On the existence of maximal elements: An impossibility theorem*

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Abstract

All sufficient (as well as necessary) conditions for a binary relation on a metric space to admit a maximal element on every nonempty compact subset of its domain that have been found so far in the literature can be expressed as the prohibition of certain "configurations." No condition of this form could be necessary and sufficient simultaneously.

Keywords: Binary relation; Maximal element; Necessary and sufficient condition; Potential game

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1 Introduction

The choice of maximal elements of a binary relation is often used to model decision making in various contexts (Fishburn, 1973; Sen, 1984; Aizerman and Aleskerov, 1995). For a maximal element to be chosen, however, it must exist in the first place. Accordingly, there is a considerable literature studying what conditions should be imposed on a binary relation to ensure the existence of maximal elements on potential feasible sets (Gillies, 1959; Smith, 1974; Bergstrom, 1975; Mukherji, 1977; Walker, 1977; Kiruta et al., 1980; Danilov and Sotskov, 1983; Campbell and Walker, 1990). This paper demonstrates that there may be limits to such undertakings, independent of the ingenuity of particular authors.

As is well known, a binary relation admits a maximal element on every finite subset of its domain if and only if it is acyclic. Here we assume that every compact subset may emerge as the set of feasible alternatives. This, admittedly stylized, setting is a natural second step from a purely technical viewpoint, even though a narrower class of feasible sets may be justified in this or that particular context. Acyclicity obviously remains necessary, but not sufficient. Acyclicity plus open lower contours is sufficient, but too exacting as (weak) Pareto dominance, or lexicographic orders, show.

Smith (1974) found a condition necessary and sufficient for a *preference relation* (i.e., complete preorder) to attain a maximum on every compact subset of its domain. The restriction, however, need not be justified in every potential application, especially when the relation in question is supposed to reflect the preferences of several agents as well as their abilities to influence the outcome. For instance, a Nash equilibrium in a strategic game is naturally perceived as a maximal element of the individual improvement relation.

Here we put no *a priori* restriction on the relation. It turns out that no condition of a reasonable form can be equivalent to the existence of maximal elements on every nonempty compact subset of the universal domain.

The simplest admissible condition consists of, loosely speaking, one quantifier, either there exists or there does not exist, followed by a description of a "configuration," i.e., a list of points with the relation (or its absence) fixed between some pairs, and with the convergence (or the absence of it) fixed for some sequences. The exact definitions are semantic rather than syntactic. Then we allow conjunctions of such "elementary" conditions and, finally, disjunctive forms.

One justification for this particular notion of an admissible condition is its relative simplicity (necessary and sufficient conditions more complicated than the original property hardly make any sense). Another lies in the fact that all sufficient, as well as necessary, conditions found so far in our setting can be represented in the form. Still, our theorem does not cover more involved conditions (e.g., with several quantifiers), which are also used for various purposes in the literature.

The next section contains necessary formal definitions; Section 3, the main theorem and its proof. Implications for the theory of topological potential games are considered in Section 4. Section 5 shows that the same impossibility theorem holds for transitive relations. A discussion of related questions completes the paper.

2 Formulations

A *binary relation* on a set X is a Boolean function on $X \times X$; as usual, we write $y \rhd x$ whenever the relation $\rhd$ is true on a pair (y, x) and $y \ntriangleright x$ whenever it is false. Let $Y \subseteq X$; $x \in Y$ is a *maximizer* for $\rhd$ on Y if $y \ntriangleright x$ for every $y \in Y$. We denote $I\!N = \{0, 1, \dots\}$ the chain of natural numbers starting from zero.

An *abstract configuration* consists of $P_=, N_=, P_\rhd, N_\rhd \subseteq I\!N \times I\!N$ and $P_\to, N_\to \subseteq I\!N^{I\!N}$, where $I\!N^{I\!N}$ means the set of mappings $I\!N \to I\!N$, i.e., sequences in $I\!N$. Let $\rhd$ be a binary relation on a metric space X and C be an abstract configuration. A *realization* of C in X for $\rhd$ is a mapping $\mu : I\!N \to X$ such that: $\mu(k') = \mu(k)$ whenever $(k', k) \in P_=$; $\mu(k') \neq \mu(k)$ whenever $(k', k) \in N_=$; $\mu(k') \rhd \mu(k)$ whenever $(k', k) \in P_\rhd$; $\mu(k') \ntriangleright \mu(k)$ whenever $(k', k) \in N_\rhd$; $\mu(\nu(k)) \to \mu(\nu(0))$ whenever $\nu \in P_\to$; $\mu(\nu(k)) \not\to \mu(\nu(0))$ whenever $\nu \in N_\to$.

