on a metric space (a first countable Hausdorff topological space would do as well: what we actually need is that the topology on A be adequately described by convergent sequences). The set of all nonempty compact subsets of A is denoted $\mathfrak{C} \subset \mathfrak{B}$. Theorem 1 below shows, in particular, that all the implications in Proposition 1 can be reversed when $\mathfrak{B}' = \mathfrak{C}$.

Given a binary relation $\succ$, an *improvement path* is a (finite or infinite) sequence $\langle x^k \rangle_{k=0,1,\dots}$ such that $x^{k+1} \succ x^k$ whenever both sides are defined. A relation $\succ$ is *acyclic* if it admits no *finite improvement cycle*, i.e., no improvement path such that $x^m = x^0$ for an $m > 0$. A relation is *strongly acyclic* if it admits no infinite improvement path.

These two statements are well known (and easy to check anyway): a relation $\succ$ is acyclic if and only if $M_\succ(X) \neq \emptyset$ for every finite $X \in \mathfrak{B} \setminus \{\emptyset\}$; a relation $\succ$ is strongly acyclic if and only if $M_\succ(X) \neq \emptyset$ for every $X \in \mathfrak{B} \setminus \{\emptyset\}$.

A binary relation $\succ$ on a metric space is called *ω-transitive* if it is transitive and, whenever $\langle x^k \rangle_{k=0,1,\dots}$ is an infinite improvement path and $x^k \to x^\omega$, there holds $x^\omega \succ x^0$. It is worth noting that $x^\omega \succ x^k$ is valid for all $k = 0, 1, \dots$ in this situation, once $\succ$ is ω-transitive: $x^k, x^{k+1}, \dots$ is also an infinite improvement path converging to the same x^ω.

The property seems to have been first considered by Gillies (1959), who proved its sufficiency for the existence of maximal elements on compact sets. Smith (1974) gave it the name of "σ-transitivity" and proved that it is *necessary* and sufficient for the existence of maximal elements on all compact subsets provided $\succ$ is a strict ordering. However, the prefix "σ" traditionally refers to the cardinal concept of a countable set whereas what matters here is the order type of the chain of natural numbers, usually referred to as ω.

A binary relation $\succ$ is called *ω-acyclic* if it is acyclic and, whenever $\langle x^k \rangle_{k=0,1,\dots}$ is an infinite improvement path and $x^k \to x^\omega$, there holds $x^\omega \neq x^0$. It is worth noting that $x^k \not\succ x^\omega$ for any k in this situation once $\succ$ is ω-acyclic:

otherwise, $x^\omega, x^k, x^{k+1}, \ldots$ would form an infinite improvement cycle. The prohibition of such cycles was introduced by Mukherji (1977) as "Condition (A5)."

Remark. An anonymous referee observed that we could define a *weak improvement path* as an infinite sequence $\langle x^k \rangle_{k\in\mathbb{N}}$ such that, for each $k \in \mathbb{N}$, there holds either $x^{k+1} \succ x^k$ or $x^{k+1} = x^k$, while $x^{k+1} \succ x^k$ holds for at least one $k \in \mathbb{N}$. Then ω-transitivity could be expressed as "$y \succ x$ whenever there is a weak improvement path starting at x and converging to y"; ω-acyclicity, as "no weak improvement path can converge to its origin." From a technical viewpoint, however, the notion of an improvement path as defined in this paper is more convenient.

3 Transitivity

Theorem 1. *Let $\succ$ be a binary relation on a metric space. Then the following statements are equivalent.*

1. *$\succ$ is irreflexive and ω-transitive.*
2. *$\succ$ has the NM property on every $X \in \mathfrak{C}$.*
3. *The choice function $M_\succ$ satisfies* (3) *and* (5) *for all $X, X' \in \mathfrak{C}$.*

Proof. Let us prove the implication [Statement 1 $\Rightarrow$ Statement 2] first. Let Statement 1 hold and $X \in \mathfrak{C}$; for each $x \in X$, we denote $G(x) = \{y \in X \mid y \succ x\}$. The key role is played by the following auxiliary statement:

$$\forall x^* \in X\, [G(x^*) \neq \emptyset \Rightarrow M_\succ(G(x^*)) \neq \emptyset]. \tag{6}$$

