Acyclicity and Aggregation in Strategic Games*

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November 2, 2006

Abstract

Connections between aggregation in preferences and the acyclicity of improvements in strategic games are studied in two distinct contexts. The first are "games with common intermediate objectives," where: the players participate in certain "activities"; each activity generates a "level of satisfaction," shared by all participating players; the utility of each player is an aggregate of the relevant levels. The second are games with ordered strategy sets where each player's best responses are increasing in an aggregate of the partners' strategies. The necessity of scalar aggregation is shown for a stylized model, viz. an endomorphism. Certain necessity results are also obtained for two or three person games.

Journal of Economic Literature Classification Number: C 72.

Key words: Strategic game; Improvement dynamics; Acyclicity; Additive aggregation

*Financial support from a Presidential Grant for the State Support of the Leading Scientific Schools (NSh-5379.2006.1), from the Mathematics Department of the Russian Academy of Sciences (program of basic research No. 3), from the Russian Foundation for Basic Research (grant 05-01-00942), from the Spanish Ministry of Education (project SEJ2004-00968), and from the Lady Davis Foundation (a fellowship at the Technion, Haifa, in 2005) is acknowledged. I thank Francisco Marhuenda and Dov Monderer, respectively, for procuring the last two grants.

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1 Introduction

This paper continues the study of conditions for the acyclicity of improvement relations in strategic games. A systematic investigation of games where the convergence of unilateral improvement dynamics is ensured was started by Monderer and Shapely (1996); Milchtaich (1996) suggested similar treatment of best response improvements. Kukushkin (1999, 2000, 2003) showed the usefulness of the language of binary relations and developed transfinite techniques.

The main purpose of this paper is to extend further the results of Kukushkin (2003) and Kukushkin (2004b). Thus we consider two independent contexts, vaguely united by the term "aggregation," which has a distinct meaning in either case.

The first are "games with structured preferences" (a far-reaching extension of the model from Germeier and Vatel', 1974, and Kukushkin et al., 1985). The players derive "intermediate utilities" from participation in certain "activities"; "aggregation" means replacing Pareto dominance with an ordering. The description of preferences with complete relations being a well-established practice in game theory, the appearance of aggregation in this sense seems inevitable. The focus of our study is on what kind of aggregation rules ensure the acyclicity of improvements in every derivative game; in other words, it is necessity results that are most important.

The central position is occupied by Theorem 2 – an analogue of the famous Debreu–Gorman Theorem (Fleming, 1952; Debreu, 1960; Gorman, 1968; see also Wakker, 1989, and Vind, 1991) in a strategic game context: If every aggregation rule is continuous and strictly increasing in each variable, then the existence of a Nash equilibrium in every derivative game implies additivity in the same sense as in Gorman (1968). Unfortunately, a complete proof was only given for two particular cases. One of them is sufficient for the derivation of the main result of Gorman (1968, Theorem 1); probably, a closed proof of that theorem is given for the first time ever.

It seems proper to recognize (at least) two different aspects of additivity: It may refer to an algebraic object, a (semi)group, where entities can be combined producing new entities of the same kind. It may also refer to a purely ordinal framework, where an ordering on a given set is produced with the help of numeric functions. An exposition of interplay between both aspects can be found in the wonderful book of Krantz et al. (1971). Here we only consider additivity in the ordinal sense.

The second class are games with ordered strategy sets where each player's best responses are increasing in an aggregate of the partners' strategies. The presence of such aggregates is by no means mandatory. As is well known, monotonicity conditions may ensure the existence of an equilibrium and some other nice properties of strategic games, including important economic models, without any hint of aggregation in this sense (Topkis, 1979; Bulow et al., 1985; Tirole, 1988; Vives, 1990; Milgrom and Roberts, 1990; Fudenberg and Tirole, 1991; Milgrom and Shannon, 1994; Topkis, 1998). Still, it was noticed long ago that aggregation can help (McManus 1962, 1964; Novshek, 1985; see also Kukushkin, 1994a).

Here we are concerned with conditions for nice best response improvement dynamics rather than the mere existence of a Nash equilibrium. It is already known that aggregation is conducive to the acyclicity of best response improvements. In a sense, this paper supplies evidence supporting the claim "there is no acyclicity without aggregation." Taken literally, the claim is invalid: Kandori and Rob (1995; Theorem 2) showed that symmetry in a game with strategic complements may be sufficient, although S. Takahashi (personal communication) demonstrated that the exact analogue of the theorem for games with strategic substitutes is wrong.

To make the necessity of aggregation provable, we concentrate on statements whose conditions survive the application of a monotonic transformation; for instance, Theorems 1 and 2 of Kukushkin (2004a) are of this kind, while the symmetry of Theorem 3 from the same paper or Theorem 2 from Kandori and Rob (1995) is not. The most convincing results are obtained for the case of an endomorphism of a partially (pre)ordered set – a stylized image of the system of the best response correspondences in a strategic game.

The next Section 2 contains basic definitions and notation. The following two sections correspond to two contexts mentioned above: Section 3 considers games with structured preferences; Section 4, "acyclic patterns" for endomorphisms and systems of reactions.

In Subsection 3.1, basic notions concerning games with structured preferences are introduced. There is a set of players, a set of "activities," lists of activities where every player participates, sets of feasible values of "intermediate utilities" associated with every activity, and an aggregation rule, i.e., an ordering on the set of feasible vector utilities for each player. Theorem 1 shows that the existence of a potential in every strategic game consistent with a given game structure and a given list of aggregation rules is equivalent to the existence of a potential in the same sense on the level of a general scheme.

In Subsection 3.2, the definition of Gorman additive aggregation rules is given and Theorem 2 about them is formulated. It is also shown that the Gorman additivity implies that individual improvements in every derivative game are acyclic in a rather strong sense. In Subsections 3.3 and 3.4, constructions needed for a necessity proof are developed, although, as has already been mentioned, the proof is left uncompleted, except for two special cases, Propositions 3.4 and 3.6 of Subsection 3.5. In particular, Theorem 1 of Gorman (1968) is easily derived from Proposition 3.4.

In Subsection 4.1, basic definitions concerning monotonic mappings and correspondences on (pre)ordered sets are given; auxiliary results about the existence of increasing mappings are proven. Subsection 4.2 is about "acyclic patterns" for endomorphisms, i.e., endomorphisms of (pre)ordered sets that remain acyclic after any monotonic transformation. Theorems 3 and 4 show the importance of linear orders; in a sense, they explain the lack of conditions for the acyclicity of *simultaneous* best response improvements in the literature.

In Subsection 4.3, an analogue of Theorem 3 is obtained for two person games (Theorem 5); unfortunately, no analogue of Theorem 4 for this case has been obtained so far (Example 5.2 from Kukushkin, 2000, shows the difficulties here).

Subsection 4.4 contains two isolated results on three person games: three continuous and strictly increasing (in Theorem 6; decreasing in Theorem 7) functions form an acyclic pattern if and only if all the three can be transformed into simple sums by the same change of variables. The proof is based on a special case of Theorem 2 covered by Proposition 3.6. The necessity statement becomes wrong if there are more than three players (Example 4.3).

2 Basic Notions

A *binary relation* on a set X is a Boolean function on $X \times X$; as usual, we write $y \rhd x$ whenever the relation $\rhd$ is true on a pair (y, x). The most popular in mathematical literature seem to be order relations. A (partial) *strict order* is an irreflexive and transitive binary relation; a reflexive and transitive binary relation is called a *preorder*. With every preorder $\succeq$, strict orders $\succ$ and $\prec$, as well as an equivalence relation $\sim$, are naturally associated. A complete preorder is called an *ordering*; an ordering $\succeq$ is *represented* by a numeric function $f : X \to \mathbb{R}$ if $y \succeq x \iff f(y) \geq f(x)$.

Let $\rhd$ be a binary relation on $X = X_1 \times X_2$. A relation $\rhd_1$ on X_1 is a *separable projection* of $\rhd$ to X_1 (along X_2) if

$$(x_1', x_2) \rhd (x_1, x_2) \iff x_1' \rhd_1 x_1$$

for all $x_1', x_1 \in X_1$ and $x_2 \in X_2$. Usually X_2 is clear from the context and not mentioned at all. Obviously, a separable projection "inherits" all properties inherited by the restrictions to subsets (as being a preorder, strict order, ordering, etc.). The following two statements are also easy to prove. Let $X = X_1 \times X_2 \times X_3$ and $\rhd$ be a binary relation on X. Then: (1) If $\rhd$ admits separable projections to both $X_1 \times X_2$ and $X_2 \times X_3$, then it also admits a separable projection to X_2. (2) If $\rhd$ admits a separable projection $\rhd_{12}$ to $X_1 \times X_2$, then a separable projection of $\rhd_{12}$ to X_1 (if exists) is a separable projection of $\rhd$ to X_1 and vice versa.

We always assume X to be a metric space. An ordering on X is *continuous* if upper and lower contours, $\{y \in X | \ y \succ x\}$ and $\{y \in X | \ x \succ y\}$, are open for every $x \in X$. By the famous Debreu Theorem, an ordering on a separable metric space is continuous if and only if it can be represented by a continuous function. As Gorman (1968) stressed, topological separability has nothing to do with the separability of the previous paragraph.

This paper is mostly concerned with improvement relations in strategic games. As usual, a *strategic game* Γ is defined by a finite set of players N (we denote $n = \#N$), and strategy sets X_i and preference relations $\succeq_i$ on $X = \prod_{i \in N} X_i$ for all $i \in N$. We assume each $\succeq_i$ to be an ordering; its numeric representation, when exists, is called a *utility function*. We always assume that each X_i, hence X too, is a metric space; if each X_i is compact, we call Γ a compact game. The *best response correspondence* $\mathcal{R}_i : X_{-i} \to 2^{X_i}$ for each $i \in N$ is defined in the usual way:

$$\mathcal{R}_i(x_{-i}) = \{x_i \in X_i | \ \forall x_i' \in X_i \, [(x_i, x_{-i}) \succeq_i (x_i', x_{-i})]\};$$

generally, $\mathcal{R}_i(x_{-i}) = \emptyset$ is possible.

With every strategic game, a number of improvement relations on X are associated. Here we shall use two of them: the *individual improvement relation* $\rhd^{\mathrm{Ind}}$ and the *best response improvement relation* $\rhd^{\mathrm{BR}}$ (called *Cournot relation* in Kukushkin, 2004a).

$$y \rhd^{\mathrm{Ind}}{}_i \, x \iff [y_{-i} = x_{-i} \ \& \ y \succ_i x]; \tag{2.1a}$$

$$y \rhd^{\mathrm{Ind}} x \iff \exists i \in N \, [y \rhd^{\mathrm{Ind}}{}_i \, x]; \tag{2.1b}$$

$$y \rhd^{\mathrm{BR}}{}_i \, x \iff [y_{-i} = x_{-i} \ \& \ x_i \notin \mathcal{R}_i(x_{-i}) \ni y_i]; \tag{2.2a}$$

$$y \rhd^{\mathrm{BR}} x \iff \exists i \in N \, [y \rhd^{\mathrm{BR}}{}_i \, x] \tag{2.2b}$$

$(y, x \in X,\, i \in N)$.

By definition, a strategy profile $x \in X$ is a Nash equilibrium if and only if x is a *maximizer* for $\rhd^{\mathrm{Ind}}$, i.e., if $y \rhd^{\mathrm{Ind}} x$ holds for no $y \in X$. Every Nash equilibrium is a maximizer for $\rhd^{\mathrm{BR}}$; if $\mathcal{R}_i(x_{-i}) \neq \emptyset$ for all $i \in N$ and $x_{-i} \in X_{-i}$, then the converse is also true. The condition holds, for instance, when preferences are represented by utility functions each of which only takes a finite number of values; alternatively, we may require each X_i to be compact and each u_i upper semicontinuous in own strategy x_i.

For a finite game, the acyclicity of the individual improvement relation (2.1) implies that an infinite individual improvement path is impossible, hence every such path, if continued whenever possible, ends at a Nash equilibrium; the existence of the latter is thus implied. Monderer and Shapley (1996) called this the finite improvement path (FIP) property of the game. Similarly, the acyclicity of (2.2) in a finite game implies that every best response improvement path eventually leads to a Nash equilibrium (the FBRP property of Milchtaich, 1996).

Von Neumann and Morgenstern (1953) called a relation *strongly acyclic* if it admits no infinite improvement path; if the individual (best response) improvement relation in Γ is strongly acyclic, then Γ has the FIP (FBRP) property even if it is infinite. To the best of my knowledge, however, this condition does not hold in interesting infinite games.

A more promising extension of acyclicity to the infinite (topological) case is based on the notion of a *potential*, i.e., a strict order $\succ$ such that

$$y \rhd x \Rightarrow y \succ x. \tag{2.3a}$$

Clearly, $\rhd$ admits a potential in this sense if and only if it is acyclic. An *ω-potential* is a strict order which is *ω-transitive*,

$$\bigl[x^{\omega} = \lim_{k\to\infty} x^k \;\&\; \forall k \in \mathbb{N}[x^{k+1} \succ x^k]\bigr] \Rightarrow x^{\omega} \succ x^0, \tag{2.3b}$$

and satisfies (2.3a). A binary relation admits an ω-potential if and only if no path combining improvements with taking limit points can return to its origin after any (transfinite) number of steps. Since this version of acyclicity is of crucial importance, I reproduce basic formal constructions.

A linearly ordered set is *well ordered* if every subset contains a least point; Natanson (1974, Chapter XIV) can be used as a reference textbook. Considering a well ordered set Σ, we will denote 0 the least point of the whole Σ, and $\beta + 1$, for $\beta \in \Sigma$, the least point exceeding β (the latter exists unless $\beta = \max \Sigma$). A point $\beta \in \Sigma \setminus \{0\}$ is called *isolated* if $\beta = \beta' + 1$ for some $\beta' \in \Sigma$; otherwise, β is called a *limit point*. Thus, we have a partition $\Sigma = \{0\} \cup \Sigma_{\mathrm{iso}} \cup \Sigma_{\mathrm{lim}}$. It is sometimes convenient to consider a partial function $\beta - 1$ defined by the equality $\beta = (\beta - 1) + 1$ for isolated β and not defined at all on $\Sigma \setminus \Sigma_{\mathrm{iso}}$.

Let $\rhd$ be a binary relation on a metric space X (interesting for us are improvement relations in strategic games). An *improvement path* for $\rhd$ is a mapping $\pi : \Sigma \to X$, where Σ is a countable well ordered set, satisfying these two conditions:

$$\pi(\beta + 1) \rhd \pi(\beta) \text{ whenever } \beta, \beta + 1 \in \Sigma; \tag{2.4a}$$

whenever $\beta^{\omega} \in \Sigma_{\mathrm{lim}}$, there exists a sequence $\{\beta^k\}_k \subset \Sigma$ for which

$$\beta^{k+1} > \beta^k \text{ for all } k = 0, 1, \ldots,\; \beta^{\omega} = \sup_k \beta^k,\; \text{and } \pi(\beta^{\omega}) = \lim_{k\to\infty} \pi(\beta^k). \tag{2.4b}$$

An improvement path π is *narrow* if

$$\pi(\beta^\omega) = \lim_{k\to\infty} \pi(\beta^k) \text{ whenever } \beta^\omega \in \Sigma_{\lim} \text{ and a sequence } \{\beta^k\}_k \subset \Sigma$$
$$\text{are such that } \beta^\omega = \sup_k \beta^k \text{ and } \beta^{k+1} > \beta^k \text{ for all } k = 0, 1, \ldots \quad (2.5)$$

A (*narrow*) *improvement cycle* is a (narrow) improvement path π such that $\pi(\beta'') = \pi(\beta')$ for some $\beta'' > \beta' \in \Sigma$. Deleting from Σ all $\gamma < \beta'$ and $\gamma > \beta''$, we can assume $\beta' = 0$ and $\beta'' = \max \Sigma$. A relation $\triangleright$ is called (*weakly*) Ω-*acyclic* if it admits no (narrow) improvement cycle.

By Theorem 2 of Kukushkin (2003), a relation $\triangleright$ is Ω-acyclic if and only if it admits an ω-potential (2.3); an Ω-acyclic relation admits a maximizer on every compact subset of its domain. The weak Ω-acyclicity is necessary for the existence of maximizers on all compact subsets, but not sufficient for anything of interest. If strategy sets are compact and an improvement relation ($\triangleright^{\mathrm{Ind}}$ or $\triangleright^{\mathrm{BR}}$) admits an ω-potential, then every appropriate improvement path "eventually" (perhaps in a very long run) reaches a Nash equilibrium.

It is clear from (2.2) that the relation $\triangleright^{\mathrm{BR}}$ depends on the correspondences $\mathcal{R}_i$ rather that on Γ as such. It makes sense, therefore, to introduce the notion of a "system of reactions" (Kukushkin, 2000). Virtually the same object was called an "abstract game" by Vives (1990); however, he focussed attention on an endomorphism, the Cartesian product of all reactions.

A *system of reactions* $\mathcal{S}$ is defined by a finite set of indices N, and sets X_i and mappings $R_i : X_{-i} \to 2^{X_i} \setminus \{\emptyset\}$ for all $i \in N$. A point $x^0 \in X = \prod_{i\in N} X_i$ is called a *fixed point* of $\mathcal{S}$ if $x_i^0 \in R_i(x_{-i}^0)$ for all $i \in N$.

With every system $\mathcal{S}$, one can associate binary relations on X: $y \triangleright_i^{\mathcal{S}} x \iff [y_{-i} = x_{-i} \ \&\ x_i \notin R_i(x_{-i}) \ni y_i]$; $y \triangleright^{\mathcal{S}} x \iff \exists i \in N\, [y \triangleright_i^{\mathcal{S}} x]$. Clearly, $x \in X$ is a maximizer for $\triangleright^{\mathcal{S}}$ if and only if x is a fixed point of $\mathcal{S}$. Improvement paths for $\triangleright^{\mathcal{S}}$ are generated by iterating R_i's and picking limit points; we usually call them iteration paths. We call $\mathcal{S}$ (Ω-)*acyclic* if so is $\triangleright^{\mathcal{S}}$. As a rule, we omit the superscript $^{\mathcal{S}}$ at $\triangleright$. By Theorem 2 ([2.1] $\Rightarrow$ [2.6]) of Kukushkin (2003), an Ω-acyclic system of reactions with compact sets X_i has a fixed point.

It is sometimes useful to consider endomorphisms ("fixed point frameworks") as simplified analogues of systems of reactions. With every mapping $F : X \to 2^X \setminus \{\emptyset\}$ (a correspondence $X \to X$), we associate a binary relation $\triangleright^F$: $y \triangleright^F x \iff x \notin F(x) \ni y$; for the particular case of a mapping $f : X \to X$, we have $y \triangleright^f x \iff y = f(x) \neq x$. Maximizers for $\triangleright^F$ ($\triangleright^f$) are exactly the fixed points of F (or f), while improvement paths for $\triangleright^F$ ($\triangleright^f$) combine iterating F (or f) and picking limit points, so we also call them iteration paths. We call a mapping F (or f) acyclic or Ω-acyclic, if so is $\triangleright^F$ ($\triangleright^f$).

3 Games with Structured Preferences

3.1 Game structures and aggregation rules

We start with an abstract scheme closely related to the notion of a game with structured utilities (Kukushkin, 2004b).

A *game structure* is a list $\mathcal{S} = \langle N, \mathrm{A}, \langle \Upsilon_i \rangle_{i\in N}, \langle V_\alpha \rangle_{\alpha\in\mathrm{A}} \rangle$, where N and A are finite sets (of players and *activities*, respectively), and all $\Upsilon_i \subseteq \mathrm{A}$ and $V_\alpha \subseteq \mathbb{R}$ are not empty. We always assume that $\bigcup_{i\in N} \Upsilon_i = \mathrm{A}$. For each $\alpha \in \mathrm{A}$, we denote $N(\alpha) = \{i \in N |\ \alpha \in \Upsilon_i\}$; for each $\mathrm{B} \subseteq \mathrm{A}$, $V_\mathrm{B} = \prod_{\alpha\in\mathrm{B}} V_\alpha$. We always equip V_B with the topology defined by the norm $\|v_\mathrm{B}\| = \max_{\alpha\in\mathrm{B}} |v_\alpha|$. For a given game structure and $i \in N$, an *aggregation rule* is an ordering $\succeq^{\mathrm{a}}_i$ on V_{Υ_i} such that $v' \geq v \Rightarrow v' \succeq^{\mathrm{a}}_i v$ for all $v', v \in V_{\Upsilon_i}$. An aggregation rule is *strictly responsive* if

$$v' \succ^{\mathrm{a}}_i v \text{ whenever } v', v \in V_{\Upsilon_i},\ \forall \alpha \in \Upsilon_i\,[v'_\alpha \geq v_\alpha], \text{ and } \exists \alpha \in \Upsilon_i\,[v'_\alpha > v_\alpha]. \tag{3.1}$$

The property is essential in Theorem 2 below.

Given a game structure $\mathcal{S}$ and a list of aggregation rules $\mathcal{L} = \langle \succeq^{\mathrm{a}}_i \rangle_{i\in N}$, we introduce binary relations on V_A:

$$v' \triangleright^{\mathrm{a}}_i v \iff [v'_{\Upsilon_i} \succ^{\mathrm{a}}_i v_{\Upsilon_i} \ \&\ v'_{\mathrm{A}\setminus\Upsilon_i} = v_{\mathrm{A}\setminus\Upsilon_i}]; \tag{3.2a}$$

$$v' \triangleright^{\mathrm{a}} v \iff \exists i \in N\,[v' \triangleright^{\mathrm{a}}_i v]. \tag{3.2b}$$

We call $\triangleright^{\mathrm{a}}$ the aggregate improvement relation. By the general results cited in the previous section, the aggregate improvement relation $\triangleright^{\mathrm{a}}$ is $(\Omega\text{-})$acyclic if and only if it admits a(n ω-)potential.

A strategic game Γ is *consistent* with a game structure $\mathcal{S}$ and a list $\mathcal{L}$ if N is the set of players in Γ and there is a continuous mapping $\varphi_\alpha : X_{N(\alpha)} \to V_\alpha$ for each $\alpha \in \mathrm{A}$ such that

$$x' \succeq_i x \iff \varphi_{\Upsilon_i}(x') \succeq^{\mathrm{a}}_i \varphi_{\Upsilon_i}(x), \tag{3.3}$$

where $\varphi_\mathrm{B}(x) = \langle \varphi_\alpha(x_{N(\alpha)}) \rangle_{\alpha\in\mathrm{B}} \in V_\mathrm{B}$ for every $\mathrm{B} \subseteq \mathrm{A}$ and $x \in X$.

Remark. One could think that considering $N' \subset N$ would widen the scope of games consistent with $\mathcal{S}$ and $\mathcal{L}$; however, a singleton X_i is equivalent to the exclusion of player i.

Proposition 3.1. *Let a game structure $\mathcal{S}$ and a list of aggregation rules $\mathcal{L}$ be given. Then every strategic game Γ consistent with $\mathcal{S}$ and $\mathcal{L}$ has the FIP property if and only if the relevant aggregate improvement relation is strongly acyclic.*

Proof. If $x^0, x^1, \ldots$ is an individual improvement path in Γ, then $\varphi_\mathrm{A}(x^0), \varphi_\mathrm{A}(x^1), \ldots$ is an improvement path for the aggregate improvement relation by (3.3); therefore, the strong acyclicity of the latter relation implies the FIP property of Γ.

Conversely, let the aggregate improvement relation not be strongly acyclic and $v^0, v^1, \ldots$ be an infinite improvement path for $\triangleright^{\mathrm{a}}$. For every $k \in \mathbb{N}$, we pick $\iota(k)$ such that $v^k \triangleright^{\mathrm{a}}_{\iota(k)} v^{k-1}$, obtaining a mapping $\iota : \mathbb{N} \to N$. (Since we have made no assumption on $\mathcal{S}$, the choice of $\iota(k)$ need not be unique.) For each $i \in N$, we define $X_i = \{0\} \cup \iota^{-1}(i)$; they will be strategy sets (with the discrete topology). For each $k \in \mathbb{N}$, $i \in N$ and $I \subseteq N$, we define $\xi_i(k) = \max\{h \in X_i |\ h \leq k\}$ and $\xi_I(k) = \langle \xi_i(k) \rangle_{i\in I} \in X_I$. For each $\alpha \in \mathrm{A}$, we define $X^\alpha = \{\xi_{N(\alpha)}(k)\}_{k\in\mathbb{N}} \subseteq X_{N(\alpha)}$. Whenever $\xi_{N(\alpha)}(h) = \xi_{N(\alpha)}(k)$, we have $v^h_\alpha = v^k_\alpha$: the proof consists in iteration of (3.2a). Therefore, we may define $\varphi_\alpha(\xi_{N(\alpha)}(k)) = v^k_\alpha$, obtaining a mapping $\varphi_\alpha : X^\alpha \to V_\alpha$; then we extend it to a mapping $X_{N(\alpha)} \to V_\alpha$ in an arbitrary way (since we assumed discrete topology, there is no problem with continuity). Finally, we perceive (3.3) as the definition of preferences, obtaining a strategic game Γ, which is obviously consistent with $\mathcal{S}$ and $\mathcal{L}$. It is easy to see that $(0, \ldots, 0), \xi_N(1), \xi_N(2), \ldots$ is an infinite improvement path in Γ. □

Proposition 3.2. *Let a game structure $\mathcal{S}$ and a list of aggregation rules $\mathcal{L}$ be given. Then the individual improvement relation in every strategic game Γ consistent with $\mathcal{S}$ and $\mathcal{L}$ is acyclic if and only if so is the relevant aggregate improvement relation.*

Proof. The sufficiency part is the same as in Proposition 3.1; the necessity is almost the same, but there is a subtle point. Assuming the existence of an aggregate improvement cycle $v^0, v^1, \ldots, v^m = v^0$, we pick $\iota(k)$, for $k \in \{1, \ldots, m\}$, such that $v^k \triangleright^{\mathrm{a}}_{\iota(k)} v^{k-1}$. Then we "double" our cycle, defining $M = \{1, \ldots, 2m\}$ and $v^{m+k} = v^k$ $(k = 1, \ldots, m)$, and extend ι to a mapping $M \to N$ by $\iota(m+k) = \iota(k)$. For each $i \in N$, we define $X_i = \iota^{-1}(i)$ and, for each $k \in M$, $\Xi_i(k) = \{h \in X_i | \ h \leq k\}$. If $X_i = \emptyset$, player i is not needed for the cycle; formally, he can be given a singleton strategy set and forgotten about. We assume $\emptyset \neq X_i = \Xi_i(2m)$ for all $i \in N$. We define $\xi_i(k) = \max \Xi_i(k)$ if $\Xi_i(k) \neq \emptyset$ and $\xi_i(k) = \max \Xi_i(2m) = \max X_i$ otherwise; we denote $\xi_I(k) = \langle \xi_i(k) \rangle_{i \in I} \in X_I$ for all $I \subseteq N$.

Let $\alpha \in \mathrm{A}$ and $h, k \in M$; we assume $h < k$. An important intermediate statement is: $v^h_\alpha = v^k_\alpha$ whenever $\xi_{N(\alpha)}(h) = \xi_{N(\alpha)}(k)$. A similar claim in the proof of Proposition 3.1 was virtually obvious because of the monotonicity of $\xi(\cdot)$ in that case; let us prove it now. If $\iota^{-1}(N(\alpha)) \cap]h, k] = \emptyset$, then $v^h_\alpha = v^k_\alpha$ by (3.2a) applied $k - h$ times. Let there be $i \in N(\alpha)$ and $k' \in X_i$ such that $h < k' \leq k$; without restricting generality, $\xi_i(k) = k'$. Now $\xi_i(h) = k'$ is only possible if $\iota^{-1}(i) \cap [1, h] = \emptyset = \iota^{-1}(i) \cap]k', 2m]$, which implies $k' > h + m$ (by the extension of ι, $\iota(k' \pm m) = i$ provided $(k' \pm m) \in M$), hence $k > h + m$. Therefore, for every $j \in N(\alpha)$, we have $\iota^{-1}(j) \cap]h, k] \neq \emptyset$, hence $\xi_j(h) = \xi_j(k)$ is only possible if $\iota^{-1}(j) \cap [1, h] = \emptyset = \iota^{-1}(j) \cap]k, 2m]$. Thus, $\iota^{-1}(N(\alpha)) \cap [1, h] = \emptyset = \iota^{-1}(N(\alpha)) \cap]k, 2m]$ and we obtain $v^h_\alpha = v^k_\alpha$ applying (3.2a) $2m + h - k$ times.

The rest of the proof is essentially the same as in Proposition 3.1: we define φ_α on $\{\xi_{N(\alpha)}(x^k)\}_{k \in M}$, extend it to a mapping $X_{N(\alpha)} \to V_\alpha$ in an arbitrary way, and perceive (3.3) as the definition of preferences, obtaining a strategic game Γ, which is obviously consistent with $\mathcal{S}$ and $\mathcal{L}$, but admits an individual improvement cycle. $\square$

Remark. If an aggregate improvement cycle contains just one improvement by a player i, then the cycle obviously cannot be generated by an individual improvement cycle in a strategic game. Thus the doubling of the cycle may be indispensable indeed (cf. the proof of Lemma 3.3.1 below).

Theorem 1. *Let a game structure $\mathcal{S}$ and a list of aggregation rules $\mathcal{L}$ be given. Then the individual improvement relation in every strategic game Γ consistent with $\mathcal{S}$ and $\mathcal{L}$ is Ω-acyclic if and only if so is the relevant aggregate improvement relation.*

Proof. The sufficiency part is essentially the same as in Proposition 3.1. The necessity is much more complicated. Unfortunately, I cannot insist that the theorem justifies the effort; however, it would look even sillier to leave it as an open problem.

Let us assume that there is an aggregate improvement cycle, i.e., a mapping $\hat{\pi} : \Sigma' \to V_{\mathrm{A}}$ (where Σ' is a countable well ordered set) satisfying (2.4) and such that $\hat{\pi}(0) = \hat{\pi}(\bar{\beta})$. Taking into account Proposition 3.2, we may assume that Σ' is infinite; without restricting generality, $\bar{\beta} = \max \Sigma' \in \Sigma'_{\mathrm{lim}}$. For every $\beta \in \Sigma'_{\mathrm{iso}}$, we pick $\hat{\imath}(\beta)$ such that $\hat{\pi}(\beta) \triangleright^{\mathrm{a}}_{\hat{\imath}(\beta)} \hat{\pi}(\beta - 1)$.