Many properties of binary relations can be expressed as the impossibility to realize a certain configuration. For example, to define the reflexivity of $\rhd$, we can prohibit the realization of a configuration with $N_\rhd = \{(0, 0)\}$ and other sets empty; to define irreflexivity, with $P_\rhd = \{(0, 0)\}$; transitivity, with $P_\rhd = \{(1, 0), (2, 1)\}$ and $N_\rhd = \{(2, 0)\}$. Open lower contours (lower continuity) are described by the prohibition of a configuration with $P_\rhd = \{(0, 1)\}$, $N_\rhd = \{(0, k)\}_{k \geq 2}$, and $P_\to = \{\nu^+\}$, where $\nu^+(k) = k + 1$; weak lower continuity (Campbell and Walker, 1990), by $P_\rhd = \{(0, 1)\} \cup \{(k, 0)\}_{k \geq 2}$ and $P_\to = \{\nu^+\}$ with the same $\nu^+(k)$. To formalize acyclicity in this style, we prohibit the realization of each of a countable set of configurations parameterized with $m \in I\!N$: $P_\rhd^{(m)} = \{(1, 0), (2, 1), \dots, (m + 1, m)\}$ and $P_=^{(m)} = \{(0, m + 1)\}$. The list can easily be extended.

Remark. One can always dispense with $P_=$, but a symmetric definition seems preferable.

A *simplest configurational condition* is defined by a set of abstract configurations $\mathcal{N}$. We say that such a condition *holds* on a metric space X for a binary relation $\rhd$ if no configuration $C \in \mathcal{N}$ admits a realization in X for $\rhd$. The class of all simplest configurational conditions is denoted $\mathcal{S}_0$.

Every condition from $\mathcal{S}_0$ is "inherited" (Walker, 1977): if such a condition holds on X for $\triangleright$, then it also holds on every $X' \subseteq X$ for the restriction of $\triangleright$ to X'. It seems natural, therefore, to use such conditions when trying to characterize properties of binary relations which are inherited by their nature (like the existence of a maximizer on every compact subset). It is also worth noting that, e.g., all the properties of binary relations considered in Duggan (1999) belong to $\mathcal{S}_0$.

Our impossibility result can be proven almost as easily for a wider class of conditions, viz. for logical combinations of "negative" and "positive" conditions. We formalize such combinations as "(infinite) disjunctive forms" made of (infinite) conjunctions of positive and negative conditions.

A *general configurational condition* (*C-condition*) consists of a set A of indices, and two sets of abstract configurations, $\mathcal{P}(\alpha)$ and $\mathcal{N}(\alpha)$, for every $\alpha \in \mathrm{A}$. We say that such a condition $\mathcal{C}$ *holds* on X for $\triangleright$ if there is $\alpha \in \mathrm{A}$ such that every configuration $C \in \mathcal{P}(\alpha)$, and no configuration $C \in \mathcal{N}(\alpha)$, admits a realization in X for $\triangleright$. The class of all general configurational conditions is denoted $\mathcal{S}_1$; obviously, $\mathcal{S}_0 \subset \mathcal{S}_1$.

Remark. We put no restriction on the cardinality, or the complexity, of the sets involved in the definition of a C-condition: the more freedom is allowed, the more convincing our negative result is.