Let us prove (6). Statement 1 means that $\succ$ is a strict order. Therefore, we may apply Zorn's Lemma (see, e.g., Kuratowski, 1966, p. 27) to $G(x^*)$ ordered by $\succ$: a maximal element of $\succ$ on $G(x^*)$ exists if every chain in $G(x^*)$ [i.e., $C \subseteq G(x^*)$ such that, for every $x, y \in C$, either $y \succ x$, or $x \succ y$, or $y = x$] admits an upper bound in $G(x^*)$ [i.e., $y \in G(x^*)$ such that, for every $x \in C$, either $y \succ x$ or $y = x$].

We fix a chain $C \subseteq G(x^*)$ and, for every $x \in C$, denote $G^*(x) = G(x) \cap C$ and $F(x)$ the closure of $G^*(x)$ in X. If there happens to be $y \in C$ such that

$G^*(y) = \emptyset$, then, obviously, $y \succ x$ for every $x \in C \setminus \{y\}$, hence y is an upper bound needed. Therefore, we may assume that $G^*(x) \neq \emptyset$ for every $x \in C$.

We denote $F = \bigcap_{x \in C} F(x) \subseteq X$. Since C is a chain, all the sets $G^*(x)$ ($x \in C$), hence $F(x)$ too, contain each other; therefore, every intersection of a finite number of $F(x)$ is not empty. Since X is compact, $F \neq \emptyset$. We pick $y \in F$; let us show that $y \succ x$ for every $x \in C$. Supposing the contrary, we can pick $x^0 \in C$ for which $y \not\succ x^0$. In the next paragraph, we recursively define a sequence $x^k \in G^*(x^0)$ ($k = 1, 2 \dots$) such that $x^{k+1} \succ x^k$ and $x^k \to y$; then the ω-transitivity of $\succ$ will imply that $y \succ x^0$, i.e., a contradiction.

We have $y \in F(x^0) = \operatorname{cl} G^*(x^0)$; the assumption $y \not\succ x^0$ implies $y \notin G^*(x^0)$. Therefore, we can pick $x^1 \in G^*(x^0)$ such that $0 < \rho(y, x^1) < \rho(y, x^0)/2$. Since $x^1 \succ x^0$ and $y \not\succ x^0$, we must have $y \not\succ x^1$, hence $y \notin G^*(x^1)$. Now the procedure to be recursively executed for each $k = 1, 2, \dots$ should be clear. Having $x^k \in G^*(x^0)$ already defined, we notice that $y \in F(x^k) \setminus G^*(x^k)$; therefore, we can pick $x^{k+1} \in G^*(x^k) \subseteq G^*(x^0)$ such that $0 < \rho(y, x^{k+1}) < \rho(y, x^k)/2$. On each step, we have $x^{k+1} \succ x^k$; the condition on distances ensures $x^k \to y$. Therefore, the ω-transitivity of $\succ$ implies that $y \succ x^0$, which contradicts our assumption, as was planned at the end of the previous paragraph.

Since $C \subseteq G(x^*)$ and $\succ$ is transitive, we have $y \in G(x^*)$; in other words, y is an upper bound of C in $G(x^*)$. We see that Zorn's Lemma can, indeed, be applied, ensuring $M_\succ(G(x^*)) \neq \emptyset$. Thus, (6) is proven.

Let $x^* \in X \setminus M_\succ(X)$; then $G(x^*) \neq \emptyset$, hence (6) applies, producing a maximal element y^* of $\succ$ on $G(x^*)$. By the definition of $G(x^*)$, we have $y^* \succ x^*$. Let us show that $y^* \in M_\succ(X)$. Indeed, $z \succ y^*$ would imply $z \succ x^*$ as well, hence $z \in G(x^*)$, which clearly contradicts the fact that $y^* \in M_\succ(G(x^*))$.

To summarize, we have proved that, for every $X \in \mathfrak{C}$ and $x^* \in X \setminus M_\succ(X)$, there is $y^* \in M_\succ(X)$ such that $y^* \succ x^*$, i.e., Statement 2 is valid.

Given Statement 2, we immediately obtain Statement 3 by Proposition 1.