We define $\Sigma = (\Sigma' \times \{0, 1\}) \setminus \{(0, 1)\}$ with a lexicographic order: $(\beta'', \theta'') \geq (\beta', \theta')$ $(\beta'', \beta' \in \Sigma', \theta'', \theta' \in \{0, 1\})$ iff $\theta'' > \theta'$ or $\theta'' = \theta'$ and $\beta'' \geq \beta'$. When $\beta \in \Sigma$, we assume $\beta = (\beta_1, \beta_2)$ with $\beta_1 \in \Sigma'$ and $\beta_2 \in \{0, 1\}$. Clearly, Σ is also a well ordered set, which is compact in its intrinsic (as a chain) topology (Birkhoff, 1967); since Σ is countable, the topology can be defined with a metric. We extend $\hat{\pi}$ to a mapping $\pi : \Sigma \to V_{\mathrm{A}}$ by $\pi(\beta, 1) = \pi(\beta, 0) = \hat{\pi}(\beta)$ [thus $\pi(0,0) = \pi(\bar{\beta}, 1)$], and $\hat{\iota}$ to a mapping $\iota : \Sigma_{\text{iso}} \to N$ by $\iota(\beta, 1) = \iota(\beta, 0) = \hat{\iota}(\beta)$.

For each $i \in N$, we denote $Y_i^0 = \iota^{-1}(i) \subseteq \Sigma_{\text{iso}}$ and Y_i its closure in Σ, which is automatically compact. As in the proof of Proposition 3.2, if $Y_i = Y_i^0 = \emptyset$, then player i can be given a singleton strategy set and forgotten about. Clearly, $Y_i = Y_i^0 \cup Y_i^\infty$, where $Y_i^\infty = Y_i \cap \Sigma_{\text{lim}}$ consists of the least upper bounds of infinite, strictly increasing sequences in Y_i^0. Each Y_i is well ordered itself; its limit points are exactly $\beta \in Y_i^\infty$ and its intrinsic topology coincides with that induced from Σ.

For each $\beta \in \Sigma$ and $i \in N$, we denote $\Xi_i(\beta) = \{\gamma \in Y_i |\ \gamma \leq \beta\}$; obviously, $\Xi_i(\beta) \neq \emptyset$ whenever $\beta_2 = 1$, in particular, for $\beta = (\bar{\beta}, 1)$. If $\Xi_i(\beta) \neq \emptyset$, we define $\xi_i(\beta) = \max \Xi_i(\beta) \in Y_i$. If $\Xi_i(\beta) = \emptyset$, we define $\xi_i(\beta) = \xi_i(\bar{\beta}, 1)$. We denote $\xi_I(\beta) = \langle \xi_i(\beta) \rangle_{i \in I} \in X_I$ for every $I \subseteq N$.

Lemma 3.1.1. *Let $\alpha \in \mathrm{A}$, $\beta'', \beta' \in \Sigma$, and $\xi_{N(\alpha)}(\beta'') = \xi_{N(\alpha)}(\beta')$; then $\pi_\alpha(\beta'') = \pi_\alpha(\beta')$.*

Proof. The scheme is the same as in Proposition 3.2; without restricting generality, $\beta'' > \beta'$. If $\iota^{-1}(N(\alpha)) \cap \{\beta \in \Sigma |\ \beta' < \beta < \beta''\} = \emptyset$, then a straightforward inductive process (generally, transfinite) based on (3.2a) shows $\pi_\alpha(\beta'') = \pi_\alpha(\beta')$.

Let there be $i \in N(\alpha)$ such that $\xi_i(\beta'') > \beta'$. By the condition of the lemma, $\xi_i(\beta') = \xi_i(\beta'')$, hence $\Xi_i(\beta') = \emptyset = Y_i \cap \{\beta \in \Sigma |\ \beta > \beta''\}$; by the definition of ι, $(\xi_i(\beta')_1, 0) \in Y_i \ni (\xi_i(\beta')_1, 1)$. Therefore, $\beta''_1 > \beta'_1$, $\beta'_2 = 0$, and $\beta''_2 = 1$.

If there are $j \in N(\alpha)$ and $\gamma \in Y_j$ such that $\gamma \leq \beta'$, then $\xi_j(\beta') \leq \beta'$, while $\xi_j(\beta'') \geq (\gamma_1, 1) > \beta'$, hence $\xi_j(\beta'') \neq \xi_j(\beta')$, contradicting the condition of the lemma. Therefore, $\Xi_j(\beta') = \emptyset$ for all $j \in N(\alpha)$, hence $\xi_{N(\alpha)}(\beta') = \xi_{N(\alpha)}(\bar{\beta}, 1)$ and $\pi_\alpha(\beta') = \pi_\alpha(0,0)$ by straightforward induction based on (3.2a).

If there are $j \in N(\alpha)$ and $\gamma \in Y_j$ such that $\beta'' < \gamma$, then $(\gamma_1, 0) \in Y_j$ and $(\gamma_1, 0) < \beta''$, hence $\Xi_j(\beta'') \neq \emptyset$, hence $\xi_j(\beta'') \leq \beta''$; on the other hand, $\xi_j(\bar{\beta}, 1) \geq \gamma > \beta''$. Taking into account $\xi_j(\beta') = \xi_j(\bar{\beta}, 1)$ established in the previous paragraph, we again obtain a contradiction. Thus, $\iota^{-1}(N(\alpha)) \cap \{\beta \in \Sigma |\ \beta > \beta''\} = \emptyset$. The same induction based on (3.2a) shows that $\pi_\alpha(\beta'') = \pi_\alpha(\bar{\beta}, 1)$; since $\pi(\bar{\beta}, 1) = \pi(0,0)$, we are home. □

Let $\beta \in \Sigma_{\text{lim}}$; we denote $I^*(\beta) = \{i \in N |\ \beta \in Y_i\}$ and $I^0(\beta) = N \setminus I^*(\beta)$. By (2.4b), there is an increasing sequence $\beta^k \to \beta$ such that $\pi(\beta^k) \to \pi(\beta)$. Deleting superfluous members from the sequence if necessary, we obtain two sequences $b_k(\beta), c_k(\beta) \in \Sigma$ $(k \in \mathbb{N})$ such that: $\beta = \sup_k b_k(\beta)$; $c_{k+1}(\beta) > b_k(\beta) > c_k(\beta)$ and $\|\pi(\beta) - \pi(b_k(\beta))\| < 2^{-k}$ for all k; $\xi_i \circ c_{k+1}(\beta) > \xi_i \circ b_k(\beta) > \xi_i \circ c_k(\beta)$ for all k and $i \in I^*(\beta)$; $\xi_i(\beta) = \xi_i \circ c_0(\beta)$ for all $i \in I^0(\beta)$.

For each $i \in N$, we define a binary relation on Σ:

$$\beta \rhd\!\!\rhd_i \gamma \iff \big[\beta \in Y_i^\infty \text{ and there are } m_i(\beta, \gamma) \geq 1$$

(to the end of the formula, we write just m_i)

and corteges $\gamma_i(\beta, \gamma) = \langle \gamma_i^0, \gamma_i^1, \ldots, \gamma_i^{m_i} \rangle \in \Sigma^{m_i+1}$

and $s_i(\beta, \gamma) = \langle s_i^0, \ldots, s_i^{m_i-1} \rangle \in \mathbb{N}^{m_i}$ such that

$$\gamma_i^0 = \beta, \tag{3.4a}$$

$$\gamma_i^{h+1} = b_{s_i^h} \circ \xi_i(\gamma_i^h) \text{ and} \tag{3.4b}$$

$$\xi_i \circ c_{s_i^{h+1}} \circ \xi_i(\gamma_i^{h+1}) > c_{s_i^h} \circ \xi_i(\gamma_i^h) \tag{3.4c}$$

$$\text{for } h = 0, 1, \ldots, m_i - 1, \text{ and } \xi_i(\gamma_i^{m_i}) = \xi_i(\gamma)\,\big].$$

Obviously, if $\beta \rhd\!\!\rhd_i \gamma$, then $\beta > \gamma$. It is important to stress that

$$\beta \rhd\!\!\rhd_i \gamma \iff \beta \rhd\!\!\rhd_i \xi_i(\gamma).$$

Lemma 3.1.2. *Let $i, j \in N$ and $\beta, \gamma \in \Sigma$. If $\beta \rhd\!\!\rhd_i \gamma$, then $m_i(\beta,\gamma)$, $\gamma_i(\beta,\gamma)$, and $s_i(\beta,\gamma)$ are unique. If $\beta \rhd\!\!\rhd_i \gamma$, $\beta \rhd\!\!\rhd_j \gamma$, and $m_i(\beta,\gamma) \geq m_j(\beta,\gamma)$, then $\gamma_i(\beta,\gamma) = \langle \gamma_j(\beta,\gamma), \gamma_i^{m_j+1}, \ldots, \gamma_i^{m_i}\rangle$, $s_i(\beta,\gamma) = \langle s_j(\beta,\gamma), s_i^{m_j}, \ldots, s_i^{m_i-1}\rangle$, and $\xi_j(\gamma) = \xi_j(\gamma_i^{m_i})$.*

Proof. Let $\beta \rhd\!\!\rhd_i \gamma$ and $\beta \rhd\!\!\rhd_j \gamma$; we argue by induction in $\min\{m_i(\beta,\gamma), m_j(\beta,\gamma)\}$. By definition, $\gamma_i^1 = b_{s_i^0}(\beta)$ and $\gamma_j^1 = b_{s_j^0}(\beta)$. Let $\gamma_i^1 \neq \gamma_j^1$; then, without restricting generality, $\gamma_i^1 > \gamma_j^1$, hence $s_i^0 > s_j^0$, $\xi_i(\gamma_i^1) > \xi_i \circ c_{s_i^0}(\beta)$, and $\xi_j \circ c_{s_i^0}(\beta) > \xi_j(\gamma_j^1)$. From the inequality for ξ_i and (3.4c), we easily obtain $\xi_i(\gamma_i^k) > \xi_i \circ c_{s_i^0}(\beta)$ for all $k \leq m_i$; therefore, $\xi_i(\gamma_i^{m_i}) = \xi_i(\gamma)$ is only possible when $\gamma > c_{s_i^0}(\beta)$. From the inequality for ξ_j and straightforward inequalities $\gamma_j^{k+1} < \gamma_j^k$ for all $k < m_j$, we see that $\xi_j(\gamma_j^{m_j}) = \xi_j(\gamma)$ is only possible when $\gamma < c_{s_i^0}(\beta)$. Therefore, $s_i^0 = s_j^0$ and $\gamma_i^1 = \gamma_j^1$.

Now let $\langle s_i^0, \ldots, s_i^{h-1}\rangle = \langle s_j^0, \ldots, s_j^{h-1}\rangle$ and $\langle \gamma_i^0, \ldots, \gamma_i^h\rangle = \langle \gamma_j^0, \ldots, \gamma_j^h\rangle$. If $m_i = h = m_j$, we are home; suppose the contrary. Let us assume first that $\xi_i(\gamma_i^h) > \xi_j(\gamma_j^h)$. If $m_i = h < m_j$, then from $\xi_i(\gamma) = \xi_i(\gamma_i^h)$ we obtain $\gamma \geq \xi_i(\gamma_i^h) > \xi_j(\gamma_j^h)$, while from $\xi_j(\gamma) = \xi_j(\gamma_j^{m_j})$ we obtain $\gamma < \xi_j(\gamma_j^h)$. Thus, $m_i > h$; if $m_j = h$, we are home again. If $m_i > h < m_j$, we see that $\xi_i(\gamma) = \xi_i(\gamma_i^{m_i})$ is only possible for $\gamma > c_{s_i^h} \circ \xi_i(\gamma_i^h) > \xi_j(\gamma_j^h)$ because of (3.4c) and $j \in I^0(\xi_i(\gamma_i^h))$. On the other hand, $\xi_j(\gamma) = \xi_j(\gamma_j^{m_j})$ is only possible for $\gamma < \xi_j(\gamma_j^h)$.

Finally, if $\xi_i(\gamma_i^h) = \xi_j(\gamma_j^h) = \beta^*$ (which must hold if $i = j$!), we have $\beta^* \in Y_i \cap Y_j$. If $\beta^* \in \Sigma_{\text{iso}}$, then $m_i = h = m_j$ and we are home again; otherwise, we have $\beta^* \rhd\!\!\rhd_i \gamma$ and $\beta^* \rhd\!\!\rhd_j \gamma$, so the induction hypothesis applies. $\square$

For each $i \in N$, we denote $L_i = [0,1]^{Y_i^\infty}$ with the metric $\rho(w'', w') = \sum_{\beta \in Y_i^\infty} \left| w''_\beta - w'_\beta \right| \cdot 2^{-\lambda(\beta)}$, where $\lambda : \Sigma_{\text{lim}} \to \mathbb{N}$ is an injection ("arbitrary, but fixed"). Then we define a mapping $\eta^i : Y_i \to L_i$ coordinate-wise: $\eta^i_\beta(\gamma) = 0$ whenever $\beta \leq \gamma$; $\eta^i_\beta(\gamma) = \sum_{k=0}^{m_i(\beta,\gamma)-1} 2^{-s_i^k(\beta,\gamma)}$ whenever $\beta \rhd\!\!\rhd_i \gamma$; and $\eta^i_\beta(\gamma) = 1$ otherwise. Now we can define the strategy set: $X_i = \{(y_i, \eta^i(y_i))\}_{y_i \in Y_i}$ (simply put, X_i is the graph of η^i). For technical convenience, we define $\bar{\eta}^i : Y_i \to X_i$ ($i \in N$) as the Cartesian product of the identity mapping on Y_i and η^i; as usual, $\bar{\eta}^I : Y_I \to X_I$ is the Cartesian product of $\bar{\eta}^i$ for $i \in I \subseteq N$.

Lemma 3.1.3. *Let $\alpha \in \mathrm{A}$ and there be a sequence $\{\beta^k\}_{k \in \mathbb{N}}$ such that, for each $i \in N(\alpha)$, $\xi_i(\beta^k) \to y_i \in \Sigma$ (hence $y_i \in Y_i$) and $\eta^i \circ \xi_i(\beta^k) \to \eta^i(y_i)$. We denote I_0 the set of $i \in N(\alpha)$ for which the sequence $\xi_i(\beta^k)$ stabilizes after a finite number of steps. Then either $I_0 = N(\alpha)$, or there are a subsequence $\{\beta^{k_s}\}_{s \in \mathbb{N}}$ and $\beta^\omega \in \Sigma$ such that $I_0 = I^0(\beta^\omega) \cap N(\alpha)$, $\beta^{k_{s+1}} > \beta^{k_s}$ ($s \in \mathbb{N}$), $\beta^{k_s} \to \beta^\omega$, and $\pi_\alpha(\beta^{k_s}) \to \pi_\alpha(\beta^\omega)$.*

Proof. Since Σ is well ordered, we may assume, without restricting generality, that $\beta^{k+1} \geq \beta^k \geq \xi_i(\beta^k)$ for all $k \in \mathbb{N}$ and $i \in N(\alpha)$. Denoting $\beta^\omega = \sup_k \beta^k$, we have $\beta^k \to \beta^\omega$. Let $i \in N(\alpha) \setminus I_0$; without restricting generality, $\xi_i(\beta^{k+1}) > \xi_i(\beta^k)$, hence $\xi_i(\beta^{k+1}) > \beta^k$, for all $k \in \mathbb{N}$, which implies $y_i \geq \beta^\omega$. Taking into account $\xi_i(\beta^k) \leq \beta^k$, we obtain $y_i \leq \beta^\omega$, hence $\beta^\omega = y_i \in Y_i$, i.e., $i \in I^*(\beta^\omega)$. Conversely, if $i \in I^*(\beta^\omega)$, then $\xi_i(\beta^k)$ has to jump up at an infinite number of steps k, hence $i \notin I_0$.

Without restricting generality, we may assume that $\xi_i(\beta^\omega) = \xi_i(\beta^k)$ for all $i \in I_0 = I^0(\beta^\omega) \cap N(\alpha)$ and all $k \in \mathbb{N}$. By the conditions of the lemma, $\eta^i \circ \xi_i(\beta^k) \to \eta^i(\beta^\omega)$ for every $i \in I^*(\beta^\omega)$; therefore, $\eta^i_{\beta^\omega} \circ \xi_i(\beta^k) \to \eta^i_{\beta^\omega}(\beta^\omega)$. Since $\eta^i_{\beta^\omega}(\beta^\omega) = 0$, for every $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that $\eta^i_{\beta^\omega} \circ \xi_i(\beta^k) < \varepsilon$ for every $i \in I^*(\beta^\omega)$. By the definition of η^i, we have $\beta^\omega \rhd\!\!\rhd_i \xi_i(\beta^k)$, hence $\beta^\omega \rhd\!\!\rhd_i \beta^k$, with $\sum_{h=0}^{m_i(\beta^\omega,\beta^k)-1} 2^{-s_i^h} < \varepsilon$.

Picking the minimal $m_i(\beta^\omega, \beta^k)$, and invoking Lemma 3.1.2, we see that there is $\gamma_i^{m_i}$ such that $\xi_j(\gamma_i^{m_i}) = \xi_j(\beta^k)$ for all $j \in I^*(\beta^\omega)$; for $j \in I_0$, the same equality was assumed in the previous paragraph. Therefore, $\pi_\alpha(\gamma_i^{m_i}) = \pi_\alpha(\beta^k)$ by Lemma 3.1.1. For each $h < m_i$, we denote $\bar\gamma_i^h = \xi_i(\gamma_i^h)$; by Lemmas 3.1.2 and 3.1.1, we have $\pi_\alpha(\bar\gamma_i^h) = \pi_\alpha(\gamma_i^h)$. Taking into account the definition of functions b_k, we obtain $\left|\pi_\alpha(\bar\gamma_i^h) - \pi_\alpha(\gamma_i^{h+1})\right| < 2^{-s_i^h}$. Therefore, we have $\left|\pi_\alpha(\beta^\omega) - \pi_\alpha(\beta^k)\right| \le |\pi_\alpha(\beta^\omega) - \pi_\alpha(\gamma_i^1)| + \sum_{h=1}^{m_i-1} \left|\pi_\alpha(\bar\gamma_i^h) - \pi_\alpha(\gamma_i^{h+1})\right| < \sum_{h=0}^{m_i-1} 2^{-s_i^h} < \varepsilon$. □

For each $\alpha \in \mathrm{A}$, we denote $X^*_{N(\alpha)} = \bar\eta^{N(\alpha)} \circ \xi_{N(\alpha)}(\Sigma)$; the set $X^*_{N(\alpha)}$ is closed in $X_{N(\alpha)}$ by Lemma 3.1.3. By Lemma 3.1.1, there is a mapping $\varphi^*_\alpha : X^*_{N(\alpha)} \to V_\alpha$ such that $\pi_\alpha = \varphi^*_\alpha \circ \bar\eta^{N(\alpha)} \circ \xi_{N(\alpha)}$; the mapping is continuous by Lemma 3.1.3. Therefore, we can extend it to a continuous mapping $\varphi_\alpha : X_{N(\alpha)} \to V_\alpha$, just as in the proof of Proposition 3.1. Perceiving (3.3) as the definition of preferences, we obtain a strategic game Γ, which is obviously consistent with $\mathcal{S}$ and $\mathcal{L}$.

Let us show that $\bar\eta^N \circ \xi_N : \Sigma \to X$ defines an individual improvement cycle in Γ. We have $\xi_N(0,0) = \xi_N(\bar\beta, 1)$ by the definition of ξ_i. Whenever $\beta \in \Sigma_{\text{iso}}$, we have $\beta \in Y_i$ for a unique $i \in N$ and $\xi_{-i}(\beta - 1) = \xi_{-i}(\beta)$. Therefore, (3.2a) implies that $\bar\eta^N \circ \xi_N(\beta)$ is the result of a unilateral improvement over $\bar\eta^N \circ \xi_N(\beta - 1)$ by player i.

Finally, let us check (2.4b). Let $\beta^\omega \in \Sigma_{\lim}$; we have to produce a sequence $\{\beta^k\}_k$ in Σ' such that $\beta^k \to \beta^\omega$ and $\bar\eta^N \circ \xi_N(\beta^k) \to \bar\eta^N \circ \xi_N(\beta^\omega)$. For every $i \in I^*(\beta^\omega)$ and $\beta \in \Sigma_{\lim}$ such that $\beta > \beta^\omega$, we define $\zeta_i(\beta) < \beta^\omega$ by the following inductive process. We define $\bar\gamma_i^0(\beta) = \beta$ and execute the following procedure for each $h = 0, 1, \ldots$ First, we define $s_i^h = \min\{k \in \mathbb{N} |\ b_k(\bar\gamma_i^h(\beta)) \ge \beta^\omega\}$ and $\bar\gamma_i^{h+1}(\beta) = \xi_i \circ b_{s_i^h}(\bar\gamma_i^h(\beta))$. If $\bar\gamma_i^{h+1}(\beta) = \beta^\omega$ or $s_i^h = 0$, we set $\zeta_i(\beta) = (0,0)$; if $\bar\gamma_i^{h+1}(\beta) \in \Sigma_{\text{iso}}$ or $c_{s_i^h}(\bar\gamma_i^h(\beta)) \ge \beta^\omega$, we set $\zeta_i(\beta) = \xi_i \circ b_{s_i^h-1}(\bar\gamma_i^h(\beta))\, [< \beta^\omega]$. In either case, we stop the process; if neither condition holds, we proceed to $h+1$. Since Σ is well ordered and $\bar\gamma_i^{h+1}(\beta) < \bar\gamma_i^h(\beta)$ for all h, the process must stop at some stage.

For every $k \in \mathbb{N}$, we denote $\Lambda_k = \lambda^{-1}(\{0, 1, \ldots, k+1\})$, $\Lambda_k^- = \{\beta \in \Lambda_k |\ \beta < \beta^\omega\}$, and $\Lambda_k^+ = \{\beta \in \Lambda_k |\ \beta \ge \beta^\omega\}$; note that each of them is finite. Then we pick $s_k \in \mathbb{N}$ inductively, satisfying, at each step, the following requirements:

$$s_k \ge k + 2 \ \&\ s_k > s_{k-1}; \tag{3.5a}$$

$$\forall i \in I^*(\beta^\omega)\, [\xi_i \circ b_{s_k}(\beta^\omega) > \max_{\beta \in \Lambda_k^+} \zeta_i(\beta)]; \tag{3.5b}$$

$$\forall i \in I^*(\beta^\omega)\, [\xi_i \circ b_{s_k}(\beta^\omega) > \max \Lambda_k^-]. \tag{3.5c}$$

Finally, we define $\beta^k = b_{s_k}(\beta^\omega)$.

Now $\beta^k \to \beta^\omega$ by (3.5a) and the definition of the functions b_{s_k}, hence $\xi_i(\beta^k) \to \xi_i(\beta^\omega) = \beta^\omega$ for all $i \in I^*(\beta^\omega)$. For all $i \in I^0(\beta^\omega)$, we have $\xi_i(\beta^k) = \xi_i(\beta^\omega)$ again by the definition of b_{s_k}. Therefore, we only have to check $\eta^i \circ \xi_i(\beta^k) \to \eta^i(\beta^\omega)$ for every $i \in I^*(\beta^\omega)$. It is sufficient to show that

$$\rho(\eta^i \circ \xi_i(\beta^k), \eta^i(\beta^\omega)) = \sum_{\beta \in Y_i^\infty} \left|\eta^i_\beta \circ \xi_i(\beta^k) - \eta^i_\beta(\beta^\omega)\right| \cdot 2^{-\lambda(\beta)} < 2^{-k}. \tag{3.6}$$

Obviously, the contribution of all $\beta \notin \Lambda_k$ to the left hand side of (3.6) is less than 2^{-k-1}; it is enough to be able to assert the same about the contribution of all $\beta \in \Lambda_k$. If $\beta \in \Lambda_k^-$, then we have $\eta^i_\beta(\beta^\omega) = 0 = \eta^i_\beta \circ \xi_i(\beta^k)$ by (3.5c), so there is no contribution at all here.

Let us note that $\beta^\omega \rhd\!\!\rhd_i \beta^k$ with $m_i(\beta^\omega, \beta^k) = 1$ and $s_i^0(\beta^\omega, \beta^k) = s_k \geq k+2$; therefore, $\left|\eta^i_{\beta^\omega} \circ \xi_i(\beta^k) - \eta^i_{\beta^\omega}(\beta^\omega)\right| < 2^{-k-2}$. Let $\beta \in \Lambda_k^+ \setminus \{\beta^\omega\}$. If $\beta \rhd\!\!\rhd_i \beta^\omega$, then $\beta \rhd\!\!\rhd_i \beta^k$ as well, $m_i(\beta, \beta^k) = m_i(\beta, \beta^\omega) + 1$, and $s_i^{m_i}(\beta, \beta^k) = s_k$; therefore, $\left|\eta^i_\beta \circ \xi_i(\beta^k) - \eta^i_\beta(\beta^\omega)\right| < 2^{-k-2}$. Finally, if $\beta \rhd\!\!\rhd_i \beta^\omega$ does not hold for $\beta > \beta^\omega$, then $\beta \rhd\!\!\rhd_i \beta^k$ also cannot hold because of (3.5b) and the definition of $\zeta_i(\beta)$, hence $\eta^i(\beta^\omega) = 1 = \eta^i \circ \xi_i(\beta^k)$. Summing up the terms associated with $\beta \in \Lambda_k^+$ in (3.6), we obtain less than 2^{-k-1}. □

Proposition 3.2 and Theorem 1 can be reformulated as follows. Given a game structure $\mathcal{S}$ and a list of aggregation rules $\mathcal{L}$, the individual improvement relation in every strategic game Γ consistent with $\mathcal{S}$ and $\mathcal{L}$ admits a(n ω-)potential if and only if a potential in the same sense exists on the level of $\mathcal{S}$ and $\mathcal{L}$. It is easy to see that the Ω-acyclicity of the aggregate improvement relation implies that the strict component of each aggregation rule, $\succ^{\mathrm{a}}_i$, is ω-transitive.

The aggregate improvement relation is acyclic whenever there is an order on V_{A} such that the asymmetric component of each player's aggregation rule is the separable projection of the order to V_{Υ_i}. If the order is ω-transitive, the aggregate improvement relation is Ω-acyclic. A trivial example is when $\{\Upsilon_i\}_{i\in N}$ is a partition of A and each $\succ^{\mathrm{a}}_i$ is ω-transitive; the strict component of the Cartesian product of all aggregation rules (Pareto dominance) then becomes an ω-potential. More interesting examples are provided by the cases when $\succeq^{\mathrm{a}}_i$ is leximin$_{\alpha\in\Upsilon_i} v_\alpha$, or when $\succeq^{\mathrm{a}}_i$ is represented by a function on V_{Υ_i} of the form $\sum_{\alpha\in\Upsilon_i} \lambda_\alpha(v_\alpha)$. It is even sufficient that the strict component of each player's aggregation rule be a coarsening of the separable projection of a strict order on V_{A} to V_{Υ_i} (see Kukushkin, 2004b, Proposition 5.1) as in the case when $\succeq^{\mathrm{a}}_i$ is represented by a function on V_{Υ_i} of the form $\min_{\alpha\in\Upsilon_i} \lambda_\alpha(v_\alpha)$. However, there is no necessity result here.

Example 3.1. Let $N = \{1, 2\}$, $\mathrm{A} = \{1, 2, N\}$, $\Upsilon_i = \{i, N\}$ and $V_i = \{0, 1\}$ for both $i \in N$, while $V_N = \{0, 1, 2, 3, 4, 5\}$. The aggregation rules for both players are defined by numeric functions U_i; we depict triples $(U_1(v_{-2}), U_2(v_{-1}); P(v))$, where P is a numeric potential for the aggregate improvement relation (the existence of which is thus established), in a single table assuming that the abscissae axis corresponds to the private objective v_1, the ordinates axis to the public one, v_N, while two possible values of v_2 determine which table is appropriate:

$$\begin{bmatrix} (6,9;11) & (8,9;15) \\ (5,6;8) & (7,6;12) \\ (4,5;7) & (5,5;8) \\ (3,2;3) & (4,2;4) \\ (2,1;2) & (3,1;3) \\ (0,0;0) & (1,0;1) \end{bmatrix} \qquad \begin{bmatrix} (6,11;15) & (8,11;17) \\ (5,10;14) & (7,10;16) \\ (4,8;10) & (5,8;14) \\ (3,7;9) & (4,7;13) \\ (2,4;6) & (3,4;7) \\ (0,3;4) & (1,3;5) \end{bmatrix}.$$

Nonetheless, even the strict components of the aggregation rules cannot be represented as separable projections of the same strict order on V_{A}. Indeed, if $\succ$ were the strict component of such a strict order and $\succ_1$ and $\succ_2$ were its separable projections to $V_i \times V_N$, we would have $(0,0;3) \succ (0,1;1) \succ (1,1;0) \succ (1,0;2)$, hence $(0;3) \succ_1 (1;2)$; on the other hand, $(1,1;2) \succ (1,0;4) \succ (0,0;5) \succ (0,1;3)$, hence $(1;2) \succ_1 (0;3)$.

Let a game structure $\mathcal{S}$ be given; if a list of aggregation rules $\mathcal{L}$ is such that the aggregate improvement relation is acyclic (Ω-acyclic), then every finite (compact) game consistent with $\mathcal{S}$ and $\mathcal{L}$ admits a Nash equilibrium. Remarkably, neither statement can be reversed.

Example 3.2. Let $N = \{1, 2\}$, $\mathrm{A} = \{1, 2, N\}$, $\Upsilon_i = \{i, N\}$ for both $i \in N$, and $V_\alpha = \{0, 1, 2\}$ for all $\alpha \in \mathrm{A}$. Let the aggregation rules for both players be "isomorphic" and defined by a numeric function U_i depicted under the convention that the abscissae axis corresponds to the private objective v_i and the ordinates axis to the public one, v_N:

$$\begin{array}{ccc} 5 & 6 & 8 \\ 1 & 3 & 7. \\ 0 & 2 & 4 \end{array} \tag{3.7}$$

Assuming that V_{A} consists of triples $(v_1, v_2; v_N)$, we demonstrate that the aggregate improvement relation is not acyclic:

$$(1,0;2) \xrightarrow{1} (2,0;1) \xrightarrow{2} (2,1;0) \xrightarrow{1} (0,1;2) \xrightarrow{2} (0,2;1) \xrightarrow{1} (1,2;0) \xrightarrow{2} (1,0;2).$$

Proposition 3.3. *If a strategic game Γ is consistent with the game structure and the list of aggregation rules from Example 3.2, then Γ possesses a Nash equilibrium.*

Proof. Let us assume first that $\varphi_i(x_i) = 2$ is feasible for both i and pick a strategy profile $x^0 = (x_1^0, x_2^0)$ where $\varphi_N(x^0)$ is maximized subject to the restrictions $\varphi_i(x_i^0) = 2$ for both i. If x^0 is a Nash equilibrium, we are home. Let there be x_1' such that $(x_1', x_2^0) \succ_1 x^0$, hence $U_1(\varphi_1(x_1'), \varphi_N(x_1', x_2^0)) > U_1(2, \varphi_N(x^0)) \geq 4$. It is clear from (3.7) that $\varphi_N(x^0) = 0$ and $\varphi_N(x_1', x_2^0) = 2$ hence $U_2(\varphi_2(x_2^0), \varphi_N(x_1', x_2^0)) = 8$ and cannot be improved upon. Therefore, if $x_1' \in \mathcal{R}_1(x_2^0)$, then (x_1', x_2^0) is obviously a Nash equilibrium.