3 Main Result

Theorem 1. *There exists no condition $\mathcal{C} \in \mathcal{S}_1$ such that $\mathcal{C}$ would hold on a subset X of a finite-dimensional Euclidean space for a binary relation $\triangleright$ on X if and only if $\triangleright$ admits a maximizer on every nonempty compact subset of X.*

Proof. Let $\mathcal{C}$ be such a condition. We consider $X' = \{e^{\mathrm{i}t} | \; t \in I\!\!R\}$ (where $\mathrm{i} = \sqrt{-1}$); geometrically, X' is a circle embedded into the plane of complex numbers. We define a binary relation $y \triangleright x \iff y = e^{\mathrm{i}} \cdot x$. Clearly, there is no maximizer for $\triangleright$ on X'; since X' is compact itself, $\mathcal{C}$ must not hold on X'.

Picking $x^0 \in X'$, we denote $X'' = X' \setminus \{x^0\}$. Let us show that $\triangleright$ admits a maximizer on every nonempty compact $Y \subseteq X''$. Supposing the contrary, we pick $y^0 \in Y$; since y^0 is not a maximizer, we can pick $y^1 \in Y$ such that $y^1 \triangleright y^0$; since y^1 is not a maximizer, we can pick $y^2 \in Y$ such that $y^2 \triangleright y^1$; etc. Since Y is compact, every limit point of $\{y^k\}_{k \in I\!\!N}$ must belong to Y. On the other hand, we have $y^{k+1} = e^{\mathrm{i}} \cdot y^k$; by the Jacobi theorem (see, e.g., Billingsley, 1965), $\{y^k\}_{k \in I\!\!N}$ is dense in X'. Therefore, x^0 is a limit point, hence $x^0 \in Y \subset X' \setminus \{x^0\}$. The contradiction proves our claim.

Therefore, $\mathcal{C}$ must hold on X'', i.e., there is $\alpha \in \mathrm{A}$ such that every configuration $C \in \mathcal{P}(\alpha)$, and no configuration $C \in \mathcal{N}(\alpha)$, admits a realization in X'' for $\rhd$. Every realization $\mu : \mathbb{N} \to X'' \subset X'$ being simultaneously a realization in X', and $\mathcal{C}$ not holding on X', there must be $C \in \mathcal{N}(\alpha)$ admitting a realization μ in X'. We pick $r \in X' \setminus \{x^0/\mu(k)\}_{k\in\mathbb{N}}$ and define $\mu^* : \mathbb{N} \to X'$ by $\mu^*(k) = r \cdot \mu(k)$. Clearly, $\mu^*(k) = \mu^*(h) \iff \mu(k) = \mu(h)$, $\mu^*(k) \rhd \mu^*(h) \iff \mu(k) \rhd \mu(h)$, and $\mu^*(\nu(k)) \to \mu^*(\nu(0)) \iff \mu(\nu(k)) \to \mu(\nu(0))$ for all $k, h \in \mathbb{N}$ and $\nu \in \mathbb{N}^{\mathbb{N}}$; besides, $x^0 \notin \mu^*(\mathbb{N})$ by the choice of r. Thus, μ^* is a realization of C in X'', contradicting the choice of α. $\square$

Remark. An anonymous referee raised a question of whether Theorem 1 would survive if the hypothetical C-condition were supposed only to work on compact domains (the non-compactness of X'' is inherent in the proof). Actually, we can delete a pair from the graph of the relation rather than a point from the domain (i.e., pick $x^0 \in X'$ and declare $e^{\mathrm{i}} \cdot x^0 \not\rhd x^0$), and again obtain the existence of a maximizer on every compact subset. Moreover, every realization for the original relation can be transformed, after an appropriate turn, into a realization for the "reduced" relation, exactly as in the above proof. Unfortunately, the converse is wrong. Therefore, we have an impossibility theorem for $\mathcal{C} \in \mathcal{S}_0$, but nothing is clear about $\mathcal{C} \in \mathcal{S}_1$.

4 Implications for the Theory of Potential Games

As usual, a *strategic game* Γ is defined by a finite set of players N, and strategy sets X_i and preference relations $\succeq_i$ on $X = \prod_{i\in N} X_i$ for all $i \in N$. With every strategic game, the *individual improvement relation* $\rhd^{\mathrm{Ind}}$ on X is associated ($y, x \in X$, $i \in N$):

$$y \rhd^{\mathrm{Ind}}{}_i\, x \iff [y_{-i} = x_{-i} \,\&\, y \succ_i x]; \tag{1a}$$

$$y \rhd^{\mathrm{Ind}} x \iff \exists i \in N\, [y \rhd^{\mathrm{Ind}}{}_i\, x]. \tag{1b}$$

By definition, a strategy profile $x \in X$ is a Nash equilibrium if and only if x is a maximizer for $\rhd^{\mathrm{Ind}}$ on X.

