

Improvement Dynamics in Games with Strategic Complementarities*

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Abstract

In a finite game with strategic complementarities, every strategy profile is connected to a Nash equilibrium with a finite individual improvement path. If, additionally, the strategies are scalar, then every strategy profile is connected to a Nash equilibrium with a finite individual best response improvement path.

Key words: Nash equilibrium, Better response dynamics, Best response dynamics, Game with strategic complementarities

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1 Introduction

Learning and adaptation in strategic games is an important and interesting topic, attracting much attention (Milgrom and Roberts, 1990, 1991; Young, 1993; Kandori and Rob, 1995; Monderer and Shapley, 1996; Milchtaich, 1996; Fudenberg and Levine, 1998; Friedman and Mezzetti, 2001; Kukushkin, 2004). Quite often, the analysis of sophisticated scenarios involving random moves crucially depends on simple properties of the game, expressed in terms of improvement paths. It is essential, therefore, to understand when we can expect which property to hold.

For instance, when Young (1993) found the importance of the condition that every strategy profile should be connected to a Nash equilibrium with a best response path, he was unable to point out a single natural class of games where the property had been established. Since then, much work has been done; in particular, that property holds in every finite potential game as defined by Monderer and Shapley (1996), who provide quite a list of such games. Still, we cannot say that the subject is thoroughly investigated.

Convergence properties of improvement dynamics in games with strategic complementarities have been considered since the very beginning: Topkis (1979) and Vives (1990) showed that every best response improvement path starting from a certain region in the outcomes space eventually reaches a Nash equilibrium. Milgrom and Roberts (1990) studied rather general adaptive scenarios without explicitly considering their convergence. Kandori and Rob (1995) established the convergence to a Nash equilibrium of every best response path in every finite, symmetric, and strictly supermodular game with scalar strategies; Kukushkin (2004) established the same property of best response improvement paths in games with additive aggregation. Friedman and Mezzetti (2001) showed that every strategy profile in a finite game with strategic complementarities and one-dimensional strategy sets can be connected to a Nash equilibrium with an improvement path.

This paper strengthens the last result considerably: it turns out that Friedman and Mezzetti's statement holds in a multidimensional context as well, whereas their conditions ensure the possibility to reach a Nash equilibrium with a *best response* improvement path. Both extensions are more than purely technical hair-splitting. Although most games with strategic complementarities in the literature have scalar strategies, multidimensional models are also important: e.g., in the context of industrial organization, it is natural for a firm to choose several parameters (Vives, 2003, Section 4.4). The difference between better response and best response dynamics is crucial, e.g., for the scenario of Young (1993) mentioned above.

Strategic complementarities are interpreted in the broadest sense available in the literature: we assume single crossing (Milgrom and Shannon, 1994) and pseudosupermodularity (Agliardi, 2000) conditions. The proofs use the technique of binary relations suggested in Kukushkin (1999, 2004).

The attention here is restricted to finite games. Similar results can be obtained for the infinite (topological) case too; however, the proofs need rather heavy tools such as the notion of improvement paths parameterized with transfinite numbers, outlined in Kukushkin (2000).

Section 2 contains basic formal definitions. Section 3 reproduces conditions usually associated with the term “strategic complementarities.” Section 4 contains the main theorems and examples showing the impossibility of easy generalizations. The proofs are deferred to Section 5.

2 Basic Notions

A *strategic game* is defined by a finite set of players N , and strategy sets X_i and ordinal utility functions u_i on $X = \prod_{i \in N} X_i$ for all $i \in N$. In this paper, we assume that each X_i is a finite set.

For every player $i \in N$, the best response correspondence $R_i(\cdot)$ is defined in the usual way:

$$R_i(x_{-i}) = \{y_i \in X_i \mid \forall z_i \in X_i [u_i(y_i, x_{-i}) \geq u_i(z_i, x_{-i})]\},$$

where $x_{-i} \in X_{-i} = \prod_{j \in N \setminus \{i\}} X_j$.