Supposing that $\varphi_2(x_2) < 2$ for all $x_2 \in X_2$, we notice that Γ is consistent with a modification of the game structure from Example 3.2 where $v_2 = 2$ is deleted from V_2. In the new situation, however, the aggregate improvement relation admits a potential. We depict triples $(U_1(v_{-2}), U_2(v_{-1}); P(v))$, where P is a numeric representation of the potential, in a single table assuming that the abscissae axis corresponds to the private objective v_1, the ordinates axis to the public one, v_N, while two possible values of v_2 determine which table is appropriate:

$$\begin{bmatrix} (5,5;5) & (6,5;6) & (8,5;12) \\ (1,1;1) & (3,1;3) & (7,1;7) \\ (0,0;0) & (2,0;2) & (4,0;4) \end{bmatrix} \quad \begin{bmatrix} (5,6;9) & (6,6;10) & (8,6;13) \\ (1,3;3) & (3,3;5) & (7,3;11) \\ (0,2;2) & (2,2;4) & (4,2;8) \end{bmatrix}.$$

Therefore, the aggregate improvement relation is strongly acyclic, hence Γ has the FIP property by Proposition 3.2, hence possesses a Nash equilibrium. $\square$

Remark. There is a gap between Theorem 1 and Example 3.2: Ω-acyclicity of individual improvements is only helpful in a compact game, but the game constructed in the proof of Theorem 1 need not be compact. A plausible hypothesis is this: The individual improvement relation in every compact strategic game Γ consistent with $\mathcal{S}$ and $\mathcal{L}$ is Ω-acyclic if and only if so are restrictions of the relevant aggregate improvement relation to all $W_{\mathrm{A}} = \prod_{\alpha \in \mathrm{A}} W_\alpha$ with compact $W_\alpha \subseteq V_\alpha$. The sufficiency is again straightforward; as to the necessity, there are plenty of technical details that have not yet received enough attention (see the first paragraph of the proof of Theorem 1).

3.2 Gorman additive aggregation rules

For an arbitrary game structure, we define an important class of aggregation rules. The definition needs auxiliary constructions. We denote $\mathfrak{B} = 2^{\mathrm{A}} \setminus \{\emptyset\}$, $N(\mathrm{B}) = \bigcap_{\alpha \in \mathrm{B}} N(\alpha)$ for $\mathrm{B} \in \mathfrak{B}$, and $\Upsilon_I = \bigcup_{i \in I} \Upsilon_i$ for $I \subseteq N$.

Let $i, j \in N$; we say that Υ_i and Υ_j *overlap* if $\Upsilon_i \setminus \Upsilon_j$, $\Upsilon_j \setminus \Upsilon_i$, and $\Upsilon_i \cap \Upsilon_j$ are not empty; the notation $i \bowtie j$ will be used in this case. An *overlap path* is a sequence $i_0, i_1, \ldots, i_m$ with $m > 0$ such that $i_{k-1} \bowtie i_k$ for each $k = 1, \ldots, m$. A nonempty subset $I \subseteq N$ is *connected* if for every $i, j \in I$ there is an overlap path $i = i_0, i_1, \ldots, i_m = j$ with $i_k \in I$ for each $k = 1, \ldots, m$. It follows immediately that $\#I \geq 2$.

We say that i and j are *contiguous*, $i \approx j$, if either $\Upsilon_i = \Upsilon_j$ or there is an overlap path with $i_0 = i$ and $i_m = j$. Clearly, $\approx$ is an equivalence relation on N; its equivalence classes are called *clusters*. We denote $\mathfrak{N}$ the set of clusters; for each $i \in N$, we denote $\nu(i) \in \mathfrak{N}$ the cluster containing i. A cluster $I \in \mathfrak{N}$ is *proper* if it contains i and j such that $\Upsilon_i \neq \Upsilon_j$. Every proper $I \in \mathfrak{N}$ is connected.

For each connected $I \subseteq N$, we define $\mathcal{Y}_I$ as the least subset of $\mathfrak{B}$ which is closed under intersection ($\mathrm{B}, \mathrm{B}' \in \mathcal{Y}_I \Rightarrow [\mathrm{B} \cap \mathrm{B}' \in \mathcal{Y}_I]$) and contains all Υ_i ($i \in I$) and $\Upsilon_i \setminus \Upsilon_j$ for $i, j \in I$ such that $i \bowtie j$. We define $\mathcal{F}_I$ as the least subset of $\mathfrak{B}$ including $\mathcal{Y}_I$ and closed under set difference ($\mathrm{B}, \mathrm{B}' \in \mathcal{F}_I \Rightarrow [\mathrm{B} \setminus \mathrm{B}' \in \mathcal{F}_I]$); it is easily checked that $\mathcal{F}_I$ is closed under intersection too. The set $\mathcal{A}_I \subseteq \mathfrak{B}$ consists of minimal (w.r.t. set inclusion) members of $\mathcal{F}_I$; it is easy to see that each $\mathrm{B} \in \mathcal{A}_I$ is associated with a partition $I = I^+ \cup I^-$ such that $\mathrm{B} = (\bigcap_{i \in I^+} \Upsilon_i) \setminus (\bigcup_{i \in I^-} \Upsilon_i)$. Every $\mathrm{B} \in \mathcal{F}_I$ is the union of some members of $\mathcal{A}_I$. We define $\mathcal{B}_I \subseteq \mathfrak{B}$ as the set of all unions of members of $\mathcal{A}_I$; it is easy to see that $\Upsilon_I = \bigcup_{\mathrm{B} \in \mathcal{A}_I} \mathrm{B}$. There holds $\mathcal{Y}_I \subseteq \mathcal{F}_I \subseteq \mathcal{B}_I \subseteq \mathfrak{B}$.

For technical convenience, we also define $\mathcal{A}_I = \mathcal{B}_I = \{\Upsilon_i\}$ ($i \in I$) for non-proper $I \in \mathfrak{N}$; then we denote $\mathcal{B} = \bigcup_{I \in \mathfrak{N}} \mathcal{B}_I$ and $\mathcal{A} = \bigcup_{I \in \mathfrak{N}} \mathcal{A}_I$. We define $\mathcal{M}(\mathrm{B}) = \{\mathrm{B}' \in \mathcal{B} | \ \mathrm{B}' \subset \mathrm{B} \ \& \ \nexists \mathrm{B}'' \in \mathcal{B} \, [\mathrm{B}' \subset \mathrm{B}'' \subset \mathrm{B}]\}$ and $\Delta(\mathrm{B}) = \mathrm{B} \setminus \bigcup_{\mathrm{B}' \in \mathcal{M}(\mathrm{B})} \mathrm{B}'$ for each $\mathrm{B} \in \mathcal{A}$.

To clarify the structure of $\mathcal{B}$, we define a strict order on $\mathfrak{N}$:

$$I < J \iff \big[I \neq J \ \& \ \exists \mathrm{B} \in \mathcal{A}_J \, \forall i \in I \, [\Upsilon_i \subseteq \mathrm{B}]\big]. \tag{3.8}$$

Lemma 3.2.1. *If $I, J \in \mathfrak{N}$, $I \neq J$, $i \in I$, $j \in J$, and $\Upsilon_i \subset \Upsilon_j$, then $\Upsilon_{i'} \subset \Upsilon_j$ for each $i' \in I$.*

Proof. Otherwise, we pick the first violation of the strict inclusion in the overlap path $i = i_0, i_1, \ldots, i_m = i'$: there are $k-1$ and k such that $\Upsilon_{i_{k-1}} \subset \Upsilon_j$, but not $\Upsilon_{i_k} \subset \Upsilon_j$. Since $\emptyset \neq \Upsilon_{i_{k-1}} \setminus \Upsilon_{i_k} \subset \Upsilon_j$, the equality $\Upsilon_{i_k} = \Upsilon_j$ is impossible; therefore, $\Upsilon_{i_k} \setminus \Upsilon_j \neq \emptyset$. Since $\emptyset \neq \Upsilon_{i_{k-1}} \cap \Upsilon_{i_k} \subseteq \Upsilon_j \cap \Upsilon_{i_k}$ and $\Upsilon_{i_{k-1}} \setminus \Upsilon_{i_k} \subseteq \Upsilon_j \setminus \Upsilon_{i_k}$, we have $i_k \bowtie j$, hence $I = J$, contradicting our assumption. □

Lemma 3.2.2. *Let $I, J \in \mathfrak{N}$ and $I \neq J$; then exactly one of the following alternatives holds: (1) $I > J$; (2) $J > I$; (3) $\mathrm{B}_I \cap \mathrm{B}_J = \emptyset$ for all $\mathrm{B}_I \in \mathcal{B}_I$ and $\mathrm{B}_J \in \mathcal{B}_J$.*

Proof. The incompatibility of the alternatives is obvious. Let (3) not hold, i.e., there be $\mathrm{B}_I \in \mathcal{B}_I$ and $\mathrm{B}_J \in \mathcal{B}_J$ such that $\mathrm{B}_I \cap \mathrm{B}_J \neq \emptyset$. Since $\mathrm{B}_I \subseteq \Upsilon_I$ and $\mathrm{B}_J \subseteq \Upsilon_J$, there are $i \in I$ and $j \in J$ such that $\Upsilon_i \cap \Upsilon_j \neq \emptyset$. If i and j overlap or $\Upsilon_i = \Upsilon_j$, then $I = J$, contradicting the assumption.

If $\Upsilon_i \subset \Upsilon_j$, then $\Upsilon_{i'} \subset \Upsilon_j$ for each $i' \in I$ by Lemma 3.2.1. Now if $\Upsilon_{i'} \cap \Upsilon_{j'} \neq \emptyset$ for $i' \in I$ and $j' \in J$, then, again, $\Upsilon_{i'} \subset \Upsilon_{j'}$; otherwise, we would have $\Upsilon_j \subset \Upsilon_{i'}$ by Lemma 3.2.1, contradicting $\Upsilon_{i'} \subset \Upsilon_j$. In other words, for every $j \in J$ either $\Upsilon_i \subset \Upsilon_j$ for all $i \in I$, or $\Upsilon_i \cap \Upsilon_j = \emptyset$ for all $i \in I$. Denoting $J^+ = \{j \in J | \ \forall i \in I \, [\Upsilon_i \subset \Upsilon_j]\}$, $J^- = J \setminus J^+$, and $\mathrm{B} = (\bigcap_{j \in J^+} \Upsilon_j) \setminus \Upsilon_{J^-}$, we have $\mathrm{B} \in \mathcal{A}_J$ and $\Upsilon_i \subset \mathrm{B}$ for each $i \in I$. $\square$

Thus, for every $I \in \mathfrak{N}$, the set of successors, $\{J \in \mathfrak{N} | \ I \leq J\}$ is a chain. For every $\mathrm{B} \in \mathcal{A}_J$, the set $\mathcal{M}(\mathrm{B})$ consists of sets Υ_I for some "immediate predecessors" of J in the sense of (3.8). For each connected $I \subseteq N$ and $i \in I$, we denote $\mathcal{A}_I^i = \{\mathrm{B} \in \mathcal{A}_I | \ \mathrm{B} \subseteq \Upsilon_i\}$; clearly, $\Upsilon_i = \bigcup_{\mathrm{B} \in \mathcal{A}_I^i} \mathrm{B}$.

A list of aggregation rules $\mathcal{L} = \langle \succeq^{\mathrm{a}}_i \rangle_{i \in N}$ is called *Gorman additive* if for each $\mathrm{B} \in \mathcal{B}$ there is a continuous function $\mu_{\mathrm{B}} : V_{\mathrm{B}} \to \mathbb{R}$ such that

1. μ_{Υ_i} represents $\succeq^{\mathrm{a}}_i$ for each $i \in N$;
2. if $I \in \mathfrak{N}$ is proper, $\mathrm{B}, \mathrm{B}' \in \mathcal{B}_I$, and $\mathrm{B} \cap \mathrm{B}' = \emptyset$, then
$$\mu_{\mathrm{B} \cup \mathrm{B}'}(v_{\mathrm{B}}, v_{\mathrm{B}'}) = \mu_{\mathrm{B}}(v_{\mathrm{B}}) + \mu_{\mathrm{B}'}(v_{\mathrm{B}'}) \tag{3.9a}$$
for every $v_{\mathrm{B}} \in V_{\mathrm{B}}$ and $v_{\mathrm{B}'} \in V_{\mathrm{B}'}$;
3. for every $\mathrm{B} \in \mathcal{A}$, there is a continuous function $\varkappa_{\mathrm{B}} : \left[\prod_{\mathrm{B}' \in \mathcal{M}(\mathrm{B})} \mu_{\mathrm{B}'}(V_{\mathrm{B}'})\right] \times V_{\Delta(\mathrm{B})} \to \mathbb{R}$ such that
$$\mu_{\mathrm{B}}(v_{\mathrm{B}}) = \varkappa_{\mathrm{B}}(\langle \mu_{\mathrm{B}'}(v_{\mathrm{B}'}) \rangle_{\mathrm{B}' \in \mathcal{M}(\mathrm{B})}, v_{\Delta(\mathrm{B})}) \tag{3.9b}$$
for every $v_{\mathrm{B}} \in V_{\mathrm{B}}$.

Remark. When $\langle \Upsilon_i \rangle_{i \in N}$ is a partition of A, every list $\mathcal{L}$ is Gorman additive.

"Theorem" 2. *Let $\mathcal{S}$ be a game structure where each set V_α ($\alpha \in \mathrm{A}$) is an open interval (bounded or not), and let $\mathcal{L}$ be a list of aggregation rules each of which is continuous and strictly responsive in the sense of* (3.1). *Then the following statements are equivalent.*

1. *The aggregate improvement relation is Ω-acyclic.*
2. *Every strategic game which is consistent with $\mathcal{S}$ and $\mathcal{L}$, and where every player has at most two strategies, possesses a Nash equilibrium.*
3. *The list $\mathcal{L}$ is Gorman additive.*

Sufficiency proof. Let $\mathcal{L}$ be Gorman additive. For every $I \in \mathfrak{N}$, we have $\Upsilon_I \in \mathcal{B}_I$ and $\mu_{\Upsilon_I}(v_{\Upsilon_I}) = \sum_{\mathrm{B} \in \mathcal{A}_I} \mu_{\mathrm{B}}(v_{\mathrm{B}})$. Denoting $\mathfrak{N}^+$ the set of maximizers on $\mathfrak{N}$ of the order (3.8), we derive, from Lemma 3.2.2 and our assumption $\Upsilon_N = \mathrm{A}$, that $\{\Upsilon_I\}_{I \in \mathfrak{N}^+}$ is a partition of A. Now we define $\mu^* : \mathbb{R}^{\mathrm{A}} \to \mathbb{R}$ by $\mu^*(v) = \sum_{I \in \mathfrak{N}^+} \mu_{\Upsilon_I}(v_{\Upsilon_I})$. Clearly, μ^* is continuous, hence the order $v' \succ^{\mathrm{a}} v \iff \mu^*(v') > \mu^*(v)$ is ω-transitive.

Let $i \in N$, $v', v \in \mathbb{R}^{\mathrm{A}}$, and $v' \rhd^{\mathrm{a}}_i v$; a straightforward inductive reasoning shows that $\mu_{\Upsilon_I}(v'_{\Upsilon_I}) > \mu_{\Upsilon_I}(v_{\Upsilon_I})$ for every $I \geq \nu(i)$, in particular for the unique $I \in \mathfrak{N}^+$ among them; since $v'_{\Upsilon_I} = v_{\Upsilon_I}$ for all other $I \in \mathfrak{N}^+$, we have $\mu^*(v') > \mu^*(v)$. Therefore, the order $\succ^{\mathrm{a}}$ is an ω-potential for the aggregate improvement relation, hence the latter is Ω-acyclic. $\square$

In the next two subsections, constructions needed for a necessity proof are developed. As was already mentioned, Lemma 3.4.2 is left without a general proof. Two special cases are treated in Subsection 3.5.

The term "Gorman additivity," naturally, refers to Gorman (1968). To see a connection clearer, let us assume that there is a finite set A of indices ("sectors"), an open interval V_α (bounded or not) for each $\alpha \in \mathrm{A}$, and a continuous ordering $\succeq_*$ on $V_{\mathrm{A}} = \prod_{\alpha \in \mathrm{A}} V_\alpha$ strictly responsive in the sense of (3.1). Let there be a finite list of subsets $\Upsilon_i \subseteq \mathrm{A}$ indexed by $i \in N$ such that $\succeq_*$ admits a separable projection $\succeq^{\mathrm{a}}_i$ to each V_{Υ_i} (which is inevitably a continuous and strictly responsive ordering itself). Then the main result of Gorman (1968) states, if one reads 1.7 through 1.13 of that paper attentively enough, that the list $\langle \succeq^{\mathrm{a}}_i \rangle_{i \in N} \cup \{\succeq_*\}$ is Gorman additive.

The statement easily follows from our Theorem 2 (assuming it proven): Perceiving each $\succeq^{\mathrm{a}}_i$ as the aggregation rule of a player $i \in N$, we add one "fictitious" player $*$ to N with $\Upsilon_* = \mathrm{A}$ and $\succeq_*$ as the aggregation rule. Now $\succ_*$ is obviously an ω-potential for the complete list of aggregation rules, hence Statement 2 of Theorem 2 holds, hence Statement 1 holds as well.

It is by no means obvious that *all* Gorman's results, especially those from Section 6 of that paper, could be derived from our Theorem 2. However, the question goes well beyond the intended scope of this paper. On the other hand, there is no way to derive our Theorem 2 from Gorman (1968): there is no "global" ordering on V_{A} in the conditions of the theorem; it only emerges at the end of the proof.

It is worth noting that neither continuity without strict monotonicity, nor strict monotonicity without continuity, ensure the necessity of Gorman additivity: consider, e.g., an aggregation rule represented by the function $U_i(v) = \min_{\alpha \in \Upsilon_i} v_\alpha$ in the first case (Theorem 1 of Kukushkin, 2004a), and $\operatorname{leximin}_{\alpha \in \Upsilon_i} v_\alpha$ in the second (Proposition 5.1 of Kukushkin, 2004b).

We end this subsection with an auxiliary result.

Lemma 3.2.3. *If $I \subseteq N$ is connected, then there exists an order on I, i.e., a one-to-one mapping $\sigma : \{1, \ldots, \#I\} \to I$, such that each set $\sigma(\{1, \ldots, s\})$ for $s \geq 2$ is connected and, whenever $s' > s$ and $\Upsilon_{\sigma(s')} \cap \Upsilon_{\sigma(s)} \neq \emptyset$, either $\sigma(s') \bowtie \sigma(s)$, or $\Upsilon_{\sigma(s')} \subseteq \Upsilon_{\sigma(s)}$.*

Proof. First of all, we define $I_0^{\max} \subseteq I$ by the condition that $\{\Upsilon_i\}_{i \in I_0^{\max}}$ is the subset of maximal (w.r.t. set inclusion) members of $\{\Upsilon_i\}_{i \in I}$. Note that $\#I_0^{\max} \geq 2$. We pick $i_0 \in I_0^{\max}$ arbitrarily and define $\sigma(1) = i_0$. For every $i \in I_0^{\max} \setminus \{i_0\}$, we define $m(i)$ as the minimal $m \in \mathbb{N}$ for which there is an overlap path $i_0, \ldots, i_m = i$ such that $i_k \in I$ for each k. It is important to note that we may assume $i_k \in I_0^{\max}$ for each $k = 1, \ldots, m-1$. Otherwise, let k be the least for which $i_k \notin I_0^{\max}$; then $\Upsilon_{i_k} \subset \Upsilon_j$ with $j \in I_0^{\max}$. Since $\Upsilon_j \cap \Upsilon_{i_{k-1}} \neq \emptyset$ and $i_{k-1} \in I_0^{\max}$, we have $i_{k-1} \bowtie j$. If $i_{k+1} \bowtie j$, then we can replace i_k with j; otherwise $\Upsilon_{i_{k+1}} \subset \Upsilon_j$, hence, taking the first $h > k+1$ for which $\Upsilon_{i_h} \not\subset \Upsilon_j$, we see that $i_h \bowtie j$, so the overlap path $i_0, \ldots, i_m$ was not minimal.

Then we define $\sigma : \{2, \ldots, \#I_0^{\max}\} \to I_0^{\max} \setminus \{i_0\}$ so that $m(i) > m(j) \Rightarrow \sigma^{-1}(i) > \sigma^{-1}(j)$; there is no problem with existence. By definition, for each $i \in I_0^{\max} \setminus \{i_0\}$, there is an overlap path $i_0, \ldots, i_{m(i)} = i$ in $I_0^{\max}$. Clearly, $m(i_k) \leq k < m(i)$ for all $k < m(i)$, hence $\sigma^{-1}(i_k) < \sigma^{-1}(i)$. Now for each s such that $2 \leq s < \#I_0^{\max}$, and each $i, j \in \sigma(\{1, \ldots, s\})$, there is an overlap path $i = i_{m(i)}, \ldots, i_1, i_0, j_1, \ldots, j_{m(j)} = j$ in $\sigma(\{1, \ldots, s\})$. If $s < s' \leq \#I_0^{\max}$ and $\Upsilon_{\sigma(s')} \cap \Upsilon_{\sigma(s)} \neq \emptyset$, then both $\Upsilon_{\sigma(s')} \subset \Upsilon_{\sigma(s)}$

and $\Upsilon_{\sigma(s)} \subset \Upsilon_{\sigma(s')}$ are incompatible with maximality; therefore, either $\sigma(s') \bowtie \sigma(s)$, or $\Upsilon_{\sigma(s')} = \Upsilon_{\sigma(s)}$.

Now we define $I_1^{\max} \subseteq I$ by the condition that $\{\Upsilon_i\}_{i \in I_1^{\max}}$ is the subset of maximal (w.r.t. set inclusion) members of $\{\Upsilon_i\}_{i \in I \setminus I_0^{\max}}$. Since I is connected, we can pick $i^0 \in I_1^{\max}$ such that $i^0 \bowtie i$ for an $i \in I_0^{\max}$; we define $\sigma(\# I_0^{\max} + 1) = i^0$. For every $i \in I_1^{\max} \setminus \{i^0\}$, we define $m(i)$ as the minimal $m \in \mathbb{N}$ for which there is an overlap path $i^0, \ldots, i^m = i$ such that $i^k \in I$ for each k. As in the first paragraph of the proof, we may assume $i^k \in I_0^{\max} \cup I_1^{\max}$ for each $k = 1, \ldots, m-1$. We define σ on $\{\# I_0^{\max} + 1, \ldots, \# I_0^{\max} + \# I_1^{\max}\}$ so that $m(i) > m(j) \Rightarrow \sigma^{-1}(i) > \sigma^{-1}(j)$ and then repeat the same procedure for $I \setminus (I_0^{\max} \cup I_1^{\max})$.

The conditions on σ are checked in the same way as in the second paragraph of the proof. □

3.3 Necessity of Gorman additivity: Basic lemmas

Let $\mathcal{L} = \langle \succeq^{\mathrm{a}}_i \rangle_{i \in N}$ be a list of continuous and strictly increasing aggregation rules such that every strategic game which is consistent with the list and where every player has at most two strategies possesses a Nash equilibrium. Then we have to prove that $\mathcal{L}$ is Gorman additive. All information to be derived from the equilibrium existence condition is contained in four technical lemmas.

Lemma 3.3.1. *Let* $i, j \in N$, $\Upsilon_i \cap \Upsilon_j = \mathrm{B} \neq \emptyset$, $\mathrm{B}_i = \Upsilon_i \setminus \mathrm{B}$, $\mathrm{B}_j = \Upsilon_j \setminus \mathrm{B}$, $\bar{\mathrm{B}} = \Upsilon_i \cup \Upsilon_j$, $v_{\bar{\mathrm{B}}} \in V_{\bar{\mathrm{B}}}$, $v'_{\mathrm{B}} \in V_{\mathrm{B}}$, *and*

$$(v_{\mathrm{B}}, v_{\mathrm{B}_i}) \sim^{\mathrm{a}}_i (v'_{\mathrm{B}}, v_{\mathrm{B}_i}). \tag{3.10a}$$

Then

$$(v_{\mathrm{B}}, v_{\mathrm{B}_j}) \sim^{\mathrm{a}}_j (v'_{\mathrm{B}}, v_{\mathrm{B}_j}). \tag{3.10b}$$

Proof. Suppose, without restricting generality, that $(v_{\mathrm{B}}, v_{\mathrm{B}_j}) \succ^{\mathrm{a}}_j (v'_{\mathrm{B}}, v_{\mathrm{B}_j})$. By continuity, there is $v''_{\mathrm{B}} < v_{\mathrm{B}}$ such that $(v''_{\mathrm{B}}, v_{\mathrm{B}_j}) \succ^{\mathrm{a}}_j (v'_{\mathrm{B}}, v_{\mathrm{B}_j})$; by monotonicity from (3.10a), $(v'_{\mathrm{B}}, v_{\mathrm{B}_i}) \succ^{\mathrm{a}}_i (v''_{\mathrm{B}}, v_{\mathrm{B}_i})$. Now we define a strategic game Γ consistent with $\mathcal{S}$ and $\mathcal{L}$: $X_i = X_j = \{0, 1\}$, $X_k = \{0\}$ for $k \neq i, j$; for each $\alpha \in \bar{\mathrm{B}} \setminus \mathrm{B}$ and each $x_{N(\alpha)} \in X_{N(\alpha)}$, $\varphi_\alpha(x_{N(\alpha)}) = v_\alpha$; for each $\alpha \in \mathrm{B}$ and $x_{N(\alpha)} \in X_{N(\alpha)}$, $\varphi_\alpha(x_{N(\alpha)}) = v'_\alpha$ whenever $x_i = x_j$, while $\varphi_\alpha(x_{N(\alpha)}) = v''_\alpha$ otherwise. Clearly, Γ possesses no Nash equilibrium: if $x_i = x_j$, then player j chooses $x'_j \neq x_i$ and improves; if $x_i \neq x_j$, player i chooses $x'_i = x_j$ and improves. □

Lemma 3.3.2. *Let* $i, j \in N$, *the sets* $\mathrm{B}, \mathrm{B}_i, \mathrm{B}_j, \bar{\mathrm{B}} \subseteq \mathrm{A}$ *have the same meaning as in Lemma 3.3.1,* $\mathrm{B} \neq \emptyset$, $v_{\bar{\mathrm{B}}}, v'_{\bar{\mathrm{B}}} \in V_{\bar{\mathrm{B}}}$,

$$(v_{\mathrm{B}}, v'_{\mathrm{B}_i}) \sim^{\mathrm{a}}_i (v_{\mathrm{B}}, v_{\mathrm{B}_i}), \tag{3.11a}$$

and

$$(v_{\mathrm{B}}, v'_{\mathrm{B}_j}) \sim^{\mathrm{a}}_j (v'_{\mathrm{B}}, v_{\mathrm{B}_j}). \tag{3.11b}$$

Then

$$(v'_{\mathrm{B}}, v'_{\mathrm{B}_i}) \sim^{\mathrm{a}}_i (v'_{\mathrm{B}}, v_{\mathrm{B}_i}). \tag{3.12}$$

Proof. Suppose, without restricting generality, that $(v'_{\mathrm{B}}, v'_{\mathrm{B}_i}) \succ^{\mathrm{a}}_i (v'_{\mathrm{B}}, v_{\mathrm{B}_i})$. By continuity, there are $v^+_{\mathrm{B}} > v'_{\mathrm{B}}$ and $v^+_{\mathrm{B}_i} > v_{\mathrm{B}_i}$ such that

$$(v'_{\mathrm{B}}, v'_{\mathrm{B}_i}) \succ^{\mathrm{a}}_i (v^+_{\mathrm{B}}, v^+_{\mathrm{B}_i}). \tag{3.13a}$$

By monotonicity from (3.11a), we have

$$(v_{\mathrm{B}}, v^{+}_{\mathrm{B}_i}) \succ^{\mathrm{a}}_i (v_{\mathrm{B}}, v'_{\mathrm{B}_i}); \tag{3.13b}$$

by monotonicity from (3.11b), $(v^{+}_{\mathrm{B}}, v_{\mathrm{B}_j}) \succ^{\mathrm{a}}_j (v_{\mathrm{B}}, v'_{\mathrm{B}_j})$. By the continuity from the last relation, there is $v^{+}_{\mathrm{B}_j} > v'_{\mathrm{B}_j}$ such that

$$(v^{+}_{\mathrm{B}}, v_{\mathrm{B}_j}) \succ^{\mathrm{a}}_j (v_{\mathrm{B}}, v^{+}_{\mathrm{B}_j}). \tag{3.13c}$$

Finally, we have

$$(v_{\mathrm{B}}, v^{+}_{\mathrm{B}_j}) \succ^{\mathrm{a}}_j (v'_{\mathrm{B}}, v_{\mathrm{B}_j}) \tag{3.13d}$$

by monotonicity from (3.11b).

Now we define a strategic game Γ consistent with $\mathcal{S}$ and $\mathcal{L}$: $X_i = X_j = \{0,1\}$, $X_k = \{0\}$ for $k \neq i, j$; for each $\alpha \in \mathrm{B}_i$ and $x_{N(\alpha)} \in X_{N(\alpha)}$, $\varphi_\alpha(x_{N(\alpha)}) = v'_\alpha$ whenever $x_i = 1$, while $\varphi_\alpha(x_{N(\alpha)}) = v^{+}_\alpha$ whenever $x_i = 0$; for each $\alpha \in \mathrm{B}_j$ and $x_{N(\alpha)} \in X_{N(\alpha)}$, $\varphi_\alpha(x_{N(\alpha)}) = v_\alpha$ whenever $x_j = 1$, while $\varphi_\alpha(x_{N(\alpha)}) = v^{+}_\alpha$ whenever $x_j = 0$; for each $\alpha \in \mathrm{B}$ and $x_{N(\alpha)} \in X_{N(\alpha)}$, $\varphi_\alpha(x_{N(\alpha)}) = v_\alpha$ whenever $x_j = 0$, $\varphi_\alpha(x_{N(\alpha)}) = v^{+}_\alpha$ whenever $x_j = 1$ and $x_i = 0$, and $\varphi_\alpha(x_{N(\alpha)}) = v'_\alpha$ whenever $x_j = 1$ and $x_i = 1$.