Finally, let us prove that Statement 3 implies Statement 1. If $x \succ x$, then $M_\succ(\{x\}) = \emptyset$. If $z \succ y \succ x$, then (5) with $X = \{x, y, z\}$ and $X' = \{x, z\}$ gives us $M_\succ(\{x, z\}) = M_\succ(X) = \{z\}$, hence $z \succ x$. Let $x^k \to x^\omega$ and $x^{k+1} \succ x^k$ for each k; applying (5) to $X = \{x^\omega\} \cup \{x^k\}_{k=0,1,\dots}$ and $X' = \{x^0, x^\omega\}$, we again obtain $M_\succ(\{x^0, x^\omega\}) = \{x^\omega\}$, hence $x^\omega \succ x^0$. $\square$

Remark. The conditions listed in Theorem 1 do not imply that $M_{\succ}(X) \in \mathfrak{C}$ for all $X \in \mathfrak{C}$: consider (weak) Pareto dominance on $\mathbb{R}^n$ with $n \geq 3$.

Theorem 1 gives us a sufficient condition for the existence of maximal elements. A *potential* for $\succ$ is an irreflexive and ω-transitive relation $\ggcurly$ finer than $\succ$, i.e., satisfying $y \succ x \Rightarrow y \ggcurly x$ for all $y, x \in A$. The notion was introduced in Kukushkin (1999).

Corollary. *If $\succ$ admits a potential, then $M_{\succ}(X) \neq \emptyset$ for each $X \in \mathfrak{C}$.*

Proof. Obviously, $M_{\ggcurly}(X) \subseteq M_{\succ}(X)$ for any potential $\ggcurly$ for $\succ$. □

The Corollary often helps to establish the existence of maximal elements. It is easy to see that a binary relation admits a potential if and only if its ω-transitive closure is irreflexive. Despite the well-known fact that a relation is acyclic if and only if its transitive closure is irreflexive, ω-acyclicity is *not* sufficient for the existence of a potential, see Examples 1 and 2 below.

The following result is included for completeness; it can be proven in essentially the same way as its well-known finite analogues (no need for Zorn's Lemma).

Theorem 2. *Let $\succ$ be a binary relation on a set A. Then the following statements are equivalent.*

1. *$\succ$ is transitive and strongly acyclic.*
2. *$\succ$ has the NM property on every $X \in \mathfrak{B}$.*
3. *The choice function $M_{\succ}$ is nonempty-valued on $\mathfrak{B} \setminus \{\emptyset\}$ and satisfies* (5) *for all $X, X' \in \mathfrak{B}$.*

4 Acyclicity

Proposition 3. *If $M_{\succ}(X) \neq \emptyset$ for each $X \in \mathfrak{C}$, then $\succ$ is ω-acyclic on A.*

Proof. If $\langle x^k \rangle_{k \in \mathbb{N}}$ is an improvement path converging to x^0, then it is a compact subset without maximal elements itself. (This argument was present in the proof of Corollary 3 from Mukherji, 1977, although the formulation was different.) □

The condition is by no means sufficient.

Example 1. Let A be a circle perceived as the result of identifying the end points of the closed interval $[-\pi, \pi]$. We define $\succ$ by

$$y \succ x \rightleftharpoons \big[\, \pi > y > x \geq 0 \;\text{ or }\; 0 > y > x \geq -\pi \,\big]. \tag{7}$$

The relation $\succ$ is obviously transitive; it is ω-acyclic because every improvement path is strictly increasing and contained in a half-circle. On the other hand, the key condition in the definition of ω-transitivity is violated whenever an improvement path converges to 0 or π. The set A itself is compact, but there is no maximal element on A.

Actually, the relation in (7) admits an infinite cycle a bit different from those prohibited in the definition of ω-acyclicity. Let $x^0 = 0$, $x^k = (1 - 1/(k+1))\pi$ ($k \in \mathbb{N}$), $x^\omega = -\pi$, and $x^{\omega+k} = -1/(k+1)\pi$ ($k \in \mathbb{N}$). Clearly, $x^{k+1} \succ x^k$ and $x^{\omega+k+1} \succ x^{\omega+k}$ for all $k \in \mathbb{N}$, while $x^k \to x^\omega$ and $x^{\omega+k} \to x^0$. In other words, $\succ$ admits an infinite improvement cycle parameterized by ordinal numbers from $\{0, 1, \dots \omega, \omega+1, \dots \omega+\omega\}$. Arguing exactly as in Proposition 3, it is easy to show that the absence of such cycles is also necessary for (3) to hold for all $X \in \mathfrak{C}$.