We introduce a number of binary relations on X ($y, x \in X$ and $i \in N$):

$$y \triangleright_i x \iff [y_{-i} = x_{-i} \ \& \ u_i(y) > u_i(x)];$$

$$y \triangleright x \iff \exists i \in N [y \triangleright_i x];$$

$$y \triangleright_i^* x \iff [y_{-i} = x_{-i} \ \& \ u_i(y) > u_i(x) \ \& \ y_i \in R_i(x_{-i})];$$

$$y \triangleright^* x \iff \exists i \in N [y \triangleright_i^* x].$$

A strategy profile $x \in X$ is a Nash equilibrium if and only if x is a *maximizer* for \triangleright , i.e., if $y \triangleright x$ is impossible for any $y \in X$; equivalently, $x \in X$ is a Nash equilibrium if and only if x is a maximizer for \triangleright^* .

An (individual) *improvement path* is a finite or infinite sequence $\{x^k\}_{k=0,1,\dots}$, such that $x^{k+1} \triangleright x^k$ whenever $k \geq 0$ and x^{k+1} is defined. A *best response improvement path* is a

finite or infinite sequence $\{x^k\}_{k=0,1,\dots}$, such that $x^{k+1} \triangleright^* x^k$ whenever $k \geq 0$ and x^{k+1} is defined.

Putting together the terminology of Monderer and Shapley (1996), Milchtaich (1996), and Friedman and Mezzetti (2001), we introduce the following definitions.

A game has the *finite improvement path (FIP)* property if there exists no infinite improvement path. A game has the *finite best response improvement path (FBRP)* property if there exists no infinite best response improvement path. A game has the *weak FIP* property if, for every $x \in X$, there exists a finite improvement path $\{x^0, \dots, x^m\}$ such that $x^0 = x$ and x^m is a Nash equilibrium, i.e., if every strategy profile is connected to a Nash equilibrium with a finite improvement path. A game has the *weak FBRP* property if, for every $x \in X$, there exists a finite best response improvement path $\{x^0, \dots, x^m\}$ such that $x^0 = x$ and x^m is a Nash equilibrium.

It is easy to see that the following chain of implications holds:

$$\text{FIP} \Rightarrow \text{FBRP} \Rightarrow \text{weak FBRP} \Rightarrow \text{weak FIP}.$$

Each of these properties characterizes the global convergence property in a corresponding class of learning and adaptive dynamics. A stationary Markov process on X is said to be a better-reply (respectively best-reply) dynamic if (1) for each strategy profile x that is not a Nash equilibrium, there exists x' such that $x' \triangleright x$ (respectively $x' \triangleright^* x$) and the one step transition probability from x to x' is positive, and (2) every Nash equilibrium is absorbing. It is easy to show that a game has the FIP (respectively FBRP) property if and only if, under any better-reply (respectively best-reply) dynamic, the sequence of strategy profiles from any initial strategy profile almost surely converges to a Nash equilibrium. Alternatively, if we assume those dynamics to have full support, i.e., for *any* x' such that $x' \triangleright x$ (respectively $x' \triangleright^* x$), the one step transition probability from x to x' is positive, then we obtain the weak versions of the above statements. See Friedman and Mezzetti (2001), Kandori and Rob (1995), and Milchtaich (1996) for examples of better- and best-reply dynamics.

3 Games with Strategic Complementarities

For the term to be applicable to a game, the latter must satisfy several requirements, which can be formalized in somewhat different ways.

First, we assume that each X_i is a lattice; therefore, X and every X_{-i} are lattices too.