Putting x_i on the abscissae axis and x_j on the ordinates, we can depict the values of functions $(\varphi_{\mathrm{B}_i}, \varphi_{\mathrm{B}}, \varphi_{\mathrm{B}_j})$ in the following matrix:

$$\begin{matrix} (v^{+}_{\mathrm{B}_i}, v^{+}_{\mathrm{B}}, v_{\mathrm{B}_j}) & (v'_{\mathrm{B}_i}, v'_{\mathrm{B}}, v_{\mathrm{B}_j}) \\ (v^{+}_{\mathrm{B}_i}, v_{\mathrm{B}}, v^{+}_{\mathrm{B}_j}) & (v'_{\mathrm{B}_i}, v_{\mathrm{B}}, v^{+}_{\mathrm{B}_j}) \end{matrix}.$$

It follows immediately from relations (3.13) that Γ possesses no Nash equilibrium. $\square$

Lemma 3.3.3. *Let $i, j \in N$, $i \bowtie j$, the sets $\mathrm{B}, \mathrm{B}_i, \mathrm{B}_j, \bar{\mathrm{B}} \subseteq \mathrm{A}$ have the same meaning as in Lemma 3.3.1, $v_{\bar{\mathrm{B}}}, v'_{\bar{\mathrm{B}}} \in V_{\bar{\mathrm{B}}}$, $v''_{\mathrm{B}} \in V_{\mathrm{B}}$,*

$$(v'_{\mathrm{B}}, v_{\mathrm{B}_i}) \sim^{\mathrm{a}}_i (v_{\mathrm{B}}, v'_{\mathrm{B}_i}), \tag{3.14a}$$

$$(v''_{\mathrm{B}}, v_{\mathrm{B}_i}) \sim^{\mathrm{a}}_i (v'_{\mathrm{B}}, v'_{\mathrm{B}_i}), \tag{3.14b}$$

and

$$(v'_{\mathrm{B}}, v_{\mathrm{B}_j}) \sim^{\mathrm{a}}_j (v_{\mathrm{B}}, v'_{\mathrm{B}_j}). \tag{3.14c}$$

Then

$$(v''_{\mathrm{B}}, v_{\mathrm{B}_j}) \sim^{\mathrm{a}}_j (v'_{\mathrm{B}}, v'_{\mathrm{B}_j}). \tag{3.15}$$

Proof. Suppose first that $(v''_{\mathrm{B}}, v_{\mathrm{B}_j}) \succ^{\mathrm{a}}_j (v'_{\mathrm{B}}, v'_{\mathrm{B}_j})$. By continuity, there are $v^{-}_{\mathrm{B}} < v''_{\mathrm{B}}$ and $v^{+}_{\mathrm{B}_j} > v'_{\mathrm{B}_j}$ such that

$$(v^{-}_{\mathrm{B}}, v_{\mathrm{B}_j}) \succ^{\mathrm{a}}_j (v'_{\mathrm{B}}, v^{+}_{\mathrm{B}_j}). \tag{3.16a}$$

By monotonicity from (3.14b), $(v'_{\mathrm{B}}, v'_{\mathrm{B}_i}) \succ^{\mathrm{a}}_i (v^{-}_{\mathrm{B}}, v_{\mathrm{B}_i})$; therefore, there is $v^{-}_{\mathrm{B}_i} < v'_{\mathrm{B}_i}$ such that

$$(v'_{\mathrm{B}}, v^{-}_{\mathrm{B}_i}) \succ^{\mathrm{a}}_i (v^{-}_{\mathrm{B}}, v_{\mathrm{B}_i}). \tag{3.16b}$$

By monotonicity from (3.14c) and (3.14a), respectively, we have

$$(v_{\mathrm{B}}, v^{+}_{\mathrm{B}_j}) \succ^{\mathrm{a}}_j (v'_{\mathrm{B}}, v_{\mathrm{B}_j}) \tag{3.16c}$$

and

$$(v'_{\mathrm{B}}, v_{\mathrm{B}_i}) \succ^{\mathrm{a}}_i (v_{\mathrm{B}}, v^{-}_{\mathrm{B}_i}). \tag{3.16d}$$

Now we define a strategic game Γ consistent with $\mathcal{S}$ and $\mathcal{L}$: $X_i = X_j = \{0,1\}$, $X_k = \{0\}$ for $k \neq i, j$; for each $\alpha \in \mathrm{B}_i$ and $x_{N(\alpha)} \in X_{N(\alpha)}$, $\varphi_\alpha(x_{N(\alpha)}) = v_\alpha^-$ whenever $x_i = 1$, while $\varphi_\alpha(x_{N(\alpha)}) = v_\alpha$ whenever $x_i = 0$; for each $\alpha \in \mathrm{B}_j$ and $x_{N(\alpha)} \in X_{N(\alpha)}$, $\varphi_\alpha(x_{N(\alpha)}) = v_\alpha^+$ whenever $x_j = 1$, while $\varphi_\alpha(x_{N(\alpha)}) = v_\alpha$ whenever $x_j = 0$; for each $\alpha \in \mathrm{B}$ and $x_{N(\alpha)} \in X_{N(\alpha)}$, $\varphi_\alpha(x_{N(\alpha)}) = v_\alpha$ whenever $x_i = x_j = 1$, $\varphi_\alpha(x_{N(\alpha)}) = v'_\alpha$ whenever $x_i + x_j = 1$, and $\varphi_\alpha(x_{N(\alpha)}) = v_\alpha^-$ whenever $x_i = x_j = 0$. Putting x_i on the abscissae axis and x_j on the ordinates, we can depict the values of functions $(\varphi_{\mathrm{B}_i}, \varphi_{\mathrm{B}}, \varphi_{\mathrm{B}_j})$ in the following matrix:

$$\begin{matrix} (v_{\mathrm{B}_i}, v'_{\mathrm{B}}, v^+_{\mathrm{B}_j}) & (v^-_{\mathrm{B}_i}, v_{\mathrm{B}}, v^+_{\mathrm{B}_j}) \\ (v_{\mathrm{B}_i}, v^-_{\mathrm{B}}, v_{\mathrm{B}_j}) & (v^-_{\mathrm{B}_i}, v'_{\mathrm{B}}, v_{\mathrm{B}_j}) \end{matrix}.$$

It follows immediately from relations (3.16) that Γ possesses no Nash equilibrium.

If $(v'_{\mathrm{B}}, v'_{\mathrm{B}_j}) \succ^{\mathrm{a}}_j (v''_{\mathrm{B}}, v_{\mathrm{B}_j})$, then we follow the same scheme, changing all signs. To be more precise, we pick $v^-_{\mathrm{B}} > v''_{\mathrm{B}}$, $v^+_{\mathrm{B}_j} < v'_{\mathrm{B}_j}$, and $v^-_{\mathrm{B}_i} > v'_{\mathrm{B}_i}$ such that the relations opposite to (3.16) hold. Now a strategic game defined in the same way as above possesses no Nash equilibrium. □

Lemma 3.3.4. *Let $m \geq 2$, $i_0, i_1, \ldots, i_{m-1} \in N$, $\emptyset \neq \mathrm{B}_k \subseteq \mathrm{A}$ ($k = 0, \ldots, m-1$; for notational convenience, we add $i_m = i_0$ and $\mathrm{B}_m = \mathrm{B}_0$, and also denote $\bar{\mathrm{B}} = \bigcup_k \mathrm{B}_k$), $v', v \in V_{\bar{\mathrm{B}}}$, and $w \in V_{\mathrm{A}}$ be such that: $v'_\alpha > v_\alpha$ for every $\alpha \in \bar{\mathrm{B}}$; $\mathrm{B}_k \cup \mathrm{B}_{k+1} \subseteq \Upsilon_{i_k}$; $\mathrm{B}_h \cap \Upsilon_{i_k} = \emptyset$ whenever $h \notin \{k, k+1\}$ (hence $\mathrm{B}_h \cap \mathrm{B}_k = \emptyset$ whenever $h \neq k$, and $i_k \bowtie i_{k+1}$ for every $k = 0, 1, \ldots, m-1$);*

$$(v'_{\mathrm{B}_{k+1}}, v_{\mathrm{B}_k}, w_{\Upsilon_{i_k} \setminus \{\mathrm{B}_{k+1} \cup \mathrm{B}_k\}}) \sim^{\mathrm{a}}_{i_k} (v_{\mathrm{B}_{k+1}}, v'_{\mathrm{B}_k}, w_{\Upsilon_{i_k} \setminus \{\mathrm{B}_{k+1} \cup \mathrm{B}_k\}}) \tag{3.17}$$

for all $k = 0, \ldots, m-2$. Then

$$(v'_{\mathrm{B}_{m-1}}, v_{\mathrm{B}_0}, w_{\Upsilon_{i_{m-1}} \setminus \{\mathrm{B}_{m-1} \cup \mathrm{B}_0\}}) \sim^{\mathrm{a}}_{i_{m-1}} (v_{\mathrm{B}_{m-1}}, v'_{\mathrm{B}_0}, w_{\Upsilon_{i_{m-1}} \setminus \{\mathrm{B}_{m-1} \cup \mathrm{B}_0\}}). \tag{3.18}$$

Proof. Suppose the contrary; changing the numeration if needed, we may assume that

$$(v'_{\mathrm{B}_{m-1}}, v_{\mathrm{B}_0}, w_{\Upsilon_{i_{m-1}} \setminus \{\mathrm{B}_{m-1} \cup \mathrm{B}_0\}}) \succ^{\mathrm{a}}_{i_{m-1}} (v_{\mathrm{B}_{m-1}}, v'_{\mathrm{B}_0}, w_{\Upsilon_{i_{m-1}} \setminus \{\mathrm{B}_{m-1} \cup \mathrm{B}_0\}}).$$

By continuity, there is $\delta_0 > 0$ such that, defining $v''_\alpha = v'_\alpha + \delta_0$ for all $\alpha \in \mathrm{B}_0$ and $v''_\alpha = v'_\alpha$ for all $\alpha \in \mathrm{B}_{m-1}$, we still have

$$(v''_{\mathrm{B}_{m-1}}, v_{\mathrm{B}_0}, w_{\Upsilon_{i_{m-1}} \setminus \{\mathrm{B}_{m-1} \cup \mathrm{B}_0\}}) \succ^{\mathrm{a}}_{i_{m-1}} (v_{\mathrm{B}_{m-1}}, v''_{\mathrm{B}_0}, w_{\Upsilon_{i_{m-1}} \setminus \{\mathrm{B}_{m-1} \cup \mathrm{B}_0\}}).$$

By monotonicity from (3.17) with $k = 0$, we have

$$(v_{\mathrm{B}_1}, v''_{\mathrm{B}_0}, w_{\Upsilon_{i_0} \setminus \{\mathrm{B}_1 \cup \mathrm{B}_0\}}) \succ^{\mathrm{a}}_{i_0} (v'_{\mathrm{B}_1}, v_{\mathrm{B}_0}, w_{\Upsilon_{i_0} \setminus \{\mathrm{B}_1 \cup \mathrm{B}_0\}});$$

by continuity, there is $\delta_1 > 0$ such that, defining $v''_\alpha = v'_\alpha + \delta_1$ for all $\alpha \in \mathrm{B}_1$, we still have

$$(v_{\mathrm{B}_1}, v''_{\mathrm{B}_0}, w_{\Upsilon_{i_0} \setminus \{\mathrm{B}_1 \cup \mathrm{B}_0\}}) \succ^{\mathrm{a}}_{i_0} (v''_{\mathrm{B}_1}, v_{\mathrm{B}_0}, w_{\Upsilon_{i_0} \setminus \{\mathrm{B}_1 \cup \mathrm{B}_0\}}).$$

A straightforward inductive process based on the monotonicity and continuity of preferences shows the existence, for every $\alpha \in \bar{\mathrm{B}}$, of $v''_\alpha \geq v'_\alpha > v_\alpha$ such that

$$(v_{\mathrm{B}_{k+1}}, v''_{\mathrm{B}_k}, w_{\Upsilon_{i_k} \setminus \{\mathrm{B}_{k+1} \cup \mathrm{B}_k\}}) \succ^{\mathrm{a}}_{i_k} (v''_{\mathrm{B}_{k+1}}, v_{\mathrm{B}_k}, w_{\Upsilon_{i_k} \setminus \{\mathrm{B}_{k+1} \cup \mathrm{B}_k\}}) \tag{3.19}$$

for all $k = 0, \ldots, m-1$.

Now we define a strategic game Γ consistent with $\mathcal{S}$ and $\mathcal{L}$: $X_{i_k} = \{0, 1\}$ for each $k = 0, 1, \dots, m-1$, $X_j = \{0\}$ for all other $j \in N$; for each $k = 1, \dots, m-1$ and $\alpha \in \mathrm{B}_k$, there holds $\varphi_\alpha(x_{N(\alpha)}) = v''_\alpha$ whenever $x_{i_k} = x_{i_{k-1}}$ and $\varphi_\alpha(x_{N(\alpha)}) = v_\alpha$ whenever $x_{i_k} \neq x_{i_{k-1}}$; for each $\alpha \in \mathrm{B}_0$, there holds $\varphi_\alpha(x_{N(\alpha)}) = v''_\alpha$ whenever $x_{i_0} \neq x_{i_{m-1}}$ and $\varphi_\alpha(x_{N(\alpha)}) = v_\alpha$ whenever $x_{i_0} = x_{i_{m-1}}$; for each $\alpha \in \mathrm{A} \setminus \bar{\mathrm{B}}$, there holds $\varphi_\alpha(x_{N(\alpha)}) = w_\alpha$ for all $x_{N(\alpha)} \in X_{N(\alpha)}$.

Clearly, for every $x \in X$, there is k such that $\varphi_\alpha(x_{N(\alpha)}) = v_\alpha$ for all $\alpha \in \mathrm{B}_k$. Therefore, player i_k can change his strategy and obtain a better outcome by (3.1) or (3.19). It follows immediately that Γ possesses no Nash equilibrium. $\square$

There is a point about Lemma 3.3.4 worth discussion. In the games constructed in the proofs of all other lemmas of this subsection, all players but two were dummies. Here, on the contrary, the number of active players may be arbitrary (between 2 and n). The fact is that the existence of a Nash equilibrium (or even Ω-acyclicity of individual improvements) in every game consistent with $\mathcal{S}$ and $\mathcal{L}$ and with no more than two active players does *not* imply the existence of a Nash equilibrium (or Ω-acyclicity of improvements) in *every* game consistent with $\mathcal{S}$ and $\mathcal{L}$. In Kukushkin (2006), the opposite was true.

Example 3.3. Let $N = \{1, 2, 3\}$, $\mathrm{A} = \{0, 1, 2\}$, $\oplus$ mean addition modulo 3, $\Upsilon_i = \{i-1, (i-1) \oplus 1\}$ for every $i \in N$, and $V_\alpha = \mathbb{R}$ for all $\alpha \in \mathrm{A}$; let each $\succeq^{\mathrm{a}}_i$ be represented by the function $v_{i-1} + 2v_{(i-1)\oplus 1}$. It is easy to see that Lemma 3.3.4 does not hold: $(4_0, 0_1) \sim^{\mathrm{a}}_1 (0_0, 2_1)$, $(2_1, 0_2) \sim^{\mathrm{a}}_2 (0_1, 1_2)$, but $(4_0, 0_2) \succ^{\mathrm{a}}_3 (0_0, 1_2)$. On the other hand, if we delete any one player from N, obvious transformations will make both utility functions just sums; therefore, every two person game consistent with $\mathcal{S}$ and $\mathcal{L}$ possesses a Nash equilibrium (hence, the other three lemmas hold).

Lemmas 3.3.1 and 3.3.2 have important implications in terms of separability. Let us denote $\mathcal{E}$ the set of $\mathrm{B} \in \mathfrak{B}$ such that there is an ordering $\succeq^{\mathrm{a}}_{\mathrm{B}}$ on V_{B} which is a separable projection to V_{B} of $\succeq^{\mathrm{a}}_i$ whenever $\mathrm{B} \subseteq \Upsilon_i$ $(i \in N)$.

Lemma 3.3.5. *The following five statements hold:*

1. *If $i, j \in N$, $\mathrm{B}' \subseteq \Upsilon_i$, $\mathrm{B}'' = \Upsilon_j \cap \mathrm{B}' \neq \emptyset$, and $\succeq^{\mathrm{a}}_i$ admits a separable projection $\succeq^{\mathrm{a}}_{\mathrm{B}'}$ to $V_{\mathrm{B}'}$, then $\succeq^{\mathrm{a}}_{\mathrm{B}'}$ and $\succeq^{\mathrm{a}}_j$ admit a common separable projection to $V_{\mathrm{B}''}$;*

2. *$\Upsilon_i \setminus \Upsilon_j \in \mathcal{E}$ whenever $i \bowtie j$;*

3. *$\mathcal{Y}_I \subseteq \mathcal{E}$ whenever $I \subseteq N$ is connected;*

4. *$\mathrm{B} \setminus \mathrm{B}' \in \mathcal{E}$ whenever $\mathrm{B}, \mathrm{B}' \in \mathcal{E}$, B and B' overlap, and $\mathrm{B} \cup \mathrm{B}' \subseteq \Upsilon_i$ for an $i \in N$;*

5. *$\mathcal{A}_I \subseteq \mathcal{E}$ whenever $I \subseteq N$ is connected.*

Remark. Theorem 2 itself implies that $\Upsilon_i \cup \Upsilon_j \in \mathcal{E}$ whenever $\nu(i) = \nu(j)$ (cf. Theorem 1 of Gorman, 1968); however, this assertion can only be proven at the end of a rather long way.

Proof. (1) Assuming the contrary, we must have $v_{\mathrm{B}' \cup \Upsilon_j} \in V_{\mathrm{B}' \cup \Upsilon_j}$ and $v''_{\mathrm{B}''} \in V_{\mathrm{B}''}$ such that

$$v_{\mathrm{B}'} \succeq^{\mathrm{a}}_{\mathrm{B}'} (v_{\mathrm{B}' \setminus \mathrm{B}''}, v''_{\mathrm{B}''}), \tag{3.20a}$$

but

$$(v''_{\mathrm{B}''}, v_{\Upsilon_j \setminus \mathrm{B}''}) \succ^{\mathrm{a}}_j v_{\Upsilon_j}. \tag{3.20b}$$

In the case of a strict preference in (3.20a) and indifference in (3.20b), we can increase $v''_{\mathrm{B}''}$ a bit, obtaining (3.20). If we replace each v''_α, $\alpha \in \mathrm{B}''$, with $\max\{v''_\alpha, v_\alpha\}$, the relation in (3.20a) will be reversed; therefore, there is $v'_{\mathrm{B}''} \geq v''_{\mathrm{B}''}$ such that $v_{\mathrm{B}'} \sim^{\mathrm{a}}_{\mathrm{B}'} (v_{\mathrm{B}' \setminus \mathrm{B}''}, v'_{\mathrm{B}''})$; picking $v_{\Upsilon_i \setminus (\mathrm{B}' \cup \Upsilon_j)}$ arbitrarily, we obtain $v_{\Upsilon_i} \sim^{\mathrm{a}}_i (v'_{\mathrm{B}''}, v_{\Upsilon_i \setminus \mathrm{B}''})$. Now we can invoke Lemma 3.3.1 with $v'_{\Upsilon_i \cap \Upsilon_j} = (v'_{\mathrm{B}''}, v_{\Upsilon_i \cap \Upsilon_j \setminus \mathrm{B}''})$, obtaining $v_{\Upsilon_j} \sim^{\mathrm{a}}_j (v'_{\mathrm{B}''}, v_{\Upsilon_j \setminus \mathrm{B}''})$, but this contradicts (3.20b).

(2) In the light of Statement 1, it is sufficient to prove that $\succeq^{\mathrm{a}}_i$ admits a separable projection on $V_{\Upsilon_i \setminus \Upsilon_j}$ provided $i \bowtie j$. The following argument is similar to, but shorter than, that of Gorman (1968, 2.6-2.13). In the notation of Lemmas 3.3.2 or 3.3.1, the contrary would mean the existence of $v_{\bar{\mathrm{B}}}$ and $v'_{\bar{\mathrm{B}}}$ such that

$$(v_{\mathrm{B}}, v'_{\mathrm{B}_i}) \succeq^{\mathrm{a}}_i (v_{\mathrm{B}}, v_{\mathrm{B}_i}), \tag{3.21a}$$

but

$$(v'_{\mathrm{B}}, v_{\mathrm{B}_i}) \succ^{\mathrm{a}}_i (v'_{\mathrm{B}}, v'_{\mathrm{B}_i}). \tag{3.21b}$$

Since (3.21b) is impossible when $v'_{\mathrm{B}} = v_{\mathrm{B}}$, we can shift v_{B} towards v'_{B} along a straight line and mark the last point where (3.21b) does not hold. Therefore, we can assume that (3.21a) holds as an equivalence and v'_{B} is close to v_{B}, so there is v'_{B_j} such that $(v_{\mathrm{B}}, v'_{\mathrm{B}_j}) \sim^{\mathrm{a}}_j (v'_{\mathrm{B}}, v_{\mathrm{B}_j})$. Finally, we can apply Lemma 3.3.2 in the same way as Lemma 3.3.1 was applied above, obtaining a contradiction with (3.21b).

(3) An easy induction based on Statements 1 and 2.

(4) We consider another game structure, where $N = \{1, 2\}$, A is the same, $\Upsilon_1 = \mathrm{B}$, $\Upsilon_2 = \mathrm{B}'$ (thus the condition $\bigcup_i \Upsilon_i = \mathrm{A}$ is violated, but this can be fixed easily), $\succeq^{\mathrm{a}}_1$ is $\succeq^{\mathrm{a}}_{\mathrm{B}}$, and $\succeq^{\mathrm{a}}_2$ is $\succeq^{\mathrm{a}}_{\mathrm{B}'}$. Clearly, the strict aggregate preferences $\succ^{\mathrm{a}}_i$ from the original list $\mathcal{L}$ is an aggregate ω-potential for the new situation. Therefore, we can apply Statement 2 of this same Lemma.

(5) We order I in the way described in Lemma 3.2.3, and show by recursion that $\mathcal{A}_{\sigma(\{1,\ldots,s\})} \subseteq \mathcal{E}$. Since $\sigma(1) \bowtie \sigma(2)$, we have $\mathcal{A}_{\{\sigma(1),\sigma(2)\}} = \mathcal{Y}_{\{\sigma(1),\sigma(2)\}} \subseteq \mathcal{E}$ by Statement 3. At each consecutive step, some $\mathrm{B} \in \mathcal{A}_{\sigma(\{1,\ldots,s\})}$ are replaced with $\mathrm{B} \cap \Upsilon_{\sigma(s+1)}$ and $\mathrm{B} \setminus \Upsilon_{\sigma(s+1)}$ (whenever both are not empty); the first term obviously belongs to $\mathcal{E}$. If not empty, $\mathrm{B}' = \Upsilon_{\sigma(s+1)} \setminus \big(\bigcup_{\mathrm{B} \in \mathcal{A}_{\sigma(\{1,\ldots,s\})}} \mathrm{B}\big)$ is added as well. Clearly, $\mathrm{B}' = \bigcap_{h \in \{1,\ldots,s\}} \big(\Upsilon_{\sigma(s+1)} \setminus \Upsilon_{\sigma(h)}\big)$; for every h, either $\sigma(s+1) \bowtie \sigma(h)$ or $\Upsilon_{\sigma(s+1)} \setminus \Upsilon_{\sigma(h)} = \Upsilon_{\sigma(s+1)}$ by Lemma 3.2.3. Therefore, $\mathrm{B}' \in \mathcal{Y}_{\sigma(\{1,\ldots,s+1\})} \subseteq \mathcal{E}$ by Statement 3.

In the case of $\sigma(s+1) \in I_0^{\max}$ (in the notation of the proof of Lemma 3.2.3), we add another statement to be carried through the recursion: $\mathcal{F}_{\sigma(\{1,\ldots,s\})} = \mathcal{Y}_{\sigma(\{1,\ldots,s\})}$; it is obvious when $s = 2$. Generally, we have $\mathrm{B} = (\bigcap_{i \in I^+} \Upsilon_i) \setminus (\bigcup_{i \in I^-} \Upsilon_i)$, where $I^+ \cap I^- = \emptyset$ and $I^+ \cup I^- \subseteq \sigma(\{1, \ldots, s\})$; we denote $I^* = \{i \in I^- | \ \Upsilon_i \cap \Upsilon_{\sigma(s+1)} \neq \emptyset\}$. Since $\sigma(s+1) \in I_0^{\max}$, the inclusion $\Upsilon_{\sigma(s+1)} \subset \Upsilon_{\sigma(h)}$ is impossible for any $h \leq s$. The case of $\Upsilon_{\sigma(s+1)} = \Upsilon_{\sigma(h)}$ for an $h \leq s$ being trivial, we may assume that $\sigma(s+1) \bowtie i$ for every $i \in I^+ \cup I^*$ (again, by Lemma 3.2.3). Now we have $\mathrm{B} \cap \Upsilon_{\sigma(s+1)} = \bigcap_{i \in I^+} (\Upsilon_i \cap \Upsilon_{\sigma(s+1)}) \cap \bigcap_{i \in I^*} (\Upsilon_{\sigma(s+1)} \setminus \Upsilon_i) \in \mathcal{Y}_{\sigma(\{1,\ldots,s+1\})}$ and $\mathrm{B} \setminus \Upsilon_{\sigma(s+1)} = \mathrm{B} \cap \bigcap_{i \in I^+} (\Upsilon_i \setminus \Upsilon_{\sigma(s+1)}) \in \mathcal{Y}_{\sigma(\{1,\ldots,s+1\})}$, since $\mathrm{B} \in \mathcal{Y}_{\sigma(\{1,\ldots,s\})}$ by the induction hypothesis. Now Statement 3 applies.

In the case of $\sigma(s+1) \notin I_0^{\max}$, we only have to consider $\mathrm{B} \in \mathcal{A}_{\sigma(\{1,\ldots,s\})}$, hence $I^+ \cup I^- = \sigma(\{1, \ldots, s\})$. There is $i \in \{1, \ldots, s\}$ such that $\Upsilon_{\sigma(s+1)} \subset \Upsilon_i$, hence $i \in I^+$, hence $\mathrm{B} \subseteq \Upsilon_i$. There also exists $j \in \{1, \ldots, s\}$ such that $\sigma(s+1) \bowtie j$; if $j \in I^+$, then

$\emptyset \neq \Upsilon_{\sigma(s+1)} \setminus \Upsilon_j \subseteq \Upsilon_{\sigma(s+1)} \setminus \mathrm{B}$; if $j \in I^-$, then $\emptyset \neq \Upsilon_{\sigma(s+1)} \cap \Upsilon_j \subseteq \Upsilon_{\sigma(s+1)} \setminus \mathrm{B}$. In either case, $\Upsilon_{\sigma(s+1)}$ and B overlap, hence Statement 4 applies. $\square$

3.4 Necessity of Gorman additivity: Main construction

From now on, we assume $V_\alpha = \mathbb{R}$ for each $\alpha \in \mathrm{A}$. This assumption inflicts no loss of generality – there always exists an order-preserving homeomorphism to the whole line – but it makes our notation much less cumbersome. For each $\mathrm{B} \in \mathfrak{B}$, we denote $0_\mathrm{B} \in V_\mathrm{B}$ ($1_\mathrm{B} \in V_\mathrm{B}$) a vector each component of which is 0 (1).

Let $I \subseteq N$ be connected. An *integer net* on I is a collection of mappings $\psi_\mathrm{B} : M_\mathrm{B} \to \mathbb{R}$, $\mathrm{B} \in \mathcal{A}_I$, where M_B is the set of integers m satisfying $m_\mathrm{B}^- < m < m_\mathrm{B}^+$ ($m_\mathrm{B}^- \in \{-\infty, \ldots, -2, -1\}$, $m_\mathrm{B}^+ \in \{2, 3, \ldots, +\infty\}$), such that

1. for every $\mathrm{B} \in \mathcal{A}_I$,
$$\psi_\mathrm{B}(0) = 0; \tag{3.22a}$$

2. whenever $i \in I$ and there are $m_\mathrm{B} \in M_\mathrm{B}$ for all $\mathrm{B} \in \mathcal{A}_I^i$ and $\mathrm{B}', \mathrm{B}'' \in \mathcal{A}_I^i$ such that $(m_{\mathrm{B}'} + 1) \in M_{\mathrm{B}'}$ and $(m_{\mathrm{B}''} + 1) \in M_{\mathrm{B}''}$, there holds
$$\big(\langle \psi_\mathrm{B}(m_\mathrm{B}) \cdot 1_\mathrm{B} \rangle_{\mathrm{B} \in \mathcal{A}_I^i \setminus \{\mathrm{B}'\}}, \psi_{\mathrm{B}'}(m_{\mathrm{B}'} + 1) \cdot 1_{\mathrm{B}'}\big) \sim^{\mathrm{a}}_i \big(\langle \psi_\mathrm{B}(m_\mathrm{B}) \cdot 1_\mathrm{B} \rangle_{\mathrm{B} \in \mathcal{A}_I^i \setminus \{\mathrm{B}''\}}, \psi_{\mathrm{B}''}(m_{\mathrm{B}''} + 1) \cdot 1_{\mathrm{B}''}\big); \tag{3.22b}$$

3. whenever $i \in I$, $\mathrm{B}, \mathrm{B}' \in \mathcal{A}_I^i$, $\mathrm{B} \neq \mathrm{B}'$, $m_\mathrm{B}^+ < +\infty$, and $v_\mathrm{B} \in V_\mathrm{B}$, there holds
$$\big(\psi_\mathrm{B}(m_\mathrm{B}^+ - 1) \cdot 1_\mathrm{B}, \psi_{\mathrm{B}'}(1) \cdot 1_{\mathrm{B}'}, 0_{\Upsilon_i \setminus (\mathrm{B} \cup \mathrm{B}')}\big) \succcurlyeq^{\mathrm{a}}_i \big(v_\mathrm{B}, 0_{\mathrm{B}'}, 0_{\Upsilon_i \setminus (\mathrm{B} \cup \mathrm{B}')}\big); \tag{3.22c}$$

4. whenever $i \in I$, $\mathrm{B}, \mathrm{B}' \in \mathcal{A}_I^i$, $\mathrm{B} \neq \mathrm{B}'$, $m_\mathrm{B}^- > -\infty$, and $v_\mathrm{B} \in V_\mathrm{B}$, there holds
$$\big(v_\mathrm{B}, \psi_{\mathrm{B}'}(1) \cdot 1_{\mathrm{B}'}, 0_{\Upsilon_i \setminus (\mathrm{B} \cup \mathrm{B}')}\big) \succcurlyeq^{\mathrm{a}}_i \big(\psi_\mathrm{B}(m_\mathrm{B}^- + 1) \cdot 1_{\mathrm{B}'}, 0_{\mathrm{B}'}, 0_{\Upsilon_i \setminus (\mathrm{B} \cup \mathrm{B}')}\big). \tag{3.22d}$$

Lemma 3.4.1. *Let ψ be an integer net on a connected $I \subseteq N$. Then whenever $i \in I$ and m'_B, $m_\mathrm{B} \in M_\mathrm{B}$ for every $\mathrm{B} \in \mathcal{A}_I^i$, we have*
$$\big\langle \psi_\mathrm{B}(m'_\mathrm{B}) \cdot 1_\mathrm{B} \big\rangle_{\mathrm{B} \in \mathcal{A}_I^i} \succcurlyeq^{\mathrm{a}}_i \big\langle \psi_\mathrm{B}(m_\mathrm{B}) \cdot 1_\mathrm{B} \big\rangle_{\mathrm{B} \in \mathcal{A}_I^i} \iff \sum_{\mathrm{B} \in \mathcal{A}_I^i} m'_\mathrm{B} \geq \sum_{\mathrm{B} \in \mathcal{A}_I^i} m_\mathrm{B}. \tag{3.23}$$

Proof. First, an equality in the right hand side of (3.23) implies an equivalence in the left hand side by a straightforward inductive reasoning based on (3.22b).