We can divide the circle A into three parts and modify (7) accordingly, obtaining an ω-acyclic relation which admits no improvement cycle of the length $\omega + \omega$, and still admits no maximal element on the compact A. This time, the absence of maximal elements can be explained by the presence of an improvement cycle of the length $\omega + \omega + \omega$; the absence of such cycles can also be added to the statement of Proposition 3.

A reader familiar with the notion of ordinal numbers easily recognizes now that there is an uncountable number of logically independent conditions, each of which is necessary for the existence of maximal elements on all compact subsets. What is most interesting (or most troubling) is that even the conjunction of all those conditions, i.e., the prohibition of an improvement cycle parameterized by *any* (countable) ordinal, does not constitute a sufficient condition.

Example 2. Let us consider a circle represented as the set of complex numbers with $|z| = 1$; formally, $A = \{\mathrm{e}^{\mathrm{i}t} \mid t \in \mathbb{R}\}$. We define a binary relation by $y \succ x \rightleftharpoons y = \mathrm{e}^{\mathrm{i}} \cdot x$. The relation is acyclic because 1 is incommensurable with 2π. It is ω-acyclic by default because no infinite improvement path, i.e., a sequence $\langle x^k \rangle_{k\in\mathbb{N}}$ such that $x^k = \mathrm{e}^{k\mathrm{i}} \cdot x^0$, can converge. Cycles similar to that from Example 1, or of a greater length, are impossible for the same reason. On the other hand, A itself is compact, but $M_{\succ}(A) = \emptyset$.

Here we have cycles in a weaker sense: every infinite improvement path is dense in A (the Jacobi theorem, see, e.g., Billingsley, 1965), hence its origin is among its limit points. The absence of *such* cycles, however, is not necessary for (3) to hold for all $X \in \mathfrak{C}$.

It turns out that these complications disappear when $\succ$ is an interval order. To be more precise, ω-acyclicity is then sufficient for the existence of a potential.

With any binary relation $\succ$ on A, we associate these two relations:

$$y \succsim^{\omega} x \rightleftharpoons \exists \langle x^k \rangle_{k\in\mathbb{N}} \big[x^0 = x \;\&\; \forall k \in \mathbb{N}\, [x^{k+1} \succ x^k] \;\&\; x^k \to y\big]; \tag{8a}$$

$$y \succ\!\!\succ x \rightleftharpoons \big[y \succ x \;\text{ or }\; y \succsim^{\omega} x\big]. \tag{8b}$$

Remark. Whenever $\succ$ is transitive, $y \succ\!\!\succ x$ if and only if there is a weak improvement path, as defined in the remark at the end of Section 2, starting at x and converging to y.

Proposition 4. *If $\succ$ is an ω-acyclic interval order, then $\succ\!\!\succ$ defined by* (8) *is irreflexive and ω-transitive.*

Proof. Irreflexivity of $\succ\!\!\succ$ immediately follows from the ω-acyclicity of $\succ$. To prove ω-transitivity, this auxiliary statement is needed:

$$\forall x, y, z \in A\, [z \succ y \succsim^{\omega} x \Rightarrow z \succ x]. \tag{9}$$

Let $\langle x^k \rangle_{k\in\mathbb{N}}$ be an improvement path such that $x^0 = x$ and $x^k \to y$. Applying (1) to $z \succ y$ and $x^1 \succ x$, we obtain that either $z \succ x$ or $x^1 \succ y$. The latter relation would contradict the ω-acyclicity of $\succ$ (consider the sequence $y, x^1, x^2, \dots$). Thus $z \succ x$ and (9) is proven.

Let $z \succ\!\!\succ y \succ\!\!\succ x$. If $y \succ x$, then obviously $z \succ\!\!\succ x$; let $y \succsim^{\omega} x$. If $z \succ y$, then (9) applies immediately. Let $z \succsim^{\omega} y$ and $\langle y^k \rangle_{k\in\mathbb{N}}$ be an appropriate sequence. (9) implies that $y^1 \succ x$; therefore, $z \succsim^{\omega} x$. Thus, $\succ\!\!\succ$ is transitive.