Second, there must be a complementarity condition on strategies of different players. The basic choice is between Topkis's (1979) cardinal *increasing differences* condition:

$$[y_i \geq x_i \ \& \ y_{-i} \geq x_{-i}] \Rightarrow [u_i(y) - u_i(x_i, y_{-i}) \geq u_i(y_i, x_{-i}) - u_i(x)] \quad (3.1)$$

and Milgrom and Shannon's (1994) ordinal *single crossing* condition:

$$[y_i \geq x_i \ \& \ y_{-i} \geq x_{-i}] \Rightarrow [\text{sign}(u_i(y) - u_i(x_i, y_{-i})) \geq \text{sign}(u_i(y_i, x_{-i}) - u_i(x))], \quad (3.2)$$

where $i \in N$, $x_i, y_i \in X_i$, $x_{-i}, y_{-i} \in X_{-i}$, and $\text{sign}(t)$ is -1 if $t < 0$, 0 if $t = 0$, and 1 if $t > 0$ (although subtraction is used in the second definition, the property itself is purely ordinal).

Finally, if strategies of a given player i are multidimensional, there must be a complementarity condition on different dimensions. Here we have even three versions: the cardinal *supermodularity* condition

$$u_i(x_i \vee y_i, z_{-i}) - u_i(x_i, z_{-i}) \geq u_i(y_i, z_{-i}) - u_i(x_i \wedge y_i, z_{-i}); \quad (3.3)$$

the ordinal *quasisupermodularity* condition (Milgrom and Shannon, 1994)

$$\text{sign}(u_i(x_i \vee y_i, z_{-i}) - u_i(x_i, z_{-i})) \geq \text{sign}(u_i(y_i, z_{-i}) - u_i(x_i \wedge y_i, z_{-i})); \quad (3.4)$$

the, also ordinal, *pseudosupermodularity* condition (Agliardi, 2000)

$$\begin{aligned} \text{sign}(\max\{u_i(x_i \vee y_i, z_{-i}) - u_i(x_i, z_{-i}), u_i(x_i \vee y_i, z_{-i}) - u_i(y_i, z_{-i})\}) \geq \\ \text{sign}(\max\{u_i(x_i, z_{-i}) - u_i(x_i \wedge y_i, z_{-i}), u_i(y_i, z_{-i}) - u_i(x_i \wedge y_i, z_{-i})\}). \end{aligned} \quad (3.5)$$

Again, $i \in N$, $x_i, y_i \in X_i$, and $z_{-i} \in X_{-i}$; $\text{sign}(t)$ has the same meaning as above.

The following implications are easy to check: (3.1) \Rightarrow (3.2); (3.3) \Rightarrow (3.4) \Rightarrow (3.5).

Lemma 3.1. *If a utility function u_i satisfies (3.2) and (3.5) for each $x_i, y_i \in X_i$ and $x_{-i}, y_{-i}, z_{-i} \in X_{-i}$, then*

$$[y_{-i} \geq x_{-i} \ \& \ y_i \in R_i(y_{-i}) \ \& \ x_i \in R_i(x_{-i})] \Rightarrow [y_i \vee x_i \in R_i(y_{-i}) \ \& \ y_i \wedge x_i \in R_i(x_{-i})].$$

The statement means that $R_i(x_{-i})$ is a sublattice of X_i (pick $y_{-i} = x_{-i}$) and $R_i(\cdot)$ is increasing w.r.t. the strong set order defined by Veinott (see Topkis, 1979).

Proof. Indeed, $x_i \in R_i(x_{-i})$ implies $u_i(x) \geq u_i(x_i \wedge y_i, x_{-i})$ and $u_i(x) \geq u_i(y_i, x_{-i})$, hence, by (3.5), $u_i(x_i \vee y_i, x_{-i}) \geq u_i(y_i, x_{-i})$, hence, by (3.2), $u_i(x_i \vee y_i, y_{-i}) \geq u_i(y)$, hence $x_i \vee y_i \in R_i(y_{-i})$. On the other hand, $y_i \in R_i(y_{-i})$ implies that $u_i(y) \geq u_i(x_i \vee y_i, y_{-i})$, hence, by (3.2), $u_i(y_i, x_{-i}) \geq u_i(x_i \vee y_i, x_{-i})$, hence, by (3.5), $u_i(x_i \wedge y_i, x_{-i}) \geq u_i(x)$, hence $x_i \wedge y_i \in R_i(x_{-i})$. \square

Remark. The lemma is obviously inspired by Proposition 3 of Agliardi (2000), but is formally independent of it.

