Improvement paths in strategic games: A topological approach

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Abstract

An abstract theory of improvement dynamics for binary relations in metric spaces is developed, providing a general framework for studying various improvement relations and tâtonnement processes in strategic games. Special attention is paid to interrelationships between finite, infinite, and transfinite tâtonnement paths.

Key words: binary relation; improvement dynamics; acyclicity; equilibrium

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1 Introduction

Quite often, solution concepts in game theory are defined through the absence of "objections" or "blocking." For instance, Nash equilibrium assumes the absence of profitable individual deviations. A dynamic scenario of consecutively putting forward and implementing such objections naturally emerges in such situations, and a question arises of whether such a process leads to an equilibrium.

A positive answer to the question ensures the existence of a solution. But we then have much more than pure existence: we have a natural procedure leading to a solution from any initial point; we have reasonable grounds to believe that a solution will be adopted by the players themselves.

The study of such "tâtonnement" processes can be traced back to Cournot (1838). Similar research in various contexts was done by, e.g., Topkis (1979), Bernheim (1984), Moulin (1984), Vives (1990), Milgrom and Roberts (1990), Kandori and Rob (1995), Monderer and Shapley (1996a), and Milchtaich (1996). Numerous classes of strategic games have been found where consecutive improvements (in one sense or another) lead to equilibria. This paper strives to push forward an approach started in Kukushkin (2000), which is characterized by the following principal features.

First, we work in a purely ordinal framework. Mixed extensions of (finite or infinite) games can be included, naturally, by specific assumptions about strategy sets (simplexes) and preferences (polylinear utilities). However, those assumptions are not conducive to good behavior of our processes; a different kind of adaptation is much more promising under them (Monderer and Shapley, 1996b; Huang, 2002).

Second, we try to treat various notions of equilibrium and improvement (Nash or strong equilibrium; best response or better response dynamics) as uniformly as possible. Accordingly, considerable attention is paid to abstract binary relations without any reference to games as such.

Thirdly, we embrace tâtonnement paths parameterized with transfinite numbers. Whenever an infinite number of steps have been made, a limit point can be taken and, if the point is still not a maximizer (equilibrium), the process can continue further. The main justification is that transfinite paths seem to allow a simpler theory. If we cannot reach an equilibrium with a finite path in a *finite* game, there would be no point in going beyond natural numbers. However, in an infinite game, a failure to reach an equilibrium in the first limit does not mean that the situation is hopeless. If one accepts the idea of actual infinity, why not accept transfinite convergence as well? All hope is only lost when the whole range of countable ordinals has been exhausted.

Since Kukushkin (2000) is not available on the Net (there is no pdf version), all results from that paper quoted here are given complete proofs. There have been considerable changes in terminology, but they are not discussed here. A number of results are quoted from Kukushkin (2003); since that paper *is* available on the Net, their proofs are not reproduced.

Two features distinguish this paper from its predecessors. First, we describe preferences by binary relations (without any *a priori* restrictions) rather than by utility functions. This fact allows us, in particular, to include ε-improvement and ε-equilibria into the same general scheme. Second, we pay considerable attention to conditions under which transfinite paths become redundant, in one sense or another; typically, continuity of some sort is needed.

In Section 2, the basic definitions are given: improvement relations in a strategic game; finite tâtonnement paths of an abstract binary relation. Section 3 is about tâtonnement paths of abstract binary relations on a metric space. First, we consider the possibility to approximate a maximizer with a finite tâtonnement path; then we proceed to paths parameterized with countable transfinite numbers. The last Subsection 3.5 deals with relations associated with endomorphisms, i.e., mappings (or correspondences) from a set to itself.

Section 4 starts with a list of properties of strategic games defined in terms of improvement relations. Then we study how, and to what extent, (quasi)continuity assumptions may help to dispense with transfinite (individual or coalition) improvement paths. Subsection 4.3 deals with specific properties of best response dynamics.

The last two sections consider improvement dynamics in strategic games with certain structural properties: dominance solvability in Section 5 and strategic complementarities in Section 6.

2 Basic notions

2.1 Preliminaries

A *binary relation* on a set X is a Boolean function on $X \times X$. As usual, we write $y \rhd x$ when the relation $\rhd$ is true on a pair (y, x) and $y \not\rhd x$ when it is false; we also write $z \rhd y \rhd x$ instead of $z \rhd y$ and $y \rhd x$. Given a binary relation $\rhd$ on a set X and $Y \subseteq X$, we denote

$$M(Y, \rhd) := \{x \in Y \mid \nexists y \in Y\, [y \rhd x]\}, \tag{2.1}$$

the set of *maximizers* of $\rhd$ on Y. Given $x \in X$, we denote $\mathrm{Up}(x, \rhd) := \{y \in X \mid y \rhd x\}$ and $\mathrm{Lo}(x, \rhd) := \{y \in X \mid x \rhd y\}$ – upper and lower contours of $\rhd$. Clearly, $x \in M(Y, \rhd) \iff \mathrm{Up}(x, \rhd) \cap Y = \emptyset$. A relation $\succ$ on X is an *extension* of $\rhd$ if

$$\forall x, y \in X\, [y \rhd x \Rightarrow y \succ x]. \tag{2.2}$$

The most popular in mathematical literature seem to be order relations. A *preorder* is a reflexive and transitive binary relation. If $\succeq$ is a preorder, then its *asymmetric component*, $y \succ x \rightleftharpoons y \succeq x \;\&\; x \not\succeq y$ is irreflexive and transitive ("*strict order*"); its *symmetric component*, $y \sim x \rightleftharpoons y \succeq x \;\&\; x \succeq y$ is an equivalence relation. If a preorder $\succeq$ is *total*, i.e., either $y \succeq x$ or $x \succeq y$ holds for all $x, y \in X$, its asymmetric component is called an *ordering*; equivalently, $\succ$ is an ordering if it is irreflexive, transitive, and *negatively* *transitive* ($z \not\succ y \not\succ x \Rightarrow z \not\succ x$). A *linear order* is an ordering such that every two distinct points are comparable.

An antisymmetric preorder ($y \sim x \Rightarrow y = x$) is called a *partial order.* A set with a given partial order is called a *poset*; when the order is linear, the poset is called a *chain.* A poset is *well ordered* if every subset contains a least point (then the poset obviously must be a chain). When X is a poset and $a, b \in X$, we use notations $[a, b] := \{x \in X \mid a \leq x \leq b\}$, $[a, b[:= \{x \in X \mid a \leq x < b\}$, etc.

The Axiom of Choice, which we adopt throughout, implies the Zermelo Theorem: Every set can be well ordered. Therefore, a well ordered set of an arbitrary cardinality can be found whenever needed. Considering any well ordered set Λ, we denote 0 the least point of the whole Λ. Given $\alpha \in \Lambda \setminus \{\max \Lambda\}$, we define the *successor* of α as $\alpha + 1 := \min\{\beta \in \Lambda \mid \beta > \alpha\}$. $\alpha \in \Lambda \setminus \{0\}$ is called a *limit* if it is not a successor to any $\beta \in \Lambda$.

Besides finite chains, these three well ordered sets are most important for us here: the chain $\mathbb{N}$ of natural numbers (starting from 0) with the standard order; the chain $\mathbb{N} \cup \{\omega\}$ with the standard order on $\mathbb{N}$ and $\omega > k$ for each $k \in \mathbb{N}$; the "chain of all countable ordinals," denoted Ω. Technically, Ω can be defined (up to isomorphism) as an uncountable well ordered set such that every $[0, \alpha]$ ($\alpha \in \Omega$) is countable. Natanson (1974, Chapter XIV), can be used as a reference book. Readers unfamiliar with the concept and suspicious of it may be reassured by the fact that every $[0, \alpha]$ ($\alpha \in \Omega$) can be interpreted as a subset of rational numbers with their natural order (Natanson, 1974, Chapter XIV, Section 1, Theorem 4).

It is convenient to assume $\mathbb{N} \cup \{\omega\} \subset \Omega$. Then ω is the least limit in Ω; ω and greater ordinals are called *transfinite numbers.* Every countable subset of Ω has a least upper bound in Ω (Natanson, 1974, Chapter XIV, Section 5, Theorem 2). Every limit $\alpha \in \Omega$ is the least upper bound of a strictly increasing infinite sequence in Ω (Natanson, 1974, Chapter XIV, Section 5, Theorem 4).

