Maximizing a Binary Relation on Compact Subsets*

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Abstract

The sets of maximal elements of a binary relation on compact subsets of a metric space define a choice function. Possibilities to characterize natural properties of the choice function (path independence, nonempty values, closed values, etc.) by algebraic and topological conditions on the underlying relation are investigated. The latter are formulated in terms of "configurations" realizable or not for the relation. The existence of maximal elements on all compact subsets exceeds every other property in the complexity of conditions needed for its characterization.

Keywords: Binary relation; Maximal element; Choice function; Path independence; The existence of maximal elements

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1 Introduction

The notion of a choice function plays a central role in the decision theory (Fishburn, 1973; Sen, 1984; Aizerman and Aleskerov, 1995), the most important being the case of a choice function defined by a binary relation. The subject of this paper are connections between properties of such a choice function and properties of the underlying binary relation.

Our general framework is this. A binary relation is given on a metric space; undominated points of every subset define a choice function. We only consider the behaviour of the choice function on compact (nonempty) subsets. There is a more or less standard list of requirements, which can be divided into two subclasses: "point-wise" (non-emptiness, closedness, etc.) and "functional" ones (e.g., path independence). For each desirable property of the choice function, we try to find an equivalent condition on the relation, expressed in algebraic and topological terms. Some of the results obtained have a "heuristic" value as well, e.g., helping to establish the existence of maximizers (fixed points or Nash equilibria) in certain classes of models (Kukushkin 2003, 2005). However, here we concentrate on a purely formal aspect.

Were our attention focused on finite subsets, all our questions could be answered easily. For compact subsets, there is a considerable literature studying conditions for the existence of maximal elements (Smith, 1974; Bergstrom, 1975; Mukherji, 1977; Walker, 1977; Kiruta et al., 1980; Danilov and Sotskov, 1985; Campbell and Walker, 1990); however, a characterization result was only obtained by Smith (1974) under restriction to *preference relations*, which class is too narrow for many purposes.

It turns out that the existence of maximal elements on every compact subset is the most resistant to characterization among all properties from our list. In each of the other cases, a necessary and sufficient condition is found in the form of prohibition of certain "configurations" (e.g., a violation of transitivity can be viewed as a configuration consisting of three points, not necessarily different). In the former case, however, no such condition exists.

To be more precise, we define a hierarchy of classes of "configurational properties" of binary relations, which can be expressed in terms of possibility or impossibility to extend one configuration to another (to another). The non-emptiness of choice is proven not to belong to the simplest classes in the hierarchy. On the other hand, it can be characterized by a combination of configurational tests involving, at most, two checks whether a configuration could be extended to another; the equivalence is essentially tautological.

The next section contains necessary formal definitions. Section 3, main "positive" results: Theorem 1 characterizes path independence plus nonempty values; Theorem 2, the possibility to approximate the choice function from below with a path independent one having nonempty values; Theorem 3 provides a condition for the sets of maximal elements on nonempty compact subsets to be nonempty and compact themselves. The section ends with a modification of our basic problem, when some *a priori* restrictions are imposed on the relation; Theorem 4 characterizes interval orders for which the non-emptiness of choice is ensured.

Section 4 contains "negative" results. We define an "abstract configuration" and its realization in a metric space for a binary relation, and develop a formal theory of configurational tests and properties. Theorems 5 and 6 show that simplest configurational tests cannot distinguish between binary relations for which the non-emptiness of choice is ensured, and for which it is not. The last Section 5 contains a funny "positive" result, Theorem 7: a more sophisticated configurational test is described which does the job.

2 Basic Notions

A *binary relation* on a set A is a Boolean function on $A \times A$; as usual, we write $y \rhd x$ whenever the relation $\rhd$ is true on a pair (y, x). Henceforth, we assume that our binary relations are defined on a metric space A (a first countable Hausdorff topological space would do as well: what we actually need is that the topology on A be adequately described by convergent sequences). We denote $\mathfrak{B}$ the lattice of all subsets of A and $\mathfrak{C} \subset \mathfrak{B}$ the set of all nonempty compact subsets of A; when necessary, notation like $\mathfrak{C}(A)$ is used.

A *choice function* on A is a mapping $F : \mathfrak{B} \to \mathfrak{B}$ satisfying $F(X) \subseteq X$ for every $X \in \mathfrak{B}$. Let $X \subseteq A$; a point $x \in X$ is a *maximizer* for $\rhd$ on X if $y \rhd x$ does not hold for any $y \in X$. The set of all maximizers for $\rhd$ on X is denoted $M_{\rhd}(X)$; we omit the subscript when the relation is clear from the context. $M_{\rhd}(\cdot)$ defines a choice function on $\mathfrak{B}$; we are interested in the properties of this function on $\mathfrak{C}$. A choice function is *path independent* on $\mathfrak{C}$ if

$$F(X \cup Y) = F(F(X) \cup Y) \quad \text{whenever } X \cup Y \in \mathfrak{C}. \tag{1}$$

Remark. Plott's (1973) original definition was a bit different from (1), but both are equivalent. Theorems 1 and 2 below would remain valid if we restricted (1) to $X, Y \in \mathfrak{C}$.

A binary relation $\rhd$ is *acyclic* if there is no "finite improvement cycle", i.e., $x^0, x^1, \dots, x^{m+1} \in A$ such that $x^{k+1} \rhd x^k$ for $k = 0, \dots, m$ and $x^{m+1} = x^0$. When dealing with relations on (infinite) topological spaces, more general notions of acyclicity, based on improvement paths with well ordered sets of parameters, prove useful (Kukushkin, 2003).

A partially ordered set is *well ordered* if every subset contains a least point (then the set obviously must be a chain). Actually, there exists a "universal" well ordered set such that every countable, well ordered set is isomorphic to an initial segment of it (Natanson, 1974, Chapter XIV). To simplify the terminology, we will not refer to the concept without necessity. Considering a well ordered set Δ, we will denote 0 the least point of the whole Δ, and $\alpha + 1$, for $\alpha \in \Delta$, the least point exceeding α (the latter exists unless $\alpha = \max \Delta$). A point $\alpha \in \Delta \setminus \{0\}$ is called *isolated* if $\alpha = \beta + 1$ for some $\beta \in \Delta$; otherwise, α is called a *limit point*. It is sometimes convenient to consider a partial function $\alpha - 1$ defined by the equality $\alpha = (\alpha - 1) + 1$ for isolated α and not defined at all for limit points and for $\alpha = 0$.

Let $\rhd$ be a binary relation on A. An *improvement path* for $\rhd$ (the relation will not be mentioned when clear from the context) is a mapping $\pi : \Delta \to A$, where Δ is a countable, well ordered set, satisfying these two conditions:

$$\pi(\alpha + 1) \rhd \pi(\alpha) \text{ whenever } \alpha, \alpha + 1 \in \Delta; \tag{2a}$$

whenever $\alpha \in \Delta$ is a limit point, there exists a sequence $\{\beta^k\}_k \subset \Delta$ for which

$$\beta^{k+1} > \beta^k \text{ for all } k = 0, 1, \dots, \ \alpha = \sup_k \beta^k, \text{ and } \pi(\alpha) = \lim_{k\to\infty} \pi(\beta^k). \tag{2b}$$

An improvement path π is *narrow* if

$$\pi(\alpha) = \lim_{k\to\infty} \pi(\beta^k) \text{ whenever } \alpha \in \Delta \text{ is a limit point, and a sequence } \{\beta^k\}_k \subset \Delta \\ \text{is such that } \alpha = \sup_k \beta^k \text{ and } \beta^{k+1} > \beta^k \text{ for all } k = 0, 1, \dots \tag{2c}$$

Remark. By Theorem 4, Section 5, Chapter XIV of Natanson (1974), every limit point $\alpha \in \Delta$ is the least upper bound of a strictly increasing infinite sequence in Δ; therefore, (2c) implies (2b).

Thus, every point $\pi(\alpha)$ from an improvement path dominates the preceding point (if α is isolated) or is a limit point of the preceding points (if α is a limit point). When π is narrow, every $\pi(\alpha)$ with a limit α is the limit of the preceding points. An *improvement sequence* is an improvement path parameterized with the set $I\!N$ of natural numbers (or with a set isomorphic to $I\!N$).

A (*narrow*) *improvement cycle* for $\rhd$ is a (narrow) improvement path π such that $\pi(\alpha) = \pi(0)$ for an $\alpha > 0$. A relation $\rhd$ is called (*weakly*) Ω-*acyclic* if there is no (narrow) improvement cycle for $\rhd$.

At the end of the section, we prove a purely technical lemma indispensable in some proofs below.

Proposition 1. *If $\pi : \Delta \to A$ is an improvement path and $\alpha \in \Delta$ is a limit point, then the sequence $\{\beta^k\}_{k=0,1,\dots}$ in* (2b) *can be picked so that each β^k is isolated.*

Proof. Supposing the contrary, we can pick the least limit point $\alpha \in \Delta$ for which there exists no appropriate sequence. By (2b), there exists an infinite sequence γ^k such that $\gamma^{k+1} > \gamma^k$ for all $k = 0, 1, \dots$, $\beta = \sup_k \gamma^k$, and $\pi(\beta) = \lim_{k\to\infty} \pi(\gamma^k)$. Now we may construct the sequence $\{\beta^k\}$ by the following "algorithm": fix a numeric sequence $r_h \to 0$ (e.g., $r_h = 1/h$); pick the first k_1 for which $\rho(\pi(\beta), \pi(\gamma^{k_1})) < r_1$; if γ^{k_1} is isolated, define $\beta^1 = \gamma^{k_1}$; otherwise, pick $\beta' < \gamma^{k_1}$ which is isolated and satisfies $\rho(\pi(\beta), \pi(\beta')) < r_1$ (the statement of the lemma holds for $\gamma^{k_1} < \alpha$), and define $\beta^1 = \beta'$. Then repeat the same procedure with just two additional conditions: each new k_{h+1} must be greater than k_h chosen on the previous step and, when β' is being chosen at the step $h+1$, it must also satisfy $\beta' > \gamma^{k_h}$. Clearly, $\pi(\beta^k) \to \pi(\beta)$, $\beta = \sup_k \beta^k$, $\beta^{k+1} > \beta^k$ for all k, and each β^k is isolated. Thus, having assumed our statement wrong for α, we derived it, i.e., obtained a contradiction. □

3 Characterization Results

3.1 Path Independence

A binary relation $\succ$ on A is called ω-*transitive* if it is transitive and the conditions $x^\omega = \lim_{k\to\infty} x^k$ and $x^{k+1} \succ x^k$ for all $k = 0, 1, \dots$ always imply $x^\omega \succ x^0$.