All the above properties admit strict versions; we only use one of them. A utility function u_i is *strictly pseudosupermodular* if it satisfies (3.5) and, besides, whenever x_i and y_i are incomparable and the right hand side of (3.5) is 0, the left hand side is 1 (when x_i and y_i are comparable, (3.5) becomes an equality).

Remark. In a sense, a generic pseudosupermodular function on a finite set is strictly pseudosupermodular.

Lemma 3.2. *If a utility function u_i is strictly pseudosupermodular and satisfies (3.2) for each $x_i, y_i \in X_i$ and $x_{-i}, y_{-i} \in X_{-i}$, then*

$$[y_{-i} \geq x_{-i} \ \& \ y_i \in R_i(y_{-i}) \ \& \ x_i \in R_i(x_{-i})] \Rightarrow [y_i \leq x_i \ \text{or} \ y_i \geq x_i]$$

(i.e., $R_i(x_{-i}) \cup R_i(y_{-i})$ is a chain).

Proof. Suppose to the contrary that x_i and y_i are incomparable. Lemma 3.1 implies $x_i \wedge y_i \in R_i(x_{-i})$; therefore, the right hand side of (3.5) with $z_{-i} = x_{-i}$ is 0, hence the left hand side must be 1, i.e., $u_i(x_i \vee y_i, x_{-i}) > u_i(y_i, x_{-i})$. Now (3.2) implies $u_i(x_i \vee y_i, y_{-i}) > u_i(y_i, y_{-i})$, contradicting $y_i \in R_i(y_{-i})$. \square

4 Theorems and Examples

Theorem 1. *If a game satisfies (3.2) and (3.5), then it has the weak FIP property.*

If all strategies are scalar (i.e., each X_i is a chain), then (3.5) is satisfied automatically and may be dropped. Otherwise, it is essential as the following example shows.

Example 1. Let us consider a two person game with $X_1 = X_2 = \{(0,0), (0,1), (1,0), (1,1)\} \subseteq \mathbb{R}^2$ (with the standard order) and utilities described by these “matrices” (where player 1 chooses a 2×2 “block,” and player 2 a position in the block; the axes are directed upwards and to the right):

$$\begin{bmatrix} \underline{(9, 8)} & (6, 6) \\ (6, 6) & \underline{(8, 9)} \end{bmatrix} \quad \begin{bmatrix} (6, 6) & (7, 7) \\ (2, 2) & (6, 6) \end{bmatrix}$$

$$\begin{bmatrix} (6, 6) & (2, 2) \\ (7, 7) & (6, 6) \end{bmatrix} \quad \begin{bmatrix} \underline{(8, 9)} & (6, 6) \\ (6, 6) & \underline{(9, 8)} \end{bmatrix}$$

Although the statement of Lemma 3.1 (as well as the condition (3.1)) holds for the game, the condition (3.5) does not. There exist two Nash equilibria: $\langle(0, 0), (0, 0)\rangle$ and $\langle(1, 1), (1, 1)\rangle$. However, no strategy profile from $\{(0, 1), (1, 0)\} \times \{(0, 1), (1, 0)\}$ can be connected to an equilibrium with an improvement path.