2.2 Strategic games

Our basic model is a *strategic game with ordinal preferences.* It is defined by a finite set of players N (we denote $n = \#N$), and strategy sets X_i and *preference relations* $\succ_i$ on $X_N := \prod_{i \in N} X_i$ for all $i \in N$. We denote $\mathcal{N} := 2^N \setminus \{\emptyset\}$ (the set of potential coalitions) and $X_I := \prod_{i \in I} X_i$ for each $I \in \mathcal{N}$; instead of $X_{N \setminus \{i\}}$ and $X_{N \setminus I}$, we write X_{-i} and X_{-I}, respectively. If $n = 2$, then $-i$ refers to the partner/rival of player i.

Each $\succ_i$ is interpreted as *strict preference. Weak preference* relation $\succcurlyeq_i$ may be defined by $y_N \succcurlyeq_i x_N \rightleftharpoons x_N \not\succ_i y_N$; however, we invoke that relation only when $\succ_i$ is an ordering, hence $\succcurlyeq_i$ is a total preorder. This happens, in particular, when the preferences are described by a *utility function* $u_i \colon X_N \to \mathbb{R}$, i.e.,

$$y_N \succ_i x_N \rightleftharpoons u_i(y_N) > u_i(x_N); \tag{2.3}$$

then $y_N \succcurlyeq_i x_N \iff u_i(y_N) \geq u_i(x_N)$. Natural examples of preference relations that are *not* orderings are provided by ε-*dominance*,

$$y_N \succ\!\!\succ_i x_N \rightleftharpoons u_i(y_N) > u_i(x_N) + \varepsilon \quad (\varepsilon > 0), \tag{2.4}$$

or *Pareto dominance*

$$y_N \succ\!\!\succ_i x_N \rightleftharpoons \forall \alpha \in \mathrm{A}\,[u_i^\alpha(y_N) \geq u_i^\alpha(x_N)] \ \&\ \exists \alpha \in \mathrm{A}\,[u_i^\alpha(y_N) > u_i^\alpha(x_N)]. \tag{2.5}$$

With every strategic game, a number of improvement relations on X_N are associated ($i \in N$, $I \in \mathcal{N}$, y_N, $x_N \in X_N$):

$$y_N \triangleright_i^{\mathrm{Ind}} x_N \rightleftharpoons [y_{-i} = x_{-i} \ \&\ y_N \succ\!\!\succ_i x_N]; \tag{2.6a}$$

$$y_N \triangleright^{\mathrm{Ind}} x_N \rightleftharpoons \exists i \in N\,[y_N \triangleright_i^{\mathrm{Ind}} x_N] \tag{2.6b}$$

(*individual improvement* relation);

$$y_N \triangleright_I^{\mathrm{sCo}} x_N \rightleftharpoons \big[y_{-I} = x_{-I} \ \&\ \forall i \in I\,[y_N \succ\!\!\succ_i x_N]\big]; \tag{2.7a}$$

$$y_N \triangleright^{\mathrm{sCo}} x_N \rightleftharpoons \exists I \in \mathcal{N}\,[y_N \triangleright_I^{\mathrm{sCo}} x_N] \tag{2.7b}$$

(*strict coalition improvement* relation);

$$y_N \triangleright_I^{\mathrm{wCo}} x_N \rightleftharpoons \big[y_{-I} = x_{-I} \ \&\ \exists i \in I\,[y_N \succ\!\!\succ_i x_N] \ \&\ \nexists i \in I\,[x_N \succ\!\!\succ_i y_N]\big]; \tag{2.8a}$$

$$y_N \triangleright^{\mathrm{wCo}} x_N \rightleftharpoons \exists I \in \mathcal{N}\,[y_N \triangleright_I^{\mathrm{wCo}} x_N] \tag{2.8b}$$

(*weak coalition improvement* relation).

It is often convenient to speak of just "an improvement relation" $\triangleright$ without specifying which of the above-defined relations is meant. A maximizer of an improvement relation $\triangleright$ is an equilibrium: a Nash equilibrium if $\triangleright$ is $\triangleright^{\mathrm{Ind}}$; a ("very") strong equilibrium if $\triangleright$ is $\triangleright^{\mathrm{sCo}}$ ($\triangleright^{\mathrm{wCo}}$). A strategy profile $x_N \in X_N$ is a strong (weak) Pareto optimum if and only if it is a maximizer of $\triangleright_N^{\mathrm{wCo}}$ ($\triangleright_N^{\mathrm{sCo}}$).

A *subgame* Γ' of Γ is a strategic game with the same set of players N, nonempty subsets $\emptyset \neq X'_i \subseteq X_i$ for all $i \in N$, and the restrictions of the same preference relations to $X'_N = \prod_{i\in N} X'_i$.

Proposition 2.1. *Let Γ' be a subgame of Γ. Then each improvement relation $\triangleright^{\mathrm{Ind}}$, $\triangleright^{\mathrm{sCo}}$, or $\triangleright^{\mathrm{wCo}}$ in Γ' coincides with the restriction of the appropriate relation in Γ to X'_N.*

Proof. Straightforward. □

It is often convenient to have a specific notation, $\succ\!\!\succ_i^{x_{-i}}$, for the projection to X_i of the restriction of $\succ\!\!\succ_i$ to $X_i \times \{x_{-i}\}$. Defining the *best response correspondence* $\mathcal{R}_i \colon X_{-i} \to 2^{X_i}$ for each $i \in N$ in the usual way,

$$\mathcal{R}_i(x_{-i}) := M(X_i, \succ\!\!\succ_i^{x_{-i}}),$$

we may introduce two more relations:

$$y_N \rhd_i^{\mathrm{BR}} x_N \rightleftharpoons [y_N \rhd^{\mathrm{Ind}} x_N \;\&\; y_i \in \mathcal{R}_i(x_{-i})]; \tag{2.9a}$$

$$y_N \rhd^{\mathrm{BR}} x_N \rightleftharpoons \exists i \in N\, [y_N \rhd_i^{\mathrm{BR}} x_N] \tag{2.9b}$$

(*best response improvement* relation);

$$y_N \rhd^{\mathrm{sBR}} x_N \rightleftharpoons \big[y_N \neq x_N \;\&\; \forall i \in N\, [y_i = x_i \in \mathcal{R}_i(x_{-i}) \text{ or } (y_i, x_{-i}) \rhd_i^{\mathrm{BR}} x_N]\big] \tag{2.10}$$

(*simultaneous best response* relation). There is a principal difference between the last relation and all others: $y_N \rhd^{\mathrm{sBR}} x_N$ is compatible with $x_N \succ_i y_N$ for all $i \in N$.

Every Nash equilibrium is a maximizer of both $\rhd^{\mathrm{BR}}$ and $\rhd^{\mathrm{sBR}}$. We call a game *BR-consistent* if, for every $x_N \in X_N$ and $i \in N$, either $x_i \in \mathcal{R}_i(x_{-i})$ or there is $y_i \in \mathcal{R}_i(x_{-i})$ such that $y_i \succ_i^{x_{-i}} x_i$. The property implies that $\mathcal{R}_i(x_{-i}) \neq \emptyset$ for all $i \in N$ and $x_{-i} \in X_{-i}$, but is stronger than that. In a BR-consistent game, every maximizer of $\rhd^{\mathrm{BR}}$ or $\rhd^{\mathrm{sBR}}$ is a Nash equilibrium.

A game is called *strongly BR-consistent* if, for each $i \in N$, $\mathcal{R}_i(x_{-i}) \neq \emptyset$ for all $x_{-i} \in X_{-i}$, and there holds $y_i \succ_i^{x_{-i}} x_i$ whenever $x_N \in X_N$ and $x_i \notin \mathcal{R}_i(x_{-i}) \ni y_i$. Evidently, a strongly BR-consistent game is BR-consistent. In a strongly BR-consistent game, (2.9a) can be replaced with

$$y_N \rhd_i^{\mathrm{BR}} x_N \rightleftharpoons [y_{-i} = x_{-i} \;\&\; x_i \notin \mathcal{R}_i(x_{-i}) \ni y_i], \tag{2.9a$'$}$$

while (2.10) with

$$y_N \rhd^{\mathrm{sBR}} x_N \rightleftharpoons \big[y_N \neq x_N \;\&\; \forall i \in N\, [y_i = x_i \in \mathcal{R}_i(x_{-i}) \text{ or } x_i \notin \mathcal{R}_i(x_{-i}) \ni y_i]\big]. \tag{2.10$'$}$$

When the preferences of the players are defined by semiorders, e.g., by ε-dominance (2.4), the existence of the best responses (which is automatically ensured in a finite game) is equivalent to BR-consistency. When the preferences are defined by orderings, e.g., by utility functions, the existence of the best responses is equivalent to strong BR-consistency. If the game is not even BR-consistent, the relations $\rhd^{\mathrm{BR}}$ and $\rhd^{\mathrm{sBR}}$ provide a dubious framework for studying adaptive dynamics.