Remark. It is worth noting that $x^\omega \succ x^k$ is valid for all $k = 0, 1, \dots$ in the above situation, once $\succ$ is ω-transitive.

The notion seems to have been first considered by Smith (1974) under the name of "σ-transitivity." However, the prefix "σ" traditionally refers to the cardinal concept of a countable set whereas what matters here is the order type of the set of natural numbers, usually referred to as ω.

Theorem 1. *Let $\succ$ be a binary relation on a metric space A. Then the operator $M_\succ$ is nonempty-valued and path independent on $\mathfrak{C}$ if and only if $\succ$ is irreflexive and ω-transitive.*

Proof. Let us prove the necessity first. If $x \succ x$, then $M(\{x\}) = \emptyset$. If $z \succ y \succ x$, then $\{z\} = M(\{x, y, z\}) = M(\{y, z\} \cup \{x\}) = M(\{x, z\})$, hence $z \succ x$. Finally, let $x^k \to x^\omega$ and $x^{k+1} \succ x^k$ for each k. Denoting $X = \{x^\omega\} \cup \{x^k\}_{k=0,1,\dots}$ and $X' = X \setminus \{x^0\}$, we have $x^k \notin M(X)$ for each k, hence $M(X) = \{x^\omega\}$; similarly, $M(X') = \{x^\omega\}$. Now $\{x^\omega\} = M(X' \cup \{x^0\}) = M(M(X') \cup \{x^0\}) = M(\{x^\omega, x^0\})$, hence $x^\omega \succ x^0$.

The key role in the sufficiency proof is played by the following

Lemma 1.1. *Let $\succ$ be an irreflexive and ω-transitive binary relation on a compact metric space X and $x^* \in X$. Then there is $z \in M_{\succ}(X)$ such that either $z = x^*$ or $z \succ x^*$.*

Remark. The statement immediately follows from Theorem 1 of Kukushkin (2003); however, the current proof is much shorter and needs weaker topological assumptions.

Proof. For each $x \in X$, we denote $G(x) = \{y \in X | \ y \succ x\}$. If x^* is not a maximizer, then $G(x^*) \neq \emptyset$; if we show that $M_{\succ}(G(x^*)) \neq \emptyset$, then we are obviously home.

To apply Zorn's Lemma (see, e.g., Kuratowski, 1966, p. 27), we have to consider an arbitrary chain $C \subseteq G(x^*)$ and show the existence of $y \in X$ such that $y \succ x$ or $y = x$ for each $x \in C$ (hence $y \in G(x^*)$). If C contains a greatest element, there is nothing to prove; otherwise $G(x) \neq \emptyset$ for each $x \in C$. We denote $F(x) = \operatorname{cl} G(x)$ and $F = \bigcap_{x \in C} F(x)$. Since C is a chain, all the sets $G(x)$ ($x \in C$), hence $F(x)$ too, contain each other; therefore, every finite intersection of $F(x)$ is not empty. Since X is compact, $F \neq \emptyset$.

Pick $y \in F$; let us prove that $y \succ x$ for each $x \in C$. Supposing the contrary, we pick $x^0 \in C$ for which $y \succ x^0$ does not hold and define a sequence $x^k \in G(x^0)$ ($k = 1, 2 \dots$) inductively so that $x^{k+1} \succ x^k$ and $x^k \to y$; then the ω-transitivity of $\succ$ implies $y \succ x^0$: a contradiction. Having $x^k \in G(x^0)$ already defined, we notice that $y \in F(x^k) \setminus G(x^k)$; therefore, we can pick $x^{k+1} \in G(x^k) \subset G(x^0)$ such that $0 < \rho(y, x^{k+1}) < \rho(y, x^k)/2$. Obviously, $x^{k+1} \succ x^k$ and $x^k \to y$. $\square$

The inclusion $M(X \cup Y) \subseteq M(M(X) \cup Y)$ holds for maximizers of every binary relation and for all sets X and Y. Let $x \in X \cup Y \setminus M(X \cup Y)$ and $X \cup Y \in \mathfrak{C}$; by Lemma 1.1, there is $z \in M(X \cup Y) \subseteq M(M(X) \cup Y)$ such that $z \succ x$. Therefore, even if $x \in M(X) \cup Y$, $x \notin M(M(X) \cup Y)$. $\square$

Remark. It is not difficult to characterize the path independence of $M_{\triangleright}$ alone, without non-emptiness; however, there is nothing interesting in that result.

3.2 Existence

Theorem 1 gives us a sufficient condition for the existence of maximizers. A *potential* for $\triangleright$ is an irreflexive and ω-transitive relation $\succ$ finer than $\triangleright$, i.e., satisfying $y \triangleright x \Rightarrow y \succ x$ for all $y, x \in A$.

Corollary. *If $\triangleright$ admits a potential, then $M_{\triangleright}(X) \neq \emptyset$ for each $X \in \mathfrak{C}$.*

Proof. Obviously, $M_{\succ}(X) \subseteq M_{\triangleright}(X)$ for any potential $\succ$ for $\triangleright$. $\square$

The Corollary often helps to establish the existence of maximizers. From the formal viewpoint, however, it is disputable whether the existence of a potential should be viewed as an "internal" property of the relation. Fortunately, the condition can be reformulated without mentioning other relations: Theorem 2 ([2.1] $\iff$ [2.2]) of Kukushkin (2003) states that a binary relation admits a potential if and only if it is Ω-acyclic. Therefore, the Corollary to Theorem 1 can be reformulated as $M_{\triangleright}(X) \neq \emptyset$ for each $X \in \mathfrak{C}$ if $\triangleright$ is Ω-acyclic.

Theorem 2. *Let $\rhd$ be a binary relation on a metric space A. Then a choice function F on A path independent on $\mathfrak{C}$ and satisfying the condition*

$$\emptyset \neq F(X) \subseteq M_{\rhd}(X) \quad \text{for each } X \in \mathfrak{C} \tag{3}$$

exists if and only if $\rhd$ is Ω-acyclic.

Proof. The sufficiency easily follows from Theorem 1: if $\succ$ is a potential for $\rhd$, then (3) holds for $F = M_{\succ}$, which is path independent on $\mathfrak{C}$.

Let an F exist and π be an improvement path for $\rhd$. As a first step, we prove that

$$F(\{\pi(\beta), \pi(\alpha)\}) = \{\pi(\alpha)\} \neq \{\pi(\beta)\} \tag{4}$$

for each $\beta < \alpha$.

Supposing the contrary, let $(\alpha, \beta) \in \Delta^2$ be a pair for which (4) does not hold and where α is the least possible. If α is isolated, we have $F(\{\pi(\alpha-1), \pi(\alpha)\}) \subseteq M_{\rhd}(\{\pi(\alpha-1), \pi(\alpha)\}) = \{\pi(\alpha)\}$, hence $F(\{\pi(\alpha-1), \pi(\alpha)\}) = \{\pi(\alpha)\}$; $\pi(\alpha) = \pi(\alpha-1)$ would imply $M_{\rhd}(\{\pi(\alpha-1)\}) = \emptyset$. Thus, (4) holds for $\beta = \alpha - 1$. For $\beta < \alpha - 1$, we have $F(\{\pi(\beta), \pi(\alpha-1)\}) = \{\pi(\alpha-1)\} \neq \{\pi(\beta)\}$ because α is the least possible, hence $\pi(\beta) \neq \pi(\alpha)$. Thus, $F(\{\pi(\beta), \pi(\alpha-1), \pi(\alpha)\}) = F(F(\{\pi(\beta), \pi(\alpha-1)\}) \cup \{\pi(\alpha)\}) = F(\{\pi(\alpha-1), \pi(\alpha)\}) = \{\pi(\alpha)\}$; on the other hand, $F(\{\pi(\beta), \pi(\alpha-1), \pi(\alpha)\}) = F(F(\{\pi(\alpha-1), \pi(\alpha)\}) \cup \{\pi(\beta)\}) = F(\{\pi(\alpha), \pi(\beta)\})$, i.e., (4) holds. Therefore, α cannot be isolated.

If α is a limit point, we invoke (2b), assuming, without restricting generality, that $\beta^0 = \beta$. The set $X = \{\pi(\beta^k)\}_{k=0,1,\ldots} \cup \{\pi(\alpha)\}$ is compact, hence $F(X) \neq \emptyset$. For each k, we have $F(X) = F(\{\pi(\beta^k), \pi(\beta^{k+1})\} \cup [X \setminus \{\pi(\beta^k)\}]) = F(F(\{\pi(\beta^k), \pi(\beta^{k+1})\}) \cup [X \setminus \{\pi(\beta^k)\}]) = F(X \setminus \{\pi(\beta^k)\})$ because (4) holds for $\beta^k, \beta^{k+1} < \alpha$; therefore, $\pi(\beta^k) \notin F(X)$ for each k, hence $F(X) = \{\pi(\alpha)\} \neq \{\pi(\beta)\}$. Denoting $X' = X \setminus \{\pi(\beta)\}$, we obtain $F(X) = \{\pi(\alpha)\}$ in the same way. Now $F(X) = F(F(X') \cup \{\pi(\beta)\}) = F(\{\pi(\beta), \pi(\alpha)\})$, hence (4) holds.