Theorem 2. *If each strategy set X_i is a chain, except one, say X_1 , which is a lattice, (3.2) holds for all $i \in N$, and (3.5) for $i = 1$, then the game has the weak FBRP property.*

Example 2. Let us consider a two person game with the same strategy sets as in Example 1, but with different utilities (depicted under the same conventions):

$$\begin{array}{cc} \left[\begin{array}{cc} (\underline{4}, 0) & (3, 3) \\ (2, 2) & (0, \underline{4}) \end{array} \right] & \left[\begin{array}{cc} (3, 2) & (\underline{6}, \underline{6}) \\ (0, 0) & (2, 3) \end{array} \right] \\ \\ \left[\begin{array}{cc} (2, 3) & (0, 0) \\ (\underline{6}, \underline{6}) & (3, 2) \end{array} \right] & \left[\begin{array}{cc} (0, \underline{4}) & (2, 2) \\ (3, 3) & (\underline{4}, 0) \end{array} \right] \end{array}$$

The game satisfies both conditions of Theorem 1; actually, even strict versions of (3.1) and (3.3). However, no strategy profile from $\{(0, 1), (1, 0)\} \times \{(0, 1), (1, 0)\}$ can be connected to an equilibrium with a best response improvement path. (The best responses of both players are underlined in the “matrix.”)

Theorem 3. *If, in a two person game, X_1 is a lattice, X_2 is a chain, (3.2) holds for both i , and u_1 is strictly pseudosupermodular, then the game has the FBRP property.*

Remark. If both X_i are chains, the FBRP property immediately follows from Theorem 1 of Kukushkin (2004): condition (2) there is even weaker than our (3.2).

Example 3. Let us consider a two person game with $X_1 = \{(0,0), (0,1), (1,0), (1,1)\} \subseteq \mathbb{R}^2$, $X_2 = \{0, 1\} \subseteq \mathbb{R}$ (with the standard order) and utilities described by these “matrices” (where player 1 chooses a block enclosed by brackets and player 2 a position in the block; the axes are directed upwards and to the right):

$$\begin{array}{cc} \left[\begin{array}{cc} (2, \underline{3}) & (\underline{3}, 2) \end{array} \right] & \left[\begin{array}{cc} (2, 2) & (\underline{3}, \underline{3}) \end{array} \right] \\ \\ \left[\begin{array}{cc} (\underline{3}, \underline{3}) & (2, 2) \end{array} \right] & \left[\begin{array}{cc} (\underline{3}, 2) & (2, \underline{3}) \end{array} \right] \end{array}$$

The condition (3.1) holds for both players; u_1 is supermodular, but not strictly pseudosupermodular. There is a best response improvement cycle:

$$\langle(1, 0), 1\rangle \xrightarrow{1} \langle(0, 1), 1\rangle \xrightarrow{2} \langle(0, 1), 0\rangle \xrightarrow{1} \langle(1, 0), 0\rangle \xrightarrow{2} \langle(1, 0), 1\rangle.$$

Example 4. Let us consider a three person game with $X_i = \{0, 1\}$ for all i and utilities $u_i(x) = -|x_i - x_{\varphi(i)}|$, where $\varphi(1) = 2$, $\varphi(2) = 3$, $\varphi(3) = 1$. The condition (3.1) obviously holds; all strategy sets are chains. However, there is a best response improvement cycle:

$$\langle 1, 0, 1 \rangle \xrightarrow{1} \langle 0, 0, 1 \rangle \xrightarrow{2} \langle 0, 1, 1 \rangle \xrightarrow{3} \langle 0, 1, 0 \rangle \xrightarrow{1} \langle 1, 1, 0 \rangle \xrightarrow{2} \langle 1, 0, 0 \rangle \xrightarrow{3} \langle 1, 0, 1 \rangle.$$

Finally, the example from Section 7.6 of Kukushkin (2004) shows that we cannot expect the FIP property even in a two person game with scalar strategies and strictly increasing differences.

5 Proofs

5.1 Proof of Theorem 1

To produce an improvement path from an arbitrary strategy profile to an equilibrium, we impose the following rules: (1) If, at a current profile, there exist profitable deviations upwards (i.e., with $y_i > x_i$), one of them must be chosen. (2) Otherwise, one of the most profitable (for the deviating player) deviations downwards (with $y_i < x_i$) must be chosen. (3) If neither (1) nor (2) can be applied, stop. It turns out that a path abiding by these rules cannot cycle and only stops at a Nash equilibrium.