Let $x^k \to x^{\omega}$ and $x^{k+1} \succ\!\!\succ x^k$ for all k. If $x^{k+1} \succ x^k$ for all k except for a finite number, a straightforward backward induction based on (9) shows that

$x^\omega \succcurlyeq^\omega x^0$. Otherwise, we may assume that $x^{k+1} \succcurlyeq^\omega x^k$ for all k. Let $\langle x_k^h \rangle_{k,h\in\mathbb{N}}$ be such that $x_k^0 = x^k$, $\langle x_k^h \rangle_{h\in\mathbb{N}}$ is an improvement path, and $x_k^h \underset{h}{\to} x^{k+1}$ for each $k \in \mathbb{N}$. (9) implies that $x_{k'}^{h'} \succ x_k^h$ whenever $k' > k$ and $h' > 0$. We denote $y^0 = x^0$, and, for each $k > 0$, pick $h(k) > 0$ such that $\rho(x_k^{h(k)}, x^{k+1}) < 1/k$ and denote $y^k = x_k^{h(k)}$. Now we have $y^k \to x^\omega$ and $y^{k+1} \succ y^k$ for all k; therefore, $x^\omega \succcurlyeq^\omega x^0$ and $\ggcurly$ is ω-transitive. $\square$

Theorem 3. *Let $\succ$ be an interval order on a metric space A. Then $M_\succ(X) \neq \emptyset$ for every $X \in \mathfrak{C}$ if and only if $\succ$ is ω-acyclic.*

Proof. The necessity immediately follows from Proposition 3; the sufficiency, from Proposition 4 and Corollary to Theorem 1. $\square$

Campbell and Walker (1990) called a relation $\succ$ "weakly lower continuous" if $y \succ x$ implies the existence of an open neighborhood U of x such that $z \not\succ y$ for any $z \in U$. Obviously, the weak lower continuity of a transitive relation implies its ω-acyclicity; therefore, Theorem 1 of Campbell and Walker (when restricted to metric spaces) immediately follows from our Theorem 3. Weak lower continuity is not necessary for an interval order to admit a maximal element on every $X \in \mathfrak{C}$: consider a lexicographic order on a plane with fixed coordinates.

Every ω-acyclic interval order has an "ε-version" of the NM property on every $X \in \mathfrak{C}$.

Proposition 5. *Let $\succ$ be an ω-acyclic interval order on a compact metric space X and $x^* \in X \setminus M_\succ(X)$. Then there is $z \in M_\succ(X)$ for which either $z \succ x^*$ or there is an infinite improvement path $\langle z^k \rangle_{k\in\mathbb{N}}$ such that $z^0 = x^*$ and $z^k \to z$ (in the last case, clearly, $z^k \succ x^*$ for each k).*

Proof. Applying Theorem 1 to $\ggcurly$ defined by (8), we obtain $z \in M_\succ(X)$ such that $z \ggcurly x^*$. A reference to (8b) completes the proof. $\square$

Theorem 4. *Let $\succ$ be a semiorder; then the following statements are equivalent.*

1. *$M_\succ(X) \neq \emptyset$ for every $X \in \mathfrak{C}$.*
2. *$M_\succ(\cdot)$ has the NM property on every $X \in \mathfrak{C}$.*

3. $\succ$ *is* ω*-transitive.*

4. $\succ$ *is* ω*-acyclic.*

Proof. The implications [Statement 2 $\Rightarrow$ Statement 1] and [Statement 3 $\Rightarrow$ Statement 4] are obvious; [Statement 1 $\Rightarrow$ Statement 4] follows from Proposition 3; [Statement 2 $\iff$ Statement 3], from Theorem 1. Thus, it is sufficient to prove that Statement 4 implies Statement 3. Let $\succ$ be an ω-acyclic semiorder, $x^k \to x^\omega$, and $x^{k+1} \succ x^k$ for all k. Applying (2) to $x^2 \succ x^1 \succ x^0$, we obtain that either $x^\omega \succ x^0$ or $x^2 \succ x^\omega$. The latter would contradict the ω-acyclicity (with the sequence $x^\omega, x^2, x^3, \dots$). $\square$

The restriction of the equivalence [Statement 3 $\iff$ Statement 1] to strict orderings renders Theorem 4.1 of Smith (1974).