If there is a strong inequality in the right hand side of (3.23), we can find $\langle m''_\mathrm{B} \rangle_{\mathrm{B} \in \mathcal{A}_I^i}$ such that $\sum_{\mathrm{B} \in \mathcal{A}_I^i} m''_\mathrm{B} = \sum_{\mathrm{B} \in \mathcal{A}_I^i} m_\mathrm{B}$ while $\langle m'_\mathrm{B} \rangle_{\mathrm{B} \in \mathcal{A}_I^i}$ Pareto dominates $\langle m''_\mathrm{B} \rangle_{\mathrm{B} \in \mathcal{A}_I^i}$. Now $\big\langle \psi_\mathrm{B}(m'_\mathrm{B}) \big\rangle_{\mathrm{B} \in \mathcal{A}_I^i} \succ^{\mathrm{a}}_i \big\langle \psi_\mathrm{B}(m''_\mathrm{B}) \big\rangle_{\mathrm{B} \in \mathcal{A}_I^i} \sim^{\mathrm{a}}_i \big\langle \psi_\mathrm{B}(m_\mathrm{B}) \big\rangle_{\mathrm{B} \in \mathcal{A}_I^i}$ by the strict monotonicity of $\succcurlyeq^{\mathrm{a}}_i$ and the findings of the previous paragraph. Finally, an "(in)equality" in the left hand side of (3.23) implies the same (in)equality in the right hand side because $\sum_{\mathrm{B} \in \mathcal{A}_I^i} m'_\mathrm{B}$ and $\sum_{\mathrm{B} \in \mathcal{A}_I^i} m_\mathrm{B}$ are always comparable. $\square$

Let ψ be an integer net on a connected $I \subseteq N$. The *even half* of ψ is a collection of mappings $\hat{\psi}_\mathrm{B} : \hat{M}_\mathrm{B} \to \mathbb{R}$ for $\mathrm{B} \in \mathcal{A}_I$ such that $\hat{\psi}_\mathrm{B}(m) = \psi_\mathrm{B}(2m)$ and $\hat{M}_\mathrm{B} = \{m \in M_\mathrm{B} |\, (2m) \in M_\mathrm{B}\}$. It is easily checked that, for each $\mathrm{B} \in \mathcal{A}_I$, $\hat{m}_\mathrm{B}^\pm = \pm\infty \iff m_\mathrm{B}^\pm = \pm\infty$;

$\hat{\psi}$ itself is an integer net on I if and only if $1 \in \hat{M}_{\mathrm{B}}$ for each $\mathrm{B} \in \mathcal{A}_I$. Let ψ and $\bar{\psi}$ be two integer nets on I; we call $\bar{\psi}$ a *doubling* of ψ if ψ is the even half of $\bar{\psi}$. A *binary net* on I is an infinite sequence of integer nets $\psi^0, \psi^1, \ldots$ on I such that each ψ^{d+1} is a doubling of ψ^d.

Unproven Lemma 3.4.2. *Whenever $I \subseteq N$ is connected, there exists a binary net on I.*

Two special cases are proven in Subsection 3.5 (Propositions 3.4 and 3.6).

Lemma 3.4.3. *Let ψ be an integer net on I. Then, whenever $\mathrm{B} \neq \mathrm{B}'$ and $m \in M_{\mathrm{B}'}$ are such that $\mathrm{B}, \mathrm{B}' \in \mathcal{A}_I^i$ for some $i \in N$, and $(m+1) \in M_{\mathrm{B}'}$, there are continuous and strictly increasing mappings $e^m_{\mathrm{BB}'}, f^m_{\mathrm{BB}'} : [0, \psi_{\mathrm{B}}(1)] \to [\psi_{\mathrm{B}'}(m), \psi_{\mathrm{B}'}(m+1)]$ such that $e^m_{\mathrm{BB}'}(0) = f^m_{\mathrm{BB}'}(0) = \psi_{\mathrm{B}'}(m)$, $e^m_{\mathrm{BB}'}(\psi_{\mathrm{B}}(1)) = f^m_{\mathrm{BB}'}(\psi_{\mathrm{B}}(1)) = \psi_{\mathrm{B}'}(m+1)$,*

$$\big(t \cdot 1_{\mathrm{B}}, \psi_{\mathrm{B}'}(m) \cdot 1_{\mathrm{B}'}, 0_{\Upsilon_i \setminus (\mathrm{B} \cup \mathrm{B}')}\big) \sim^{\mathrm{a}}_i \big(0_{\mathrm{B}}, e^m_{\mathrm{BB}'}(t) \cdot 1_{\mathrm{B}'}, 0_{\Upsilon_i \setminus (\mathrm{B} \cup \mathrm{B}')}\big) \tag{3.24a}$$

and

$$\big(t \cdot 1_{\mathrm{B}}, \psi_{\mathrm{B}'}(m+1) \cdot 1_{\mathrm{B}'}, 0_{\Upsilon_i \setminus (\mathrm{B} \cup \mathrm{B}')}\big) \sim^{\mathrm{a}}_i \big(\psi_{\mathrm{B}}(1) \cdot 1_{\mathrm{B}}, f^m_{\mathrm{BB}'}(t) \cdot 1_{\mathrm{B}'}, 0_{\Upsilon_i \setminus (\mathrm{B} \cup \mathrm{B}')}\big) \tag{3.24b}$$

for every $t \in [0, \psi_{\mathrm{B}}(1)]$. If $\mathrm{B} \cup \mathrm{B}' \subseteq \Upsilon_j$ for a $j \in N$, then both equivalences (3.24) *hold with i replaced with j. If $\bar{\psi}$ is a doubling of ψ, then $e^m_{\mathrm{BB}'}(\bar{\psi}_{\mathrm{B}}(1)) = \bar{\psi}_{\mathrm{B}'}(2m+1) = f^m_{\mathrm{BB}'}(\bar{\psi}_{\mathrm{B}}(1))$.*

Proof. The validity of (3.24a) for $t = 0$ and of (3.24b) for $t = \psi_{\mathrm{B}}(1)$ is tautological; (3.24a) for $t = \psi_{\mathrm{B}}(1)$ and (3.24b) for $t = 0$ immediately follow from (3.22b). Since both mappings $t \mapsto t \cdot 1_{\mathrm{B}}$ and $t \mapsto t \cdot 1_{\mathrm{B}'}$ are obviously continuous and strictly increasing, the existence and uniqueness of solutions $e^m_{\mathrm{BB}'}(t), f^m_{\mathrm{BB}'}(t)$ to (3.24) immediately follows from the continuity of $\succeq^{\mathrm{a}}_i$. Each function $e^m_{\mathrm{BB}'}$ or $f^m_{\mathrm{BB}'}$ is strictly increasing and maps $[0, \psi_{\mathrm{B}}(1)]$ onto $[\psi_{\mathrm{B}'}(m), \psi_{\mathrm{B}'}(m+1)]$ because (3.24) can be resolved in the opposite direction as well. Therefore, both are continuous.

The second statement immediately follows from Lemma 3.3.1; the third, from (3.22b) and the uniqueness of a solution to (3.24). □

Remark. It follows from the uniqueness statement that $e^0_{\mathrm{BB}'} = (e^0_{\mathrm{B}'\mathrm{B}})^{-1}$ whenever one side is well defined.

Lemma 3.4.4. *Let $\psi^0, \psi^1, \ldots$ be a binary net on a connected $I \subseteq N$ and $\mathrm{B} \in \mathcal{A}_I$; then $\Psi = \{\psi^d_{\mathrm{B}}(m)\}_{d \in \mathbb{N}, m \in M^d_{\mathrm{B}}}$ is dense in $\mathbb{R}$.*

Proof. By definition, $\mathrm{B} \subseteq \Upsilon_i$ for some $i \in I$. Since I is connected, there is $j \in I$ such that $i \bowtie j$. Then either $\mathrm{B} \subseteq \Upsilon_i \cap \Upsilon_j$, or $\mathrm{B} \subseteq \Upsilon_i \setminus \Upsilon_j$; we pick $\mathrm{B}' \in \mathcal{A}_I$ included in the other "half." Thus $\mathrm{B} \cup \mathrm{B}' \subseteq \Upsilon_i$ and $\mathrm{B} \cap \mathrm{B}' = \emptyset$.

Suppose the contrary: there are $u_- < u_+$ such that $\Psi \cap [u_-, u_+] = \emptyset$. We denote $\Psi_- = \{t \in \Psi |\ t \leq u_-\}$ and $\Psi_+ = \{t \in \Psi |\ t \geq u_+\}$. There are three alternatives:

1. $\Psi_- \neq \emptyset \neq \Psi_+$;
2. $\Psi_+ = \emptyset$;
3. $\Psi_- = \emptyset$.

Step 3.4.4.1. *The first alternative cannot hold.*

Proof. If $\psi_{\mathrm{B}}^{d_-}(m_-) \in \Psi_-$ and $\psi_{\mathrm{B}}^{d_+}(m_+) \in \Psi_+$, then, defining $d = \max\{d_-, d_+\}$, we see that $\{\psi_{\mathrm{B}}^{d}(m)\}_{m \in M_{\mathrm{B}}^{d}}$ intersects both Ψ_- and Ψ_+. Deleting all $\psi^{d'}$ with $d' < d$ if needed, we may assume $d = 0$; clearly, there is $\bar{m} \in M_{\mathrm{B}}^{0} \ni (\bar{m}+1)$ such that $\psi_{\mathrm{B}}^{0}(\bar{m}) \in \Psi_-$ while $\psi_{\mathrm{B}}^{0}(\bar{m}+1) \in \Psi_+$. Denoting $\upsilon_- = \sup \Psi_-$ and $\upsilon_+ = \inf \Psi_+$, we have $\upsilon_- \le u_- < u_+ \le \upsilon_+$ and $\Psi \cap]\upsilon_-, \upsilon_+[= \emptyset$. Denoting $\upsilon'_{\pm} = (e_{\mathrm{BB}'}^{\bar{m}})^{-1}(\upsilon_{\pm})$ and $\Psi' = \{\psi_{\mathrm{B}'}^{d}(m)\}_{d \in \mathbb{N},\, m \in M_{\mathrm{B}'}^{d}}$, we have $\Psi' \cap]\upsilon'_-, \upsilon'_+[= \emptyset$ as well.

By monotonicity,

$$(\upsilon_+ \cdot 1_{\mathrm{B}}, \upsilon'_+ \cdot 1_{\mathrm{B}'}) \succ^{\mathrm{a}}_i (\upsilon_- \cdot 1_{\mathrm{B}}, \upsilon'_+ \cdot 1_{\mathrm{B}'}) \succ^{\mathrm{a}}_i (\upsilon_- \cdot 1_{\mathrm{B}}, \upsilon'_- \cdot 1_{\mathrm{B}'});$$

by continuity, there are open intervals $U_+, U_-, U'_+, U'_- \subset \mathbb{R}$ containing $\upsilon_+, \upsilon_-, \upsilon'_+, \upsilon'_-$, respectively, and such that the strict preferences are retained on $U_+ \cdot 1_{\mathrm{B}} \times U'_+ \cdot 1_{\mathrm{B}'}$, $U_- \cdot 1_{\mathrm{B}} \times U'_+ \cdot 1_{\mathrm{B}'}$, and $U_- \cdot 1_{\mathrm{B}} \times U'_- \cdot 1_{\mathrm{B}'}$. Without restricting generality, $U'_{\pm} = (e_{\mathrm{B}'\mathrm{B}}^{\bar{m}})^{-1}(U_{\pm})$.

Essentially the same argument that led us to $\bar{m}$ above shows that there are $d \in \mathbb{N}$ and $m \in M_{\mathrm{B}}^{d} \ni (m+1)$ such that $\psi_{\mathrm{B}}^{d}(m) \in U_-$ and $\psi_{\mathrm{B}}^{d}(m+1) \in U_+$, hence $\psi_{\mathrm{B}'}^{d}(m - \bar{m} \cdot 2^d) = (e_{\mathrm{B}'\mathrm{B}}^{\bar{m}})^{-1}(\psi_{\mathrm{B}}^{d}(m)) \in U'_-$ and $\psi_{\mathrm{B}'}^{d}(m + 1 - \bar{m} \cdot 2^d) = (e_{\mathrm{B}'\mathrm{B}}^{\bar{m}})^{-1}(\psi_{\mathrm{B}}^{d}(m+1)) \in U'_+$. Let us note that $\psi_{\mathrm{B}}^{d}(m) = \psi_{\mathrm{B}}^{d+1}(2m) < \psi_{\mathrm{B}}^{d+1}(2m+1) < \psi_{\mathrm{B}}^{d+1}(2m+2) = \psi_{\mathrm{B}}^{d}(m+1)$ and $\psi_{\mathrm{B}}^{d+1}(2m+1) \notin]\upsilon_-, \upsilon_+[$.

Let $\psi_{\mathrm{B}}^{d+1}(2m+1) < \upsilon_-$; then $\psi_{\mathrm{B}}^{d+1}(2m+1) \in U_-$, hence $\psi_{\mathrm{B}'}^{d+1}(2m+1-\bar{m} \cdot 2^{d+1}) \in U'_-$. Therefore, $\bigl(\psi_{\mathrm{B}}^{d+1}(2m) \cdot 1_{\mathrm{B}}, \psi_{\mathrm{B}'}^{d+1}(2m+2-\bar{m} \cdot 2^{d+1}) \cdot 1_{\mathrm{B}'}\bigr) \succ^{\mathrm{a}}_i \bigl(\psi_{\mathrm{B}}^{d+1}(2m+1) \cdot 1_{\mathrm{B}}, \psi_{\mathrm{B}'}^{d+1}(2m+1-\bar{m} \cdot 2^{d+1}) \cdot 1_{\mathrm{B}'}\bigr)$, but this contradicts (3.22b).

The assumption $\psi_{\mathrm{B}}^{d+1}(2m+1) > \upsilon_+$ is refuted dually. □

Step 3.4.4.2. *The second alternative cannot hold.*

Proof. We denote $\upsilon_+ = \sup \Psi < +\infty$; by the continuity and strict monotonicity of $\succeq^{\mathrm{a}}_i$, there are $\delta, \delta' > 0$ such that

$$((\upsilon_+ + \delta) \cdot 1_{\mathrm{B}}, 0_{\mathrm{B}'}) \sim^{\mathrm{a}}_i (\upsilon_+ \cdot 1_{\mathrm{B}}, \delta' \cdot 1_{\mathrm{B}'}).$$

Now we note that the conditions of this lemma are satisfied for B', hence Step 3.4.4.1 holds for B' as well. Therefore, there is $d \in \mathbb{N}$ such that $\psi_{\mathrm{B}'}^{d}(1) < \delta'$; without restricting generality, $d = 0$. We consider two alternatives.

If $m_{\mathrm{B}}^{+} < +\infty$, then $((\upsilon_+ + \delta) \cdot 1_{\mathrm{B}}, 0_{\mathrm{B}'}) \succ^{\mathrm{a}}_i (\upsilon_+ \cdot 1_{\mathrm{B}}, \psi_{\mathrm{B}'}^{0}(1) \cdot 1_{\mathrm{B}'}) \succ^{\mathrm{a}}_i (\psi_{\mathrm{B}}^{0}(m_{\mathrm{B}}^{+} - 1) \cdot 1_{\mathrm{B}}, \psi_{\mathrm{B}'}^{0}(1) \cdot 1_{\mathrm{B}'})$, contradicting (3.22c).

Let $m_{\mathrm{B}}^{+} = +\infty$; then $\upsilon_+ = \sup\{\psi_{\mathrm{B}}^{0}(m)\}_{m \in \mathbb{N}}$. By continuity from $(\upsilon_+ \cdot 1_{\mathrm{B}}, \psi_{\mathrm{B}'}^{0}(1) \cdot 1_{\mathrm{B}'}) \succ^{\mathrm{a}}_i (\upsilon_+ \cdot 1_{\mathrm{B}}, 0_{\mathrm{B}'})$, there is $\upsilon_* < \upsilon_+$ such that $(t \cdot 1_{\mathrm{B}}, \psi_{\mathrm{B}'}^{0}(1) \cdot 1_{\mathrm{B}'}) \succ^{\mathrm{a}}_i (t' \cdot 1_{\mathrm{B}}, 0_{\mathrm{B}'})$ whenever $t > \upsilon_*$ and $t' < \upsilon_+$. On the other hand, if $\psi_{\mathrm{B}}^{0}(m) > \upsilon_*$, then $\psi_{\mathrm{B}}^{0}(m+1) < \upsilon_+$, but $(\psi_{\mathrm{B}}^{0}(m) \cdot 1_{\mathrm{B}}, \psi_{\mathrm{B}'}^{0}(1) \cdot 1_{\mathrm{B}'}) \sim^{\mathrm{a}}_i (\psi_{\mathrm{B}}^{0}(m+1) \cdot 1_{\mathrm{B}}, 0_{\mathrm{B}'})$. □

Step 3.4.4.3. *The third alternative cannot hold.*

The proof is dual to that of Step 3.4.4.2. The lemma is proved. □

Lemma 3.4.5. *Let* $\mathrm{B} \in \mathcal{A}$ *and* $v_{\mathrm{B}} \in V_{\mathrm{B}}$; *then there is a unique* $\tau_{\mathrm{B}}(v_{\mathrm{B}}) \in \mathbb{R}$ *such that*

$$v_{\mathrm{B}} \sim^{\mathrm{a}}_{\mathrm{B}} \tau_{\mathrm{B}}(v_{\mathrm{B}}) \cdot 1_{\mathrm{B}}. \tag{3.25}$$

Proof. We denote $t^+ = \max_{\alpha\in\mathrm{B}} v_\alpha$ and $t^- = \min_{\alpha\in\mathrm{B}} v_\alpha$; by (3.1), $t^+ \cdot 1_\mathrm{B} \succeq^{\mathrm{a}}_\mathrm{B} v_\mathrm{B} \succeq^{\mathrm{a}}_\mathrm{B} t^- \cdot 1_\mathrm{B}$. Therefore, there is $\tau_\mathrm{B}(v_\mathrm{B}) \in [t^-, t^+]$ satisfying (3.25). It is unique because of (3.1). □

Lemma 3.4.6. *Let $\psi^0, \psi^1, \ldots$ be a binary net on a connected $I \subseteq N$ and $\mathrm{B} \in \mathcal{A}_I$. Then for every $d, d' \in \mathbb{N}$, $m \in M^d_\mathrm{B}$, and $m' \in M^{d'}_\mathrm{B}$, there holds $\psi^{d'}_\mathrm{B}(m') \geq \psi^d_\mathrm{B}(m) \iff m'/2^{d'} \geq m/2^d$.*

Proof. When $d = d'$, this is just the monotonicity of ψ^d_B, which immediately follows from (3.22b). Then a straightforward inductive process in $\max\{d', d\} - \min\{d', d\}$ based on the definition of a doubling works. □

Now we are prepared to define the functions μ_B for $\mathrm{B} \in \mathcal{A}$. If $I \in \mathfrak{N}$ is not proper, then $\mathcal{A}_I = \mathcal{B}_I$ consists of a single set $\Upsilon_i \in \mathcal{E}$. By Lemma 3.3.5, all preferences $\succeq^{\mathrm{a}}_i$ $(i \in I)$ coincide. By the Debreu Theorem, there is a continuous function representing $\succeq^{\mathrm{a}}_{\Upsilon_i}$ on V_{Υ_i}; by (3.1), it is strictly increasing in each argument. Now Condition 1 holds by definition; (3.9a) holds by default.

Let $I \in \mathfrak{N}$ be proper and $\psi^0, \psi^1, \ldots$ be a binary net on I. For every $\mathrm{B} \in \mathcal{A}_I$ and $d \in \mathbb{N}$, we denote $q^{+d}_\mathrm{B} = (m^{+d}_\mathrm{B} - 1)/2^d$ and $q^{-d}_\mathrm{B} = (m^{-d}_\mathrm{B} + 1)/2^d$. Then we define $Q_\mathrm{B} = \bigcup_{d\in\mathbb{N}} [q^{-d}_\mathrm{B}, q^{+d}_\mathrm{B}] \setminus \{+\infty, -\infty\}$. Clearly, Q_B is a non-degenerate interval (actually, open). For every $v_\mathrm{B} \in V_\mathrm{B}$, we define

$$\mu_\mathrm{B}(v_\mathrm{B}) = \sup\{m/2^d |\ d \in \mathbb{N} \ \&\ m \in M^d_\mathrm{B} \ \&\ \psi^d_\mathrm{B}(m) \leq \tau(v_\mathrm{B})\}. \tag{3.26}$$

By Lemmas 3.4.4 and 3.4.6, μ_B is strictly increasing and $\mu_\mathrm{B}(v_\mathrm{B}) \in Q_\mathrm{B}$ for every $v \in V_\mathrm{B}$. Conversely, if $w \in Q_\mathrm{B}$, we define $t = \sup\{\psi^d_\mathrm{B}(h) |\ d \in \mathbb{N} \ \&\ m \in M^d_\mathrm{B} \ \&\ m/2^d \leq w\}$ and $v_\mathrm{B} = t \cdot 1_\mathrm{B}$; then we easily derive from Lemma 3.4.4 that $w = \mu(v_\mathrm{B})$. Thus, we have a strictly increasing mapping onto a non-degenerate interval; therefore, μ_B is continuous.

Lemma 3.4.7. *Let $I \in \mathfrak{N}$ be proper, $i \in I$, $v'_\mathrm{B}, v_\mathrm{B} \in V_\mathrm{B}$ for all $\mathrm{B} \in \mathcal{A}^i_I$, and μ_B be defined by* (3.26)*. Then*

$$\langle v'_\mathrm{B} \rangle_{\mathrm{B}\in\mathcal{A}^i_I} \succeq^{\mathrm{a}}_i \langle v_\mathrm{B} \rangle_{\mathrm{B}\in\mathcal{A}^i_I} \iff \sum_{\mathrm{B}\in\mathcal{A}^i_I} \mu_\mathrm{B}(v'_\mathrm{B}) \geq \sum_{\mathrm{B}\in\mathcal{A}^i_I} \mu_\mathrm{B}(v_\mathrm{B}). \tag{3.27}$$

Proof. We denote $H = \#\mathcal{A}^i_I$. Suppose first that $\delta = \sum_\mathrm{B} \mu(v'_\mathrm{B}) - \sum_\mathrm{B} \mu(v_\mathrm{B}) > 0$. By Lemma 3.4.4, for every $\mathrm{B} \in \mathcal{A}^i_I$, there is $d'_\mathrm{B} \in \mathbb{N}$ such that $\psi^{d'_\mathrm{B}}_\mathrm{B}(m) \leq \tau(v'_\mathrm{B})$ for some $m \in M^{d'_\mathrm{B}}_\mathrm{B}$ and $d_\mathrm{B} \in \mathbb{N}$ such that $\psi^{d_\mathrm{B}}_\mathrm{B}(m) > \tau(v_\mathrm{B})$ for some $m \in M^{d_\mathrm{B}}_\mathrm{B}$. Let us pick $d \in \mathbb{N}$ such that $d \geq \max_{\mathrm{B}\in\mathcal{A}^i_I} \max\{d'_\mathrm{B}, d_\mathrm{B}\}$ and $2^{d-1} \geq H/\delta$. For every $\mathrm{B} \in \mathcal{A}^i_I$, we denote $m'_\mathrm{B} = \max\{m \in M^d_\mathrm{B} |\ \psi^d_\mathrm{B}(m) \leq \tau(v'_\mathrm{B})\}$ and $m_\mathrm{B} = \min\{m \in M^d_\mathrm{B} |\ \psi^d_\mathrm{B}(m) > \tau(v_\mathrm{B})\}$. Clearly, we have $m'_\mathrm{B}/2^d \leq \mu(v'_\mathrm{B}) < (m'_\mathrm{B}+1)/2^d$ and $(m_\mathrm{B}-1)/2^d \leq \mu(v_\mathrm{B}) < m_\mathrm{B}/2^d$ for all $\mathrm{B} \in \mathcal{A}^i_I$. Therefore, $\sum_\mathrm{B}(m'_\mathrm{B}/2^d) > \sum_\mathrm{B} \mu(v'_\mathrm{B}) - H/2^d$ and $\sum_\mathrm{B}(m_\mathrm{B}/2^d) < \sum_\mathrm{B} \mu(v_\mathrm{B}) + H/2^d$, hence $\sum_\mathrm{B}(m'_\mathrm{B}/2^d) - \sum_\mathrm{B}(m_\mathrm{B}/2^d) > \delta - 2H/2^d \geq 0$, hence $\sum_\mathrm{B} m'_\mathrm{B} > \sum_\mathrm{B} m_\mathrm{B}$. Now Lemma 3.4.1 (for ψ^d_B) and strict monotonicity of $\succeq^{\mathrm{a}}_i$ imply a strict preference in the left hand side of (3.27).

An equality in the right hand side of (3.27) means that we have both strict inequalities in any open neighbourhood, hence the same strict preferences in the left hand side of (3.27), hence an equivalence. The opposite implication is proven exactly as in Lemma 3.4.1. □

For every $\mathrm{B} \in \mathcal{B}_I$ and $v_{\mathrm{B}} \in V_{\mathrm{B}}$, we define

$$\mu_{\mathrm{B}}(v_{\mathrm{B}}) = \sum_{\mathrm{B}' \in \mathcal{A}_I \colon \mathrm{B}' \subseteq \mathrm{B}} \mu_{\mathrm{B}'}(v_{\mathrm{B}'}).$$

Then (3.9a) holds trivially, while Condition 1 immediately follows from Lemma 3.4.1. Thus, we only have to check (3.9b).

Lemma 3.4.8. *Let ψ be an integer net on a connected $I \subseteq N$, $i \in N$, $\Upsilon_I \subseteq \Upsilon_i$, $\mathrm{B}', \mathrm{B}'' \in \mathcal{A}_I$, $m_{\mathrm{B}} \in M_{\mathrm{B}}$ for every $\mathrm{B} \in \mathcal{A}_I$, $(m_{\mathrm{B}'} + 1) \in M_{\mathrm{B}'}$, $(m_{\mathrm{B}''} + 1) \in M_{\mathrm{B}''}$, and $v \in V_{\Upsilon_i \setminus \Upsilon_I}$. Then*

$$\begin{multline}\left(\left\langle \psi_{\mathrm{B}}(m_{\mathrm{B}}) \cdot 1_{\mathrm{B}} \right\rangle_{\mathrm{B} \in \mathcal{A}_I \setminus \{\mathrm{B}'\}}, \psi_{\mathrm{B}'}(m_{\mathrm{B}'} + 1) \cdot 1_{\mathrm{B}'}, v\right) \overset{\mathrm{a}}{\rightsquigarrow}_i \\ \left(\left\langle \psi_{\mathrm{B}}(m_{\mathrm{B}}) \cdot 1_{\mathrm{B}} \right\rangle_{\mathrm{B} \in \mathcal{A}_I \setminus \{\mathrm{B}''\}}, \psi_{\mathrm{B}''}(m_{\mathrm{B}''} + 1) \cdot 1_{\mathrm{B}''}, v\right). \quad (3.28)\end{multline}$$

Proof. If $\mathrm{B}' \cup \mathrm{B}'' \subseteq \Upsilon_j$ for a $j \in I$, then (3.28) immediately follows from (3.22b) and from $\Upsilon_j \in \mathcal{E}$ (Lemma 3.3.5). Supposing the contrary, let $i_0, \ldots, i_{\bar{m}}$ be an overlap path in I of the minimal length such that $\mathrm{B}' \subseteq \Upsilon_{i_0}$ and $\mathrm{B}'' \subseteq \Upsilon_{i_{\bar{m}}}$. We denote $K = \{0, 1, \ldots, \bar{m} + 1\}$, $\mathrm{B}^0 = \mathrm{B}'$ and $\mathrm{B}^{\bar{m}+1} = \mathrm{B}''$, and then pick $\mathrm{B}^k \in \mathcal{A}_I$ for $k \in K \setminus \{0, \bar{m} + 1\}$ such that $\mathrm{B}^k \subseteq \Upsilon_{i_{k-1}} \cap \Upsilon_{i_k}$. The minimality of the path implies that $\mathrm{B}^k \cap \mathrm{B}^h = \emptyset$ whenever $k \neq h$; by definition, $\mathrm{B}^k \cup \mathrm{B}^{k+1} \subseteq \Upsilon_{i_k}$ for all $k \in K$.