Monderer and Shapley (1996) introduced several classes of *potential games*, the "most ordinal" being that of a "generalized potential game." For a finite game, the property amounts to the acyclicity of the individual improvement relation; every such game obviously possesses a Nash equilibrium. Moreover, the existence of an equilibrium is preserved if arbitrary restrictions are imposed on feasible choices of each player, although Takahashi and Yamamori (2002) showed that the existence of a Nash equilibrium under arbitrary restrictions on strategies does not imply the acyclicity of individual improvements.

An equivalence between the acyclicity of individual improvements and persistent existence of equilibria holds if a wider class of modifications of the original game is allowed (strictly speaking, a modification of the concept of Nash equilibrium is needed as well). We assume that any strategy may prove infeasible to the relevant player, and any strategy profile may prove unacceptable to all players (say, entail a global nuclear conflict).

A *finite restriction* Γ^0 of Γ is defined by a finite (nonempty) set of *acceptable* profiles $\emptyset \neq X^0 \subseteq X$. An *acceptable Nash equilibrium* is $x^0 \in X^0$ such that $x^0 \succeq_i (x^0_{-i}, x_i)$ for every $i \in N$ and $x_i \in X_i$ such that $(x^0_{-i}, x_i) \in X^0$. Acceptable Nash equilibria can be viewed as maximizers of the individual improvement relation in Γ^0 if we modify the definition (1), restricting (1a) to $y, x \in X^0$ and replacing (1b) with

$$y \rhd^{\mathrm{Ind}} x \iff \left[x \in X \setminus X^0 \ \text{ or } \ \exists i \in N\,[y \rhd^{\mathrm{Ind}}{}_i\, x]\right].$$

Proposition 1. *Every finite restriction of Γ possesses an acceptable Nash equilibrium if and only if the individual improvement relation in Γ is acyclic (i.e., Γ is a generalized potential game).*

Proof. Let $\rhd^{\mathrm{Ind}}$ be acyclic and Γ^0 be a finite restriction of Γ. Then $\rhd^{\mathrm{Ind}}$ admits a maximizer on X^0, which is obviously an acceptable Nash equilibrium. Conversely, if there is a cycle $x^0, x^1, \ldots, x^m = x^0$ such that $x^{k+1} \rhd^{\mathrm{Ind}} x^k$, we define Γ^0 by $X^0 = \{x^0, x^1, \ldots, x^{m-1}\}$. Clearly, there is no acceptable Nash equilibrium in Γ^0. $\square$

The proposition seems to justify the following way to extend the notion of a "generalized potential game" to infinite (topological) games: Replace "finite" with "compact" in the definition of a restriction of Γ, and consider strategic games every compact restriction of which possesses an acceptable Nash equilibrium. However, Theorem 1 suggests that the property may resist reformulation in simpler terms.

Let us consider a two person game Γ' where $X_1 = X_2$ is X' from the proof of Theorem 1 whereas the preferences are given by utility functions: $u_1(x_1, x_2) = 1$ whenever $x_1 = x_2$, while $u_1(x_1, x_2) = 0$ otherwise; $u_2(x_1, x_2) = 1$ whenever $x_2 = e^{\mathrm{i}} \cdot x_1$, while $u_2(x_1, x_2) = 0$ otherwise. Actually, Γ' is a slight modification of Example 2 from Kukushkin (1999). Clearly, there is no Nash equilibrium in Γ' even though the set of strategy profiles is compact. We also consider another game, Γ'', where $X_1 = X_2$ is X'' from the same proof while the utilities are the same as in Γ'.