Now if $\rhd$ were not Ω-acyclic, we would have an improvement path with $\pi(\alpha) = \pi(0)$ for $\alpha > 0$; but this clearly contradicts (4). $\square$

As is well understood, the "usual" acyclicity is an infinite conjunction of the prohibition of an improvement cycle of the length m for every natural m, and all members of the conjunction are mutually independent. Similarly, (weak) Ω-acyclicity is an uncountable conjunction of the prohibition of particular cycles parameterized by the order types of countable, well ordered sets, i.e., countable ordinal numbers (Natanson, 1974, Chapter XIV). All the conditions are mutually independent.

Example 1. Let Δ be a countable, well ordered set such that $\alpha = \max \Delta$ exists. We introduce intrinsic topology on Δ (Birkhoff, 1967) and define the relation $\rhd$ on Δ by $\beta'' \rhd \beta'$ iff $\beta'' = \beta' + 1$; then the identity mapping $\Delta \to \Delta$ becomes a narrow improvement path. Let X be the result of the identification of $0 \in \Delta$ with $\alpha \in \Delta$; then X inherits the topology and relation $\rhd$ from Δ. Since Δ can be homeomorphically embedded into the real line (e.g., by the Debreu Theorem), X can be homeomorphically embedded into a circle. It is easy to see that X contains a narrow improvement cycle of the "length" α, but no "shorter" improvement cycle.

Proposition 2. *If $M_{\rhd}(X) \neq \emptyset$ for each $X \in \mathfrak{C}$, then $\rhd$ is weakly Ω-acyclic on A.*

This is the implication [2.6]⇒[2.7] from Theorem 2 of Kukushkin (2003).

A binary relation $\triangleright$ is called *ω-acyclic* if it is acyclic and the conditions $x^{k+1} \triangleright x^k$ for all $k = 0, 1, \dots$ and $x^0 = \lim_{k\to\infty} x^k$ are incompatible. A weakly Ω-acyclic relation (in particular, ω-transitive and irreflexive) is obviously ω-acyclic.

Corollary. *If $M_{\triangleright}(X) \neq \emptyset$ for each $X \in \mathfrak{C}$, then $\triangleright$ is ω-acyclic on A.*

Remark. The statement was proved by Mukherji (1977, Corollary 3).

Example 2. Let us consider a circle represented as the set of complex numbers with $|z| = 1$; formally, $B = \{e^{\mathrm{i}t} | \ t \in I\!R\}$. We define a binary relation $y \triangleright x \iff y = e^{\mathrm{i}} \cdot x$. Then we define $B' = B \setminus \{1\}$.

Proposition 3. *The relation $\triangleright$ defined in Example 2 is weakly Ω-acyclic, but not Ω-acyclic on both B and B'. It admits a maximizer on every nonempty compact subset of B', but not of B.*

Proof. The relation is acyclic because 1 is incommensurable with 2π. Let an infinite sequence $\{x^k\}_{k\in I\!N}$ be such that $x^{k+1} \triangleright x^k$ for all k, i.e., $x^k = e^{k\mathrm{i}} \cdot x^0$. By the Jacobi theorem (see, e.g., Billingsley, 1965), $\{x^k\}_{k\in I\!N}$ is dense in X; therefore, x^0 is a limit point, and we obtain a cycle. By the same token, $\{x^k\}_{k\in I\!N}$ cannot be convergent, so every narrow improvement path is finite; therefore, acyclicity implies weak Ω-acyclicity. Clearly, there is no maximizer for $\triangleright$ on B, which is compact itself.

Finally, let us show that every nonempty compact $X \subseteq B'$ admits a maximizer for $\triangleright$. Supposing the contrary, we would have a compact $X \subseteq B'$ containing an infinite sequence $\{x^k\}_{k\in I\!N}$ such that $x^{k+1} \triangleright x^k$ for all k. Again invoking the Jacobi theorem, we see that 1 is a limit point. On the other hand, X, being compact, must contain every limit point of $\{x^k\}_{k\in I\!N}$; therefore, $1 \in X \subset B \setminus \{1\}$. The contradiction completes the proof. $\square$

3.3 Closedness

The following condition, more or less tautologically, characterizes binary relations with closed sets $M_{\triangleright}(X)$:

there is no sequence $x^k \to x^\omega$ and $y \in A$ such that

$$\left[\ y \triangleright x^\omega,\ y \not\triangleright x^k \text{ for any } k,\ x^\omega \not\triangleright x^k \text{ for any } k, \text{ and } x^h \not\triangleright x^k \text{ for any } h, k\ \right]. \quad (5)$$

Proposition 4. *For every binary relation $\triangleright$, the following conditions are equivalent:*

4.1. *$M_{\triangleright}(X)$ is closed for every $X \in \mathfrak{C}$;*

4.2. *$\triangleright$ satisfies* (5)*;*

4.3. *$M_{\triangleright}(X)$ is closed in X for every $X \in \mathfrak{B}$.*

Proof. If (5) does not hold, we denote $X = \{x^k\}_{k=0,1,\dots} \cup \{x^\omega, y\} \in \mathfrak{C}$; obviously, $x^k \in M(X)$ for each k, but $x^\omega \notin M(X)$, i.e., $M(X)$ is not closed. If there is $X \in \mathfrak{B}$ such that $M(X)$ is not closed in X, there must be a sequence $x^k \to x^\omega$ such that $x^k \in M(X)$ and $x^\omega \notin M(X)$, hence there is $y \in X$ for which $y \triangleright x^\omega$; thus, we obtain a configuration prohibited by (5). The implication [4.3] $\Rightarrow$ [4.1] is trivial. $\square$

Theorem 3. *A binary relation $\triangleright$ on a metric space has the property that $M_{\triangleright}(X) \in \mathfrak{C}$ for every $X \in \mathfrak{C}$ if and only if $\triangleright$ is weakly Ω-acyclic and satisfies* (5)*.*

Proof. The necessity immediately follows from Propositions 2 and 4.

Lemma 3.1. *If $\rhd$ is weakly Ω-acyclic and satisfies* (5), *then for every improvement path $\pi : \Delta \to A$, every isolated $\alpha \in \Delta$, and every $\beta < \alpha$, there is a narrow improvement path from $\pi(\beta)$ to $\pi(\alpha)$.*

Proof. Supposing the contrary, we pick a pair $(\alpha, \beta) \in \Delta^2$ for which no such path exists; we also assume that (isolated) α is the least possible. If $\alpha - 1$ were isolated, there would exist a narrow improvement path from $\pi(\beta)$ to $\pi(\alpha - 1)$ because α is the least; adding to it $\pi(\alpha) \rhd \pi(\alpha - 1)$, we would obtain a narrow improvement path from $\pi(\beta)$ to $\pi(\alpha)$.

Let $\alpha - 1$ be a limit point. Invoking Proposition 1, we pick a sequence $\{\beta^k\}_{k=0,1,\dots}$ such that $\beta^{k+1} > \beta^k$ for all k, $\alpha - 1 = \sup_k \beta^k$, $\pi(\alpha - 1) = \lim_{k\to\infty} \pi(\beta^k)$, and each β^k is isolated; without restricting generality, $\beta^0 > \beta$. Since α is the least, there exist narrow improvement paths from $\pi(\beta)$ to each $\pi(\beta^k)$ (the paths need not be consistent in any sense). We call a subsequence $\{\beta^{k_h}\}_{h=0,1,\dots}$ *tight* if $k_{h+1} > k_h$ and $\pi(\beta^{k_{h+1}}) \rhd \pi(\beta^{k_h})$ for every h.

Let us suppose first that there is no infinite tight subsequence. Then we define an infinite sequence $\gamma^k \in \Delta$ by an inductive process. First, we pick a maximal finite tight subsequence starting at β^0: $\beta^0 = \beta^{k_0} < \beta^{k_1} < \cdots < \beta^{k_r}$ $(r \geq 0)$. For every $\beta^h > \beta^{k_r}$, the relation $\pi(\beta^{k_r}) \rhd \pi(\beta^h)$ would contradict the weak Ω-acyclicity of $\rhd$ because there exists a narrow improvement path from $\pi(\beta^{k_r})$ to $\pi(\beta^h)$, whereas $\pi(\beta^h) \rhd \pi(\beta^{k_r})$ would contradict the maximality of the tight subsequence. Therefore, $\pi(\beta^{k_r})$ is incomparable with $\pi(\beta^h)$ for any $h > k_r$. We set $\gamma^0 = \beta^{k_r}$ and repeat the same procedure replacing β^0 with $\beta^{k_r} + 1$. Eventually, we obtain a sequence $\{\gamma^k\}_{k=0,1,\dots}$ such that $\gamma^{k+1} > \gamma^k$ for all k, $\pi(\gamma^k) \to \pi(\alpha-1)$, and $\pi(\gamma^k)$ is incomparable with $\pi(\gamma^h)$ whenever $k \neq h$. The conditions (5) and $\pi(\alpha) \rhd \pi(\alpha - 1)$ imply that either $\pi(\alpha) \rhd \pi(\gamma^k)$ or $\pi(\alpha - 1) \rhd \pi(\gamma^k)$ must hold for some k. Now, taking a narrow improvement path connecting $\pi(\beta)$ to $\pi(\gamma^k)$ and adding to it $\pi(\alpha)$ in the first case or both $\pi(\alpha - 1)$ and $\pi(\alpha)$ in the second, we obtain a narrow improvement path connecting $\pi(\beta)$ to $\pi(\alpha)$.