2.3 Finite tâtonnement paths

Let $\rhd$ be a binary relation on a set X. Having in mind improvement relations from the previous subsection, we interpret $y \rhd x$ as an indication that somebody prefers y to x and is able to replace x with y. A dynamic process (or rather, a family of potential dynamic processes) suggests itself.

A *simple tâtonnement path* of $\rhd$ is a (finite or infinite) sequence $\langle x^k \rangle_{k=0,1,\ldots}$ such that $x^{k+1} \rhd x^k$ whenever $k \geq 0$ and x^{k+1} is defined. If x^k is defined exactly for $k = 0, \ldots, m$, we call $m \geq 0$ the *length of the path* (thus a path of length 0 is just a single point). A *finite tâtonnement cycle* is a finite tâtonnement path of length $m > 0$ for which $x^0 = x^m$.

A relation is *acyclic* if it admits no finite tâtonnement cycle. Obviously, $\rhd$ is acyclic if and only if its transitive closure is irreflexive.

A relation $\rhd$ is *strictly acyclic* if it admits no infinite tâtonnement path; as a synonym, we sometimes say that $\rhd$ has the *finite tâtonnement property* (*FTP*). The property implies that every tâtonnement path, if continued whenever possible, reaches a maximizer in a finite number of steps. A relation $\rhd$ is *weakly acyclic*, or has the *weak FTP*, if every $x \in X$ is connected to a maximizer of $\rhd$ with a tâtonnement path, i.e., there is a finite tâtonnement path $\langle x^0, \ldots, x^m \rangle$ ($m \geq 0$) such that $x^0 = x$ and $x^m \in M(X, \rhd)$. $\rhd$ has the *von Neumann-Morgenstern* (*NM*) property if the previous condition holds with $m \leq 1$; this actually means that the set $M(X, \rhd)$ is NM-stable (hence the unique NM-solution). Either of NM and FTP implies the weak FTP; generally, they do not imply each other. Each of the three properties implies the existence of a maximizer of $\rhd$.

The weak FTP does not exclude the possibility that a tâtonnement process may continue indefinitely without reaching a rest point. When X is finite, however, this is improbable under reasonable assumptions. Let a (time-homogeneous) Markov chain on X be defined by transition probabilities $\Pi_{xx'}$ such that $\left[\Pi_{xx'} > 0 \Rightarrow [x' = x \text{ or } x' \rhd x]\right]$ and $\left[x' \rhd x \Rightarrow \Pi_{xx'} > 0\right]$. Once the process reaches a maximizer, it stays there forever. The weak FTP of $\rhd$ implies that every non-maximizer is a transient state, hence the tâtonnement process will end at a maximizer with the probability one. If X is not finite, such a strong statement cannot be justified, but the property still deserves attention.

Regardless of dynamic interpretations, both strict and weak acyclicity are relevant to the existence of maximizers.

Proposition 2.2. *A binary relation $\rhd$ on a set X has the property that $M(Y, \rhd) \neq \emptyset$ for every nonempty subset $Y \subseteq X$ if and only if $\rhd$ is strictly acyclic.*

Proof. An infinite tâtonnement path would be a subset of X without a maximizer. Conversely, the absence of maximizers on $Y \subset X$ would allow us to construct an infinite tâtonnement path of $\rhd$. ∎

A subset $Y \subseteq X$ is *undominated* if $y \in Y$ whenever $y \in X$, $y \rhd x$, and $x \in Y$. Whenever $x \in Y$ is a maximizer of $\rhd$ on an undominated Y, it is a maximizer of $\rhd$ on X. Note that X itself is undominated by definition.

Proposition 2.3. *A binary relation $\rhd$ on a set X has the property that $M(Y, \rhd) \neq \emptyset$ for every nonempty undominated subset $Y \subseteq X$ if and only if $\rhd$ is weakly acyclic.*

Proof. If $\rhd$ is not weakly acyclic, then there is $x^0 \in X$ such that no finite tâtonnement path starting at x^0 reaches a maximizer of $\rhd$ on X. We denote $Y \subseteq X$ the union of all finite tâtonnement paths starting at x^0; clearly, Y is undominated. On the other hand, $M(Y, \rhd) = \emptyset$ by the choice of x^0.

Conversely, let $\rhd$ be weakly acyclic, $x^0 \in Y \subseteq X$, and Y be undominated. Let $x^0, x^1, \ldots, x^m$ be a tâtonnement path and x^m be a maximizer of $\rhd$ on X. A straightforward inductive reasoning shows that $x^m \in Y$, hence $x^m \in M(Y, \rhd)$. ∎

The difference between strict and weak acyclicity disappears if we demand either property to hold on every finite subset.

Proposition 2.4. *Let $\rhd$ be a binary relation on a set X. Then the following statements are equivalent:*

$$\rhd \text{ is strictly acyclic on every finite subset } Y \subseteq X; \tag{2.11a}$$

$$\rhd \text{ has the weak FTP on every nonempty finite subset } Y \subseteq X; \tag{2.11b}$$

$$M(Y, \rhd) \neq \emptyset \text{ for every nonempty finite subset } Y \subseteq X; \tag{2.11c}$$

$$\rhd \text{ is acyclic on } X. \tag{2.11d}$$

Proof. Straightforward. □

Remark. If X itself is not finite, the strict acyclicity on X cannot be added to the list: consider, e.g., the standard order on $\mathbb{R}$.

A *strict order potential* of $\rhd$ is a strictly acyclic and transitive extension, in the sense of (2.2), of $\rhd$. A *strict numeric potential* of $\rhd$ is a function $P \colon X \to (-\mathbb{N})$ such that

$$\forall x, y \in X \left[y \rhd x \Rightarrow P(y) > P(x) \right]. \tag{2.12}$$

Proposition 2.5. *A binary relation is strictly acyclic if and only if it admits a strict order potential. If a binary relation $\rhd$ admits a strict numeric potential, then it is strictly acyclic.*

Proof. If $\succ$ is a strict order potential of $\rhd$, then (2.2) implies that every tâtonnement path of $\rhd$ is a tâtonnement path of $\succ$; therefore, a relation admitting a strict order potential has the FTP. Conversely, if $\rhd$ is strictly acyclic, then we denote $\succ$ its transitive closure; clearly, $\succ$ is strictly acyclic as well. The second statement is straightforward. □

Remark. On a finite set, every strictly acyclic relation admits a strict numeric potential (Kukushkin, 2004, Proposition 6.1). On the other hand, if we consider a well-order on an uncountable set, say, $[0, 1]$, and define $\rhd$ as its reverse, then no real-valued function P could satisfy (2.12).

Proposition 2.6 ("Szpilrajn's Theorem for strictly acyclic relations"). *Let $\rhd$ be a strictly acyclic binary relation on a set X. Then it can be extended to a strictly acyclic linear order on X.*

Proof. Let Λ be a well ordered set of a cardinality greater than that of X. By (transfinite) recursion, we construct a mapping $\lambda \colon \Lambda \to X \cup \{x^*\}$, where $x^* \notin X$. First, we pick $\lambda(0) \in M(X, \rhd)$. Having $\lambda(\beta)$ defined for all $\beta < \alpha \in \Lambda$, we denote $X(\alpha) := X \setminus \{\lambda(\beta)\}_{\beta<\alpha}$. If $X(\alpha) = \emptyset$, we set $\lambda(\alpha) := x^*$; otherwise, we pick $\lambda(\alpha) \in M(X(\alpha), \rhd)$ arbitrarily.