We denote $K^+ = \{k \in K |\ (m_{\mathrm{B}^k} + 1) \in M_{\mathrm{B}^k}\}$, $K^- = K \setminus K^+$, $K^+_{\mathrm{bgn}} = \{k \in K^+ |\ (k - 1) \in K^-\}$, $K^-_{\mathrm{bgn}} = \{k \in K^- |\ (k + 1) \in K^+\}$, $K^+_{\mathrm{end}} = \{k \in K^+ |\ (k + 1) \in K^-\}$, and $K^-_{\mathrm{end}} = \{k \in K^- |\ (k - 1) \in K^+\}$. The definition of an integer net implies that $(m_{\mathrm{B}^k} - 1) \in M_{\mathrm{B}^k}$ whenever $k \in K^-$. Therefore, for every collection of $h_{\mathrm{B}} \in M_{\mathrm{B}}$, $\mathrm{B} \in \mathcal{A}_I$, and each $k \in K$, at least one of the following equivalences is valid:

$$\begin{multline}\left(\left\langle \psi_{\mathrm{B}}(h_{\mathrm{B}}) \cdot 1_{\mathrm{B}} \right\rangle_{\mathrm{B} \in \mathcal{A}_I^{i_k} \setminus \{\mathrm{B}^k, \mathrm{B}^{k+1}\}}, \psi_{\mathrm{B}^k}(m_{\mathrm{B}^k} + 1) \cdot 1_{\mathrm{B}^k}, \psi_{\mathrm{B}^{k+1}}(m_{\mathrm{B}^{k+1}}) \cdot 1_{\mathrm{B}^{k+1}}\right) \overset{\mathrm{a}}{\rightsquigarrow}_{i_k} \\ \left(\left\langle \psi_{\mathrm{B}}(h_{\mathrm{B}}) \cdot 1_{\mathrm{B}} \right\rangle_{\mathrm{B} \in \mathcal{A}_I^{i_k} \setminus \{\mathrm{B}^k, \mathrm{B}^{k+1}\}}, \psi_{\mathrm{B}^k}(m_{\mathrm{B}^k}) \cdot 1_{\mathrm{B}^k}, \psi_{\mathrm{B}^{k+1}}(m_{\mathrm{B}^{k+1}} + 1) \cdot 1_{\mathrm{B}^{k+1}}\right); \quad (3.29a)\end{multline}$$

$$\begin{multline}\left(\left\langle \psi_{\mathrm{B}}(h_{\mathrm{B}}) \cdot 1_{\mathrm{B}} \right\rangle_{\mathrm{B} \in \mathcal{A}_I^{i_k} \setminus \{\mathrm{B}^k, \mathrm{B}^{k+1}\}}, \psi_{\mathrm{B}^k}(m_{\mathrm{B}^k}) \cdot 1_{\mathrm{B}^k}, \psi_{\mathrm{B}^{k+1}}(m_{\mathrm{B}^{k+1}}) \cdot 1_{\mathrm{B}^{k+1}}\right) \overset{\mathrm{a}}{\rightsquigarrow}_{i_k} \\ \left(\left\langle \psi_{\mathrm{B}}(h_{\mathrm{B}}) \cdot 1_{\mathrm{B}} \right\rangle_{\mathrm{B} \in \mathcal{A}_I^{i_k} \setminus \{\mathrm{B}^k, \mathrm{B}^{k+1}\}}, \psi_{\mathrm{B}^k}(m_{\mathrm{B}^k} + 1) \cdot 1_{\mathrm{B}^k}, \psi_{\mathrm{B}^{k+1}}(m_{\mathrm{B}^{k+1}} - 1) \cdot 1_{\mathrm{B}^{k+1}}\right); \quad (3.29b)\end{multline}$$

$$\begin{multline}\left(\left\langle \psi_{\mathrm{B}}(h_{\mathrm{B}}) \cdot 1_{\mathrm{B}} \right\rangle_{\mathrm{B} \in \mathcal{A}_I^{i_k} \setminus \{\mathrm{B}^k, \mathrm{B}^{k+1}\}}, \psi_{\mathrm{B}^k}(m_{\mathrm{B}^k}) \cdot 1_{\mathrm{B}^k}, \psi_{\mathrm{B}^{k+1}}(m_{\mathrm{B}^{k+1}}) \cdot 1_{\mathrm{B}^{k+1}}\right) \overset{\mathrm{a}}{\rightsquigarrow}_{i_k} \\ \left(\left\langle \psi_{\mathrm{B}}(h_{\mathrm{B}}) \cdot 1_{\mathrm{B}} \right\rangle_{\mathrm{B} \in \mathcal{A}_I^{i_k} \setminus \{\mathrm{B}^k, \mathrm{B}^{k+1}\}}, \psi_{\mathrm{B}^k}(m_{\mathrm{B}^k} - 1) \cdot 1_{\mathrm{B}^k}, \psi_{\mathrm{B}^{k+1}}(m_{\mathrm{B}^{k+1}} + 1) \cdot 1_{\mathrm{B}^{k+1}}\right); \quad (3.29c)\end{multline}$$

$$\begin{multline}\left(\left\langle \psi_{\mathrm{B}}(h_{\mathrm{B}}) \cdot 1_{\mathrm{B}} \right\rangle_{\mathrm{B} \in \mathcal{A}_I^{i_k} \setminus \{\mathrm{B}^k, \mathrm{B}^{k+1}\}}, \psi_{\mathrm{B}^k}(m_{\mathrm{B}^k} - 1) \cdot 1_{\mathrm{B}^k}, \psi_{\mathrm{B}^{k+1}}(m_{\mathrm{B}^{k+1}}) \cdot 1_{\mathrm{B}^{k+1}}\right) \overset{\mathrm{a}}{\rightsquigarrow}_{i_k} \\ \left(\left\langle \psi_{\mathrm{B}}(h_{\mathrm{B}}) \cdot 1_{\mathrm{B}} \right\rangle_{\mathrm{B} \in \mathcal{A}_I^{i_k} \setminus \{\mathrm{B}^k, \mathrm{B}^{k+1}\}}, \psi_{\mathrm{B}^k}(m_{\mathrm{B}^k}) \cdot 1_{\mathrm{B}^k}, \psi_{\mathrm{B}^{k+1}}(m_{\mathrm{B}^{k+1}} - 1) \cdot 1_{\mathrm{B}^{k+1}}\right). \quad (3.29d)\end{multline}$$

Taking into account that $\Upsilon_{i_k} \in \mathcal{E}$ by Lemma 3.3.5, we see that (3.29) remain valid if we replace $\overset{\mathrm{a}}{\rightsquigarrow}_{i_k}$ with $\overset{\mathrm{a}}{\rightsquigarrow}_i$, and $\mathrm{B} \in \mathcal{A}_I^{i_k} \setminus \{\mathrm{B}^k, \mathrm{B}^{k+1}\}$ with $\mathrm{B} \in \mathcal{A}_I \setminus \{\mathrm{B}^k, \mathrm{B}^{k+1}\}$, and add v.

By our assumptions, both 0 and $\bar{m} + 1$ belong to K^+. Therefore, each "connected component" of K^- has a $k \in K^-_{\mathrm{end}}$ at its left end and a $k \in K^-_{\mathrm{bgn}}$ at its right end. Similarly, each "connected component" of K^+ has a $k \in K^+_{\mathrm{bgn}}$ or $k = 0$ at its left end and a $k \in K^+_{\mathrm{end}}$

or $k = \bar{m}+1$ at its right end. Putting together all equivalences of the type (3.29c) with $k \in K^-_{\text{bgn}}$ and $(k+1) \in K^+_{\text{bgn}}$, we obtain

$$\begin{aligned}\big(\langle \psi_{\mathrm{B}}(m_{\mathrm{B}}) \cdot 1_{\mathrm{B}} \rangle_{\mathrm{B}\in\mathcal{A}_I\setminus\{\mathrm{B}'\}}, \psi_{\mathrm{B}'}(m_{\mathrm{B}'}+1) \cdot 1_{\mathrm{B}'}, v\big) \overset{\mathrm{a}}{\sim}_i \\ \big(\langle \psi_{\mathrm{B}}(m_{\mathrm{B}}) \cdot 1_{\mathrm{B}} \rangle_{\mathrm{B}\in\mathcal{A}_I\setminus(\{\mathrm{B}'\}\cup\{\mathrm{B}^k\}_{k\in K^-_{\text{bgn}}\cup K^+_{\text{bgn}}})}, \psi_{\mathrm{B}'}(m_{\mathrm{B}'}+1) \cdot 1_{\mathrm{B}'}, \\ \langle \psi_{\mathrm{B}}(m_{\mathrm{B}^k}-1) \cdot 1_{\mathrm{B}^k} \rangle_{k\in K^-_{\text{bgn}}}, \langle \psi_{\mathrm{B}}(m_{\mathrm{B}^k}+1) \cdot 1_{\mathrm{B}^k} \rangle_{k\in K^+_{\text{bgn}}}, v\big). \quad (3.30\text{a})\end{aligned}$$

Then we consecutively move along each component of K^+ from the left to the right, applying equivalences of the type (3.29a) with $k, (k+1) \in K^+$, and along each component of K^- in the opposite direction, applying equivalences of the type (3.29d) with $k, (k+1) \in K^-$. Eventually, we obtain

$$\begin{aligned}\big(\langle \psi_{\mathrm{B}}(m_{\mathrm{B}}) \cdot 1_{\mathrm{B}} \rangle_{\mathrm{B}\in\mathcal{A}_I\setminus(\{\mathrm{B}'\}\cup\{\mathrm{B}^k\}_{k\in K^-_{\text{bgn}}\cup K^+_{\text{bgn}}})}, \psi_{\mathrm{B}'}(m_{\mathrm{B}'}+1) \cdot 1_{\mathrm{B}'}, \\ \langle \psi_{\mathrm{B}}(m_{\mathrm{B}^k}-1) \cdot 1_{\mathrm{B}^k} \rangle_{k\in K^-_{\text{bgn}}}, \langle \psi_{\mathrm{B}}(m_{\mathrm{B}^k}+1) \cdot 1_{\mathrm{B}^k} \rangle_{k\in K^+_{\text{bgn}}}, v\big) \overset{\mathrm{a}}{\sim}_i \\ \big(\langle \psi_{\mathrm{B}}(m_{\mathrm{B}}) \cdot 1_{\mathrm{B}} \rangle_{\mathrm{B}\in\mathcal{A}_I\setminus(\{\mathrm{B}''\}\cup\{\mathrm{B}^k\}_{k\in K^-_{\text{end}}\cup K^+_{\text{end}}})}, \psi_{\mathrm{B}''}(m_{\mathrm{B}''}+1) \cdot 1_{\mathrm{B}''}, \\ \langle \psi_{\mathrm{B}}(m_{\mathrm{B}^k}-1) \cdot 1_{\mathrm{B}^k} \rangle_{k\in K^-_{\text{end}}}, \langle \psi_{\mathrm{B}}(m_{\mathrm{B}^k}+1) \cdot 1_{\mathrm{B}^k} \rangle_{k\in K^+_{\text{end}}}, v\big). \quad (3.30\text{b})\end{aligned}$$

Then we apply all equivalences of the type (3.29b) with $k \in K^+_{\text{end}}$ and $(k+1) \in K^-_{\text{end}}$, obtaining

$$\begin{aligned}\big(\langle \psi_{\mathrm{B}}(m_{\mathrm{B}}) \cdot 1_{\mathrm{B}} \rangle_{\mathrm{B}\in\mathcal{A}_I\setminus(\{\mathrm{B}''\}\cup\{\mathrm{B}^k\}_{k\in K^-_{\text{end}}\cup K^+_{\text{end}}})}, \psi_{\mathrm{B}''}(m_{\mathrm{B}''}+1) \cdot 1_{\mathrm{B}''}, \\ \langle \psi_{\mathrm{B}}(m_{\mathrm{B}^k}-1) \cdot 1_{\mathrm{B}^k} \rangle_{k\in K^-_{\text{end}}}, \langle \psi_{\mathrm{B}}(m_{\mathrm{B}^k}+1) \cdot 1_{\mathrm{B}^k} \rangle_{k\in K^+_{\text{end}}}, v\big) \overset{\mathrm{a}}{\sim}_i \\ \big(\langle \psi_{\mathrm{B}}(m_{\mathrm{B}}) \cdot 1_{\mathrm{B}} \rangle_{\mathrm{B}\in\mathcal{A}_I\setminus\{\mathrm{B}''\}}, \psi_{\mathrm{B}''}(m_{\mathrm{B}''}+1) \cdot 1_{\mathrm{B}''}, v\big). \quad (3.30\text{c})\end{aligned}$$

Finally, we notice that the three equivalences (3.30) give us just (3.28). □

Lemma 3.4.9. *Let ψ be an integer net on a connected $I \subseteq N$, $i \in N$, $\Upsilon_I \subseteq \Upsilon_i$, and $v \in V_{\Upsilon_i\setminus\Upsilon_I}$. Then whenever $m'_{\mathrm{B}}, m_{\mathrm{B}} \in M_{\mathrm{B}}$ for every $\mathrm{B} \in \mathcal{A}_I$, we have*

$$\big(\langle \psi_{\mathrm{B}}(m'_{\mathrm{B}}) \cdot 1_{\mathrm{B}} \rangle_{\mathrm{B}\in\mathcal{A}_I}, v\big) \succeq_i \big(\langle \psi_{\mathrm{B}}(m_{\mathrm{B}}) \cdot 1_{\mathrm{B}} \rangle_{\mathrm{B}\in\mathcal{A}_I}, v\big) \iff \sum_{\mathrm{B}\in\mathcal{A}_I} m'_{\mathrm{B}} \geq \sum_{\mathrm{B}\in\mathcal{A}_I} m_{\mathrm{B}}. \quad (3.31)$$

Proof. The proof is based on (3.28) exactly in the same way as the proof of Lemma 3.4.1 is based on (3.22b). □

Lemma 3.4.10. *Let $I \in \mathfrak{N}$ be proper, $i \in N$, $\Upsilon_I \subseteq \Upsilon_i$, $v \in V_{\Upsilon_i\setminus\Upsilon_I}$, and $v'_{\mathrm{B}}, v_{\mathrm{B}} \in V_{\mathrm{B}}$ for all $\mathrm{B} \in \mathcal{A}_I$; let each μ_{B} be defined by (3.26). Then*

$$\big(\langle v'_{\mathrm{B}} \rangle_{\mathrm{B}\in\mathcal{A}_I}, v\big) \overset{\mathrm{a}}{\succeq}_i \big(\langle v_{\mathrm{B}} \rangle_{\mathrm{B}\in\mathcal{A}_I}, v\big) \iff \sum_{\mathrm{B}\in\mathcal{A}_I} \mu_{\mathrm{B}}(v'_{\mathrm{B}}) \geq \sum_{\mathrm{B}\in\mathcal{A}_I} \mu_{\mathrm{B}}(v_{\mathrm{B}}). \quad (3.32)$$

Proof. The proof is based on (3.31) exactly in the same way as the proof of Lemma 3.4.7 is based on (3.23) □

We see that, in the conditions of Lemma 3.4.10, we have $\Upsilon_I \in \mathcal{E}$. Therefore, (3.9b) immediately follows from Lemma 1 of Gorman (1968), so a reference to Murphy (1981) suffices (the latter paper gave a final resolution to a dispute between W. Gorman and K. Vind concerning the above-mentioned Lemma 1).

3.5 Necessity of Gorman additivity: Proven cases

A special case of Theorem 2 is the main result of Kukushkin (1994b): $\mathrm{A} = N \cup \{N\}$ and $\Upsilon_i = \{i, N\}$ there; Gorman additivity means that the preferences can be represented by functions $\mu_i(\varphi_i(x_i)) + \mu_N(\varphi_N(x))$. In a sense, Concluding Remark 2 from that paper can be viewed as a precursor of this theorem.

Proposition 3.4. *Lemma 3.4.2 is valid if $\#I = 2$.*

Proof. To simplify notation, we assume $I = \{1, 2\}$ and denote $\mathrm{B}^0 = \Upsilon_1 \cup \Upsilon_2$, $\mathrm{B}^1 = \Upsilon_1 \setminus \Upsilon_2$, and $\mathrm{B}^2 = \Upsilon_2 \setminus \Upsilon_1$. All the three sets are not empty because Υ_1 and Υ_2 overlap.

Lemma 3.5.1. *Let $t_0, t_1, t_2 \in \mathbb{R}$ be such that $(t_0 \cdot 1_{\mathrm{B}^0}, 0_{\mathrm{B}^1}) \sim^{\mathrm{a}}_1 (0_{\mathrm{B}^0}, t_1 \cdot 1_{\mathrm{B}^1})$ and $(t_0 \cdot 1_{\mathrm{B}^0}, 0_{\mathrm{B}^2}) \sim^{\mathrm{a}}_2 (0_{\mathrm{B}^0}, t_2 \cdot 1_{\mathrm{B}^2})$. Then there exists a unique integer net ψ on I such that $\psi_{\mathrm{B}^k}(1) = t_k$ for $k = 0, 1, 2$.*

Proof. For each $k \in \{0, 1, 2\}$, we set $\psi_{\mathrm{B}^k}(0) = 0$ and $\psi_{\mathrm{B}^k}(1) = t_k$. Then for each $k \in \{1, 2\}$ we define $\psi_{\mathrm{B}^k}(m+1)$ for integer $m \geq 1$ inductively, by the relations

$$(0_{\mathrm{B}^0}, \psi_{\mathrm{B}^k}(m+1) \cdot 1_{\mathrm{B}^k}) \sim^{\mathrm{a}}_k (\psi_{\mathrm{B}^0}(1) \cdot 1_{\mathrm{B}^0}, \psi_{\mathrm{B}^k}(m) \cdot 1_{\mathrm{B}^k}). \tag{3.33a}$$

There are two alternatives: either no solution $\psi_{\mathrm{B}^k}(m+1)$ to (3.33a) can be found at a stage $m \geq 1$, in which case we stop the process and set $m^+_{\mathrm{B}^k} = m + 1$; or $\psi_{\mathrm{B}^k}(m)$ will be defined for all $m \geq 0$, in which case we set $m^+_{\mathrm{B}^k} = +\infty$.

For $m \leq 0$, $\psi_{\mathrm{B}^k}(m-1)$ is also defined inductively, by the relations

$$(\psi_{\mathrm{B}^0}(1) \cdot 1_{\mathrm{B}^0}, \psi_{\mathrm{B}^k}(m-1) \cdot 1_{\mathrm{B}^k}) \sim^{\mathrm{a}}_k (0_{\mathrm{B}^0}, \psi_{\mathrm{B}^k}(m) \cdot 1_{\mathrm{B}^k}). \tag{3.33b}$$

Note that (3.33a) and (3.33b) only differ in their viewpoint. Again, if no solution $\psi_{\mathrm{B}^k}(m-1)$ to (3.33b) can be found at a stage $m \leq 0$, we stop the process and set $m^-_{\mathrm{B}^k} = m - 1$; if $\psi_{\mathrm{B}^k}(m)$ is defined for all $m \leq 0$, we set $m^-_{\mathrm{B}^k} = -\infty$.

Then we define $\psi_{\mathrm{B}^0}(m)$ also by two inductions, satisfying the relations

$$(\psi_{\mathrm{B}^0}(m+1) \cdot 1_{\mathrm{B}^0}, 0_{\mathrm{B}^1}) \sim^{\mathrm{a}}_1 (\psi_{\mathrm{B}^0}(m) \cdot 1_{\mathrm{B}^0}, \psi_{\mathrm{B}^1}(1) \cdot 1_{\mathrm{B}^1}) \tag{3.34a}$$

for $m \geq 1$, and

$$(\psi_{\mathrm{B}^0}(m-1) \cdot 1_{\mathrm{B}^0}, \psi_{\mathrm{B}^1}(1) \cdot 1_{\mathrm{B}^1}) \sim^{\mathrm{a}}_1 (\psi_{\mathrm{B}^0}(m) \cdot 1_{\mathrm{B}^0}, 0_{\mathrm{B}^1}) \tag{3.34b}$$

for $m \leq 0$. Each of the inductive processes either stops at some stage m, defining $m^\pm_{\mathrm{B}^0}$, or continues forever, in which case we set $m^\pm_{\mathrm{B}^0} = \pm\infty$.

Turning to the definition of an integer net, we notice that the condition (3.22a) is satisfied automatically. All equivalences (3.33) and (3.34) follow from (3.22b), hence the uniqueness of ψ. Let us check (3.22b) itself; it is convenient to reproduce it here:

$$(\psi_{\mathrm{B}^0}(m+1) \cdot 1_{\mathrm{B}^0}, \psi_{\mathrm{B}^k}(h) \cdot 1_{\mathrm{B}^k}) \sim^{\mathrm{a}}_k (\psi_{\mathrm{B}^0}(m) \cdot 1_{\mathrm{B}^0}, \psi_{\mathrm{B}^k}(h+1) \cdot 1_{\mathrm{B}^k}) \tag{3.35}$$

for both $k = 1, 2$ and all $m^-_{\mathrm{B}^0} < m < m^+_{\mathrm{B}^0} - 1$ and $m^-_{\mathrm{B}^k} < h < m^+_{\mathrm{B}^k} - 1$. Note that we already have it for both k and all h if $m = 0$. For each k and each h, we organize two inductive processes in m: upwards and downwards. It is essential to execute both processes for player 2 first. Each step consists in an application of Lemma 3.3.3.

On an "upward" induction step, we assume $i = 1$; $j = 2$; $v_\alpha = \psi_{B^0}(m-1)$ for $\alpha \in B^0$; $v_\alpha = 0$ for $\alpha \in B^1$; $v_\alpha = \psi_{B^2}(h)$ for $\alpha \in B^2$; $v'_\alpha = \psi_{B^0}(m)$ for $\alpha \in B^0$; $v'_\alpha = \psi_{B^1}(1)$ for $\alpha \in B^1$; $v'_\alpha = \psi_{B^2}(h+1)$ for $\alpha \in B^2$; $v''_\alpha = \psi_{B^0}(m+1)$ for $\alpha \in B^0$; Taking into account $(\psi_{B^0}(m-1) \cdot 1_{B^0}, \psi_{B^1}(1) \cdot 1_{B^1}) \sim^{a}_1 (\psi_{B^0}(m) \cdot 1_{B^0}, 0_{B^1})$ and $(\psi_{B^0}(m) \cdot 1_{B^0}, \psi_{B^1}(1) \cdot 1_{B^1}) \sim^{a}_1 (\psi_{B^0}(m+1) \cdot 1_{B^0}, 0_{B^1})$ from the definition of ψ_{B^0} and $(\psi_{B^0}(m-1) \cdot 1_{B^0}, \psi_{B^2}(h+1) \cdot 1_{B^2}) \sim^{a}_2 (\psi_{B^0}(m) \cdot 1_{B^0}, \psi_{B^2}(h) \cdot 1_{B^2})$ from the induction hypothesis, we obtain $(\psi_{B^0}(m) \cdot 1_{B^0}, \psi_{B^2}(h+1) \cdot 1_{B^2}) \sim^{a}_2 (\psi_{B^0}(m+1) \cdot 1_{B^0}, \psi_{B^2}(h) \cdot 1_{B^2})$ from (3.15).

On a "downward" step, we again assume $i = 1$ and $j = 2$, but the order of everything else is reversed: $v_\alpha = \psi_{B^0}(m+1)$ for $\alpha \in B^0$; $v_\alpha = \psi_{B^1}(1)$ for $\alpha \in B^1$; $v_\alpha = \psi_{B^2}(h+1)$ for $\alpha \in B^2$; $v'_\alpha = \psi_{B^0}(m)$ for $\alpha \in B^0$; $v'_\alpha = 0$ for $\alpha \in B^1$; $v'_\alpha = \psi_{B^2}(h)$ for $\alpha \in B^2$; $v''_\alpha = \psi_{B^0}(m-1)$ for $\alpha \in B^0$. This time, (3.15) implies $(\psi_{B^0}(m) \cdot 1_{B^0}, \psi_{B^2}(h) \cdot 1_{B^1}) \sim^{a}_2 (\psi_{B^0}(m-1) \cdot 1_{B^0}, \psi_{B^2}(h+1) \cdot 1_{B^2})$.

Now similar induction processes for player 1 can be organized. Each step again consists in an application of Lemma 3.3.3, but the roles of the players are reversed: $i = 2$ and $j = 1$. Conditions (3.14a) and (3.14b) follow from (3.35) with $i = 2$ and $h = 0$, which is already proven; condition (3.14c) is the induction hypothesis.

Finally, let us turn to (3.22c) and (3.22d). Suppose that $k \in \{1, 2\}$ and $m^+_{B^k} < +\infty$, but (3.22c) does not hold (only $B = B^k$ and $B' = B^0$ can be relevant), i.e., there is $v_{B^k} \in V_{B^k}$ such that $(0_{B^0}, v_{B^k}) \succeq^{a}_k (\psi_{B^0}(1) \cdot 1_{B^0}, \psi_{B^k}(m^+_{B^k} - 1) \cdot 1_{B^k})$, hence $(0_{B^0}, \tau_{B^k}(v_{B^k}) \cdot 1_{B^k}) \succeq^{a}_k (\psi_{B^0}(1) \cdot 1_{B^0}, \psi_{B^k}(m^+_{B^k} - 1) \cdot 1_{B^k})$. Therefore, a solution to (3.33a) with $m = m^+_{B^k} - 1$ exists, so our inductive process could not have stopped here. Quite similarly, the "downward" process of choosing $\psi_{B^k}(m-1)$ can only stop at a finite m if (3.22d) is satisfied.

The situation with M_{B^0} is a bit subtler. Here (3.22c) and (3.22d) mean two conditions each: one for $B = B^0$ and $B' = B^1$; the other for $B = B^0$ and $B' = B^2$. The first ones are treated exactly as above; the second deserve separate consideration. Let there be $v_{B^0} \in V_{B^0}$ such that $(v_{B^0}, 0_{B^2}) \succeq^{a}_2 (\psi_{B^0}(m^+_{B^0} - 1) \cdot 1_{B^0}, \psi_{B^2}(1) \cdot 1_{B^2})$. Invoking Lemma 3.4.5 and continuity, we obtain the existence of $t^* \in \mathbb{R}$ such that

$$(t^* \cdot 1_{B^0}, 0_{B^2}) \sim^{a}_2 (\psi_{B^0}(m^+_{B^0} - 1) \cdot 1_{B^0}, \psi_{B^2}(1) \cdot 1_{B^2}). \tag{3.36}$$

Then we invoke Lemma 3.3.3 with $i = 2$, $j = 1$, $v_\alpha = \psi_{B^0}(m^+_{B^0} - 2)$ for all $\alpha \in B^0$, $v_\alpha = 0$ for all $\alpha \in B^1 \cup B^2$, $v'_\alpha = \psi_{B^0}(m^+_{B^0} - 1)$ for all $\alpha \in B^0$, $v'_\alpha = \psi_{B^k}(1)$ for all $\alpha \in B^k$ $(k = 1, 2)$, and $v''_\alpha = t^*$ for all $\alpha \in B^0$. Conditions (3.14a) and (3.14b) follow from (3.22b); condition (3.14c), from (3.36). Now (3.15) implies that t^* as $\psi_{B^0}(m^+_{B^0})$ solves (3.34a) with $m = m^+_{B^0} - 1$, contradicting the definition of $m^+_{B^0}$. A dual argument proves (3.22d). □

Since both aggregation rules are continuous and strictly responsive, such t_0, t_1, t_2 as needed in the lemma, obviously can be found. Therefore, we have the existence of an integer net ψ on I. It is sufficient now to show that *every* integer net ψ on I admits a doubling $\bar{\psi}$.

From $(\psi_{B^0}(1) \cdot 1_{B^0}, \psi_{B^1}(1) \cdot 1_{B^1}) \succ^{a}_1 (\psi_{B^0}(1) \cdot 1_{B^0}, 0_{B^1}) \succ^{a}_1 (0_{B^0}, 0_{B^1})$ and continuity, we immediately derive the existence of $t^* \in]0, \psi_{B^0}(1)[$ such that $(t^* \cdot 1_{B^0}, e^0_{B^0 B^1}(t^*) \cdot 1_{B^1}) \sim^{a}_1 (\psi_{B^0}(1) \cdot 1_{B^0}, 0_{B^1})$ $[\sim^{a}_1 (0_{B^0}, \psi_{B^1}(1) \cdot 1_{B^1})]$. Invoking Lemma 3.5.1, we define $\bar{\psi}$ by the conditions $\bar{\psi}_{B^0}(1) = t^*$ and $\bar{\psi}_{B^k}(1) = e^0_{B^0 B^k}(t^*)$ for $k = 1, 2$.

The definition of t^* implies $\bar{\psi}_{B^0}(2) = \psi_{B^0}(1)$ and $\bar{\psi}_{B^1}(2) = \psi_{B^1}(1)$. From Lemma 3.3.3 with $i = 1$, $j = 2$, $v_\alpha = 0$ and $v'_\alpha = \psi_{B^k}(1)$ for all $\alpha \in B^0 \cup B^1 \cup B^2$, and $v''_\alpha = \psi_{B^0}(1)$ for all $\alpha \in B^0$, we obtain $(\bar{\psi}_{B^0}(1) \cdot 1_{B^0}, \bar{\psi}_{B^2}(1) \cdot 1_{B^2}) \sim^{a}_2 (0_{B^0}, \psi_{B^2}(1) \cdot 1_{B^2})$, i.e., $\bar{\psi}_{B^2}(2) = \psi_{B^2}(1)$.

Therefore, ψ coincides with the even half of $\bar{\psi}$ by the uniqueness statement of Lemma 3.5.1, i.e., $\bar{\psi}$ is a doubling of ψ. $\square$

Proposition 3.4 completes a proof of Theorem 1 from Gorman (1968) free from outside references.

Proposition 3.5. *Let $\mathcal{S}$ be such that $\#\{\Upsilon_i\}_{i\in I} \le 2$ for every $I \in \mathfrak{N}$ which is maximal w.r.t. the order* (3.8). *Then Theorem 2 holds for $\mathcal{S}$ and any list $\mathcal{L}$.*

Proof. By Lemma 3.3.5, $\succeq^{\mathrm{a}}_i$ coincides with $\succeq^{\mathrm{a}}_j$ whenever $\Upsilon_i = \Upsilon_j$; therefore, we can delete repetitions and assume $\#I \le 2$ for all maximal members of $\mathfrak{N}$. If $\#I = 1$, we pick μ_{Υ_i} representing $\succeq^{\mathrm{a}}_i$; if $\#I = 2$, we apply Proposition 3.4. In either case, we have (3.9a) for all $\mathrm{B} \in \mathcal{B}_I$. For every other $J \in \mathfrak{N}$, there is a maximal $I \in \mathfrak{N}$ and $i \in I$ such that $\Upsilon_J \subseteq \Upsilon_i$ (Lemma 3.2.2). Proposition 3.4 and Lemma 3.4.10 imply that $\succeq^{\mathrm{a}}_i$ admits a separable projection to $V_{\Upsilon_{j'}\cup\Upsilon_{j''}}$ whenever $j', j'' \in J$ and $j' \bowtie j''$. Now Gorman additivity can be derived in exactly the same way as in Gorman (1968): there was no outside reference after the proof of Theorem 1 there. $\square$

Proposition 3.6. *Lemma 3.4.2 is valid if $\#I = 3$ and $\Upsilon_i \cap \Upsilon_j = \Upsilon_i \setminus \Upsilon_k$ for all $i, j, k \in I$, $i \ne j \ne k \ne i$.*

Proof. Again, we assume $I = \{1, 2, 3\}$, $\mathrm{B}^0 = \Upsilon_1 \cap \Upsilon_2 = \Upsilon_1 \setminus \Upsilon_3 = \Upsilon_2 \setminus \Upsilon_3$, $\mathrm{B}^1 = \Upsilon_1 \cap \Upsilon_3 = \Upsilon_1 \setminus \Upsilon_2 = \Upsilon_3 \setminus \Upsilon_2$, and $\mathrm{B}^2 = \Upsilon_2 \cap \Upsilon_3 = \Upsilon_2 \setminus \Upsilon_1 = \Upsilon_3 \setminus \Upsilon_1$. Denoting $I' = \{1, 2\}$, we obviously have $\mathcal{A}_I = \mathcal{A}_{I'}$. By Proposition 3.4, there exists a binary net on I'; we only have to show that it is simultaneously a binary net on I. It is obviously sufficient to verify the statement for an integer net.

There is no problem with (3.22a). Turning to (3.22b), we only have to prove one statement:

$$(\psi_{\mathrm{B}^1}(m+1) \cdot 1_{\mathrm{B}^1}, \psi_{\mathrm{B}^2}(h) \cdot 1_{\mathrm{B}^2}) \sim^{\mathrm{a}}_3 (\psi_{\mathrm{B}^1}(m) \cdot 1_{\mathrm{B}^1}, \psi_{\mathrm{B}^2}(h+1) \cdot 1_{\mathrm{B}^2}) \tag{3.37}$$

for all $m^-_{\mathrm{B}^1} < m < m^+_{\mathrm{B}^1} - 1$ and $m^-_{\mathrm{B}^2} < h < m^+_{\mathrm{B}^2} - 1$. From (3.22b) for I', we immediately derive $(\psi_{\mathrm{B}^1}(m+1)\cdot 1_{\mathrm{B}^1}, \psi_{\mathrm{B}^0}(0)\cdot 1_{\mathrm{B}^0}) \sim^{\mathrm{a}}_1 (\psi_{\mathrm{B}^1}(m)\cdot 1_{\mathrm{B}^1}, \psi_{\mathrm{B}^0}(1)\cdot 1_{\mathrm{B}^0})$ and $(\psi_{\mathrm{B}^0}(1)\cdot 1_{\mathrm{B}^0}, \psi_{\mathrm{B}^2}(h)\cdot 1_{\mathrm{B}^2}) \sim^{\mathrm{a}}_2 (\psi_{\mathrm{B}^0}(0) \cdot 1_{\mathrm{B}^0}, \psi_{\mathrm{B}^2}(h+1) \cdot 1_{\mathrm{B}^2})$. Now Lemma 3.3.4 implies (3.37).