If there *is* an infinite tight subsequence $\{\beta^{k_h}\}_{h=0,1,\dots}$, we take a narrow improvement path connecting $\pi(\beta)$ to $\pi(\beta^{k_0})$ and add the whole sequence $\{\pi(\beta^{k_h})\}_{h=0,1,\dots}$ plus $\pi(\alpha-1)$ and $\pi(\alpha)$. Thus, in every case, our impossibility hypothesis leads to a contradiction. $\square$

Now let us show that $\rhd$ is Ω-acyclic; then a reference to Theorem 2 ([2.1]$\Rightarrow$[2.6]) of Kukushkin (2003) will suffice. Indeed, let π be an improvement cycle, $\pi(0) = \pi(\alpha^*)$ with $\alpha^* > 0$. If α^* is isolated, we immediately apply Lemma 3.1 and obtain a narrow improvement cycle, contradicting the weak Ω-acyclicity of $\rhd$. If α^* is a limit point, we (re)define $\pi(\alpha^* + 1) = \pi(1) [\rhd \pi(0) = \pi(\alpha^*)]$ and apply Lemma 3.1 with $\beta = 1$ and $\alpha = \alpha^* + 1$, again obtaining a narrow improvement cycle. $\square$

3.4 Restrictions on Relations

An irreflexive and transitive relation $\succ$ is called an *interval order* if $[y \succ x \ \& \ a \succ b] \Rightarrow [y \succ b \ \text{ or } \ a \succ x]$ (actually, transitivity follows from the other conditions). An interval order is called a *semiorder* if $z \succ y \succ x \Rightarrow \forall a \in A\, [z \succ a \ \text{ or } \ a \succ x]$. A relation $\succ$ is called a *strict preference relation* if it is irreflexive, transitive and *negatively transitive*, i.e., $z \not\succ y \ \& \ y \not\succ x \Rightarrow z \not\succ x$. It is easy to see that every strict preference relation is a semiorder.

Proposition 5. *Let $\succ$ be a semiorder; then the following properties are equivalent:*

5.1. $M_\succ(X) \neq \emptyset$ *for every $X \in \mathfrak{C}$;*

5.2. *$M_{\succ}(X) \neq \emptyset$ for every $X \in \mathfrak{C}$ and $M_{\succ}(\cdot)$ is path independent on $\mathfrak{C}$;*

5.3. *$\succ$ is ω-transitive;*

5.4. *$\succ$ is ω-acyclic.*

Proof. The implications [5.2]⇒[5.1] and [5.3]⇒[5.4] are obvious; [5.1]⇒[5.4] follows from the Corollary to Proposition 2; [5.2] $\iff$ [5.3], from Theorem 1. Thus, it is sufficient to prove [5.4]⇒[5.3]. Let $\succ$ be an ω-acyclic semiorder, $x^k \to x^\omega$, and $x^{k+1} \succ x^k$ for all k. Then $x^2 \succ x^1 \succ x^0$ implies that either $x^\omega \succ x^0$ or $x^2 \succ x^\omega$. The latter would contradict the ω-acyclicity (with the sequence $x^\omega, x^2, x^3, \ldots$). □

The restriction of Proposition 5 to strict preference relations renders the main theorem of Smith (1974, Theorem 4.1).

Example 3. Let $A = [0, 1]$ and $y \succ x \iff 1 > y > x$ for all $y, x \in A$. Then A is compact, $\succ$ is an interval order, ω-acyclic but not ω-transitive. Clearly, $M(X) \neq \emptyset$ for each $X \in \mathfrak{C}$; however, $M(\cdot)$ is not path independent on $\mathfrak{C}$: denoting $X = \{k/(k+1)\}_{k=0,1,\ldots} \cup \{1\} \in \mathfrak{C}$, we have $M(X) = \{1\} = M(X \setminus \{0\})$, but $M(M(X \setminus \{0\}) \cup \{0\}) = \{1, 0\} \neq M(X)$. Thus, Proposition 5 does not hold for interval orders.

Theorem 4. *Let $\succ$ be an interval order on a separable metric space A. Then $M_{\succ}(X) \neq \emptyset$ for every $X \in \mathfrak{C}$ if and only if $\succ$ is ω-acyclic.*

Proof. The necessity immediately follows from the Corollary to Proposition 2.

Let $\succ$ be an ω-acyclic interval order, and let π be an improvement path for $\succ$. We show that

$$\pi(\alpha + 1) \succ \pi(\beta) \text{ whenever } \alpha \geq \beta \text{ and } (\alpha + 1) \in \Delta. \tag{6}$$

Supposing the contrary, we pick a pair $(\alpha, \beta) \in \Delta^2$ violating (6) with the least possible α. For $\beta = \alpha$, (6) follows from (2a), so $\beta < \alpha$. If α is isolated, then $\pi(\alpha + 1) \succ \pi(\alpha)$ from (2a) and $\pi((\alpha - 1) + 1) = \pi(\alpha) \succ \pi(\beta)$ from the minimality of α; therefore, (6) holds.

Let α be a limit point. Invoking Proposition 1, we pick a sequence $\{\beta^k\}_{k=0,1,\ldots}$ such that $\beta^{k+1} > \beta^k$ for all k, $\alpha = \sup_k \beta^k$, $\pi(\beta^k) \to \pi(\alpha)$, and each β^k is isolated. For the simplicity of notation, we assume $\beta^0 = \beta$. For each $k = 0, 1, \ldots$, we have $\pi(\beta^{k+1}) = \pi((\beta^{k+1} - 1) + 1) \succ \pi(\beta^k)$ because α is the least possible. Since $\pi(\alpha + 1) \succ \pi(\alpha)$ and $\succ$ is an interval order, either $\pi(\alpha + 1) \succ \pi(\beta^0)$ or $\pi(\beta^1) \succ \pi(\alpha)$; the latter relation, however, would contradict the ω-acyclicity (for the sequence $\pi(\alpha), \pi(\beta^1), \ldots$). Therefore, (6) holds for all admissible α and β.

Now let π be an improvement cycle, $\pi(0) = \pi(\alpha^*)$ with $\alpha^* > 0$. If α^* is isolated, we, from (6), derive $\pi(\alpha^*) = \pi((\alpha^* - 1) + 1) \succ \pi(0) = \pi(\alpha^*)$, contradicting the irreflexivity of $\succ$. If α^* is a limit point, we, as in the proof of Theorem 3, (re)define $\pi(\alpha^* + 1) = \pi(1)$ and, applying (6) with $\beta = 1$ and $\alpha = \alpha^*$, again obtain a contradiction with the irreflexivity. Therefore, $\succ$ is Ω-acyclic; a reference to Theorem 2 ([2.1]⇒[2.6]) of Kukushkin (2003) completes the proof. □

Campbell and Walker (1990) called a relation $\succ$ "weak lower continuous" if $y \succ x$ implies the existence of an open neighbourhood U of x such that $z \not\succ y$ for every $z \in U$. Obviously, the weak lower continuity of $\succ$ implies its ω-acyclicity; therefore, Theorem 1 of Campbell and Walker (when restricted to metric spaces) immediately follows from our Theorem 4. Weak lower continuity is not necessary for an interval order to admit a maximizer on every $X \in \mathfrak{C}$: consider a lexicographic order on a plane with fixed coordinates.

With a strict preference relation $\succ$, the *nonstrict preference relation* $\succeq$ can be associated: $y \succeq x \iff x \not\succ y$; the relation $\succeq$ is reflexive, transitive and complete ("weak order"). The relation $y \sim x \iff [y \succeq x \;\&\; x \succeq y]$ is an equivalence relation.

Proposition 6. *(Smith 1974, Theorem 4.2) A strict preference relation $\succ$ has the property that $M_\succ(X) \in \mathfrak{C}$ for every $X \in \mathfrak{C}$ if and only if $\succeq$ is ω-transitive.*

Proof. Let $\succeq$ be ω-transitive; then $\succ$ is obviously ω-transitive too, hence $M_\succ(X) \neq \emptyset$ for every $X \in \mathfrak{C}$ by Theorem 1. If $x^k \to x^\omega$ and $x^{k+1} \sim x^k$ for all k, then $x^\omega \succeq x^k$ for all k, hence $y \succ x^\omega$ implies $y \succ x^k$ for all k. Therefore, the configuration prohibited by (5) is impossible. A reference to Proposition 4 completes the sufficiency proof.

As to the necessity, let $x^k \to x^\omega$ and $x^{k+1} \succeq x^k$ for all k; without restricting generality, either $x^{k+1} \succ x^k$ for all k or $x^{k+1} \sim x^k$ for all k. In the first case, we have $x^\omega \succ x^0$ because the ω-transitivity of $\succ$ is necessary by Proposition 5. In the second case, assuming $x^0 \succ x^\omega$, we obtain a configuration prohibited by (5) (with $y = x^0$); therefore, $x^\omega \succeq x^0$. ☐

4 Impossibility Results

4.1 Configurations

We denote $I\!N = \{0, 1, \dots\}$ the chain of natural numbers starting from zero. An *abstract configuration* C consists of $\operatorname{Dom} C \subseteq I\!N$, $C_=, C_{\neq}, C_>, C_{\not>} \subseteq \operatorname{Dom} C \times \operatorname{Dom} C$, and $C_\to, C_{\not\to} \subseteq (\operatorname{Dom} C)^{I\!N}$, where $(\operatorname{Dom} C)^{I\!N}$ means the set of mappings $I\!N \to \operatorname{Dom} C$, i.e., sequences in $\operatorname{Dom} C$. In the following we use indices $\varkappa \in \{=, \neq, >, \not>, \to, \not\to\}$.

Let $\rhd$ be a binary relation on a metric space A and C be an abstract configuration. A *realization* of C in A for $\rhd$ is a mapping $\mu : \operatorname{Dom} C \to A$ such that: $\mu(k') = \mu(k)$ whenever $(k', k) \in C_=$; $\mu(k') \neq \mu(k)$ whenever $(k', k) \in C_{\neq}$; $\mu(k') \rhd \mu(k)$ whenever $(k', k) \in C_>$; $\mu(k') \not\rhd \mu(k)$ whenever $(k', k) \in C_{\not>}$; $\mu(\nu(k)) \to \mu(\nu(0))$ whenever $\nu \in C_\to$; $\mu(\nu(k)) \not\to \mu(\nu(0))$ whenever $\nu \in C_{\not\to}$.