Let us introduce some notation:

$$X^+ = \{x \in X \mid \exists y \in X [y > x \ \& \ y \triangleright x]\}; \quad X^- = X \setminus X^+;$$

$$R_i^-(x) = \{y_i \in X_i \mid y_i \leq x_i \ \& \ \forall z_i \in X_i [z_i \leq x_i \Rightarrow u_i(y_i, x_{-i}) \geq u_i(z_i, x_{-i})]\};$$

$$y \triangleright_i x \iff [y \triangleright_i x \ \& \ (y_i > x_i \ \text{or} \ [x \in X^- \ \& \ y_i \in R_i^-(x)])];$$

$$y \triangleright x \iff \exists i \in N [y \triangleright_i x];$$

$$y \succ x \iff [(y \in X^- \ \& \ x \in X^+) \ \text{or} \ (x, y \in X^+ \ \& \ y > x) \ \text{or} \ (x, y \in X^- \ \& \ y < x)].$$

It is easy to check that \succ is irreflexive and transitive. By definition, $y \triangleright x \Rightarrow y \triangleright x$; an improvement path abides by the rules if and only if $x^{k+1} \triangleright x^k$ for all relevant k and the path either is infinite or ends at a maximizer for \triangleright .

Lemma 5.1.1. *If $y \triangleright x$, then $y \succ x$.*

Proof. The only point worth discussing is the incompatibility of $x \in X^-$ and $y \triangleright_j x$ with $y \in X^+$. Suppose the contrary: there are $i \in N$ and $z_i > y_i$ such that

$$u_i(z_i, y_{-i}) > u_i(y). \tag{5.1}$$

Since $x \in X^-$, $y_j < x_j$. If $i \neq j$, then $y_i = x_i$ and $y_{-i} < x_{-i}$. Now (5.1) and (3.2) imply $u_i(z_i, x_{-i}) > u_i(x)$, contradicting $x \in X^-$.

Thus, we are led to $i = j$, hence $y_{-i} = x_{-i}$. Now $y_i \in R_i^-(x)$ implies $u_i(y) \geq u_i(z_i \wedge x_i, x_{-i})$, hence, by (5.1) and (3.5), $u_i(z_i \vee x_i, x_{-i}) > u_i(x)$, hence $z_i \vee x_i > x_i$, again contradicting $x \in X^-$. \square

Lemma 5.1.2. *If $x \in X$ is a maximizer for \triangleright , then x is a Nash equilibrium (i.e., a maximizer for \triangleright).*

Proof. Suppose the contrary: $x \in X$ is a maximizer for \triangleright (i.e., there is no profitable deviation either upwards or downwards), but not a Nash equilibrium, i.e., there exist $i \in N$ and $y_i \in X_i$ such that $u_i(y_i, x_{-i}) > u_i(x)$ (then y_i must be incomparable with x_i). We have $x_i \wedge y_i < x_i$, hence $u_i(x_i \wedge y_i, x_{-i}) \leq u_i(x) < u_i(y_i, x_{-i})$; similarly, $x_i \vee y_i > x_i$, hence $u_i(x_i \vee y_i, x_{-i}) \leq u_i(x) < u_i(y_i, x_{-i})$. Now we have a contradiction with (3.5). \square

Lemma 5.1.1 implies that *every* improvement path abiding by the rules ends at a maximizer for \triangleright . Lemma 5.1.2 implies that the maximizer is a Nash equilibrium. Theorem 1 is proven.