The assumption about the cardinality of Λ ensures that $\lambda(\Lambda) \supset X$; moreover, $\lambda^{-1}(x)$ is a singleton for every $x \in X$. Now we define a relation $\succ$ on X by: $y \succ x \rightleftharpoons \lambda^{-1}(y) < \lambda^{-1}(x)$. Clearly, $\succ$ is a strictly acyclic linear order on X, and $y \rhd x \Rightarrow y \succ x$. □

Corollary. *A binary relation $\rhd$ on a set X is strictly acyclic if and only if there are a well ordered set Λ and an injective mapping $\mu\colon X \to \Lambda$ such that $[y \rhd x \Rightarrow \mu(y) < \mu(x)]$ for all $y, x \in X$.*

A *weak order potential* of $\rhd$ is a strictly acyclic and transitive binary relation $\succ$ on X such that

$$\forall x \in X \left[\exists y \in X\, [y \rhd x] \Rightarrow \exists z \in X\, [z \rhd x \;\&\; z \succ x]\right]. \tag{2.13}$$

A *weak numeric potential* of $\rhd$ is a function $P\colon X \to (-\mathbb{N})$ such that

$$\forall x \in X \left[\exists y \in X\, [y \rhd x] \Rightarrow \exists z \in X\, [z \rhd x \;\&\; P(z) > P(x)]\right].$$

Proposition 2.7. *For any binary relation $\rhd$ on a set X, the following statements are equivalent:*

$$\rhd \text{ admits a weak numeric potential on } X; \tag{2.14a}$$

$$\rhd \text{ admits a weak order potential on } X; \tag{2.14b}$$

$$\rhd \text{ has the weak FTP on } X. \tag{2.14c}$$

Proof. (2.14a) obviously implies (2.14b). Assuming that (2.14b) holds, we define the following rule for the extension of tâtonnement paths: if x^k is not a maximizer, we pick x^{k+1} such that $x^{k+1} \rhd x^k$ and $x^{k+1} \succ x^k$. Starting with an arbitrary x^0 and applying the rule, we obtain a tâtonnement path of both $\rhd$ and $\succ$, which must stop after a finite number of steps, hence reaches a maximizer of both $\rhd$ and $\succ$. Therefore, we have the weak FTP indeed.

Assuming that (2.14c) holds, we define $p(x) :=$ [the minimal length of a tâtonnement path of $\rhd$ starting at x and ending at a maximizer] and $P(x) := -p(x)$ for every $x \in X$. Now $y \rhd x$ implies $p(x) > 0$; we pick a tâtonnement path $\langle x^k \rangle_{k=0,\dots,p(x)}$ such that $x^0 = x$ and $x^{p(x)}$ is a maximizer, and define $z := x^1$. Clearly, $p(z) = p(x) - 1$, hence $P(z) > P(x)$; on the other hand, $z = x^1 \rhd x^0 = x$. □

Remark. Proposition 2.7 for finite X was obtained in Kukushkin (2004, Proposition 6.2).

Proposition 2.8. *A binary relation $\succ$ on X has the NM property on every finite subset $Y \subseteq X$ if and only if $\succ$ is irreflexive and transitive on X.*

Proposition 2.9 (Kukushkin, 2008b, Theorem 2). *A binary relation $\succ$ on a set X has the NM property on every subset $Y \subseteq X$ if and only if $\succ$ is strictly acyclic and transitive on X.*

Corollary. *A strategic game Γ has the property that every subgame of Γ is BR-consistent if and only if every relation $\succ\!\!\succ_i^{x_{-i}}$ ($i \in N$; $x_{-i} \in X_{-i}$) is strictly acyclic and transitive on X_i.*

Proposition 2.10. *A strategic game Γ has the property that every subgame of Γ is strongly BR-consistent if and only if every relation $\succ\!\!\succ_i^{x_{-i}}$ ($i \in N$; $x_{-i} \in X_{-i}$) is a strictly acyclic ordering on X_i.*

3 Binary relations on a metric space

Throughout most of the paper, X is assumed a metric space with the metric d. Quite often, an assumption that X is a first countable Hausdorff topological space would be sufficient: as is well known, topology on X is then adequately described by convergent sequences. However, the availability of distance simplifies presentation everywhere and may be indispensable for some results.

3.1 Infinite tâtonnement paths

A relation $\rhd$ has the *very weak FTP* if, whenever $x^0 \in X$, there is $y \in M(X, \rhd)$ such that for every $\varepsilon > 0$ there is a finite tâtonnement path $x^0, x^1, \ldots, x^m$ for which $d(x^m, y) < \varepsilon$. An infinite tâtonnement path $\langle x^k \rangle_{k \in \mathbb{N}}$ is *maximal* if its set of limit points is a nonempty subset of $M(X, \rhd)$. A relation $\rhd$ has the *approximate FTP* if every infinite tâtonnement path of $\rhd$ is maximal. A relation $\rhd$ has the *weak approximate FTP* if for every $x^0 \in X$, there is either a finite tâtonnement path $x^0, x^1, \ldots, x^m$ such that $x^m \in M(X, \rhd)$, or a maximal tâtonnement path starting at x^0.

Proposition 3.1. *For every binary relation on a metric space, these implications hold:*

$$\begin{array}{ccccc} \textit{FTP} & \Rightarrow & \textit{approximate FTP} & \Rightarrow & \textit{acyclicity} \\ \Downarrow & & \Downarrow & & \\ \textit{weak FTP} & \Rightarrow & \textit{weak approximate FTP} & \Rightarrow & \textit{very weak FTP.} \end{array}$$

Proof. Straightforward. □

None of the implications can be reversed: the standard order $>$ on $X := \{-1/(k+1)\}_{k \in \mathbb{N}} \cup \{0\} \subset \mathbb{R}$ has the approximate FTP, but not the FTP; the same order on $X := [0, 1]$ is acyclic and has the weak approximate FTP, but does not have the approximate FTP; the interval order in Example 3 of Kukushkin (2008b) has the weak approximate FTP, but not the weak FTP; Example 3.35 below presents a relation having the very weak FTP, but not the weak approximate FTP.

Proposition 3.2. *A binary relation $\rhd$ on a metric space X has the very weak FTP if and only if $M(X, \rhd) \cap Y \neq \emptyset$ whenever Y is the closure of a nonempty undominated subset of X.*

Proof. Straightforward. □

An *approximate potential* of $\rhd$ is an irreflexive and transitive binary relation $\succ$ satisfying (2.2) on X and strictly acyclic on every set

$$X(\delta) := \{x \in X \mid \forall x' \in X \left[d(x, x') < \delta \Rightarrow \exists y \in X \left[y \rhd x' \right] \right] \} \quad (\delta > 0). \tag{3.1}$$

Proposition 3.3. *Let $\rhd$ be a binary relation on a compact metric space X. If $\rhd$ has the approximate FTP, then it admits an approximate potential. If $\rhd$ admits an approximate potential, then $M(X, \rhd) \neq \emptyset$; if, additionally, $M(X, \rhd)$ is closed in X, then $\rhd$ has the approximate FTP.*

Proof. Assuming that $\rhd$ has the approximate FTP, we define $\succ$ as its transitive closure. Since $\rhd$ is acyclic, $\succ$ is irreflexive; (2.2) is straightforward. Suppose there is $\delta > 0$ such that $\succ$ is not strictly acyclic on $X(\delta)$, i.e., there is an infinite tâtonnement path of $\succ$ within $X(\delta)$. By the definition of $\succ$, there is an infinite tâtonnement path $\langle x^k \rangle_{k \in \mathbb{N}}$ of $\rhd$ such that for every $k \in \mathbb{N}$ there is $m \geq k$ for which $x^m \in X(\delta)$. Since $X(\delta)$ is closed in X, hence compact, there is a limit point of $\langle x^k \rangle_{k \in \mathbb{N}}$ in $X(\delta)$. Since $X(\delta) \cap M(X, \rhd) = \emptyset$, the path is not maximal.

Let $\rhd$ admit an approximate potential $\succ$. If $M(X, \rhd) = \emptyset$, then $X(\delta) = X$, hence $\succ$ is strictly acyclic on X; therefore, $\emptyset \neq M(X, \succ) \subseteq M(X, \rhd)$: a contradiction. By (2.2), every tâtonnement path of $\rhd$ is a tâtonnement path of $\succ$; therefore, every tâtonnement path of $\rhd$ leaves every $X(\delta)$ for good after a finite number of steps. Therefore, every limit point of the path belongs to $X \setminus \bigcup_{\delta > 0} X(\delta) = \operatorname{cl} M(X, \rhd)$. Thus, if $M(X, \rhd)$ is closed, then $\rhd$ has the approximate FTP indeed. ☐

A *weak approximate potential* of $\rhd$ is a function $P \colon X \to (-\mathbb{R})$ such that:

$$\forall x \in X\, [P(x) = 0 \Rightarrow x \in M(X, \rhd)]; \tag{3.2a}$$

$$\forall x \in X\, \big[P(x) < 0 \Rightarrow \exists y \in X\, [y \rhd x \;\&\; P(y) > P(x)]\big]; \tag{3.2b}$$

$$\forall \varepsilon > 0\, \big[\{v \in \mathbb{R} \mid (-v) \in P(X) \;\&\; v > \varepsilon\} \text{ is a well ordered subset of } \mathbb{R}\big]; \tag{3.2c}$$

$$\text{correspondence } v \mapsto \{x \in X \mid P(x) \geq v\} \text{ is upper hemicontinuous at } v = 0. \tag{3.2d}$$

Proposition 3.4. *A binary relation $\rhd$ on a compact metric space X has the weak approximate FTP if and only if it admits a weak approximate potential.*

Proof. Let a function P satisfying (3.2) exist. Then (3.2a) and (3.2b) immediately imply that $M(X, \rhd) = \{x \in X \mid P(x) = 0\}$. By (3.2b), there is a mapping $f \colon \{x \in X \mid P(x) < 0\} \to X$ such that $f(x) \rhd x$ and $P(f(x)) > P(x)$. Whenever $P(x^0) < 0$, we start a tâtonnement path recursively defining $x^{k+1} := f(x^k)$. If $P(x^k) = 0$ at some stage, we are home; otherwise, we have an infinite tâtonnement path along which $P(x^k)$ strictly increases. Now (3.2c) implies that $P(x^k) \to 0$ as $k \to \infty$. Therefore, every limit point of $\langle x^k \rangle_{k \in \mathbb{N}}$ belongs to $M(X, \rhd)$ by (3.2d). Finally, there *are* limit points because X is compact.