Many natural properties of binary relations, including all those considered in Section 3, can be expressed as the impossibility to realize a certain configuration (or every configuration from a certain set). For example, to define the irreflexivity of $\rhd$, we can prohibit the realization of a configuration with $\operatorname{Dom} C = \{0\}$, $C_> = \{(0, 0)\}$, and other sets empty; transitivity, with $\operatorname{Dom} C = \{0, 1, 2\}$, $C_> = \{(1, 0), (2, 1)\}$ and $C_{\not>} = \{(2, 0)\}$. To define ω-transitivity, we additionally prohibit the realization of a configuration with $\operatorname{Dom} C = I\!N$, $C_> = \{(k+1, k)\}_{k=1,2,\dots}$, $C_{\not>} = \{(0, 1)\}$, and $C_\to = \{\nu^0\}$, where $\nu^0(k) = k$. To define acyclicity, we have to prohibit the realization of each of a countable set of configurations parameterized with $m \in I\!N$: $\operatorname{Dom} C^{(m)} = \{0, \dots, m+1\}$, $C^{(m)}_> = \{(1, 0), (2, 1), \dots, (m+1, m)\}$ and $C^{(m)}_= = \{(0, m+1)\}$; to define ω-acyclicity, we additionally prohibit the realization of a configuration with $\operatorname{Dom} C^{(\omega)} = I\!N$, $C^{(\omega)}_> = \{(k+1, k)\}_{k \in I\!N}$, and $C^{(\omega)}_\to = \{\nu^0\}$.

Weak Ω-acyclicity is equivalent to the impossibility to realize each configuration from an uncountable set parameterized with isolated countable ordinal numbers α: $\operatorname{Dom} C^{(\alpha)} = I\!N$; a bijection τ between $I\!N$ and a well ordered set Δ of the type α is fixed; $C^{(\alpha)}_= = \{(\tau^{-1}(0), \tau^{-1}(\max \Delta))\}$; $C^{(\alpha)}_{\neq} = C^{(\alpha)}_{\not>} = C^{(\alpha)}_{\not\to} = \emptyset$; $(k', k) \in C^{(\alpha)}_> \iff \tau(k') = \tau(k) + 1$;

$\nu \in C^{(\alpha)}_{\to} \iff \big[\tau(\nu(k+1)) > \tau(\nu(k))$ (for $k = 1, 2, \dots$) & $\tau(\nu(0)) = \sup_k \tau(\nu(k))\big]$. It is easy to see that a realization of $C^{(\alpha)}$ is the same thing as a narrow improvement cycle of the "length" α.

A similar redefinition of Ω-acyclicity needs an even larger set of prohibited configurations: For each limit ordinal $\beta \le \alpha$, we should pick an increasing sequence $\gamma^k \in \Delta$ for which $\beta = \sup_k \gamma^k$, and then include into $C_{\to}$ the sequence ν defined by $\nu(k) = \tau^{-1}(\gamma^k)$ and $\nu(0) = \tau^{-1}(\beta)$. Thus each single configuration $C^{(\alpha)}$ is replaced with a set parameterized with all possible choices of sequences γ^k (simultaneously for all $\beta < \alpha$).

Besides "negative" conditions, prohibiting certain configurations, we will consider "positive" ones, demanding that a certain configuration must have a realization. Obviously, the negation of a "negative" condition is "positive" and vice versa.

There are also properties of binary relations in the definition of which configurations are used in a subtler way. For instance, the existence of a maximizer on the whole A, $\exists x \nexists y\, [y \triangleright x]$, clearly refers to the configuration defined by $\operatorname{Dom} C = \{0, 1\}$, $C_{>} = \{(1, 0)\}$ and every other $C_{\varkappa}$ empty. However, a realization of this configuration is neither prohibited nor requested. Rather, it must be possible to start building a realization (choosing $\mu(0)$) in such a way that the completion of the process (the choice of $\mu(1)$) is impossible. To include such (and even more complicated) conditions into our formal framework, we need a definition of one configuration or realization extending another.

Let C and C' be abstract configurations; C' is an *extension* of C (denoted $C' \ge C$) if $\operatorname{Dom} C \subseteq \operatorname{Dom} C'$ and $C_{\varkappa} \subseteq C'_{\varkappa}$ for every $\varkappa$. Let $C' \ge C$, and μ and μ' be realizations of C and C', respectively, in the same A for the same $\triangleright$; then μ' is an *extension* of μ (denoted $\mu' \ge \mu$) if μ coincides with the restriction of μ' to $\operatorname{Dom} C$.

4.2 Configurational Tests

Let C be an abstract configuration. An *object over* C is $\mathcal{A} = \langle A, \triangleright, \mu\rangle$, where A is a metric space, $\triangleright$ is a binary relation on A, and μ is a realization of C in A for $\triangleright$. Strictly speaking, all objects over C form a "class" rather than a "set." However, we may fix a "universal set" $\mathfrak{U}$ large enough for any reasonable purpose, and only consider $A \subseteq \mathfrak{U}$; this simple trick allows us to speak of the *set* $\mathfrak{A}_C$ of all objects over C.

It is important to note that our definition of an abstract configuration allows an "empty" configuration with $\operatorname{Dom} C = \emptyset$ (hence $C_{\varkappa} = \emptyset$ for every $\varkappa$). This configuration, denoted simply $\emptyset$, admits a unique, empty, realization in every metric space A for every binary relation on A. Therefore, $\mathfrak{A}_{\emptyset}$ can be viewed as consisting of pairs $\langle A, \triangleright\rangle$, each of which defining a choice function $M_{\triangleright}$ we are interested in. Other sets $\mathfrak{A}_C$ play an auxiliary role.

A *test* over C is a mapping $T : \mathfrak{A}_C \to \{0, 1\}$. The set of all tests over C will be denoted $\mathfrak{T}_C$. An object $\mathcal{A} \in \mathfrak{A}_C$ *passes* a test $T \in \mathfrak{T}_C$ if $T(\mathcal{A}) = 1$.

Let $\mathcal{T} \subseteq \mathfrak{T}_C$ and $T \in \mathfrak{T}_C$. We say that T is a *combination* of tests from $\mathcal{T}$, which fact is denoted $T \in \operatorname{Comb} \mathcal{T}$, if, whenever $\mathcal{A}_1, \mathcal{A}_2 \in \mathfrak{A}_C$ are such that $T'(\mathcal{A}_1) \ge T'(\mathcal{A}_2)$ for all $T' \in \mathcal{T}$, we have $T(\mathcal{A}_1) \ge T(\mathcal{A}_2)$ as well. Clearly, if $T(\mathcal{A}_1) = T(\mathcal{A}_2)$ for all $T \in \mathcal{T}$, then $T(\mathcal{A}_1) = T(\mathcal{A}_2)$ for all $T \in \operatorname{Comb} \mathcal{T}$. The following properties of the operator Comb are easily checked:

$$\mathcal{T} \subseteq \operatorname{Comb} \mathcal{T}; \tag{7a}$$

$$\mathcal{T}' \subseteq \operatorname{Comb} \mathcal{T} \Rightarrow \operatorname{Comb} \mathcal{T}' \subseteq \operatorname{Comb} \mathcal{T}. \tag{7b}$$

Remark. In a sense, Comb $\mathcal{T}$ consists of logical combinations which do not use negation. The point is that negation affects both formal properties and the meaning of a test so strongly that it would make no sense to view it as an innocuous operation.

Let $T \in \mathfrak{T}_C$ and $\mathcal{A}', \mathcal{A}'' \in \mathfrak{A}_C$. We say that T *distinguishes* $\mathcal{A}''$ *from* $\mathcal{A}'$ if $T(\mathcal{A}'') = 1$ while $T(\mathcal{A}') = 0$. If $\mathcal{B}', \mathcal{B}'' \subseteq \mathfrak{A}_C$, we say that T *distinguishes* $\mathcal{B}''$ *from* $\mathcal{B}'$ if $T(\mathcal{A}'') = 1$ for every $\mathcal{A}'' \in \mathcal{B}''$ while $T(\mathcal{A}') = 0$ for every $\mathcal{A}' \in \mathcal{B}'$.

Proposition 7. *For every $\mathcal{T} \subseteq \mathfrak{T}_C$ and $\mathcal{B}', \mathcal{B}'' \subseteq \mathfrak{A}_C$, the existence of a test $T \in$ Comb $\mathcal{T}$ distinguishing $\mathcal{B}''$ from $\mathcal{B}'$ is equivalent to the existence, for each $\mathcal{A}'' \in \mathcal{B}''$ and $\mathcal{A}' \in \mathcal{B}'$, of a test $T[\mathcal{A}'', \mathcal{A}'] \in \mathcal{T}$ distinguishing $\mathcal{A}''$ from $\mathcal{A}'$.*

Proof. If $T[\mathcal{A}'', \mathcal{A}']$ does not exist for a pair $\mathcal{A}'' \in \mathcal{B}''$, $\mathcal{A}' \in \mathcal{B}'$, then $T(\mathcal{A}'') \leq T(\mathcal{A}')$ for all $T \in \mathcal{T}$. By the definition of Comb $\mathcal{T}$, the same inequality holds for all $T \in$ Comb $\mathcal{T}$, hence none of them could distinguish $\mathcal{A}''$ from $\mathcal{A}'$, to say nothing of $\mathcal{B}''$ from $\mathcal{B}'$.

Conversely, if $T[\mathcal{A}'', \mathcal{A}'] \in \mathcal{T}$ exists for every pair $\mathcal{A}'' \in \mathcal{B}''$, $\mathcal{A}' \in \mathcal{B}'$, then we define

$$T^*(\mathcal{A}) = \max_{\mathcal{A}'' \in \mathcal{B}''} \min_{\mathcal{A}' \in \mathcal{B}'} T[\mathcal{A}'', \mathcal{A}'](\mathcal{A}). \tag{8}$$

Obviously, $T^* \in$ Comb $\mathcal{T}$. Now if $\mathcal{A} \in \mathcal{B}''$, we have $T[\mathcal{A}, \mathcal{A}'](\mathcal{A}) = 1$ for all $\mathcal{A}' \in \mathcal{B}'$, hence $T^*(\mathcal{A}) = 1$; if $\mathcal{A} \in \mathcal{B}'$, then $T[\mathcal{A}'', \mathcal{A}](\mathcal{A}) = 0$ for all $\mathcal{A}'' \in \mathcal{B}''$, hence $T^*(\mathcal{A}) = 0$. □

Remark. In a sense, (8) is just a disjunctive form without negations. Actually, every $T \in$ Comb $\mathcal{T}$ can be represented in the form (8) if $\mathcal{A}'' \in \mathcal{B}''$ and $\mathcal{A}' \in \mathcal{B}'$ are perceived as arbitrary parameters.