5.2 Proof of Theorem 2

We will reason similarly to the proof of Theorem 1, considering best response improvement paths abiding by the following rules: (1) If, at a current profile, $x_1 \notin R_1(x_{-1})$, then a best response improvement by player 1 must be implemented, upwards (with $y_1 > x_1$) if possible; otherwise, we choose any best response improvement by player 1 if there is a player $i \neq 1$ with a possible best response improvement upwards ($y_i > x_i$), but try to find a best response improvement $y_1 < x_1$ first if there is no such i . (2) If $x_1 \in R_1(x_{-1})$ and there exist best response improvements upwards (with $y_i > x_i$), one of them must be chosen. (3) Otherwise, one of the best response improvements downwards (with $y_i < x_i$) must be chosen. (4) If none of (1), (2), (3) can be applied, stop. Again, a path abiding by the rules cannot cycle and only stops at a Nash equilibrium.

Let us introduce some notation:

$$X^1 = \{x \in X \mid \exists y > x[y \triangleright_1^* x] \text{ or } (x_1 \in R_1(x_{-1}) \ \& \ \exists y > x[y \triangleright_1^* x])\};$$

$$X^2 = \{x \in X \mid \nexists y > x[y \triangleright_1^* x] \ \& \ (x_1 \in R_1(x_{-1}) \text{ or } \exists y < x[y \triangleright_1^* x])\};$$

$$X^0 = X \setminus (X^1 \cup X^2);$$

$$y \triangleright_i x \iff [y \triangleright_i^* x \ \& \ (x_1 \notin R_1(x_{-1}) \Rightarrow i = 1) \ \& \\ (x \in X^1 \Rightarrow y_i > x_i) \ \& \ (x \in X^2 \Rightarrow y_i < x_i)];$$

$$y \triangleright x \iff \exists i \in N [y \triangleright_i x];$$

$$y \succ x \iff [(y \in X^k \ \& \ x \in X^h \ \& \ k > h) \ \text{or} \\ (x, y \in X^1 \ \& \ y > x) \ \text{or} \ (x, y \in X^2 \ \& \ y < x)].$$

It is easy to check that \succ is irreflexive and transitive. By definition, $y \triangleright x \Rightarrow y \triangleright_i^* x$; a best response improvement path abides by the rules if and only if $x^{k+1} \triangleright x^k$ for all relevant k and the path either is infinite or ends at a maximizer for \triangleright .

The next lemma describes, in a compact way, the following properties of our paths. From $x \in X^0$, we pass to $X^1 \cup X^2$ and never return back. From $x \in X^1$, we always go upwards. From $x \in X^2$, we always go downwards and never come to $X^0 \cup X^1$.

Lemma 5.2.1. *If $y \triangleright_j x$ ($j \in N$), then $y \succ x$.*

Proof. (A) Let $x \in X^2$, then $y < x$, so $y \succ x$ provided $y \in X^2$. Suppose, to the contrary, that there are $i \in N$ and $z_i > y_i$ such that

$$y_i \notin R_i(y_{-i}) \ni z_i. \tag{5.2}$$

If $i = j$ (hence $y_{-i} = x_{-i}$), then $y_i \in R_i(y_{-i})$ and (5.2) is impossible. Thus, $i \neq j$, hence $y_i = x_i < z_i$ and $y_{-i} < x_{-i}$. Supposing $y_i \in R_i(x_{-i})$, we would have $y_i = y_i \wedge z_i \in R_i(y_{-i})$ by Lemma 3.1, contradicting (5.2); therefore, $y_i = x_i \notin R_i(x_{-i})$. Let $x'_i \in R_i(x_{-i})$; by Lemma 3.1 and (5.2), $y'_i = x'_i \vee z_i \in R_i(x_{-i})$. Since $y'_i \geq z_i > y_i$ and $(y'_i, x_{-i}) \triangleright_i^* x$, we have $x \notin X^2$, contradicting our assumption. So the first condition in the definition of X^2 is fulfilled.