Example 3. Let $A = [0, 1]$ and $y \succ x \rightleftharpoons 1 > y > x$ for all $y, x \in A$. Then $\succ$ is an interval order, ω-acyclic (because every improvement path is strictly increasing) but not ω-transitive (the basic requirement is violated whenever an improvement path converges to 1). Thus, Theorem 4 does not hold for interval orders. (Actually, there is no NM property on A itself, which is compact: any $x \in [0, 1[$ neither is a maximal element nor dominated by a maximal element.)

Theorem 4.2 of Smith (1974) provides a necessary and sufficient condition for the property that $M_\succ(X) \in \mathfrak{C}$ for every $X \in \mathfrak{C}$, again assuming that the underlying relation is an ordering; the condition is the ω-transitivity of *weak* preference relation $[y \succeq x \rightleftharpoons x \not\succ y]$. For interval orders, the weak preference relation need not be transitive in the first place, so a relevant analogue is that the limit of an "indifference path" cannot be dominated by its origin.

$$\text{There is no sequence } x^k \to x^\omega \text{ such that } \forall k, h \in \mathbb{N}\, [x^h \not\succ x^k] \text{ and } x^0 \succ x^\omega. \tag{10}$$

Theorem 5. *An interval order* $\succ$ *on a metric space has the property that* $M_\succ(X) \in \mathfrak{C}$ *for every* $X \in \mathfrak{C}$ *if and only if it is* ω*-acyclic and satisfies* (10).

Proof. The necessity of ω-acyclicity is Proposition 3. Let (10) be violated. We denote $X = \langle x^k \rangle_{k \in \mathbb{N}} \cup \{x^\omega\}$; clearly, $X \in \mathfrak{C}$ while $M_\succ(X) = X \setminus \{x^\omega\} \notin \mathfrak{C}$.

Conversely, let $\succ$ be an ω-acyclic interval order satisfying (10) and $X \in \mathfrak{C}$. By Theorem 3, $M_{\succ}(X) \neq \emptyset$; if $M_{\succ}(X)$ is closed in X, then $M_{\succ}(X) \in \mathfrak{C}$. Supposing the contrary, we must have a sequence $y^k \to x^\omega$ such that $y^k \in M_{\succ}(X)$ for all $k \in \mathbb{N}$, but $x^\omega \in X \setminus M_{\succ}(X)$. Clearly, $y^h \not\succ y^k$ for all $k, h \in \mathbb{N}$. If $y^h \succ x^\omega$ for an $h \in \mathbb{N}$, we define $x^k = y^{h+k}$ for all $k \in \mathbb{N}$ and obtain a configuration prohibited by (10). Otherwise, there must be $x^0 \in X$ such that $x^0 \succ x^\omega$; then, for all h, we have $y^h \not\succ x^0$ because $\succ$ is transitive while $x^0 \not\succ y^h$ because $y^h \in M_{\succ}(X)$. Defining $x^{k+1} = y^k$ for all $k \in \mathbb{N}$, we again obtain a prohibited configuration. □

An analogue of Theorem 5 for an arbitrary relation is obtained in Kukushkin (2005, Theorem 3). A modification of (10) should be added to the prohibition of *all* countable improvement cycles as described after Example 1. The number of independent conditions is thus, unfortunately, uncountable, although each of them is simple enough. An application of the theorem could only be envisaged to a situation where a "reasonably low" upper bound on the lengths of all improvement paths can be found.

5 Interval Representations

Let $\succ$ be a binary relation on a set A. An *interval representation* of $\succ$ consists of a chain L (i.e., a set linearly ordered by $\geq$) and two mappings $\varphi^+, \varphi^- \colon A \to L$ such that $\varphi^+(x) \geq \varphi^-(x)$ and $y \succ x \iff \varphi^-(y) > \varphi^+(x)$ for all $x, y \in A$. As is well known, $\succ$ is an interval order if and only if it admits an interval representation.