Let $C' \geq C$. We define two operators from $\mathfrak{T}_{C'}$ to $\mathfrak{T}_C$:

$$\mathrm{Ext}_{C,C'}[T](A, \triangleright, \mu) = \max_{\mu' \geq \mu} T(A, \triangleright, \mu');$$

$$\mathrm{All}_{C,C'}[T](A, \triangleright, \mu) = \min_{\mu' \geq \mu} T(A, \triangleright, \mu');$$

where μ' is a realization of C', $\max_\emptyset = 0$ and $\min_\emptyset = 1$. Clearly, both $\mathrm{Ext}_{C,C}$ and $\mathrm{All}_{C,C}$ are identity mappings.

For every configuration C, we introduce two simplest tests imaginable, $\top_C, \perp_C \in \mathfrak{T}_C$, by $\top_C(\mathcal{A}) = 1$ and $\perp_C(\mathcal{A}) = 0$ for all $\mathcal{A} \in \mathfrak{A}_C$. We denote singleton subsets $\{\top_C\}, \{\perp_C\} \subset \mathfrak{T}_C$ by T_C and $\boldsymbol{\perp}_C$ respectively.

Let a class of tests $\mathcal{L}_C \subseteq \mathfrak{T}_C$ have been defined for every abstract configuration C. Then we define

$$\forall \mathcal{L}_C = \mathrm{Comb} \bigcup_{C' \geq C} \mathrm{All}_{C,C'} \mathcal{L}_{C'} \subseteq \mathfrak{T}_C; \tag{9a}$$

$$\exists \mathcal{L}_C = \mathrm{Comb} \bigcup_{C' \geq C} \mathrm{Ext}_{C,C'} \mathcal{L}_{C'} \subseteq \mathfrak{T}_C. \tag{9b}$$

Since $C' = C$ is allowed in both equalities (9), we have $\mathcal{L} \subseteq \exists\mathcal{L} \cap \forall\mathcal{L}$ by (7a). By induction, $\sigma_m \ldots \sigma_1 \mathcal{L} \subseteq \sigma_{m'} \ldots \sigma_m \ldots \sigma_1 \mathcal{L}$ for every $m' \geq m \geq 1$ and $\sigma_k \in \{\forall, \exists\}$.

A test $T \in \mathfrak{T}_\emptyset$ is *configurational* if it belongs to one of the classes inductively defined by (9), starting with $\mathcal{L} = \boldsymbol{\perp}$ or $\mathcal{L} = \mathsf{T}$ (e.g., $\forall\exists\exists\forall\forall\exists\boldsymbol{\perp}_\emptyset$). A property of pairs $\langle A, \triangleright \rangle$ (or a condition on such pairs) is *configurational* if the characteristic function of the subset of $\mathfrak{A}_\emptyset$ defined by the property or condition is a configurational test.

The meaning of the constructions well deserves discussion. Let $\mathcal{A} = \langle A, \triangleright, \mu\rangle \in \mathfrak{A}_C$ and $C' \geq C$; by definition, $\mathrm{All}_{C,C'} \perp_{C'}(\mathcal{A}) = 1$ if and only if the realization μ of C cannot be extended to a realization μ' of C'. In particular, every test from $\{\mathrm{All}_{\emptyset,C} \perp_C\}_C$ can be interpreted as the prohibition to realize a configuration, whereas tests from $\forall\perp_\emptyset$ are combinations, "positive disjunctive forms," of such elementary negative conditions. It is easy to see that all conditions obtained in Section 3 are of this form, even without disjunctions.

Similarly, $\exists\top_\emptyset$ consists of "positive" requirements: every configuration from one of specified sets of configurations must be realizable. Here belong various non-triviality conditions, e.g., $A \neq \emptyset$, $\#A \geq 3$, "A is infinite," or "there is a pair $y, x \in A$ such that $y \triangleright x$."

It is easily checked that $\forall\top_\emptyset = \top_\emptyset$ and $\exists\perp_\emptyset = \perp_\emptyset$; therefore, we should only consider classes whose descriptions end in $\forall\perp_\emptyset$ or $\exists\top_\emptyset$. It seems $\sigma_m \dots \sigma_1\mathcal{L} \subset \sigma_{m'} \dots \sigma_m \dots \sigma_1\mathcal{L}$ for $m' > m \geq 1$, $\mathcal{L} \in \{\top_\emptyset, \perp_\emptyset\}$, and $\sigma_k \in \{\forall, \exists\}$ unless an equality can be derived from those two, although there is no formal proof for the assertion. It is also worth noting that $\perp_C \subsetneq \exists\top_C$ and $\top_C \subsetneq \forall\perp_C$ because we can demand or prohibit something impossible by itself; therefore, $\sigma_m \dots \sigma_1\exists\top_C \supsetneq \sigma_m \dots \sigma_1\perp_C$ and $\sigma_m \dots \sigma_1\forall\perp_C \supsetneq \sigma_m \dots \sigma_1\top_C$.

The existence of a maximizer on the whole A, discussed at the end of Section 4.1, is obviously a configurational property from the class $\exists\forall\perp_\emptyset$. Other natural properties belong to the same class in a less obvious way.

Let $\mathrm{Dom}\, C = \{1, 2, \dots\}$ and $C_\varkappa = \emptyset$ for all $\varkappa$; $\mathrm{Dom}\, C' = I\!N$, $C'_{\not\to} = \{\nu : I\!N \to I\!N \mid \nu(0) = 0 \ \&\ \nu(k) > 0 \text{ for all } k > 0\}$ and $C'_\varkappa = \emptyset$ for all other $\varkappa$. Then the test $\mathrm{Ext}_{\emptyset,C}\, \mathrm{All}_{C,C'} \perp_{C'} \in \exists\forall\perp_\emptyset$ characterizes topologically separable metric spaces: there is a countable subset of A, $\mu(\mathrm{Dom}\, C)$, such that there is no point, $\mu'(0)$, which is not the limit of any sequence in the subset.

Let $\mathrm{Dom}\, C = \{2, 3, \dots\}$ and $C_\varkappa = \emptyset$ for all $\varkappa$; for each $I \subseteq \mathrm{Dom}\, C$, $\mathrm{Dom}\, C'[I] = I\!N$, $C'[I]_> = \{(1, 0)\} \cup \{(k, 1)\}_{k\in I} \cup \{(0, k)\}_{k\in(\mathrm{Dom}\, C)\setminus I}$ and $C'[I]_\varkappa = \emptyset$ for all other $\varkappa$. Then the test $\mathrm{Ext}_{\emptyset,C} \min_{I\subseteq \mathrm{Dom}\, C} \mathrm{All}_{C,C'[I]} \perp_{C'[I]} \in \exists\forall\perp_\emptyset$, added (with a conjunction, i.e., minimum) to the condition that $\triangleright$ is the strict component of a linear order (which condition obviously belongs to $\forall\perp_\emptyset \subseteq \exists\forall\perp_\emptyset$), defines Cantor's condition for the possibility to imbed A with the order into the real line: there is a countable subset of A, $\mu(\mathrm{Dom}\, C)$, such that every order interval contains a point from the subset.

The class $\forall\exists\top_\emptyset$ contains, e.g., the definition of a directed set: $\forall a, b\, \exists c\, [c \triangleright a \ \&\ c \triangleright b]$ (again, the condition that $\triangleright$ is an order should be added). Another property from the class is the convergence of every infinite improvement sequence.

In principle, every configurational test, regardless of its complexity, admits a reasonable interpretation. To the best of this author's knowledge, however, the lengthiest list of quantifiers needed in the formulation of a useful property is met in the definition of a (semi)lattice order:

$$\forall a, b\, \exists c\, \big[c \triangleright a \ \&\ c \triangleright b \ \&\, \nexists d\, [d \triangleright a \ \&\ d \triangleright b \ \&\ d \not\triangleright c]\big]. \tag{10}$$

It is easy to see that the condition is characterized by a configurational test from $\forall\exists\forall\perp_\emptyset$.

4.3 Main Negative Results

There is nothing surprising in the fact that all conditions used in the results of Section 3 belong to the class $\forall\perp_\emptyset$: all the properties of choice functions considered their are "inherited" (Walker, 1977) if we replace A with a subspace; similarly inherited is every configurational property from $\forall\perp_\emptyset$, while every other class of configurational properties contains some that are not. Much more interesting is that the non-emptiness of choice, which *is* inherited, does not belong to the class, nor even to some wider classes.

Let us define $T^{\mathrm{Max}} \in \mathfrak{T}_{\emptyset}$ by $T^{\mathrm{Max}}(\mathcal{A}) = 1$ if and only if $M_{\triangleright}(X) \neq \emptyset$ for every $X \in \mathfrak{C}(\mathcal{A})$.

Theorem 5. *$T^{\mathrm{Max}} \notin \forall\exists\mathsf{T}_{\emptyset}$.*

Proof. Let us suppose, to the contrary, that

$$T^{\mathrm{Max}} \in \forall\exists\mathsf{T}_{\emptyset} = \mathrm{Comb} \bigcup_{C} \mathrm{All}_{\emptyset,C} \, \mathrm{Comb}\{\mathrm{Ext}_{C,C'} \top_{C'}\}_{C' \geq C}.$$

As in Example 2, we define $B = \{e^{\mathrm{i}t} |\ t \in I\!R\}$, $B' = B \setminus \{1\}$, and $y \triangleright x \iff y = e^{\mathrm{i}} \cdot x$; we also denote $\mathcal{A} = \langle B, \triangleright \rangle$ and $\mathcal{A}' = \langle B', \triangleright \rangle$. By Proposition 3, $T^{\mathrm{Max}}(\mathcal{A}) = 0$ whereas $T^{\mathrm{Max}}(\mathcal{A}') = 1$.