Finally, checking (3.22c) and (3.22d) (for $i = 3$) is done in essentially the same manner as in the proof of Proposition 3.4. $\square$

In a similar style, Lemma 3.4.2 can be proven for other configurations involving three players; connected sets of more than three players can also be dealt with successfully. What is lacking is a uniform procedure proving the lemma for any number of players and any configuration of Υ_i's. Most likely, one should argue by induction, ordering I in the way described in Lemma 3.2.3; it is also possible that a more detailed system of auxiliary notions and lemmas is needed.

4 Acyclic Patterns

4.1 Preliminaries

As usual, we call a set endowed with a partial order a *poset*; a set with a preorder will be called a *proset*. With every proset X, we associate its *reverse* X^{r}, which is the same set with the "reversed" preorder, $y \succeq^{\mathrm{r}} x \iff x \succeq y$, and a poset $X/\sim$ consisting of equivalence classes; there is a natural "projection" p mapping X onto $X/\sim$.

Let X and Y be two prosets. A mapping $f : X \to Y$ is *increasing* if $y \sim x \Rightarrow f(y) = f(x)$ and $y \succeq x \Rightarrow f(y) \succeq f(x)$; f is *decreasing* if it is increasing as a mapping $X \to Y^{\mathrm{r}}$ (or, the same, $X^{\mathrm{r}} \to Y$).

Let X be a proset; we denote $\mathfrak{B}_X$ the set $2^X \setminus \{\emptyset\}$ with the following preorder $\succeq^*$:

$$Y \succeq^* Z \iff [\forall y \in Y \setminus Z\ \forall x \in Y \cap Z\ \forall z \in Z \setminus Y\ \ (y \succeq x \succeq z)]. \tag{4.1}$$

It is easy to see that $Y \sim^* Z$ implies either $Y = Z$ or $y \sim z$ for all $y \in Y$, $z \in Z$, and that $\{y\} \succeq^* \{z\}$ iff $y \succeq z$. The last equivalence implies that a mapping $f : X \to Y$ is increasing if and only if it is increasing as a mapping $X \to \mathfrak{B}_Y$ which happens to be single-valued.

Remark. When X is a chain, $\succeq^*$ coincides with Veinott's order on sublattices (Topkis, 1979). If X is only a lattice, they differ. If X is just a proset, or even a poset, Veinott's order cannot be defined at all.

A *pseudochain* is a finite sequence $x^0, \dots, x^m \in X$ such that for each $k = 0, 1, \dots, m-1$ either $x^{k+1} \succeq x^k$ or $x^k \succeq x^{k+1}$. For every $a, b \in X$, we define $\deg(a,b)$, the *degree of comparability* of a and b in the following way. If a and b are comparable, $\deg(a,b) = 0$; if a and b are incomparable, but there exist $c, d \in X$ such that $c \succeq a$, $c \succeq b$, $a \succeq d$, and $b \succeq d$, then $\deg(a,b) = -1$; if there is no pseudochain such that $x^0 = a$ and $x^m = b$, then $\deg(a,b) = -\infty$. If none of the previous conditions is satisfied, then $\deg(a,b) = -$[minimal m for which there exists a pseudochain with $x^0 = a$ and $x^m = b$]. In the case of $\deg(a,b) = -2$, it is useful to distinguish two situations: when a and b have a common upper bound, we write $\deg(a,b) = -2_{\mathrm{u}}$; when they have a common lower bound, $\deg(a,b) = -2_{\mathrm{l}}$. (When $\deg(a,b) = -m$ with $m > 2$, there are also two different situations, but we never have to distinguish between them).

Thus, deg maps $X \times X$ to $\mathcal{D} = \{0, -1, -2_{\mathrm{u}}, -2_{\mathrm{l}}, -3, \dots - \infty\}$, with a natural partial order on the latter set (so that -2_{u} and -2_{l} are incomparable). Strictly speaking, a notation like $\deg_X(a,b)$ would be more accurate, but we rely on the context. When X is replaced with X^{r}, the degrees -2_{u} and -2_{l} replace each other whereas every other $\deg(a,b)$ remains the same. It is worth noting that $\mathcal{D}$ contains no infinite strictly increasing sequence, hence every subset contains a maximal point.

Lemma 4.1.1. *Let X and Y be prosets, and $\varphi : X \to Y$ be increasing; then* $\deg(x,y) \le \deg(\varphi(x), \varphi(y))$ *for all $x, y \in X$.*

Proof. If $x^0, \dots, x^m$ is a pseudochain in X, then $\varphi(x^0), \dots, \varphi(x^m)$ is a pseudochain in Y. $\square$

Lemma 4.1.2. *Let $A, B \in \mathfrak{B}_X$, $a \in A$, and $b \in B \setminus A$; then* $\deg(a,b) \ge \deg(A,B)$.

Proof. If $A \succeq^* B$ or $B \succeq^* A$, then $a \succeq b$ or $b \succeq a$, respectively, by (4.1). Let $\deg(A,B) \in \{-1, -2_{\mathrm{l}}\}$; then there is C such that $A \succ^* C$ and $B \succ^* C$. If $b \in C$, then $b \in C \setminus A$ and the previous argument implies $a \succeq b$; if $b \notin C$, then $b \succeq c$ for every $c \in C$. Now if $a \in C$, then $b \succeq a$; otherwise, $a \succeq c$ for every $c \in C$. Since $C \neq \emptyset$, we have $\deg(a,b) \ge -2_{\mathrm{l}}$. Dually, if $\deg(A,B) \ge -2_{\mathrm{u}}$, then $\deg(a,b) \ge -2_{\mathrm{u}}$. It follows immediately that $\deg(a,b) \ge -1$ whenever $\deg(A,B) = -1$. For $\deg(A,B) = -m < -2$, a straightforward induction works. Finally, if $\deg(A,B) = -\infty$, there is nothing to prove. $\square$

Lemma 4.1.3. *Let X be a poset and Y a proset; let $x'', x' \in X$, and $y'', y' \in Y$ be such that* $\deg(x'', x') \leq \min\{-1, \deg(y'', y')\}$ *and* $\deg(y'', y') \in \{0, -1, -2_{\mathrm{l}}, -2_{\mathrm{u}}, -\infty\}$*. Then there is an increasing mapping $\lambda : X \to Y$ such that $\lambda(x') = y'$ and $\lambda(x'') = y''$.*

Proof. Let $\deg(y'', y') = 0$; since the roles of x' and x'' are symmetric, we may assume $y'' \succeq y'$. We define $\lambda(x) = y''$ if $x \geq x''$ and $\lambda(x) = y'$ otherwise. Clearly, if $x^2 > x^1$, then $\lambda(x^2) = y''$ whenever $\lambda(x^1) = y''$, i.e., $\lambda : X \to Y$ is increasing.

Let $\deg(y'', y') = -1$ and $y^- \prec y', y'' \prec y^+$. We define $\lambda : X \to Y$ by $\lambda(x) = y^+$ whenever $x > x'$ and $x > x''$, $\lambda(x) = y''$ whenever $x \geq x''$ but not $x > x'$, $\lambda(x) = y'$ whenever $x \geq x'$ but not $x > x''$, and $\lambda(x) = y^-$ otherwise. It is easily checked that λ is increasing.

Let $\deg(y'', y') = -2_{\mathrm{u}} \geq \deg(x'', x')$ and $y', y'' \prec y^+$. We define $\lambda : X \to Y$ by $\lambda(x) = y''$ whenever $x'' \geq x$, $\lambda(x) = y'$ whenever $x' \geq x$, and $\lambda(x) = y^+$ otherwise. It is easily checked that λ is increasing. For $\deg(y'', y') = -2_{\mathrm{l}} \geq \deg(x'', x')$, an appropriate λ is defined dually.

Finally, let $\deg(y'', y') = -\infty = \deg(x'', x')$. We define $\lambda(x) = y''$ whenever $\deg(x'', x) > -\infty$ and $\lambda(x) = y'$ otherwise. λ is obviously increasing. □

Lemma 4.1.4. *Let X be a poset and Y a proset; let $x'', x' \in X$, and $y'', y' \in Y$ be such that $y'' \succeq y'$, but not $x' \geq x''$. Then there is an increasing mapping $\lambda : X \to Y$ such that $\lambda(x') = y'$ and $\lambda(x'') = y''$.*

Proof. We define $\lambda(x) = y''$ if $x \geq x''$ and $\lambda(x) = y'$ otherwise. Clearly, $\lambda : X \to Y$ is increasing. □

4.2 Endomorphisms

Theorem 5 of Kukushkin (2003) establishes a condition on a proset necessary and sufficient for every increasing mapping (or correspondence) of the proset to itself to be acyclic. Clearly, acyclic endomorphisms may exist even when the condition is violated and it would hardly make sense trying to describe all of them. Here we are interested in endomorphisms acyclic because of their "indifference maps"; a trivial example is when everything is mapped into the same point – it does not matter which point is chosen.

A ((*weakly*) Ω-)*acyclic pattern* for (multivalued) endomorphisms consists of a proset X and a mapping φ of X onto a poset Φ such that, for every increasing mapping $\lambda : \Phi \to X$ ($\lambda : \Phi \to \mathfrak{B}_X$), the superposition $\lambda \circ \varphi : X \to X$ ($\lambda \circ \varphi : X \to \mathfrak{B}_X$) is ((weakly) Ω-)acyclic.

Example 4.1. Let $X = [0, 1] \times [0, 1]$ with the standard order, and $f(x_1, x_2) = (x_1/2, x_2/2)$. Then $f : X \to X$ is acyclic, but there is no acyclic pattern here: defining $\lambda(x_1, x_2) = (2x_2, 2x_1)$ (for $(x_1, x_2) \in f(X)$) and $g = \lambda \circ f$, we obtain $g(x_1, x_2) = (x_2, x_1)$, which is obviously not acyclic.

Proposition 4.1. *Let φ be a mapping of a proset X onto a poset Φ. Then the following statements are equivalent.*

1. *$\langle X, \varphi, \Phi \rangle$ is an acyclic pattern for endomorphisms.*

2. *$\langle X, \varphi, \Phi \rangle$ is an acyclic pattern for multivalued endomorphisms.*

3. *The following conditions,*

$$y \succeq x \Rightarrow \varphi(y) \geq \varphi(x) \tag{4.2a}$$

and

$$\deg(x,y) < 0 \Rightarrow \deg(x,y) < \deg(\varphi(x),\varphi(y)), \tag{4.2b}$$

hold for all $x, y \in X$.

Proof. **Sufficiency.** Let (4.2) hold; we have to prove that $\langle X, \varphi, \Phi\rangle$ is an acyclic pattern for multivalued endomorphisms.

Let $\lambda : \Phi \to \mathfrak{B}_X$ be increasing and $x^0, x^1, \ldots$ be an infinite iteration path. Whenever $\deg(x^k, x^{k+1}) < 0$, we have $\deg(x^{k+1}, x^{k+2}) \geq \deg(\lambda \circ \varphi(x^k), \lambda \circ \varphi(x^{k+1})) \geq \deg(\varphi(x^k), \varphi(x^{k+1})) > \deg(x^k, x^{k+1})$ by Lemma 4.1.2 and (4.2); so there may be only a finite number of such steps. When $\deg(x^k, x^{k+1}) = 0$, we can argue exactly as in the sufficiency proof for Theorem 3 below.

Necessity. Now let $\langle X, \varphi, \Phi\rangle$ be an acyclic pattern; we have to prove (4.2).

Supposing (4.2a) violated for $a, b \in X$ such that $b \succeq a$, we apply Lemma 4.1.4, obtaining an increasing mapping $\lambda : \Phi \to X$ such that $\lambda(\varphi(a)) = b$ and $\lambda(\varphi(b)) = a$. Obviously, $\lambda \circ \varphi$ admits a cycle (actually, it admits no fixed point).

Now let (4.2a) hold for all $x, y \in X$, but (4.2b) be violated for some $a, b \in X$. If $\deg(a, b) \in \{0, -1, -2_{\mathrm{l}}, -2_{\mathrm{u}}, -\infty\}$, then $\deg(a, b) = \deg(\varphi(a), \varphi(b))$ by Lemma 4.1.1. Invoking Lemma 4.1.3, we obtain an increasing mapping $\lambda : \Phi \to X$ such that $\lambda(\varphi(a)) = b$ and $\lambda(\varphi(b)) = a$. Again, $\lambda \circ \varphi$ admits a cycle.

If $\deg(a, b) = -m < -2$, let $x^0, \ldots, x^m$ be a pseudochain with $x^0 = a$ and $x^m = b$. Then $\deg(x^0, x^2) \in \{-2_{\mathrm{l}}, -2_{\mathrm{r}}\}$, hence $\deg(\varphi(x^0), \varphi(x^2)) \geq -1$ by the previous arguments, hence either $\varphi(x^0), \varphi(x^1), \varphi(x^3), \ldots, \varphi(x^m)$ or $\varphi(x^0), \varphi(x^2), \varphi(x^3), \ldots, \varphi(x^m)$ is a pseudochain, hence $\deg(\varphi(a), \varphi(b)) \geq 1 - m$. □

Corollary. *If X contains both a greatest and a least points, then $\varphi : X \to \Phi$ is an acyclic pattern for (multivalued) endomorphisms if and only if φ is increasing and Φ is a chain.*

The corollary is applicable, in particular, to complete lattices.

Remark. Even when X is a lattice, Proposition 4.1 would not survive the replacement of $\succeq^*$ with Veinott's order.

Example 4.2. Let $X = \{a, b, c, d\}$ with the order $a < b < d$, $a < c < d$, and b and c incomparable. We define $F : X \to 2^X \setminus \{\emptyset\}$ by $F(a) = F(b) = \{a, c\}$ and $F(c) = F(d) = \{b, d\}$; clearly, there is a cycle: $b \to c \to b$. On the other hand, F can be represented as the superposition of two increasing mappings $X \xrightarrow{\varphi} \{0, 1\} \xrightarrow{\lambda} \mathcal{L}$, where $\mathcal{L}$ consists of sublattices of X with Veinott's order. Since $\{0, 1\}$ is a chain, φ obviously satisfies (4.2b).

Proposition 4.2. *For every proset X, the following statements are equivalent.*

1. *Every increasing mapping $X \to X$ is acyclic.*

2. *Every increasing mapping $X \to \mathfrak{B}_X$ is acyclic.*

3. *The preorder $\succeq$ is complete.*

Proof. It follows immediately from the definitions that Statement 1 (2) holds if and only if the projection $p : X \to X/\sim$ is an acyclic pattern for (multivalued) endomorphisms. The projection is increasing and the equality $\deg(x', x) = \deg(p(x'), p(x))$ is obvious. If $\succeq$ is an ordering, (4.2b) holds by default. Conversely, $\succeq$ must be complete because otherwise we would have to satisfy an inequality $\deg(x', x) < \deg(p(x'), p(x))$. □

Remark. Since our definition (4.1) is less exacting than the similar definition in Kukushkin (2003, Section 4), Proposition 4.2 is a bit stronger than Theorem 5 from that paper.

A proset X and a mapping φ of X onto a poset Φ is a *universal* ((*weakly*) Ω-)*acyclic pattern* for (multivalued) endomorphisms if every subset $X' \subseteq X$ with the restriction of φ to X' and $\varphi(X')$ is a ((weakly) Ω-)acyclic pattern for (multivalued) endomorphisms.

Theorem 3. *Let φ be a mapping of a proset X onto a poset Φ. Then the following statements are equivalent.*

1. *$\langle X, \varphi, \Phi\rangle$ is a universal acyclic pattern for endomorphisms.*
2. *$\langle X, \varphi, \Phi\rangle$ is a universal acyclic pattern for multivalued endomorphisms.*
3. *φ is increasing and Φ is a chain.*

Proof. **Sufficiency.** Let both conditions listed in Statement 3 hold. Since they are obviously inherited by subsets of X, we only have to prove that $\lambda \circ \varphi$ is acyclic whenever $\lambda : \Phi \to \mathfrak{B}_X$ is increasing.

Let us assume that there is an iteration cycle, i.e., a mapping $\pi : \mathbb{N} \to X$ such that $\pi(k) \notin \lambda \circ \varphi \circ \pi(k) \ni \pi(k+1)$ for all k, and $\pi(0) = \pi(m)$ for an $m > 0$. We denote

$$\Sigma^+ = \{k \in \mathbb{N}|\ \varphi \circ \pi(k+1) > \varphi \circ \pi(k)\ \&\ \pi(k+1) \succ \pi(k)\};$$

$$\Sigma^- = \{k \in \mathbb{N}|\ \varphi \circ \pi(k+1) < \varphi \circ \pi(k)\ \&\ \pi(k+1) \prec \pi(k)\}.$$

Lemma 4.2.1. *Let $k > 0$; then either $k \in \Sigma^+$ or $k \in \Sigma^-$. In the first case, $(k+1) \in \Sigma^+$; in the second, $(k+1) \in \Sigma^-$.*

Proof. Since Φ is a chain, $\varphi \circ \pi(k-1)$ and $\varphi \circ \pi(k)$ must be comparable. If they coincide, then $\lambda \circ \varphi \circ \pi(k-1) = \lambda \circ \varphi \circ \pi(k)$, hence $\pi(k) \in \lambda \circ \varphi \circ \pi(k)$ and the path could not have continued further. Let $\varphi \circ \pi(k) > \varphi \circ \pi(k-1)$, hence $\lambda \circ \varphi \circ \pi(k) \succeq^* \lambda \circ \varphi \circ \pi(k-1)$. Since $\pi(k+1) \in \lambda \circ \varphi \circ \pi(k)$ and $\pi(k) \in \lambda \circ \varphi \circ \pi(k-1) \setminus \lambda \circ \varphi \circ \pi(k)$, we have $\pi(k+1) \succeq \pi(k)$, hence $\varphi \circ \pi(k+1) \geq \varphi \circ \pi(k)$. An equality would imply the impossibility to continue the path further; therefore, $k \in \Sigma^+$. Dually, if $\varphi \circ \pi(k) < \varphi \circ \pi(k-1)$, then $k \in \Sigma^-$.

If $k \in \Sigma^+$, then $\pi(k+2) \in \lambda \circ \varphi \circ \pi(k+1) \succeq^* \lambda \circ \varphi \circ \pi(k)$ while $\pi(k+1) \in \lambda \circ \varphi \circ \pi(k) \setminus \lambda \circ \varphi \circ \pi(k+1)$; therefore, $\pi(k+2) \succeq \pi(k+1)$, hence $\varphi \circ \pi(k+2) \geq \varphi \circ \pi(k+1)$. Again, an equality would imply the impossibility to continue the path further; therefore, $(k+1) \in \Sigma^+$. The case of $k \in \Sigma^-$ is treated dually. □

Now if $1 \in \Sigma^+$, then the sequence $\{\pi(k)\}_{k \in \mathbb{N}}$ is strictly increasing; if $1 \in \Sigma^-$, it is strictly decreasing. In either case, no cycling is possible.

Necessity. Let $\langle X, \varphi, \Phi\rangle$ be a universal acyclic pattern; we have to prove Statement 3. First, φ must be increasing by (4.2a) from Proposition 4.1. If there are $x, y \in X$ such that $\varphi(x)$ and $\varphi(y)$ are incomparable in Φ, we denote $X' = \{x, y\}$. Clearly, $\deg_{X'}(y, x) = \deg_{\varphi(X')}(\varphi(y), \varphi(x)) = -\infty$. Therefore, (4.2b) is violated, hence $\langle X', \varphi|_{X'}\rangle$ cannot be an acyclic pattern by Proposition 4.1. $\square$

Theorem 4. *Let φ be a mapping of a metric proset X onto a poset Φ. Then the following statements are equivalent.*

1. *$\langle X, \varphi, \Phi\rangle$ is a universal weakly Ω-acyclic pattern for endomorphisms.*
2. *$\langle X, \varphi, \Phi\rangle$ is a universal Ω-acyclic pattern for endomorphisms.*
3. *$\langle X, \varphi, \Phi\rangle$ is a universal weakly Ω-acyclic pattern for multivalued endomorphisms.*
4. *$\langle X, \varphi, \Phi\rangle$ is a universal Ω-acyclic pattern for multivalued endomorphisms.*
5. *The following conditions hold: φ is increasing* (4.2a)*; Φ is a chain;*

$$\text{if } x^k \to x^\omega,\ x^{k+1} \succ x^k \text{ and } \varphi(x^{k+1}) > \varphi(x^k) \text{ for all } k, \text{ then } \varphi(x^\omega) \geq \varphi(x^0); \quad (4.3a)$$

$$\text{if } x^k \to x^\omega,\ x^{k+1} \prec x^k \text{ and } \varphi(x^{k+1}) < \varphi(x^k) \text{ for all } k, \text{ then } \varphi(x^\omega) \leq \varphi(x^0); \quad (4.3b)$$

there is no pair of infinite sequences $\{x^k\}_{k=0,1,\dots}$ and $\{y^h\}_{h=0,1,\dots}$ such that

$$x^k \to x^\omega,\ y^h \to y^\omega,\ y^h \succ y^{h+1} \succ x^{k+1} \succ x^k \text{ and}$$
$$\varphi(y^h) > \varphi(y^{h+1}) > \varphi(x^{k+1}) > \varphi(x^k) \text{ for all } k \text{ and } h,$$
$$\varphi(x^\omega) > \varphi(y^0), \text{ and } \varphi(y^\omega) < \varphi(x^0). \quad (4.3c)$$

Proof. **Sufficiency.** Let all the conditions listed in statement 5 hold. Since they are obviously inherited by subsets of X, we only have to prove that $\lambda \circ \varphi$ is Ω-acyclic whenever $\lambda : \Phi \to \mathfrak{B}_X$ is increasing. Since $\lambda \circ \varphi$ is acyclic by Proposition 4.1, we only have to discard the possibility of an infinite iteration cycle.

Let us assume that there is such a cycle, i.e., a mapping $\pi : \Sigma \to X$ (where Σ is a countable well ordered set) satisfying (2.4) and such that $\pi(0) = \pi(\bar\beta)$ for a $\bar\beta > 0$. Without restricting generality, $\bar\beta \in \Sigma_{\text{lim}}$ and $\pi(0) \neq \pi(\beta)$ for any $\beta < \bar\beta$. It is more convenient here *not* to assume $\bar\beta = \max\Sigma$; we assume instead that $(\bar\beta + k) \in \Sigma$ for all natural k and denote $\Sigma^* = \{\beta \in \Sigma |\ \beta \leq \bar\beta + 1\}$. Without restricting generality, $\pi(\bar\beta + 1) = \pi(1)$. We define $\Sigma^+, \Sigma^- \subseteq \Sigma$ as in the proof of Theorem 3, only replacing $k \in \mathbb{N}$ with $\beta \in \Sigma$.

Lemma 4.2.2. *Let $\beta \in \Sigma_{\text{iso}}$; then either $\beta \in \Sigma^+$ or $\beta \in \Sigma^-$. In the first case, $(\beta+1) \in \Sigma^+$; in the second, $(\beta + 1) \in \Sigma^-$.*

The proof is exactly the same as in Lemma 4.2.1.

Lemma 4.2.3. *Let $\Sigma_{\text{iso}} \cap \Sigma^* \ni \gamma > \beta \in \Sigma$. Then: $\varphi \circ \pi(\gamma) > \varphi \circ \pi(\beta)$ and $\pi(\gamma) \succ \pi(\beta)$ whenever $\beta \in \Sigma^+$; $\varphi \circ \pi(\gamma) < \varphi \circ \pi(\beta)$ and $\pi(\gamma) \prec \pi(\beta)$ whenever $\beta \in \Sigma^-$.*

Proof. We argue by (transfinite) recursion. If $\gamma = 1$, only $\beta = 0$ is admissible, and the statement immediately follows from the definition of Σ^+ and Σ^-.

Supposing the statement holding for some $\gamma \in \Sigma_{\text{iso}}$ (and all $\beta < \gamma$), let us derive it for $\gamma + 1$. If $\beta = \gamma$, the definition of Σ^+ or Σ^- suffices. Let $\Sigma^+ \ni \beta < \gamma$; by Lemma 4.2.2, $(\beta + 1) \in \Sigma^+$ too. By the induction hypothesis, $\varphi \circ \pi(\gamma) \geq \varphi \circ \pi(\beta + 1)$, hence

$$\lambda \circ \varphi \circ \pi(\gamma) \succeq^* \lambda \circ \varphi \circ \pi(\beta + 1) \succ^* \lambda \circ \varphi \circ \pi(\beta). \quad (4.4)$$

The middle term in (4.4) contains $\pi(\beta+2)$, but does not contain $\pi(\beta+1)$. Since $\pi(\beta+2) \succ \pi(\beta+1)$, there must hold $\pi(\beta+1) \notin \lambda\circ\varphi\circ\pi(\gamma)$. Since the left hand side of (4.4) contains $\pi(\gamma+1)$, while the right hand side contains $\pi(\beta+1)$, we obtain $\pi(\gamma+1) \succeq \pi(\beta+1) \succ \pi(\beta)$, hence $\varphi\circ\pi(\gamma+1) \geq \varphi\circ\pi(\beta+1) > \varphi\circ\pi(\beta)$, i.e., the statement of the lemma holds for β and $\gamma+1$ as well. The case of $\beta \in \Sigma^-$ is treated dually.

Finally, let $\gamma^\omega \in \Sigma_{\lim}$ and the statement of the lemma hold for all $\beta < \gamma < \gamma^\omega$; we have to prove it for $\gamma^\omega+1$. Let $\Sigma^+ \ni \beta < \gamma^\omega$. By Proposition 1 from Kukushkin (2005a), there is a sequence $\gamma^k \to \gamma^\omega$ such that $\pi(\gamma^k) \to \pi(\gamma^\omega)$, $\gamma^{k+1} > \gamma^k$ and $\gamma^k \in \Sigma_{\mathrm{iso}}$ for all k. Suppose first that there is an infinite number of k for which $\gamma^k \in \Sigma^+$; without restricting generality, $\gamma^k \in \Sigma^+$ for all k and $\gamma^0 > \beta+1$. By the induction hypothesis, $\pi(\gamma^{k+1}) \succ \pi(\gamma^k) \succ \pi(\beta)$ for all k. Then (4.3a) is applicable (with $x^k = \pi(\gamma^k)$), implying $\varphi\circ\pi(\gamma^\omega) \geq \varphi\circ\pi(\gamma^0) > \varphi\circ\pi(\beta+1) > \varphi\circ\pi(\beta)$ (both strict inequalities follow from the induction hypothesis). Now we derive $\pi(\gamma^\omega+1) \succ \pi(\beta)$ with a reasoning similar to that of the previous paragraph: If $\pi(\beta+1) \in \lambda\circ\varphi\circ\pi(\gamma^\omega)$, then $\pi(\gamma^\omega+1) \succeq \pi(\beta+1)$ from $\pi(\beta+1) \notin \lambda\circ\varphi\circ\pi(\beta+1) \preceq^* \lambda\circ\varphi\circ\pi(\gamma^\omega) \ni \pi(\gamma^\omega+1)$; otherwise, $\pi(\gamma^\omega+1) \succeq \pi(\beta+1)$ from $\pi(\beta+1) \notin \lambda\circ\varphi\circ\pi(\gamma^\omega) \succeq^* \lambda\circ\varphi\circ\pi(\beta+1) \ni \pi(\beta+1)$ and $\pi(\gamma^\omega+1) \in \lambda\circ\varphi\circ\pi(\gamma^\omega)$.

Let us suppose that $\gamma^k \in \Sigma^+$ only for a finite number of k; without restricting generality, $\gamma^k \in \Sigma^-$ for all k. Picking the least $\gamma > \beta$ which belongs to Σ^-, we notice from Lemma 4.2.2 that it is either a limit point or an isolated point following a limit one. In either case, there is $\gamma^\infty \in \Sigma_{\lim}$ such that $\beta < \gamma^\infty < \gamma^\omega$, $(\gamma^\infty+1) \in \Sigma^-$, and $\Sigma^- \cap \{\gamma \in \Sigma \mid \beta < \gamma < \gamma^\infty\} = \emptyset$. It is clear from the proof of Lemma 4.2.1 that $\varphi(\gamma^\infty) > \varphi(\gamma^\infty+1)$. By Proposition 1 from Kukushkin (2005a), there is a sequence $\beta^k \to \gamma^\infty$ such that $\pi(\beta^k) \to \pi(\gamma^\infty)$, $\beta^{k+1} > \beta^k$ and $\beta^k \in \Sigma_{\mathrm{iso}}$ for all k; without restricting generality, $\beta^0 > \beta+2$. By the choice of γ^∞, we have $\gamma^k \in \Sigma^+$ for all k, hence $\pi(\beta^{k+1}) \succ \pi(\beta^k)$ and $\varphi\circ\pi(\beta^{k+1}) > \varphi\circ\pi(\beta^k)$ by the induction hypothesis. Besides, $\varphi\circ\pi(\gamma^\infty) > \varphi\circ\pi(\beta)$ and $\pi(\gamma^\infty+1) > \pi(\beta)$ for the same reason as in the previous paragraph. Moreover, we have $\pi(\beta^k) \prec \pi(\gamma^h)$ and $\varphi\circ\pi(\beta^k) < \varphi\circ\pi(\gamma^h)$ for all k and h, as well as $\pi(\gamma^h) \succ \pi(\gamma^{h+1})$ and $\varphi\circ\pi(\gamma^h) > \varphi\circ\pi(\gamma^{h+1})$ for all h, by the induction hypothesis. Now an assumption that $\varphi\circ\pi(\gamma^\omega) \leq \varphi\circ\pi(\beta+1)$ would imply $\varphi\circ\pi(\gamma^\omega) < \varphi\circ\pi(\beta+2)$, hence $x^k = \pi(\beta^k)$, $y^h = \pi(\gamma^h)$, $x^\omega = \pi(\gamma^\infty)$, and $y^\omega = \pi(\gamma^\omega)$ form a configuration prohibited by (4.3c). Therefore, $\varphi\circ\pi(\gamma^\omega) > \varphi\circ\pi(\beta+1) > \varphi\circ\pi(\beta)$, hence $\pi(\gamma^\omega+1) \succeq \pi(\beta+1) \succ \pi(\beta)$ and $\varphi\circ\pi(\gamma^\omega+1) > \varphi\circ\pi(\beta)$. □

Now if $1 \in \Sigma^+$, then $\pi(\bar\beta+1) \succ \pi(1)$ by Lemma 4.2.3; if $1 \in \Sigma^-$, then $\pi(\bar\beta+1) \prec \pi(1)$. In either case, we have a contradiction with $\pi(\bar\beta+1) = \pi(1)$.