Proposition 6. *An interval order $\succ$ on a set A is strongly acyclic if it admits an interval representation $\varphi^+, \varphi^- \colon A \to \mathbb{R}$ for which φ^+ is bounded above and there is $\varepsilon > 0$ such that $\varphi^+(x) \geq \varphi^-(x) + \varepsilon$ for all $x \in A$.*

Proof. Suppose there is an infinite improvement path $x^0, x^1, \ldots$ Then $\varphi^+(x^{k+1}) \geq \varphi^-(x^{k+1}) + \varepsilon > \varphi^+(x^k) + \varepsilon$ for all k; therefore, $\varphi^+(x^k) > \varphi^+(x^0) + k \cdot \varepsilon$, hence $\sup_k \varphi^+(x^k) = +\infty$, contradicting the assumption. □

Let A be a metric space and B be a partially ordered set (with the order $\geq$). A mapping $\varphi \colon A \to B$ is called *upper ω-semicontinuous* if $\varphi(x^\omega) > \varphi(x^0)$ whenever $x^k \to x^\omega$ and $\varphi(x^{k+1}) > \varphi(x^k)$ for all $k = 0, 1, \ldots$ Evidently, every upper semicontinuous mapping is upper ω-semicontinuous; the converse is wrong: e.g., every increasing function $\varphi\colon \mathbb{R} \to \mathbb{R}$ is upper ω-semicontinuous regardless of how it jumps.

Proposition 7. *An interval order $\succ$ on a metric space A is ω-transitive if it admits an interval representation $\varphi^+, \varphi^-\colon A \to L$ such that φ^- is upper ω-semicontinuous.*

Proof. If $x^0, x^1, \ldots$ is an improvement path such that $x^k \to x^\omega$, then $\varphi^-(x^{k+1}) > \varphi^-(x^k)$ for all k, and $x^1, x^2, \ldots$ also converges to x^ω. Therefore, $\varphi^-(x^\omega) > \varphi^-(x^1) > \varphi^+(x^0)$, hence $x^\omega \succ x^0$. $\square$

The converse to Proposition 7 is wrong. For instance, if an interval order admits a representation satisfying the conditions of Proposition 6, then it is ω-transitive regardless of any discontinuities of φ^-; see Example 4 below.

Proposition 8. *An interval order $\succ$ on a metric space A is ω-acyclic if it admits an interval representation $\varphi^+, \varphi^-\colon A \to L$ such that φ^+ is upper ω-semicontinuous.*

The proof is similar to that of Proposition 7. Example 4 shows that Proposition 8 also cannot be reversed.

Proposition 9. *An interval order $\succ$ on a metric space A has the property that $M_\succ(X) \in \mathfrak{C}$ for every $X \in \mathfrak{C}$ if it admits an interval representation $\varphi^+, \varphi^-\colon A \to L$ such that φ^+ is upper semicontinuous.*

The statement immediately follows from Theorem 5 and Proposition 8. The converse does not hold: consider a lexicographic order on a plane.

A *semiorder representation* of a binary relation $\succ$ on a set A is an interval representation $\varphi^+, \varphi^-\colon\ A \to L$ for which there exists an order-preserving mapping $\lambda\colon \varphi^+(A) \to L$ [i.e., $\varphi^+(y) > \varphi^+(x) \Rightarrow \lambda \circ \varphi^+(y) \geq \lambda \circ \varphi^+(x)$ for all $x, y \in A$] such that $\varphi^-(x) = \lambda \circ \varphi^+(x)$ for all $x \in A$. As is well known, $\succ$ is a semiorder if and only if it admits a semiorder representation.

Corollary to Proposition 6. *A semiorder $\succ$ on a metric space A is strongly acyclic if it admits a semiorder representation with $L = \mathbb{R}$ for which φ^+ is bounded above and there is $\varepsilon > 0$ such that $\lambda \circ \varphi^+(x) \leq \varphi^+(x) - \varepsilon$ for all $x \in A$.*

Corollary to Proposition 7. *A semiorder $\succ$ on a metric space A is ω-transitive if it admits a semiorder representation such that φ^+ is upper ω-semicontinuous.*

Example 4. Let $A =]0, 3[$ and

$$y \succ x \rightleftharpoons [2 \neq y > x + 1 \text{ or } y \neq x = 2]. \tag{11}$$

Clearly, $\succ$ is transitive and strongly acyclic, hence ω-transitive by default; (10) is violated, e.g., by $x^k = 1 + 1/(k + 1)$ and $x^\omega = 2$. The relation admits an interval representation with functions $\varphi^+(x) = x+1$ for $x \neq 2$, $\varphi^+(2) = 0$, and $\varphi^-(x) = \varphi^+(x) - 1$ for all $x \in A$, hence is a semiorder.