By Proposition 7, there must be a configuration C and a test $T \in \mathrm{All}_{\emptyset,C} \, \mathrm{Comb}\{\mathrm{Ext}_{C,C'} \top_{C'}\}_{C' \geq C}$ distinguishing $\mathcal{A}'$ from $\mathcal{A}$. By the definition of the operator All, there must be a realization μ of C in B and a test $T' \in \mathrm{Comb}\{\mathrm{Ext}_{C,C'} \top_{C'}\}_{C' \geq C}$ such that $T'(\mathcal{A}'') = 0$ where $\mathcal{A}'' = \langle B, \triangleright, \mu \rangle \in \mathfrak{A}_C$.

Picking $r \in B \setminus \{1/\mu(k)\}_{k \in \mathrm{Dom}\, C}$, we define $\mu^* : \mathrm{Dom}\, C \to B$ by $\mu^*(k) = r \cdot \mu(k)$. Clearly, $\mu^*(k) = \mu^*(h) \iff \mu(k) = \mu(h)$, $\mu^*(k) \triangleright \mu^*(h) \iff \mu(k) \triangleright \mu(h)$, and $\mu^*(\nu(k)) \to \mu^*(\nu(0)) \iff \mu(\nu(k)) \to \mu(\nu(0))$ for all $k, h \in \mathrm{Dom}\, C$ and $\nu \in I\!N^{\mathrm{Dom}\, C}$; besides, $1 \notin \mu^*(\mathrm{Dom}\, C)$ by the choice of r. Thus, μ^* is a realization of C in B'; we denote $\mathcal{A}''' = \langle B', \triangleright, \mu^* \rangle \in \mathfrak{A}_C$. From $T(\mathcal{A}') = 1$, we obtain $T'(\mathcal{A}''') = 1$, again by the definition of the operator All.

Thus, T' distinguishes $\mathcal{A}'''$ from $\mathcal{A}''$; by Proposition 7, there must be a configuration $C' \geq C$ such that $\mathrm{Ext}_{C,C'} \top_{C'}(\mathcal{A}'') = 0$ whereas $\mathrm{Ext}_{C,C'} \top_{C'}(\mathcal{A}''') = 1$. By the definition of the operator Ext, the realization μ cannot be extended to a realization of C' in B, while there is a realization $\mu^{**} \geq \mu^*$ of C' in B'.

Now we define $\mu' : \mathrm{Dom}\, C' \to B$ by $\mu'(k) = \mu^{**}(k)/r$. For exactly the same reasons as above, μ' is a realization of C' in B. Besides, $\mu'(k) = \mu^{**}(k)/r = \mu^*(k)/r = \mu(k)$ for every $k \in \mathrm{Dom}\, C$; therefore, $\mu' \geq \mu$. The contradiction with the impossibility to extend μ proves the theorem. $\square$

Remark. A slight modification of the proof shows that $T^{\mathrm{Max}} \notin \forall\exists\ldots\exists\mathsf{T}_{\emptyset}$.

So far, it proved impossible to prove an exact analogue of Theorem 5 for the class $\exists\forall\perp_{\emptyset}$ without an additional restriction. For every abstract configuration C, we denote $\mathcal{C}^*(C)$ the set of configurations $C' \geq C$ such that $C'_{\not\to} \setminus C_{\not\to}$ is countable. Then we define

$$\mathcal{L}^* = \mathrm{Comb} \bigcup_{C} \mathrm{Ext}_{\emptyset,C} \, \mathrm{Comb}\{\mathrm{All}_{C,C'} \perp_{C'}\}_{C' \in \mathcal{C}^*(C)}. \tag{11}$$

Clearly, $\mathcal{L}^* \subseteq \exists\forall\perp_{\emptyset}$; of all properties from the latter class listed in Section 4.2, only topological separability seems not to belong to $\mathcal{L}^*$ (as usual, there is no formal proof for that).

Theorem 6. *$T^{\mathrm{Max}} \notin \mathcal{L}^*$.*

Proof. Let us suppose, to the contrary, that $T^{\mathrm{Max}} \in \mathcal{L}^*$. As in the proof of Theorem 5, we refer to Example 2, but with some elaboration: we use the same relation $\triangleright$, the same set B', and define $B'' = B' \cup B^*$, where $B^* = \{2 \cdot e^{\mathrm{i}t} | \ t \in I\!R\}$; geometrically, B'' is a disjoint union of a circle and a circle without a point. Then we define $\mathcal{A} = \langle B'', \rhd \rangle$, and $\mathcal{A}' = \langle B', \rhd \rangle$; again, $T^{\mathrm{Max}}(\mathcal{A}) = 0$ whereas $T^{\mathrm{Max}}(\mathcal{A}') = 1$.

By Proposition 7, there are a configuration C and a test $T \in \mathrm{Ext}_{\emptyset,C} \, \mathrm{Comb}\{\mathrm{All}_{C,C'} \perp_{C'}\}_{C' \in \mathcal{C}^*(C)}$ distinguishing $\mathcal{A}'$ from $\mathcal{A}$. By the definition of the operator Ext, there must be a realization μ of C in B' (hence in B'' as well) and a test $T' \in \mathrm{Comb}\{\mathrm{All}_{C,C'} \perp_{C'}\}_{C' \in \mathcal{C}^*(C)}$ such that $T'(\mathcal{A}''') = 1$, where $\mathcal{A}''' = \langle B', \rhd, \mu \rangle \in \mathfrak{A}_C$. Clearly, μ is a realization of C in B'' as well; we denote $\mathcal{A}'' = \langle B'', \rhd, \mu \rangle \in \mathfrak{A}_C$. By the definition of the operator Ext, $T'(\mathcal{A}'') = 0$.

Thus, T' distinguishes $\mathcal{A}'''$ from $\mathcal{A}''$; by Proposition 7, there must be a configuration $C' \in \mathcal{C}^*(C)$ such that $\mathrm{All}_{C,C'} \perp_{C'}(\mathcal{A}'') = 0$ whereas $\mathrm{All}_{C,C'} \perp_{C'}(\mathcal{A}''') = 1$. By the definition of the operator All, the realization μ cannot be extended to a realization of C' in B', but *can* be in B''.

Let $\mu' \geq \mu$ be a realization of C' in B''. We denote $M' = (\mu')^{-1}(B') \supseteq \mathrm{Dom}\, C$; $M^* = (\mu')^{-1}(B^*) \subseteq \mathrm{Dom}\, C' \setminus \mathrm{Dom}\, C$; $Y = \{x \in B'' | \ \exists k \in \mathrm{Dom}\, C' \, [x = \mu'(k)] \ \text{ or } \ \exists \nu \in C'_{\not\to} \setminus C_{\not\to} \, [x = \lim_{k \to \infty} \mu'(\nu(k))]\}$; $Y_1 = Y \cap B^*$; $Y_2 = (Y \cap B') \cup \{1\}$; $Z = \{2y_2/y_1, 2e^{\mathrm{i}}y_2/y_1, 2e^{-\mathrm{i}}y_2/y_1\}_{y_1 \in Y_1, \, y_2 \in Y_2}$. Since $C' \in \mathcal{C}^*(C)$, Z is countable. Picking $r \in B \setminus Z$, we define $\mu^* : \mathrm{Dom}\, C' \to B'$ by $\mu^*(k) = \mu'(k)$ for $k \in M'$ and $\mu^*(k) = r \cdot \mu'(k)/2$ for $k \in M^*$. Clearly, $\mu^*(k) = \mu(k)$ for $k \in \mathrm{Dom}\, C$. Let us check that μ^* is a realization of C'; the contradiction will prove the theorem.

The transformation of μ' into μ^* is identical on M' and isomorphic on M^* (as in the proof of Theorem 5); therefore, problems might emerge only between $k \in M'$ and $k' \in M^*$. Since B' and B^* are totally disconnected in B'' and μ' is a realization of C', neither $C'_=$, nor $C'_>$ can contain pairs (k, k') with k and k' belonging to different components of the partition $I\!N = M' \cup M^*$; similarly, if $\nu \in C'_\to$, then the number of $k \in I\!N$ for which $\nu(0)$ and $\nu(k)$ belong to different components of the partition is finite. Thus, only "negative" conditions from $C'_{\neq}$, $C'_{\not>}$, or $C'_{\not\to}$ could be violated by μ^*. However, this is impossible by the choice of r (and because the limits were included into Y). □

Remark. As in Theorem 5, we could modify the definition of $\mathcal{L}^*$, inserting several combinations $\mathrm{Comb} \bigcup_{C^{s+1} \geq C^s} \mathrm{Ext}_{C^s, C^{s+1}}$ between $\mathrm{Ext}_{\emptyset, C^1}$ and $\mathrm{Comb}\{\mathrm{All}_{C^m, C'} \perp_{C'}\}_{C' \in \mathcal{C}^*(C^m)}$ in (11), and still have a valid theorem.