Let $\rhd$ have the weak approximate FTP; we have to produce a weak approximate potential. First of all, X is partitioned into three subsets: $X^0 := M(X, \rhd)$; X^1 is the set of $x \in X$ from which $M(X, \rhd)$ can be reached with a finite tâtonnement path; $X^2 := X \setminus (X^0 \cup X^1)$. Now we define $P(x) := 0$ for $x \in X^0$ and $P(x) := -$[the minimal length of a tâtonnement path leading from x to X^0] for $x \in X^1$. If it happens that $X^2 = \emptyset$, then $\rhd$ actually has the weak FTP, and our P is a weak numeric potential as defined in the proof of Proposition 2.7.

The definition of P on X^2 is more complicated. Given $x \in X^2$ and a maximal tâtonnement path π starting at x, we denote $\varkappa(\pi) := \min\{k \in \mathbb{N} \mid \exists h \in \mathbb{N}\, [d(\pi(h), X^0) \geq 1/(k+1)]\}$ and $\eta(\pi) := \min\{k \in \mathbb{N} \mid \forall h > k\, [d(\pi(h), X^0) < 1/(\varkappa(\pi)+1)]\}$. Then we define $\varkappa^+(x)$ as the maximum of $\varkappa(\pi)$ over all maximal tâtonnement paths π starting at x, and $\eta^-(x)$ as the minimum of $\eta(\pi)$ over all maximal tâtonnement paths π starting at x such that $\varkappa(\pi) = \varkappa^+(x)$; $\varkappa^+(x)$ is well defined because $\pi(0) = x$ is the same for all π in question. Finally, we set

$$P(x) := -\frac{1}{\varkappa^+(x)+1} + \frac{1}{(\varkappa^+(x)+1)(\varkappa^+(x)+2)(\eta^-(x)+1)}$$

for all $x \in X^2$.

If $P(x) < 0$, then $x \in X^1 \cup X^2$; in the first case, (3.2b) holds for the same reason as in the proof of Proposition 2.7, while (3.2c) and (3.2d) for the restriction of P to $X^0 \cup X^1$ are trivial. Considering P on X^2, we make these straightforward observations: (1) $P(x) = -1/(\varkappa^+(x)+2)$ if $\eta^-(x) = 0$; (2) whenever $\varkappa^+(y) = \varkappa^+(x)$, $P(y) \geq P(x)$ if and only if $\eta^-(y) \leq \eta^-(x)$; (3) $-1/(\varkappa^+(x)+1) < P(x) \leq -1/(\varkappa^+(x)+2)$ for all $x \in X^2$. Since $d(x, X^0) < 1/\varkappa^+(x)$ for every $x \in X^2$ by definition, (3.2c) and (3.2d) follow immediately. To check (3.2b), we assume $x \in X^2$ given, fix a maximal tâtonnement path π starting at x such that $\varkappa(\pi) = \varkappa^+(x)$ and $\eta(\pi) = \eta^-(x)$, and set $y := \pi(1)$. We have $y \rhd x$ by definition; besides, $\varkappa^+(y) \geq \varkappa^+(x)$. If $\varkappa^+(y) > \varkappa^+(x)$, then $P(y) > P(x)$; if $\varkappa^+(y) = \varkappa^+(x)$, then $\eta^-(y) < \eta^-(x)$, hence $P(y) > P(x)$ again. □

A binary relation $\rhd$ is *upper semicontinuous* if all its lower contours are open: if $y \rhd x$, then there is $\delta > 0$ such that $y \rhd x'$ whenever $d(x, x') < \delta$. A binary relation $\rhd$ is *continuous* if its graph, $\{(x, y) \in X^2 \mid y \rhd x\}$, is open: if $y \rhd x$, then there is $\delta > 0$ such that $y' \rhd x'$ whenever $d(x, x') < \delta$ and $d(y, y') < \delta$.

Theorem 3.5. *An upper semicontinuous and acyclic binary relation $\rhd$ on a compact metric space X has the weak approximate FTP.*

Proof. For each $k \in \mathbb{N}$, we define $X^k := \{x \in X \mid \forall x' \in X\, \big[d(x, x') < 1/(k+1) \Rightarrow \exists y \in X\, [y \rhd x']\big]\}$; clearly, $X^0 \subseteq X^1 \subseteq \dots$. Since $\rhd$ is upper semicontinuous, each X^k is closed; their union, $X^\infty := \bigcup_{k \in \mathbb{N}} X^k$, is open and $X^\infty = X \setminus M(X, \rhd)$. If $M(X, \rhd) = \emptyset$, then $X^0 = X^1 = \cdots = X$; otherwise, $X^k = \{x \in X \mid d(x, M(X, \rhd)) \geq 1/(k+1)\}$.

Given $k \in \mathbb{N}$ and $x \in X^k$, we pick $y(x) \in X$ such that $y(x) \rhd x$. There is an open neighborhood $U(x)$ of x such that $y(x) \rhd x'$ for every $x' \in U(x)$. Since X^k is compact, it is covered by a finite number of such open neighborhoods $U^k_0, U^k_1, \dots, U^k_{m_k}$. Denoting $\Xi := \{(k, h) \in \mathbb{N}^2 \mid h \leq m_k\}$, we obtain a mapping $f\colon \Xi \to X$ such that $f(k, h) \rhd x$ for every $(k, h) \in \Xi$ and $x \in U^k_h$. Given $x \in X^\infty$, we pick the least $k \in \mathbb{N}$ such that $x \in X^k$ and the least $h \in \mathbb{N}$ such that $x \in U^k_h$, obtaining a mapping $g\colon X^\infty \to \Xi$.

Now we are ready to prove the weak approximate FTP. Let $x^0 \in X$. If $x^0 \in M(X, \rhd)$ we are already home; otherwise, we define $x^1 := f \circ g(x^0)$. If $x^1 \in M(X, \rhd)$ we are home; otherwise, we define $x^2 := f \circ g(x^1)$ and so on. If the process never stops, we obtain an infinite tâtonnement path $\langle x^k \rangle_{k \in \mathbb{N}}$. Given $\varepsilon > 0$, we pick $\bar{k} \in \mathbb{N}$ such that $1/(\bar{k}+1) < \varepsilon$. Since $\rhd$ is acyclic, all $g(x^k)$ must be different, hence the first component of $g(x^k)$ remains greater than $\bar{k}$ after a finite number of steps. And this means that $x^k \notin X^{\bar{k}}$, hence $d(x^k, M(X, \rhd)) < \varepsilon$, after a finite number of steps. □

Remark. Derivation from the assumptions of Theorem 3.5 of just the existence of a maximizer is a common practice (Bergstrom, 1975; Walker, 1977); the weak approximate FTP, however, is a much stronger property. (To be fair, Walker did not have to assume that the topology is defined with a metric.) It seems an assumption that $\rhd$ in Theorem 3.5 is *continuous* would not help obtain any stronger statement.

3.2 Transfinite tâtonnement paths

A deeper analysis of tâtonnement processes associated with a binary relation on a metric space becomes possible if the properties defined in Subsection 3.1 are supplemented with those concerning tâtonnement paths parameterized with countable ordinal numbers.