Necessity. Now let $\langle X, \varphi, \Phi\rangle$ be a universal weakly Ω-acyclic pattern for endomorphisms. The necessity of the first two conditions was established in Proposition 4.1, we only have to prove (4.3).

Supposing (4.3a) violated, we define $X' = \{x^k\}_k \cup \{x^\omega\} \subseteq X$ and $\Phi' = \varphi(\Phi)$; they must form a weakly Ω-acyclic pattern. Then we define $\lambda : \Phi' \to X'$ as follows: whenever $v < \varphi(x^0)$, $\lambda(v) = x^0$; otherwise, $\lambda(v) = x^1$ whenever $v < \varphi(x^1)$; ... otherwise, $\lambda(v) = x^k$ whenever $v < \varphi(x^k)$ ... By our assumption, λ is defined and increasing on the whole Φ'. Now we pick $z^0 = x^\omega$; then $z^1 = \lambda\circ\varphi(z^0) = x^0$; by induction, $z^{k+1} = \lambda\circ\varphi(z^k) = x^k$. Therefore, $z^k \to x^\omega = z^0$, i.e., $\lambda\circ\varphi$ admits a narrow cycle "of the length ω."

The proof of the necessity of (4.3b) is dual.

Supposing the existence of a pair of sequences prohibited by (4.3c), we define $X' = \{x^k\}_k \cup \{x^\omega\} \cup \{y^h\}_h \cup \{y^\omega\} \subseteq X$ and $\Phi' = \varphi(\Phi)$; they must form a weakly Ω-acyclic pattern. Then we define $\lambda : \Phi' \to X$ in a similar way. Whenever $v < \varphi(x^0)$, $\lambda(v) = x^0$; otherwise, $\lambda(v) = x^1$ whenever $v < \varphi(x^1)$; otherwise, ... $\lambda(v) = x^k$ whenever $v < \varphi(x^k)$ ... If $v \geq \varphi(x^k)$ for all k, we define $\lambda(v) = y^0$ whenever $\varphi(y^0) < v$; ... otherwise, $\lambda(v) = y^h$ whenever $\varphi(y^h) < v$; ... By our assumption, λ is defined and increasing on the whole Φ'.

Now we pick $z^0 = y^\omega$; then $z^1 = \lambda \circ \varphi(z^0) = x^0$; by induction, $z^{k+1} = \lambda \circ \varphi(z^k) = x^k$. Therefore, $z^k \to x^\omega = z^\omega$. Now $z^{\omega+1} = \lambda \circ \varphi(z^\omega) = y^0$; by induction, $z^{\omega+h+1} = \lambda \circ \varphi(z^{\omega+h}) = y^h$. Finally, $z^{\omega+h} \to y^\omega = z^0$, i.e., $\lambda \circ \varphi$ admits a narrow cycle "of the length $\omega + \omega$." □

4.3 Two players

A *pattern for reactions* is defined by a finite set N (of players), and a set X_i, a poset Φ_i and a mapping $\varphi_i : X_{-i} \to \Phi_i$ for each $i \in N$; we usually assume that φ_i is onto. Given a pattern for reactions, a *derivative system* of (multivalued) reactions is defined by subsets $X'_i \subseteq X_i$ and increasing mappings $\lambda_i : \Phi_i \to X'_i$ ($\Phi_i \to \mathfrak{B}_{X'_i}$) for all $i \in N$; in terms of the definition of Section 2, we assume $\mathcal{R}_i = \lambda_i \circ \varphi_i$. A *universal* ($\Omega$-)*acyclic pattern* for (multivalued) reactions is a pattern for reactions such that every derivative system of (multivalued) reactions is (Ω-)acyclic. If the acyclicity is only ensured when $X'_i = X_i$, we drop the adjective "universal."

Remark. One could think that considering $N' \subset N$ would widen the scope of derivative systems; however, a singleton X'_i is equivalent to the exclusion of player i.

In this subsection, we only consider $\#N = 2$, using the term "bilateral reactions." The assumptions that $N = \{1, 2\}$ while i and j are always distinct members of N simplify notation considerably.

Proposition 4.3. *Let φ_1 and φ_2 be surjective mappings $\varphi_i : X_i \to \Phi_i$, where each X_i is a proset and each Φ_i is a poset. Then the following statements are equivalent.*

1. *$\langle X_1, X_2, \varphi_1, \varphi_2, \Phi_1, \Phi_2 \rangle$ is an acyclic pattern for bilateral reactions.*
2. *$\langle X_1, X_2, \varphi_1, \varphi_2, \Phi_1, \Phi_2 \rangle$ is an acyclic pattern for multivalued bilateral reactions.*
3. *At least one of the following conditions holds:*

$$\exists i\,[\#\Phi_i = 1]; \tag{4.5a}$$

$$\exists i \left[\forall x'_i, x_i\,[\deg(\varphi_i(x'_i), \varphi_i(x_i)) > -\infty]\ \&\ \forall x'_j, x_j\,[x'_j \succeq x_j \Rightarrow \varphi_j(x'_j) = \varphi_j(x_j)]\right]; \tag{4.5b}$$

$$\forall i\, \forall x'_i, x_i\,[x'_i \succeq x \Rightarrow \varphi_i(x'_i) \geq \varphi_i(x_i)]\ \&\ \forall d \in \mathcal{D} \setminus \{0\}\, \exists i\, \forall x'_i, x_i\,[\deg(x'_i, x_i) = d \Rightarrow \deg(\varphi_i(x'_i), \varphi_i(x_i)) > d]; \tag{4.5c}$$

or (4.5c) *holds when Φ_1 is replaced with Φ_1^{r} and X_2 with X_2^{r}.* (4.5d)

Proof. **Sufficiency.** Let us prove that Statement 3 implies Statement 2; let $\lambda_1, \lambda_2 : \Phi_j \to \mathfrak{B}_{X_i}$ be fixed and $x^0, x^1, \ldots$ be an infinite iteration path. Deleting the first point from the path if needed, we may always assume that $x_i^{2k} \notin \lambda_i(\varphi_j(x_j^{2k})) \ni x_i^{2k+1}$, $x_j^{2k} = x_j^{2k+1}$, $x_i^{2k+1} = x_j^{2k+2}$, and $x_j^{2k+1} \notin \lambda_j(\varphi_i(x_i^{2k+1})) \ni x_j^{2k+2}$ for all k.

If (4.5a) holds, then x_j can change only once, hence no iteration path can include more than three steps (i.e., four points).

If (4.5b) holds, then, similarly, $\deg(\varphi_i(x_i^3), \varphi_i(x_i^1)) > -\infty$, hence, by Lemmas 4.1.1 and 4.1.2, $\deg(x_j^4, x_j^2) > -\infty$, hence the second term in (4.5b) implies that $\varphi_j(x_j^4) = \varphi_j(x_j^3)$, hence $x_i^4 = x_i^3 \in \lambda_i(\varphi_j(x_j^4))$, so the path must have stopped.

Suppose that (4.5c) holds. Then $\deg(x_i^{2k+2}, x_i^{2k}) \leq \deg(\varphi_i(x_i^{2k+2}), \varphi_i(x_i^{2k})) \leq \deg(x_j^{2k+3}, x_j^{2k+1}) \leq \deg(\varphi_j(x_j^{2k+3}), \varphi_j(x_j^{2k+1})) \leq \deg(x_i^{2k+4}, x_i^{2k+2})$ by Lemmas 4.1.1 and 4.1.2. Whenever $\deg(x_i^{2k+2}, x_i^{2k}) < 0$, the second term in (4.5c) implies that at least one of the inequalities is strict. Therefore, there may only be a finite number of steps with $\deg(x_i^{2k+2}, x_i^{2k}) < 0$. Once $\deg(x_i^{2k+2}, x_i^{2k}) = 0$, we argue exactly as in the sufficiency part of the proof of Theorem 3.

Finally, if (4.5d) holds, then $\langle X_1, X_2^{\mathrm{r}}, \varphi_1, \varphi_2, \Phi_1^{\mathrm{r}}, \Phi_2 \rangle$ is an acyclic pattern by the previous paragraph; but this is equivalent to $\langle X_1, X_2, \varphi_1, \varphi_2, \Phi_1, \Phi_2 \rangle$ being an acyclic pattern because $\lambda_2 : \Phi_1 \to X_2$ is increasing if and only if it is increasing as a mapping $\Phi_1^{\mathrm{r}} \to X_2^{\mathrm{r}}$.

Necessity. Now let $\langle X_1, X_2, \varphi_1, \varphi_2, \Phi_1, \Phi_2 \rangle$ be an acyclic pattern for bilateral reactions. We have to prove Statement 3. We assume (4.5a) does not hold, hence $\# X_i > 1$ for each i.

First, let us prove that the second conjunctive term in (4.5b) implies the first (hence the whole (4.5b)). Let there be j such that $\forall x_j', x_j\, [x_j' \succeq x_j \Rightarrow \varphi_j(x_j') = \varphi_j(x_j)]$; since (4.5a) does not hold, there are x_j and x_j' such that $\varphi_j(x_j') \neq \varphi_j(x_j)$, hence $\deg(x_j', x_j) = -\infty$. Without restricting generality, we may assume that the inequality $\varphi_j(x_j) \geq \varphi_j(x_j')$ does not hold. Suppose, to the contrary to (4.5b), that there are x_i' and x_i such that $\deg(\varphi_i(x_i'), \varphi_i(x_i)) = -\infty$. If $\deg(x_i', x_i) > -\infty$, there is a pseudochain $x_i = x_i^0, x_i^1, \ldots, x_i^m = x_i'$; clearly, there must be k such that $\deg(\varphi(x_i^{k+1}), \varphi(x_i^k)) = -\infty$. Therefore, without restricting generality, we may assume $x_i' \succeq x_i$. By Lemma 4.1.4, we have an increasing mapping $\lambda_i : \Phi_j \to X_i$ such that $\lambda_i(\varphi_j(x_j')) = x_i'$ and $\lambda_i(\varphi_j(x_j)) = x_i$. By Lemma 4.1.3, we have an increasing mapping $\lambda_j : \Phi_i \to X_j$ such that $\lambda_j(\varphi_i(x_i')) = x_j$ and $\lambda_j(\varphi_i(x_i)) = x_j'$. Clearly, this contradicts Statement 1. If $\deg(x_i', x_i) = -\infty$, we obtain λ_i and λ_j with the same properties, just applying Lemma 4.1.3 twice.

Now we may assume, for each i, the existence of x_i' and x_i such that $\deg(x_i', x_i) = 0$ and $\varphi_i(x_i') \neq \varphi_i(x_i)$. If $x_i' \succeq x_i$ and $\deg(\varphi_i(x_i'), \varphi_i(x_i)) \leq -1$, then we pick x_j' and x_j such that $\deg(x_j', x_j) = 0$ and not $\varphi_j(x_j) \geq \varphi_j(x_j')$. By Lemma 4.1.4, there exist increasing mappings $\lambda_i : \Phi_j \to X_i$ and $\lambda_j : \Phi_i \to X_j$ such that $\lambda_i(\varphi_j(x_j')) = x_i$, $\lambda_i(\varphi_j(x_j)) = x_i'$, $\lambda_j(\varphi_i(x_i')) = x_j'$, and $\lambda_j(\varphi_i(x_i)) = x_j$, which clearly contradicts Statement 1.

Thus, we have $\deg(\varphi_i(x_i'), \varphi_i(x_i)) = 0$ whenever $\deg(x_i', x_i) = 0$ (for each i). If there existed x_i', x_i, x_j', and x_j such that $x_i' \succ x_i$, $\varphi_i(x_i') > \varphi_i(x_i)$, $x_j' \succ x_j$, and $\varphi_j(x_j') < \varphi_j(x_j)$, we would apply Lemma 4.1.4 in the same way as in the previous paragraph, obtaining a contradiction with Statement 1.

Suppose the implication $x_i' \succeq x_i \Rightarrow \varphi_i(x_i') \geq \varphi_i(x_i)$ holds for each i. Then we only have to prove the second conjunctive term in (4.5c). Suppose the contrary: there is $d \in \mathcal{D} \setminus \{0\}$

such that

$$\forall i\, \exists x'_i, x_i\, [\deg(x'_i, x_i) = d = \deg(\varphi_i(x'_i), \varphi_i(x_i))]. \tag{4.6}$$

Let d^+ be a maximal $d < 0$ for which (4.6) is satisfied. If $-\infty < d^+ < -1$, we pick a pseudochain $x_i = x_i^0, x_i^1, \ldots, x_i^m = x'_i$ of the minimal length; then $\varphi_i(x_i) = \varphi_i(x_i^0), \varphi_i(x_i^1), \ldots,\ \varphi_i(x_i^m) = \varphi_i(x'_i)$ is also a pseudochain of the minimal length. Obviously, $\deg(x_i^0, x_i^2) \in \{-2_{\mathrm{l}}, -2_{\mathrm{u}}\}$ because otherwise either x_i^1 or x_i^2 could be deleted from the pseudochain. The same consideration works for $\deg(\varphi_i(x_i^0), \varphi_i(x_i^2))$, implying that we could replace x'_i with x_i^2. Therefore, $d^+ \in \{-1, -2_{\mathrm{l}}, -2_{\mathrm{u}}, -\infty\}$. Now we apply Lemma 4.1.3, obtaining increasing mappings $\lambda_i : \Phi_j \to X_i$ and $\lambda_j : \Phi_i \to X_j$ such that $\lambda_i(\varphi_j(x'_j)) = x_i$, $\lambda_i(\varphi_j(x_j)) = x'_i$, $\lambda_j(\varphi_i(x'_i)) = x'_j$, and $\lambda_j(\varphi_i(x_i)) = x_j$, which contradicts Statement 1.

Finally, if both φ_i's are decreasing, then the argument of the previous paragraph, applied to $\langle X_1, X_2^{\mathrm{r}}, \varphi_1, \varphi_2, \Phi_1^{\mathrm{r}}, \Phi_2\rangle$, establishes (4.5d). □

Corollary. *If each X_i contains both a greatest and a least points, then $\langle X_1, X_2, \varphi_1, \varphi_2, \Phi_1, \Phi_2\rangle$ is an acyclic pattern for (multivalued) bilateral reactions if and only if either* (4.5a) *holds, or at least one of Φ_i's is a chain and φ_i's are either both increasing or both decreasing.*

The corollary is applicable, in particular, to complete lattices.

Theorem 5. *Let φ_1 and φ_2 be surjective mappings $\varphi_i : X_i \to \Phi_i$, where each X_i is a proset and each Φ_i is a poset. Then the following statements are equivalent.*

1. *$\langle X_1, X_2, \varphi_1, \varphi_2, \Phi_1, \Phi_2\rangle$ is a universal acyclic pattern for bilateral reactions.*
2. *$\langle X_1, X_2, \varphi_1, \varphi_2, \Phi_1, \Phi_2\rangle$ is a universal acyclic pattern for multivalued bilateral reactions.*
3. *At least one of the following conditions holds:*

$$\exists i\, [\#\Phi_i = 1]; \tag{4.7a}$$

$$\exists i\, \big[\, [\Phi_i \textit{ is a chain}]\ \&\ \forall x'_j, x_j\, [x'_j \succeq x_j \Rightarrow \varphi_j(x'_j) = \varphi_j(x_j)]\,\big]; \tag{4.7b}$$

$$\exists i\, [\,\Phi_i \textit{ is a chain}]\ \&\ \forall i\, \forall x'_i, x_i\, [x'_i \succeq x \Rightarrow \varphi_i(x'_i) \geq \varphi_i(x_i)]; \tag{4.7c}$$

$$\exists i\, [\,\Phi_i \textit{ is a chain}]\ \&\ \forall i\, \forall x'_i, x_i\, [x'_i \succeq x \Rightarrow \varphi_i(x'_i) \leq \varphi_i(x_i)]. \tag{4.7d}$$

Proof. **Sufficiency.** Let us prove that Statement 3 implies Statement 2. If one of conditions (4.7) holds for $\langle X_1, X_2, \varphi_1, \varphi_2, \Phi_1, \Phi_2\rangle$, then the corresponding condition (4.5) obviously holds for every derivative system. Now a reference to the sufficiency part of Proposition 4.3 settles the matter.

Necessity. Let $\langle X_1, X_2, \varphi_1, \varphi_2, \Phi_1, \Phi_2\rangle$ be a universal acyclic pattern for bilateral reactions. We have to prove Statement 3.

If both Φ_i are not chains, i.e., there are $x'_i, x_i \in X_i$ for both i such that $\varphi_i(x'_i)$ and $\varphi_i(x_i)$ are incomparable, then we consider the restriction of our pattern to $X'_i = \{x'_i, x_i\}$. Conditions (4.5a) and (4.5b) do not hold. If the relation $\succeq$ holds for any pair in either X'_i, then the first conjunctive terms in (4.5c) and (4.5d) are also violated; if everything is incomparable in each X'_i, then the second conjunctive terms in (4.5c) and (4.5d) are violated for $d = -\infty$. In either case, we cannot have an acyclic pattern by Proposition 4.3.

Let Φ_i be a chain, but both (4.7a) and (4.7b) not hold: there are $x'_j \succeq x_j$ such that $\varphi_j(x'_j) \neq \varphi_j(x_j)$. We pick $X'_i = X_i$ and $X'_j = \{x'_j, x_j\}$. For the restricted pattern, (4.5a) and (4.5b) cannot hold, hence either (4.5c) or (4.5d) holds. By the way, the second conjunctive term in either condition holds by default since Φ_i is a chain; what matters is that φ_i and φ_j are either both increasing or both decreasing. Since $X'_i = X_i$ and $\#\Phi_i > 1$, the monotonicity must be the same for any other pair $\{x'''_j, x''_j\}$, i.e., we have either (4.7c) or (4.7d). □

4.4 More than two players

Theorem 6. *Let $\#N = 3$ (we always assume that i, j, and k are distinct members of N); let, for each $i \in N$, V_i be an open interval in $\mathbb{R}$ (bounded or not), $V_{-i} = V_j \times V_k$, and $\varphi_i : V_{-i} \to \mathbb{R}$ be continuous and strictly increasing in each argument. Then the following statements are equivalent.*

1. *$\langle V_i, \varphi_i, \varphi_i(V_{-i})\rangle_{i\in N}$ is a universal acyclic pattern for singleton reactions.*
2. *$\langle V_i, \varphi_i, \varphi_i(V_{-i})\rangle_{i\in N}$ is a universal acyclic pattern for multivalued reactions.*
3. *$\langle V_i, \varphi_i, \varphi_i(V_{-i})\rangle_{i\in N}$ is a universal Ω-acyclic pattern for singleton reactions.*
4. *There are continuous and strictly increasing functions $\mu_i : V_i \to \mathbb{R}$ and $\varkappa_i : \big(\mu_j(V_j)+\mu_k(V_k)\big) \to \mathbb{R}$ such that*
$$\varphi_i(v_j, v_k) = \varkappa_i\big(\mu_j(v_j) + \mu_k(v_k)\big) \tag{4.8}$$
for all $i \in N$, $v_j \in V_j$, and $v_k \in V_k$.

Proof. The sufficiency part consists of references. By Theorem 1 from Kukushkin (2004a), Statement 4 implies Statement 2. By Theorem 2 from Kukushkin (2005b), Statement 4 implies Statement 3 (actually, multivalued reactions were also allowed there, but under a stronger monotonicity condition than here). Either obviously implies Statement 1.

Let us prove that Statement 1 implies Statement 4; let $\langle V_i, \varphi_i, \varphi_i(V_{-i})\rangle_{i\in N}$ be a universal acyclic pattern for singleton reactions.

Lemma 4.4.1. *Let $v', v \in V$,*
$$\varphi_i(v'_j, v_k) = \varphi_i(v_j, v'_k) \tag{4.9a}$$
and
$$\varphi_j(v'_i, v_k) = \varphi_j(v_i, v'_k); \tag{4.9b}$$
then
$$\varphi_k(v'_i, v_j) = \varphi_k(v_i, v'_j). \tag{4.9c}$$

Proof. Without restricting generality, $v'_h > v_h$ for all $h \in N$. Suppose the contrary, say, $\varphi_k(v'_i, v_j) > \varphi_k(v_i, v'_j)$. There is $\delta_i > 0$ such that $v'_i - \delta_i > v_i$ and
$$\varphi_k(v'_i - \delta_i, v_j) > \varphi_k(v_i, v'_j). \tag{4.10a}$$
From (4.9b), we have $\varphi_j(v'_i - \delta_i, v_k) < \varphi_j(v_i, v'_k)$; by continuity, we may pick $\delta_k > 0$ such that $v'_k - \delta_k > v_k$ and
$$\varphi_j(v'_i - \delta_i, v_k) < \varphi_j(v_i, v'_k - \delta_k). \tag{4.10b}$$

By monotonicity from (4.9a),

$$\varphi_i(v'_j, v_k) > \varphi_i(v_j, v'_k - \delta_k). \tag{4.10c}$$

Now we define $X'_j = \{v_j, v'_j\}$ and $X_h = \{v_h, v'_h - \delta_h\}$ for $h = i, k$. We also define $\lambda_h : \varphi_h(V_{-h}) \to X_h$, $h \in N$, by $\lambda_i(t) = v_i$ if $t < \varphi_i(v'_j, v_k)$ and $\lambda_i(t) = v'_i - \delta_i$ otherwise; $\lambda_j(t) = v_j$ if $t < \varphi_j(v_i, v'_k - \delta_k)$ and $\lambda_j(t) = v'_j$ otherwise; $\lambda_k(t) = v_k$ if $t < \varphi_k(v'_i - \delta_i, v_j)$ and $\lambda_k(t) = v'_k - \delta_k$ otherwise. It is easy to see that the system defined by the same N, the same X_h and $r_h = \lambda_h \circ \varphi_h$ $(h \in N)$ admits an iteration cycle:

$$\begin{array}{ccccc} (v_i, v'_j, v_k) & \xrightarrow{i} & (v'_i - \delta_i, v'_j, v_k) & \xrightarrow{j} & (v'_i - \delta_i, v_j, v_k) \\ \uparrow k & & & & \downarrow k \\ (v_i, v'_j, v'_k - \delta_k) & \xleftarrow{j} & (v_i, v_j, v'_k - \delta_k) & \xleftarrow{i} & (v'_i - \delta_i, v_j, v'_k - \delta_k) \end{array}$$

□

Actually, Lemma 4.4.1 is sufficient for (4.8). This is not quite obvious, but we do not have to produce the shortest proof. Therefore, we invoke the proof of Theorem 2. Let us assume $\mathrm{A} = N$, $\Upsilon_i = N \setminus \{i\}$, and preferences $\succeq^{\mathrm{a}}_i$ represented by φ_i. There is no obvious way to check the acyclicity of the aggregate improvement relation, but we can argue in an indirect way, checking the Basic Lemmas from Subsection 3.3. Lemmas 3.3.1 and 3.3.2 hold by default. Lemma 4.4.1 is equivalent to Lemma 3.3.4 in this particular case. Let us prove Lemma 3.3.3 in our situation.

Lemma 4.4.2. *Let $v', v \in V$, $v''_i \in V_i$,*

$$\varphi_k(v'_i, v_j) = \varphi_k(v_i, v'_j), \tag{4.11a}$$

$$\varphi_k(v''_i, v_j) = \varphi_k(v'_i, v'_j), \tag{4.11b}$$

and

$$\varphi_j(v'_i, v_k) = \varphi_j(v_i, v'_k); \tag{4.11c}$$

then

$$\varphi_j(v''_i, v_k) = \varphi_j(v'_i, v'_k). \tag{4.11d}$$

Proof. Applying Lemma 4.4.1 to (4.11a) and (4.11c), we obtain

$$\varphi_i(v'_j, v_k) = \varphi_i(v_j, v'_k). \tag{4.12}$$

The same Lemma 4.4.1 applied to (4.12) and (4.11b), gives us (4.11d). □

Now the rest of the proof of Theorem 2, taking into account Proposition 3.6, works without any problem. Since all sets Υ_i overlap, Gorman additivity means exactly (4.8). □

Example 4.3. Let $N = \{1, 2, 3, 4\}$, $V_i = \mathbb{R}$ for all $i \in N$, and the functions be as follows:

$$\begin{aligned} \varphi_1(v_{-1}) &= 2v_2 + v_3 + v_4; \\ \varphi_2(v_{-2}) &= 2v_1 + v_3 + v_4; \\ \varphi_3(v_{-3}) &= v_1 + v_2 + v_4; \\ \varphi_4(v_{-4}) &= v_1 + v_2 + v_3. \end{aligned}$$

Each function φ_i is continuous and properly monotonic; they form a universal acyclic pattern by Theorem 2 from Kukushkin (2005b). (It should be noted that the proof was based on a trick invented by Huang, 2002, for the study of fictitious play, and used by Dubey et al., 2006, to produce "pseudo-potentials"). Let us show that the functions cannot be represented in the form (4.8). Supposing the contrary, we notice that $\varphi_1(1,0,0) = \varphi_1(0,1,1)$, hence $\mu_2(1) + \mu_3(0) + \mu_4(0) = \mu_2(0) + \mu_3(1) + \mu_4(1)$. Similarly, $\varphi_3(1,1,0) = \varphi_3(1,0,1)$, hence $\mu_1(1) + \mu_2(1) + \mu_4(0) = \mu_1(1) + \mu_2(0) + \mu_4(1)$. Adding $\mu_1(1)$ to both sides of the first equality and $\mu_3(1)$ to the second equality, we obtain $\mu_1(1)+\mu_2(1)+\mu_3(0)+\mu_4(0) = \mu_1(1)+\mu_2(0)+\mu_3(1)+\mu_4(1) = \mu_1(1)+\mu_2(1)+\mu_3(1)+\mu_4(0)$. Now $\mu_3(0) = \mu_3(1)$, contradicting the assumed strong monotonicity of $\mu_3(\cdot)$.

Theorem 7. *Let $\#N = 3$; let, for each $i \in N$, V_i be an open interval in $\mathbb{R}$ (bounded or not), $V_{-i} = V_j \times V_k$, and $\varphi_i : V_{-i} \to \mathbb{R}$ be continuous and strictly decreasing in each argument. Then the following statements are equivalent.*

1. *Every derivative system of $\langle V_i, \varphi_i, \varphi_i(V_{-i})\rangle_{i\in N}$ where each player has at most two strategies admits a fixed point.*
2. *$\langle V_i, \varphi_i, \varphi_i(V_{-i})\rangle_{i\in N}$ is a universal acyclic pattern for singleton reactions.*
3. *$\langle V_i, \varphi_i, \varphi_i(V_{-i})\rangle_{i\in N}$ is a universal Ω-acyclic pattern for singleton reactions.*
4. *There are continuous and strictly increasing functions $\mu_i : V_i \to \mathbb{R}$ and $\varkappa_i : \big(\mu_j(V_j)+\mu_k(V_k)\big) \to \mathbb{R}$ such that*
$$\varphi_i(v_j, v_k) = -\varkappa_i\big(\mu_j(v_j) + \mu_k(v_k)\big) \tag{4.13}$$
for all $i \in N$, $v_j \in V_j$, and $v_k \in V_k$.

Proof. The implications Statement 3 ⇒ Statement 2 ⇒ Statement 1 are straightforward. The implication Statement 4 ⇒ Statement 3 follows from Theorem 2 of Kukushkin (2005b) (again, there were multivalued reactions there under a stronger monotonicity condition, and Example 4 of the same paper showed that it could not be weakened to that of this paper).

Let us prove that Statement 1 implies Statement 4.

Lemma 4.4.3. *Let $v', v \in V$,*
$$\varphi_i(v'_j, v_k) = \varphi_i(v_j, v'_k) \tag{4.14a}$$
and
$$\varphi_j(v'_i, v_k) = \varphi_j(v_i, v'_k); \tag{4.14b}$$
then
$$\varphi_k(v'_i, v_j) = \varphi_k(v_i, v'_j). \tag{4.14c}$$

Proof. Without restricting generality, $v'_h > v_h$ for all $h \in N$. Suppose the contrary, say, $\varphi_k(v'_i, v_j) > \varphi_k(v_i, v'_j)$. There is $\delta_i > 0$ such that
$$\varphi_k(v'_i + \delta_i, v_j) > \varphi_k(v_i, v'_j). \tag{4.15a}$$
From (4.14b), we have $\varphi_j(v'_i + \delta_i, v_k) < \varphi_j(v_i, v'_k)$; by continuity, we may pick $\delta_k > 0$ such that
$$\varphi_j(v'_i + \delta_i, v_k) < \varphi_j(v_i, v'_k + \delta_k). \tag{4.15b}$$

By monotonicity from (4.14a),

$$\varphi_i(v'_j, v_k) > \varphi_i(v_j, v'_k + \delta_k). \tag{4.15c}$$

Now we define $X'_j = \{v_j, v'_j\}$ and $X_h = \{v_h, v'_h + \delta_h\}$ for $h = i, k$. We also define $\lambda_h : \varphi_h(V_{-h}) \to X_h$, $h \in N$, by $\lambda_i(t) = v_i$ if $t < \varphi_i(v'_j, v_k)$ and $\lambda_i(t) = v'_i + \delta_i$ otherwise; $\lambda_j(t) = v_j$ if $t < \varphi_j(v_i, v'_k + \delta_k)$ and $\lambda_j(t) = v'_j$ otherwise; $\lambda_k(t) = v_k$ if $t < \varphi_k(v'_i + \delta_i, v_j)$ and $\lambda_k(t) = v'_k + \delta_k$ otherwise.

Suppose that (u_i, u_j, u_k) is a fixed point of the derivative system of reactions. If $u_i = v_i$, then $\varphi_i(u_j, u_k) < \varphi_i(v'_j, v_k)$, hence $u_k = v'_k + \delta_k$, hence $\varphi_k(u_i, u_j) \geq \varphi_k(v'_i + \delta_i, v_j)$, hence $u_j = v_j$. However, $\lambda_j \circ \varphi_j(u_i, u_k) = \lambda_j \circ \varphi_j(v_i, v'_k + \delta_k) = v'_j \neq u_j$. Quite similarly, if $u_i = v'_i + \delta_i$, then $u_k = v_k$, hence $u_j = v'_j \neq \lambda_j \circ \varphi_j(u_i, u_k)$. □

Replacing each φ_i with $-\varphi_i$, we see that the rest of the proof of Theorem 6 can run without a hitch, establishing (4.13). □

Remark. An analogue of Statement 1 could not have been added to the formulation of Theorem 6: every system of increasing reactions with finite chains as the strategy sets admits a fixed point by Tarski's (1955) fixed point theorem.

Changing all the signs in Example 4.3, we see that an analogue of Theorem 7 for $\#N > 3$ is wrong.

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