If $y_1 \in R_1(y_{-1})$, then $y \in X^2$; otherwise, $j \neq 1$, hence $y_1 = x_1 \in R_1(x_{-1})$ and $y_{-1} < x_{-1}$. Picking $y'_1 \in R_1(y_{-1})$, we denote $z_1 = y'_1 \wedge y_1$; now $z_1 < y_1$ and $z_1 \in R_1(y_{-1})$ by Lemma 3.1. Therefore, $(z_1, y_{-1}) \triangleright_1^* y$, hence $y \in X^2$.

(B) It is easy to check that $x \in X^0$ implies $x_1 \notin R_1(x_{-1})$ (otherwise, either there is $y \triangleright_i^* x$ with $y > x$ or not). Therefore, $x \in X^0$ implies $j = 1$, hence $y_1 \in R_1(x_{-1}) = R_1(y_{-1})$, hence $y \notin X^0$, hence $y \succ x$.

(C) Finally, let $x \in X^1$; then $y > x$, so $y \succ x$ unless $y \in X^0$. If $y_1 \in R_1(y_{-1})$, then $y \notin X^0$ and we are home; otherwise, $j \neq 1$, hence $y_1 = x_1 \in R_1(x_{-1})$ and $y_{-1} > x_{-1}$. Picking $y'_1 \in R_1(y_{-1})$, we denote $z_1 = y'_1 \vee y_1$; now $z_1 > y_1$ and $z_1 \in R_1(y_{-1})$ by Lemma 3.1. Therefore, $(z_1, y_{-1}) \triangleright_1^* y$, hence $y \in X^1$. \square

Lemma 5.2.2. *If $x \in X$ is a maximizer for \triangleright , then x is a Nash equilibrium (i.e., a maximizer for \triangleright^*).*

Proof. Let $x_1 \notin R_1(x_{-1})$; picking $y_1 \in R_1(x_{-1})$ appropriately, we obtain $(y_1, x_{-1}) \triangleright_1 x$, contradicting our assumption. If $x_1 \in R_1(x_{-1})$ and there are $i \in N$ and $y_i > x_i$ such that $(y_i, x_{-i}) \triangleright_i^* x$, then $x \in X^1$ and $(y_i, x_{-i}) \triangleright_i x$. If $x_1 \in R_1(x_{-1})$ and there is no best response improvement upwards, then $x \in X^2$; if x is not a Nash equilibrium, there must be a best response improvement downwards (since all $X_i, i \neq 1$, are chains): $y \triangleright^* x, y < x$, but then $y \triangleright x$. \square

The end of the proof is exactly the same as in Theorem 1.

Remark. The proof remains valid without (3.2) and (3.5) provided the statement of Lemma 3.1 holds for every $i \in N$. Unlike Theorem 1 (see Example 1), Theorem 2 only hinges on the monotonicity of the best responses. To the best of our knowledge, however, there is no way to derive the monotonicity without (3.2) and (3.5).

5.3 Proof of Theorem 3

Let $\{x^0, x^1, \dots, x^m = x^0\}$ be a best response improvement cycle. All x_2^k ($k = 0, 1, \dots, m$) are linearly ordered, hence, by Lemma 3.2, all x_1^k are linearly ordered too. Without restricting generality, we may assume that $x^1 \triangleright_1^* x^0$ and $x^2 \triangleright_2^* x^1$, hence $R_2(x_1^0) \ni x_2^0 = x_2^1 \notin R_2(x_1^1)$, $x_2^2 \in R_2(x_1^1)$, $R_1(x_2^2) \ni x_1^1 = x_1^2 \notin R_1(x_2^2)$, and $x_1^3 \in R_1(x_2^2)$, and that $x_1^1 = x_1^2$ is maximal among all x_1^k . Applying Lemma 3.1 with $i = 2$ and $x_{-i} = x_1^0 \leq x_1^1 = y_{-i}$, we obtain $x_2^0 < x_2^2$; now applying the same lemma with $i = 1$ and $x_{-i} = x_2^0 < x_2^2 = y_{-i}$, we obtain $x_1^1 < x_1^3$, which contradicts our maximality assumption.

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