A couple of technical points is worth discussion. First, there is a test from $\mathcal{L}^*$ distinguishing $\langle B', \rhd \rangle$ from $\langle B, \rhd \rangle$: Let $\mathrm{Dom}\, C = \{1, 2, \dots\}$, $C_> = \{(k+1, k)\}_{k \in \mathrm{Dom}\, C}$, $\mathrm{Dom}\, C' = I\!N$, $C'_> = C_>$, $C'_\to = \{\nu^*\}$, where $\nu^*(0) = 0$ and $e^{\nu^*(k)\mathrm{i}} \to 1$ as $k \to \infty$ (such ν^* exists by the Jakobi theorem), and $C_\varkappa = \emptyset$ and $C'_\varkappa = \emptyset$ for all $\varkappa$ not mentioned. Then $\langle B', \rhd \rangle$ passes $\mathrm{Ext}_{\emptyset,C} \, \mathrm{All}_{C,C'} \perp_{C'}$, but $\langle B, \rhd \rangle$ does not. Second, there is a test from $\exists\forall\bot_\emptyset \setminus \mathcal{L}^*$, distinguishing $\langle B', \rhd \rangle$ from $\langle B'', \rhd \rangle$, e.g., an improvement sequence is dense in $\langle B', \rhd \rangle$, but no improvement sequence is dense in $\langle B'', \rhd \rangle$ (formalization is straightforward).

Finally, let us note that the possibility to imbed a chain into the real line, as well as topological separability, are also inherited, but, most likely, do not belong to $\forall\bot_\emptyset$; the former belongs to $\mathcal{L}^*$, but the latter does not seem so. (Perhaps, formal proofs for all the three assertions could be produced after some effort, but studying such properties hardly has anything to do with decision theory.) So the non-emptiness of choice may not be unique in this respect.

5 A Characterization of Existence

Test T^{Max} proves configurational after all.

Theorem 7. *There exists a test $T^* \in \forall\exists\forall\bot_\emptyset$ such that $T^*(\mathcal{A}) = 1$, for $\mathcal{A} \in \mathfrak{A}_\emptyset$, if and only if $M_\triangleright(X) \neq \emptyset$ for each $X \in \mathfrak{C}(A)$. In other words, $T^{\mathrm{Max}} \in \forall\exists\forall\bot_\emptyset$.*

Remark. It is funny to note that the definition of a (semi)lattice (10) belongs to the same class.

Proof. First, we define an abstract configuration C^0 by $\operatorname{Dom} C^0 = \{2, 3, \dots\}$ and $C^0_\varkappa = \emptyset$ for all $\varkappa$. Then we denote $\mathrm{N} = \{\nu : I\!N \to \operatorname{Dom} C^0\}$ and $\Lambda = \{\lambda : I\!N \to I\!N \mid k' > k \Rightarrow \lambda(k') > \lambda(k)\}$. For each $\nu \in \mathrm{N}$, we define an abstract configuration $C^1[\nu]$ by $\operatorname{Dom} C^1[\nu] = \{0, 2, 3, \dots\}$, $C^1[\nu]_\to = \{\nu^*\}$, where $\nu^*(0) = 0$ and $\nu^*(k) = \nu(k)$ for $k > 0$, and $C^1[\nu]_\varkappa = \emptyset$ for all other $\varkappa$. For each $\nu, \nu' \in \mathrm{N}$, we define an abstract configuration $C^2[\nu, \nu']$ by $\operatorname{Dom} C^2[\nu, \nu'] = I\!N$, $C^2[\nu, \nu']_> = \{(1, 0)\}$, $C^2[\nu, \nu']_\to = \{\nu^*, \nu^{**}\}$, where ν^* is the same as above, while $\nu^{**}(0) = 1$ and $\nu^{**}(k) = \nu'(k)$ for $k > 0$, and $C^2[\nu, \nu']_\varkappa = \emptyset$ for all other $\varkappa$. It is easily checked that $C^2[\nu, \nu'] \geq C^1[\nu] \geq C^0$ for all $\nu, \nu' \in \mathrm{N}$.

Now we define the following configurational tests:

$$T^1 = \max_{\nu \in \mathrm{N}} \min_{\lambda \in \Lambda} \mathrm{All}_{C^0, C^1[\nu \circ \lambda]} \bot_{C^1[\nu \circ \lambda]} \in \forall\bot_{C^0} \subsetneq \exists\forall\bot_{C^0};$$

$$T^2 = \max_{\nu \in \mathrm{N}} \mathrm{Ext}_{C^0, C^1[\nu]} \min_{\nu' \in \mathrm{N}} \mathrm{All}_{C^1[\nu], C^2[\nu, \nu']} \bot_{C^2[\nu, \nu']} \in \exists\forall\bot_{C^0};$$

$$T^* = \mathrm{All}_{\emptyset, C^0} \max\{T^1, T^2\} \in \forall\exists\forall\bot_\emptyset.$$

Let us prove that T^* actually coincides with T^{Max}.

Let $\mathcal{A} = \langle A, \triangleright \rangle \in \mathfrak{A}_\emptyset$, $T^*(\mathcal{A}) = 1$, and $X \in \mathfrak{C}(A)$. As a compact metric space, X includes a countable dense subset Y. Let a realization μ^0 of C^0 in A be such that $\mu^0(\operatorname{Dom} C^0) = Y$; we can be sure of its existence because all $C^0_\varkappa$ are empty. We denote $\mathcal{A}' = \langle A, \triangleright, \mu^0 \rangle \in \mathfrak{A}_{C^0}$. By the definition of the operator All, we have $\max\{T^1(\mathcal{A}'), T^2(\mathcal{A}')\} = 1$. If $T^1(\mathcal{A}') = 1$, there must be $\nu \in \mathrm{N}$ such that, for every $\lambda \in \Lambda$, the realization μ^0 cannot be extended to a realization of $C^1[\nu \circ \lambda]$, which means that the sequence $\mu^0 \circ \nu$ in X does not contain a convergent subsequence, $\mu^0 \circ \nu \circ \lambda$; however, this contradicts the compactness of X.

Therefore, $T^2(\mathcal{A}) = 1$, hence there are $\nu \in \mathrm{N}$ and a realization $\mu^1 \geq \mu^0$ of $C^1[\nu]$ such that

$$\min_{\nu' \in \mathrm{N}} \mathrm{All}_{C^1[\nu], C^2[\nu, \nu']} \bot_{C^2[\nu, \nu']}(\mathcal{A}'') = 1, \tag{12}$$

where $\mathcal{A}'' = \langle A, \triangleright, \mu^1 \rangle \in \mathfrak{A}_{C^1[\nu]}$. Denoting $x = \mu^1(0)$, we have $x = \lim_{k \to \infty} \mu^1(k)$; since X is compact, $x \in X$. If there existed $y \in X$ for which $y \triangleright x$, we could take a sequence y^k in Y converging to y and $\nu' \in \mathrm{N}$ such that $\mu^0(\nu'(k)) = y^k$ for every $k > 0$. Now defining $\mu^2(1) = y$, we would have completed the definition of a realization of $C^2[\nu, \nu']$ which extends μ^1, contradicting (12). Therefore, $x \in M_\triangleright(X) \neq \emptyset$.

Now let $M_\triangleright(X) \neq \emptyset$ for each $X \in \mathfrak{C}(A)$; we have to prove $T^*(\mathcal{A}) = 1$. Let a realization μ^0 be fixed; we denote $Y = \mu^0(\operatorname{Dom} C^0)$, $X = \operatorname{cl} Y$, and $\mathcal{A}' = \langle A, \triangleright, \mu^0 \rangle \in \mathfrak{A}_{C^0}$. Every $x \in X$ is the limit of a sequence in Y, and such sequences are defined by $\nu \in \mathrm{N}$. We consider two alternatives.

Let X be compact; then there is a maximizer $x^0 \in X$ and a sequence in Y converging to x^0. We pick ν that defines the sequence, and define $\mu^1(0) = x^0$ while $\mu^1(k) = \mu^0(k)$ for all

$k \in \operatorname{Dom} C^0$. Clearly, μ^1 is a realization of $C^1[\nu]$. If a realization μ^2 of $C^2[\nu, \nu']$ were possible for a $\nu' \in \mathrm{N}$, we would have a point $x = \mu^2(1) \in X$ such that $x \rhd x^0$; but this would contradict the choice of x^0. Thus, $T^2(\mathcal{A}') = 1$.

Let X not be compact. Then there is a sequence x^k ($k \in I\!N$) in X no subsequence of which converges in X, hence in A as well. For every $k \in I\!N$, we pick $\nu(k)$ so that $\rho(x^k, \mu^0(\nu(k))) < 1/2^k$. Each $\lambda \in \Lambda$ defines a subsequence, $\mu^0 \circ \nu \circ \lambda$, of $\mu^0 \circ \nu$. The existence of a realization μ^1 of $C^1[\nu \circ \lambda]$ extending μ^0 would imply the convergence of $\mu^0 \circ \nu \circ \lambda(k)$ to $\mu^0 \circ \nu \circ \lambda(0)$, which, in turn, would imply the convergence of $x^{\lambda(k)}$, contradicting the choice of the sequence x^k. Thus, $T^1(\mathcal{A}') = 1$.

Since μ^0 was arbitrary, $T^*(\mathcal{A}) = 1$. □

Remark. Comparing Theorem 7 with the theorems from Section 4.3, we clearly see a gap: It is not known whether $T^{\mathrm{Max}} \in \exists\forall\bot_\emptyset \setminus \mathcal{L}^*$ or whether $T^{\mathrm{Max}} \in \forall\forall\bot_\emptyset$.

A natural reaction to the construction proving Theorem 7 might be a desire to modify our definitions so that the test T^{Max} cease to be configurational. We can demand that all sets and sequences participating in the definition of an abstract configuration should be *recursive*. Then the total set of abstract configurations can be parameterized with natural numbers; defining general configurational tests, we can again impose the recursiveness restriction. "Negative" Theorems 5 and 6 will obviously remain valid in the new situation; the conditions established in "positive" Theorems 1 and 4 will remain configurational. However, the conditions established in Theorems 2, 3, and 7 will hardly remain so. The same can be said of such properties as topological separability or the existence of a countable, order dense subchain. Thus, such an attempt to distinguish between "natural" and "too complicated" conditions would clearly be less than satisfactory.

At the moment, there is no idea of how to reformulate the definition of a configurational test so that all the conditions formulated in Section 3 remain configurational, whereas T^* defined in Theorem 7 violate some requirements. Perhaps this cannot be done in a convincing way. In any case, there is plenty of room for further exploration.

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