Proposition 2. *Every compact restriction of Γ'' possesses an acceptable Nash equilibrium.*

Proof. Let a restriction be defined by $X^0 \subset X'' \times X''$; we denote $X^* = \{x'' \in X'' |\ (x'', x'') \in X^0\}$. If $X^* = \emptyset$, then $u_1(x) = 0$ for all $x \in X^0$, hence the existence of an acceptable Nash equilibrium is obvious.

Let $X^* \neq \emptyset$; then it is a nonempty compact subset of X''. Invoking the second paragraph of the proof of Theorem 1, we can pick a maximizer x^* on X^* for the relation $\rhd$ from the same proof. We have $(x^*, x^*) \in X^0$ because $x^* \in X^*$; besides, $u_1(x^*, x^*) = 1$. If $(x^*, e^{\mathrm{i}} \cdot x^*) \notin X^0$, then (x^*, x^*) is an acceptable Nash equilibrium and we are home. Otherwise, we take into account that $e^{\mathrm{i}} \cdot x^* \notin X^*$ by the choice of x^* and derive that $(x^*, e^{\mathrm{i}} \cdot x^*) \in X^0$ is an acceptable Nash equilibrium. □

Similarly to Section 2, we could develop a notion of configurations and configurational conditions suited to strategic games. However, Proposition 2, together with the proof of Theorem 1, show that we would encounter essentially the same impossibility result: our tentative class of "truly potential" games admits no simple description (e.g., it cannot be described by the absence of cycles in any sense of the individual improvement relation).

On the one hand, the fact is disappointing; on the other hand, it may be an indirect argument for the notion of a "purely ordinal" potential game from Kukushkin (1999), where, at least, the existence of an acceptable Nash equilibrium in every compact restriction is ensured.

5 Transitive Relations

Theorem 1 becomes irrelevant if a necessary and sufficient condition is only supposed to work for a narrower class of relations. As was mentioned in the Introduction, Smith (1974) characterized preference relations admitting a maximizer on every compact subset by a condition from $\mathcal{S}_0$ (actually, the prohibition of just one configuration). Another condition from $\mathcal{S}_0$ (also just one configuration) characterizes *interval orders* with the property (Kukushkin, 2005, Theorem 4). Naturally, both conditions are equivalent for preference relations, but the configurations themselves are different.

The restriction to transitive relations, on the other hand, leaves Theorem 1 intact, only requiring a bit more complicated proof.

Theorem 2. *There exists no condition $\mathcal{C} \in \mathcal{S}_1$ such that $\mathcal{C}$ would hold on a subset X of a finite-dimensional Euclidean space for a transitive binary relation $\succ$ on X if and only if $\succ$ admits a maximizer on every nonempty compact subset of X.*

Proof. Let $\mathcal{C}$ be such a condition. We consider a torus represented as a Cartesian product $X' = T \times Z$, where T is the result of the identification of the end points of the closed interval $[0, 1]$ and Z is the set of complex numbers with $|z| = 1$. The relation $\succ$ on X' will be defined together with auxiliary constructions needed further on.

Denoting $T_k = \{t \in \mathbb{R} | \; 1 - 1/(k+1) \leq t < 1 - 1/(k+2)\}$, we may assume $T = \bigcup_{k \in \mathbb{N}} T_k$.

Then we define equivalence relations on X':

$$(t', z') \approx_k (t, z) \iff z'/z = \exp\bigl((k+1)(k+2)(t'-t) \cdot \mathrm{i}\bigr)$$

and

$$(t', z') \approx (t, z) \iff \exists k \in I\!N[t' \in T_k \,\&\, t \in T_k \,\&\, (t', z') \approx_k (t, z)].$$

It is worth noting that, whenever $k \in I\!N$, $t', t \in T$, and $z \in Z$, there is a unique $z' \in Z$ such that $(t', z') \approx_k (t, z)$. Finally, we define

$$(t', z') \succ (t, z) \iff [(t', z') \approx (t, z) \,\&\, t' > t].$$

It is easy to see that $\succ$ admits no maximizer on X' itself, which is compact. Therefore, $\mathcal{C}$ must not hold on X'.