Proposition 10. *The relation $\succ$ defined by* (11) *admits no interval representation with an upper ω-semicontinuous function φ^+, nor with an upper ω-semicontinuous function φ^-.*

Proof. Given an arbitrary interval representation of $\succ$, we consider arbitrary $y, x \in]1, 2[$ such that $y > x$. We have $y + 1 \succ x$, but $y + 1 \not\succ y$; therefore, $\varphi^+(x) < \varphi^-(y + 1) \leq \varphi^+(y)$. Similarly, $y \succ x - 1$, but $x \not\succ x - 1$; therefore, $\varphi^-(y) > \varphi^+(x - 1) \geq \varphi^-(x)$. In other words, both φ^+ and φ^- are strictly increasing on $]1, 2[$ and then jump down at 2. □

6 Conclusion

6.1. Every interval order generates a nonempty-valued and path independent choice function on finite subsets of its domain. When attention is switched to infinite subsets of a metric space, interval orders are dispersed along a four-level scale. A strongly acyclic interval order admits a maximal element on every nonempty subset; the choice function thus generated is always path independent. An ω-transitive interval order generates a nonempty-valued and path independent choice function on nonempty compact subsets. If the interval order is only ω-acyclic, then it admits a maximal element on every nonempty compact subset, but the choice function need not be path independent, nor even satisfy the outcast axiom. Finally, an arbitrary interval order need not admit maximal elements even on compact subsets.

For interval orders represented by closed intervals on the real line, strong acyclicity is ensured if the upper ends of all intervals are bounded above and the lengths of all intervals are separated from zero; ω-transitivity is ensured if the lower end of the representing interval is upper semicontinuous; ω-acyclicity, if upper semicontinuous is the upper end of the representing interval.

For semiorders, ω-transitivity and ω-acyclicity are equivalent, so there are three levels. Whenever the choice from every nonempty (compact) subset is possible, it is path independent.

6.2. When compared to Smith's (1974) Theorem 4.1, Theorem 3 here is naturally seen as an indication that a simple characterization of preferences that ensure the possibility of choice from compact subsets remains obtainable after a deviation from complete rationality provided the deviation is "small enough" (i.e., the replacement of an ordering with an interval order). It may be interesting to ponder on deviations from complete rationality in different directions, for instance, to non-binary choice. Nehring (1996) extended to that context the Bergstrom-Walker sufficient condition for the existence of maximal elements on compact subsets. A natural analogue of ω-acyclicity is obtained by adding, to Nehring's A1, the assumption that $C(S) \neq \emptyset$ whenever S is a convergent sequence plus its limit. Perhaps, the condition is necessary and sufficient for the possibility of choice from every compact subset under certain "rationality" requirements, which, most likely, should include or imply Nehring's A2 and A4. Nehring's A3 will then be superfluous. Such a theorem would be appealing from a purely aesthetical, as well as technical, viewpoint; its importance for decision theory would depend on the possibility to interpret those rationality conditions in a meaningful way.

6.3. Naturally, the sufficiency parts of our Theorems 1, 3, and 5 remain valid on any subclass of $\mathfrak{C}$, which cannot be said of their necessity statements. It is well known, for instance, that a binary relation may admit a maximal element on every convex, compact subset of its domain without even being acyclic. The prospects for obtaining simple characterizations in this context are bleak; even the restriction of attention to orderings provides no immediate help.

Acknowledgments

Financial support from a Presidential Grant for the State Support of the Leading Scientific Schools (NSh-5379.2006.1), the Russian Foundation for Basic Research (grant 05-01-00942), and the Spanish Ministry of Education and Science (projects SEJ 2004-00968 and SEJ 2007-67135) is acknowledged. I thank Francisco Marhuenda for procuring the last two grants and general support. I also thank an associate editor and two anonymous referees for their helpful comment on previous versions.

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