Let $\rhd$ be a binary relation on a metric space X. A *tâtonnement path* of $\rhd$ is a mapping $\pi\colon \operatorname{Dom}\pi \to X$, where $\operatorname{Dom}\pi$ is either Ω or $[0, \bar{\alpha}[$ for $\bar{\alpha} \in \Omega$, satisfying these two conditions:

$$\pi(\alpha+1) \rhd \pi(\alpha) \text{ whenever } \alpha, \alpha+1 \in \operatorname{Dom}\pi; \tag{3.3a}$$

if $\alpha \in \operatorname{Dom}\pi$, and α is a limit, then there exists a sequence $\langle \beta^k \rangle_{k \in \mathbb{N}}$ for which

$$\beta^{k+1} > \beta^k \text{ for all } k,\ \alpha = \sup_k \beta^k, \text{ and } \pi(\alpha) = \lim_{k\to\infty} \pi(\beta^k). \tag{3.3b}$$

Let π' and π'' be tâtonnement paths (of the same $\rhd$) such that $\operatorname{Dom}\pi' = [0, \alpha']$, $\operatorname{Dom}\pi'' = [0, \alpha'']$, and $\pi'(\alpha') = \pi''(0)$. Then the *concatenation* of π' and π'' goes from $\pi'(0)$, "along π'," to $\pi'(\alpha')$, and then from $\pi'(\alpha') = \pi''(0)$, "along π''," to $\pi''(\alpha'')$. Formally speaking, we consider $\{0,1\} \times \Omega$ with the lexicographic order where the first component matters first; then we define mappings $\sigma'\colon [0, \alpha'] \to (\{0,1\} \times \Omega)$ and $\sigma''\colon [0, \alpha''] \to (\{0,1\} \times \Omega)$ by $\sigma'(\beta) := (0, \beta)$ and $\sigma''(\beta) := (1, \beta)$. Denoting $\Lambda := \sigma'([0, \alpha']) \cup \sigma''([1, \alpha'']) \subseteq (\{0,1\} \times \Omega)$, we define $\tau\colon \Lambda \to X$ by $\tau(\sigma'(\beta)) := \pi'(\beta)$ and $\tau(\sigma''(\beta)) := \pi''(\beta)$. It is easily checked that Λ is well ordered and τ is a tâtonnement path [the condition $\pi'(\alpha') = \pi''(0)$ is essential for (3.3a) to hold at $\alpha = (0, \alpha')$: $\alpha + 1$ (in Λ!) is then $(1,1) = \sigma''(1)$]. Since Λ is a countable well ordered set, there is an isomorphism $\theta\colon \Lambda \to [0, \alpha_*] \subset \Omega$ for a unique $\alpha_* \in \Omega$. Now we can define a ("standard") tâtonnement path $\pi\colon [0, \alpha_*] \to X$ by $\pi(\beta) := \tau \circ \theta^{-1}(\beta)$.

The concatenation of any finite number, or even an infinite sequence, of tâtonnement paths of the same $\rhd$ can be defined in a similar way. Let each π^k be defined on $\operatorname{Dom}\pi^k = [0, \alpha^k]$, and let $\pi^k(\alpha^k) = \pi^{k+1}(0)$ for each relevant $k \in \mathbb{N}$. Then we define mappings

$\sigma^k\colon [0,\alpha^k] \to (\mathbb{N}\times\Omega)$ (with the same lexicographic order on the latter set) by $\sigma^k(\beta) := (k,\beta)$. Denoting $\Lambda := \sigma^0([0,\alpha^0]) \cup \bigcup_{k>0}\sigma^k([1,\alpha^k])$ and defining $\tau\colon \Lambda \to X$ by $\tau(\sigma^k(\beta)) := \pi^k(\beta)$, we again see that Λ is a countable well ordered set and τ is a tâtonnement path. Therefore, there is an isomorphism $\theta\colon \Lambda \to [0,\alpha_*]$ when the number of π^k is finite, or $\theta\colon \Lambda \to [0,\alpha_*[$ otherwise. $\alpha_* \in \Omega$ is unique in either case. Again, we can define a ("standard") tâtonnement path $\pi := \tau\circ\theta^{-1}$. Finally, if, in the infinite case, there is a limit point y^* of $\langle \pi^k(\alpha^k)\rangle_{k\in\mathbb{N}}$, then we may additionally set $\pi(\alpha_*) := y^*$ – (3.3b) is easily checked – and call the extended π a *closed concatenation* of π^k.

Lemma 3.6. *Let π be the concatenation of a finite number, or an infinite sequence, of tâtonnement paths π^k of the same $\rhd$ as defined in the previous paragraph. Then $\alpha_* \geq \alpha^k$ for each relevant $k\in\mathbb{N}$.*

Proof. Supposing the contrary, $\alpha_* < \alpha^k$ for some k, we denote $\mathrm{B} := \{\beta \in [0,\alpha^k] \mid \beta > \theta\circ\sigma^k(\beta)\}$. We have $\mathrm{B} \neq \emptyset$ because $\alpha^k \in \mathrm{B}$. Let $\beta^* := \min \mathrm{B}$; clearly, $\beta^* > 0$.

If β^* is a successor, $\beta^* = \beta^{**}+1$, then we must have $\beta^{**} \leq \theta\circ\sigma^k(\beta^{**})$ by the definition of β^*, but then $\theta\circ\sigma^k(\beta^*) = \theta\circ\sigma^k(\beta^{**}+1) \geq \theta(\sigma^k(\beta^{**})+1) \geq \beta^{**}+1 = \beta^*$, contradicting $\beta^* \in \mathrm{B}$.

If β^* is a limit, we have $\beta \leq \theta\circ\sigma^k(\beta)$ for all $\beta < \beta^*$ by the definition of β^*, hence $\theta\circ\sigma^k(\beta^*) \geq \sup_{\beta<\beta^*}\theta\circ\sigma^k(\beta) \geq \beta$ for all $\beta<\beta^*$, hence $\theta\circ\sigma^k(\beta^*) \geq \sup_{\beta<\beta^*}\beta = \beta^*$ with the same contradiction. □

We say that $\rhd$ has the *countable tâtonnement property* (*CTP*) on X if there exists no tâtonnement path π of $\rhd$ with $\operatorname{Dom}\pi = \Omega$. $\rhd$ has the *weak CTP* if, for every $x\in X$, there exists a tâtonnement path π such that $\pi(0)=x$, $\operatorname{Dom}\pi \subset \Omega$, and π admits no extension, i.e., there is no tâtonnement path π' such that $\operatorname{Dom}\pi \subset \operatorname{Dom}\pi'$ and $\pi'(\alpha) = \pi(\alpha)$ for every $\alpha \in \operatorname{Dom}\pi$.

Remark. On a compact space X, where the only obstacle to extending a tâtonnement path further is the fact that it has reached a maximizer, weak CTP means that, given any point $x\in X$, a tâtonnement path π can be found such that $\pi(0)=x$ and $\pi(\alpha)\in M(X,\rhd)$ ($\alpha \in \operatorname{Dom}\pi$), cf. the implication [(3.5e)⇒(3.5f)] in Theorem 3.21 below. If X is not compact, an "unextendable" path may "go to infinity," as, e.g., on the real line with the standard order.

Proposition 3.7. *Let $\rhd$ be a binary relation on a metric space X. If $\rhd$ has the (weak) approximate FTP, then it has the (weak) CTP.*

Proof. Let $\rhd$ have the approximate FTP and π be a tâtonnement path of $\rhd$ with $\operatorname{Dom}\pi \supseteq \mathbb{N}\cup\{\omega\}$. By (3.3b), $\pi(\omega)$ is a limit point of the sequence $\pi(k)$ ($k\in\mathbb{N}$), hence $\pi(\omega)\in M(X,\rhd)$ by the approximate FTP. Therefore, $\operatorname{Dom}\pi = \mathbb{N}\cup\{\omega\}$ by (3.3a).

Let $\rhd$ have the weak approximate FTP and $x^0\in X$. If there is a finite tâtonnement path $x^0,x^1,\ldots,x^m$ such that $x^m\in M(X,\rhd)$, then the path cannot be extended because of (3.3a), and we are home. Otherwise, there is a maximal tâtonnement path $\langle x^k\rangle_{k\in\mathbb{N}}$ starting at x^0. We pick a limit point x^ω of the path and define a transfinite tâtonnement path π by $\operatorname{Dom}\pi := \mathbb{N} \cup \{\omega\}$, $\pi(k) := x^k$ for all $k \in \mathbb{N}$, and $\pi(\omega) := x^\omega$. The path π cannot be extended because of (3.3a). □

Remark. Proposition 3.7 would remain valid if we weakened the requirements in the definitions of the (weak) approximate FTP, allowing a maximal tâtonnement path to have no limit point at all. However, the last implication in Proposition 3.1, weak approximate FTP ⇒ very weak FTP, would then become wrong. For Propositions 3.3 and 3.4, nothing would change because their assumptions include the compactness of X anyway.

We finish this subsection with a technical lemma.