Picking $z^0 \in Z$, we denote $X'' = T \times (Z \setminus \{z^0\}) \subset X'$. Let us show that $\succ$ admits a maximizer on every nonempty compact $Y \subseteq X''$. Let $(t, z) \in Y$ and $t \in T_{\bar{k}}$; we denote L the equivalence class for $\approx$ containing (t, z). If the projection of $Y \cap L$ to T contains a greatest t^+, then (t^+, z^+) from $Y \cap L$ (with a unique z^+) is a maximizer for $\succ$ on Y. Otherwise, there is a (unique) $z^* \in Z$ such that $(t, z) \approx_{\bar{k}} (1 - 1/(\bar{k}+2), z^*) \in Y$. Iterating the argument, we see that the absence of a maximizer for $\succ$ on Y would imply that Y contains an infinite sequence $\{(t^k, z^k)\}_{k > \bar{k}}$ such that $t_k = 1 - 1/(k+1)$ and $z^{k+1} = z^k \cdot e^{\mathrm{i}}$. By the Jacobi theorem, z^0 is a limit point of $\{z^k\}_k$, hence $(0, z^0) \in Y \subset X' \setminus \{(0, z^0)\}$. The contradiction proves our claim.

The rest of the proof is essentially the same as in Theorem 1. On the one hand, $\mathcal{C}$ must hold on X''. On the other hand, every realization in X'' is simultaneously a realization in X', whereas every realization in X' becomes a realization in X'' after a turn of Z in an appropriate angle. $\square$

6 Concluding Remarks

6.1. In the proofs, we did not have to assume that the hypothetical C-condition works for *any* subset of a Euclidean space, however complicated. In Theorem 1 it was only applied to a circle and to an open interval; in Theorem 2, to a torus and a cylinder. Admittedly, our relations were exotic, but this property is quite usual for mathematical counterexamples. Unfortunately, the class of "natural" relations seems impossible to define.

6.2. There are quite natural conditions on binary relations admitting no (obvious) representation of the form allowed here. Consider, for instance, the existence of a maximizer on the whole X,

$$\exists x \nexists y \, [y \rhd x], \tag{2}$$

or (the key condition in the definition of) a semilattice order,

$$\forall a, b\, \exists c\, \big[c \geq a \ \&\ c \geq b \ \&\, \nexists d\, [d \geq a \ \&\ d \geq b \ \&\ \text{not } d \geq c]\big].$$

Either formulation ends with a "normal" negative condition: something is impossible. However, that "something" is preceded with one or two extra quantifiers, and, in the latter case, also with positive requirements, which was not allowed by our definitions of Section 2. Thus, Theorem 1 does not preclude the possibility that the existence of a maximizer on every compact subset could be equivalent to a (combination of) condition(s) of such a form. Moreover, if there is no restriction on the use of quantifiers and logical operations, such a combination can be written down explicitly (Kukushkin, 2005, Section 5); however, it could not claim any usefulness in any applications.

6.3. Another natural property is the existence of maximizers on convex, compact subsets. The current notion of a C-condition gives no means to express convexity, so it seems implausible that a condition of that form could be equivalent to the property. A simple modification of our notions changes the situation: let us add a ternary relation meaning "x is a convex combination of y and z." Apparently, every sufficient condition in the literature now belongs to $\mathcal{S}_0$. The question of whether a condition of the form can be *necessary* and sufficient remains open.

6.4. It may well happen in an application that only the existence of maximal elements on a single set is of interest. General sufficient conditions may still be helpful, but this could hardly be said of general necessity results. Moreover, there is virtually no room for abstract theorizing in this situation: checking (2) for a particular relation may be easy or difficult (just as checking whether $y \triangleright x$ holds for a given relation and two given points), but one cannot hope to find a *simpler* way to express the same meaning, only *different* ones. It is conceivable that, say, Conditions 1–3 from Walker (1977) may be easier to check in a particular situation than (2) itself (although, to the best of this author's knowledge, no such example was ever presented in the literature), but it is equally conceivable that the only way to verify any of them could be by pointing at a maximizer.

Since (2) is not inherited, it cannot be equivalent to a condition from $\mathcal{S}_0$; the same is true for Conditions 1–3 from Walker (1977). Similar questions concerning $\mathcal{S}_1$ admit no obvious answers, but the questions themselves can be dismissed as too formalistic.

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