Lemma 3.8. *If π is a tâtonnement path and $\beta \in \operatorname{Dom}\pi$ is a limit, then there exists an infinite sequence $\langle \beta^k \rangle_{k\in\mathbb{N}}$ such that $\beta^{k+1} > \beta^k$ for all k, $\beta = \sup_k \beta^k$, $\pi(\beta^k) \to \pi(\beta)$, and each β^k is a successor.*

Proof. By transfinite recursion in α, we prove that the statement is valid for all $\beta \le \alpha$. For $\alpha \le \omega$, it is obvious.

Suppose the statement is valid for all $\alpha' < \alpha$. If α is a successor, $\alpha = \alpha^* + 1$, then we have $\beta \le \alpha^*$ whenever $\beta \le \alpha$ and β is a limit; therefore, the statement of the lemma is true for β by the induction hypothesis.

Let α be a limit; we have to prove the statement of the lemma for $\beta = \alpha$. By the definition of an improvement path, there exists an infinite sequence $\langle \gamma^k \rangle_{k\in\mathbb{N}}$ such that $\gamma^{k+1} > \gamma^k$ for all k, $\beta = \sup_k \gamma^k$, and $\pi(\beta) = \lim_{k\to\infty} \pi(\gamma^k)$. Now we may construct the sequence $\langle \beta^k \rangle_{k\in\mathbb{N}}$ by the following "algorithm:" fix a numeric sequence $r_h \to 0$ (e.g., $r_h = 1/h$); pick the first k_1 for which $d(\pi(\beta), \pi(\gamma^{k_1})) < r_1$; if γ^{k_1} is a successor, define $\beta^1 = \gamma^{k_1}$; otherwise, invoke the induction hypothesis, pick $\beta' < \gamma^{k_1}$ which is a successor and satisfies $d(\pi(\beta), \pi(\beta')) < r_1$, and define $\beta^1 = \beta'$. Then repeat the same procedure with just two additional conditions: each new k_{h+1} must be greater than k_h chosen on the previous step and, when β' is being chosen at the step $h+1$, it must also satisfy $\beta' > \gamma^{k_h}$. Clearly, $\pi(\beta^k) \to \pi(\beta)$, $\beta = \sup_k \beta^k$, $\beta^{k+1} > \beta^k$ for all k, and each β^k is a successor. □

3.3 ω-Transitive relations

A binary relation $\succ$ on a metric space X is called *ω-transitive* if it is transitive and the conditions $x^\omega = \lim_{k\to\infty} x^k$ and $x^{k+1} \succ x^k$ for all $k \in \mathbb{N}$ always imply $x^\omega \succ x^0$.

Remark. Gillis (1959) and Smith (1974) considered this condition for orderings.

Lemma 3.9 (Kukushkin, 2003, Lemma 2.3). *Let $\succ$ be an ω-transitive relation on a metric space X and π be a tâtonnement path of $\succ$; then π satisfies the condition:*

$$\forall \alpha, \beta \in \operatorname{Dom}\pi \left[\alpha > \beta \Rightarrow \pi(\alpha) \succ \pi(\beta)\right]. \tag{3.4}$$

For a given relation $\rhd$, its *ω-transitive closure* is the conjunction of all ω-transitive extensions of $\rhd$; clearly, the ω-transitive closure is itself an ω-transitive extension of $\rhd$.

Lemma 3.10 (Kukushkin, 2003, Lemma 2.4). *Let $\rhd$ be a binary relation on X, $\trianglerighteq$ be its ω-transitive closure, and $y, x \in X$. Then $y \trianglerighteq x$ if and only if there exist a tâtonnement path π of $\rhd$ and $\alpha \in \operatorname{Dom}\pi$ satisfying $\pi(0) = x$, $\alpha > 0$, and $\pi(\alpha) = y$.*

Proposition 3.11 (Kukushkin, 2003, Proposition 2.1). *Let $\succ$ be an irreflexive and ω-transitive relation on a metric space X. Let X be second countable. Then there exists no mapping $\pi \colon \Omega \to X$ satisfying* (3.4).

Corollary. *Every irreflexive and ω-transitive relation on a separable metric space X has the CTP.*

Remark. A compact metric space is separable, hence Corollary to Proposition 3.11 applies. The set Ω with its order-induced topology is first countable, but Proposition 3.11 does not hold there; however, it is not metrizable. I have no example of a metric space where Proposition 3.11 would not hold.

Theorem 3.12 (Kukushkin, 2008b, Theorem 1). *A binary relation $\succ$ on X has the NM property on every compact subset $Y \subseteq X$ if and only if $\succ$ is irreflexive and ω-transitive on X.*

Corollary. *A strategic game Γ has the property that every subgame with compact sets X_i' of Γ is BR-consistent if and only if every relation $\ggcurly_i^{x_{-i}}$ ($i \in N$; $x_{-i} \in X_{-i}$) is irreflexive and ω-transitive on X_i.*

Proposition 3.13. *A strategic game Γ has the property that every subgame with compact sets X_i' of Γ is strongly BR-consistent if and only if every relation $\ggcurly_i^{x_{-i}}$ ($i \in N$; $x_{-i} \in X_{-i}$) is an ω-transitive ordering on X_i.*

The following simple results describe natural "mechanisms" generating ω-transitive binary relations.

Proposition 3.14. *Let $\succeq$ be a preorder on X such that every upper contour $\operatorname{Up}(x, \succeq)$ is closed. Then $\succeq$ is ω-transitive.*

Proof. Let $x^\omega = \lim_{k\to\infty} x^k$ and $x^{k+1} \succeq x^k$ for each $k \in \mathbb{N}$. Then $x^k \in \operatorname{Up}(x^h, \succeq)$ whenever $k \geq h$, hence $x^\omega \in \operatorname{Up}(x^h, \succeq)$ for all $h \in \mathbb{N}$; in particular, $x^\omega \in \operatorname{Up}(x^0, \succeq)$. □

Proposition 3.15. *Let $\succeq$ be an ω-transitive preorder on X. Then its asymmetric component $\succ$ is ω-transitive too.*

Proof. Let $x^k \to x^\omega$ and $x^{k+1} \succ x^k$ for each $k \in \mathbb{N}$. Then $x^{k+2} \succeq x^{k+1}$ for each $k \in \mathbb{N}$, hence $x^\omega \succeq x^1$ because $\succeq$ is ω-transitive, hence $x^\omega \succ x^0$ by transitivity. □

Corollary. *Let $\succeq$ be an upper semicontinuous total preorder on X. Then both $\succeq$ and its asymmetric component $\succ$ are ω-transitive.*

Proposition 3.16. *Let $\{\succeq^\alpha\}_{\alpha\in\mathrm{A}}$ be a family of ω-transitive binary relations on X. Then their conjunction, $y \succeq x \rightleftharpoons \forall\alpha \in \mathrm{A}\,[y \succeq^\alpha x]$ is ω-transitive too.*

Proof. Straightforward. □

Example 3.17. Let $\langle u^\alpha\rangle_{\alpha\in\mathrm{A}}$ be a family of upper semicontinuous functions $X \to \mathbb{R}$. Then each of them defines a total preorder on X, which is ω-transitive, together with its asymmetric component, by Corollary to Propositions 3.14 and 3.15. Therefore, both weak Pareto dominance,

$$y \succeq x \rightleftharpoons \forall\alpha \in \mathrm{A}\,[u^\alpha(y) \geq u^\alpha(x)],$$

and strong Pareto dominance,

$$y \ggcurly x \rightleftharpoons \forall\alpha \in \mathrm{A}\,[u^\alpha(y) > u^\alpha(x)],$$

are ω-transitive by Proposition 3.16. Now "normal" Pareto dominance (2.5), which is the asymmetric component of the weak Pareto dominance, is ω-transitive by Proposition 3.15.

Another common mechanism producing a new ω-transitive preorder from a family of previously given ones is lexicography (Kukushkin, 2003, Section 3.2).

Proposition 3.18. *Let $\rhd$ be a binary relation on a compact metric space X. If $\rhd$ has the very weak FTP, then it has the weak CTP.*

Proof. We denote $\trianglerighteq$ the *ω-transitive and reflexive closure* of $\rhd$ and $\succ$ the asymmetric component of $\trianglerighteq$. Similarly to Lemma 3.10, $y \trianglerighteq x$ if and only if there exist a tâtonnement path π of $\rhd$ and $\alpha \in \operatorname{Dom}\pi$ such that $\pi(0) = x$ and $\pi(\alpha) = y$; $y \succ x$ if and only if there exists a tâtonnement path π such that $\pi(0) = x$ and $\pi(\alpha) = y$, but no tâtonnement path π such that $\pi(0) = y$ and $\pi(\alpha) = x$. $\succ$ is obviously irreflexive and ω-transitive by Proposition 3.15.

Let $x^0 \in X$. By Theorem 3.12, there is $y^0 \in M(X, \succ)$ such that $y^0 \succ x^0$, hence a tâtonnement path π^0 of $\rhd$ connects x^0 to y^0. If we are lucky and $y^0 \in M(X, \rhd)$, then we are already home: π^0 cannot be extended further by (3.3a). Otherwise, we invoke the very weak FTP of $\rhd$, obtaining an appropriate $y^* \in M(X, \rhd)$. For every $k \in \mathbb{N} \setminus \{0\}$, there is a finite tâtonnement path π^k of $\rhd$ connecting y^0 to y^k such that $d(y^*, y^k) < 1/k$; therefore, $y^k \trianglerighteq y^0$. Since $y^0 \in M(X, \succ)$, we must have $y^0 \trianglerighteq y^k$ too, hence there is a finite tâtonnement path π^k_* of $\rhd$ connecting y^k to y^0.

Now we consider the concatenation of the tâtonnement paths π^0, π^1, π^1_*, π^2, π^2_*, etc. Since y^* is a limit point, the concatenation can be extended there. □

Remark. Without the compactness of X, the proof collapses. Nothing is known about the validity of the statement itself.

Open Problem 3.19 ("Szpilrajn's Theorem for ω-transitive relations"). *Let $\succ$ be an irreflexive and ω-transitive relation on a metric space X. Is it necessarily possible to extend $\succ$ to an ω-transitive linear order on X? If not, would the assumption that X is compact help?*

Proposition 3.20. *Let $\succ$ be an ω-transitive ordering on a metric space X. Then it can be extended to an ω-transitive linear order on X.*

Proof. Invoking the Zermelo Theorem, we denote $<$ a well-order on X. Then we define a relation $\gg$ on X by:

$$y \gg x \rightleftharpoons \big[y \succ x \text{ or } [y \sim x \,\&\, y < x]\big].$$

Clearly, $\gg$ is a linear order on X, and $y \succ x \Rightarrow y \gg x$. We only have to check ω-transitivity.

Let $x^k \to x^\omega$ and $x^{k+1} \gg x^k$ for all k. By the definition of $\gg$, either $x^{k+1} \succ x^k$ or $x^{k+1} \sim x^k$ and $x^{k+1} < x^k$ for each $k \in \mathbb{N}$. Since $<$ is a well order, the latter relation is only possible a finite number of times in a row; therefore, we have $x^{k_{h+1}} \succ x^{k_h}$ for a subsequence k_h. Since $x^{k_h} \to x^\omega$ as well, $x^\omega \succ x^0$ by the ω-transitivity of $\succ$. □

3.4 Ω-Acyclic relations

A *tâtonnement cycle* of $\rhd$ is a tâtonnement path π such that $\operatorname{Dom}\pi = [0, \alpha]$, $\alpha > 0$, and $\pi(\alpha) = \pi(0)$. $\rhd$ is called Ω-*acyclic* if it admits no tâtonnement cycle. A tâtonnement path $\pi\colon \operatorname{Dom}\pi \to X$ is called *narrow* if $\pi(\beta^k) \to \pi(\alpha)$ for every limit $\alpha \in \operatorname{Dom}\pi$ and every strictly increasing sequence $\langle\beta^k\rangle_{k\in\mathbb{N}}$ such that $\alpha = \sup_k \beta^k$; in other words, if each $\pi(\alpha)$ is the limit of the preceding path rather than a limit point. $\rhd$ is called *quasi-*Ω*-acyclic* if it admits no narrow tâtonnement cycle.

An ω-*potential* of $\rhd$ is an irreflexive and ω-transitive relation $\succ$ satisfying (2.2). A *weak* ω-*potential* of $\rhd$ is an irreflexive and ω-transitive relation $\succ$ satisfying (2.13).

Remark. As long as Problem 3.19 remains open, there would be no point in defining a "generalized numeric" ω-potential in the style of Corollary to Proposition 2.6.

If we revise Proposition 2.4 assuming that X is a metric space and replacing "finite" with "compact" and "FTP" with "CTP" throughout, then the equivalence of all conditions (2.11) will be replaced with a chain of implications.

Theorem 3.21 (Kukushkin, 2003, Theorem 2). *For a binary relation $\rhd$ on a metric space X, let us consider the following conditions:*

$$\rhd \text{ is } \Omega\text{-acyclic on } X; \tag{3.5a}$$

$$\rhd \text{ admits an } \omega\text{-potential on } X; \tag{3.5b}$$

$$\rhd \text{ has the CTP on } X; \tag{3.5c}$$

$$\rhd \text{ has the CTP on every compact subset } Y \subseteq X; \tag{3.5d}$$

$$\rhd \text{ has the weak CTP on every compact subset } Y \subseteq X; \tag{3.5e}$$

$$M(Y, \rhd) \neq \emptyset \text{ for every nonempty compact subset } Y \subseteq X; \tag{3.5f}$$

$$\rhd \text{ is quasi-}\Omega\text{-acyclic on } X. \tag{3.5g}$$

Then this chain of implications holds:

$$(3.5a) \iff (3.5b) \iff (3.5c) \Rightarrow (3.5d) \Rightarrow (3.5e) \iff (3.5f) \Rightarrow (3.5g).$$

None of the one-sided implications in Theorem 3.21 can be reversed (Kukushkin, 2003, Examples 2.1 – 2.4).

Comparing the formulations of Proposition 2.4 and Theorem 3.21, we see that in the new situation the acyclicity condition "splits" into several properties, each of which deserving some attention. However, none of them is equivalent to the existence of a maximizer in every compact subset. Theorem 1 of Kukushkin (2008a) shows that the latter property cannot be expressed as the prohibition of any kind of cycles.

Proposition 3.22. *If $\rhd$ admits a weak ω-potential $\succ$, then $\rhd$ has the weak CTP.*

Proof. We denote $\ggcurly$ the conjunction of $\rhd$ and $\succ$, and $\ggg$ the ω-transitive closure of $\ggcurly$. By definition, $y \succ x$ whenever $y \ggg x$, hence $\ggg$ is irreflexive too. By (2.13), $M(X, \ggg) = M(X, \ggcurly) \subseteq M(X, \rhd)$. By Theorem 3.12, $\ggg$ has the NM property, which is the same as the weak CTP of $\rhd$. ☐

Theorem 3.23. *An upper semicontinuous binary relation $\rhd$ is Ω-acyclic if and only if it is acyclic.*

Proof. The necessity is tautological. Let $\rhd$ be upper semicontinuous and acyclic.

Claim 3.23.1. *Whenever π is a tâtonnement path and $(\alpha + 1) \in \operatorname{Dom} \pi$, there exists a finite tâtonnement path π^* of length m such that $\pi^*(0) = \pi(0)$ and $\pi^*(m) = \pi(\alpha + 1)$.*

Proof. The proof goes by transfinite induction; if α is finite, the restriction of π to $[0, \alpha]$ will do. Assuming the statement valid for all $\alpha < \alpha'$, let us prove it for α'.

If α' is a successor, $\alpha' = \alpha + 1$, we have $\pi(\alpha' + 1) \rhd \pi(\alpha') = \pi(\alpha + 1)$. Applying the induction hypothesis to α, we obtain a finite tâtonnement path π^* of length m such that $\pi^*(0) = \pi(0)$ and $\pi^*(m) = \pi(\alpha + 1)$. Defining $\pi^*(m + 1) := \pi(\alpha' + 1)\, [\rhd \pi^*(m)]$, we obtain a finite tâtonnement path (of length $m + 1$) ending at $\pi(\alpha' + 1)$.

Let α' be a limit. There is $\delta > 0$ such that $\pi(\alpha' + 1) \rhd x$ whenever $d(\pi(\alpha'), x) < \delta$. We pick $\alpha < \alpha'$ such that $d(\pi(\alpha'), \pi(\alpha + 1)) < \delta$, and apply the induction hypothesis to α in the same manner as in the previous paragraph. There is a finite tâtonnement path π^* of length m such that $\pi^*(0) = \pi(0)$ and $\pi^*(m) = \pi(\alpha + 1)$. Adding $\pi^*(m + 1) := \pi(\alpha' + 1)$, we again obtain a finite tâtonnement path (of length $m + 1$) ending at $\pi(\alpha' + 1)$. ☐

To finish with the proof of the theorem, we suppose the contrary: there is a tâtonnement path π such that $\pi(\alpha) = \pi(0)$ for an $\alpha > 0$ ($\alpha \in \operatorname{Dom}\pi$). Since $\rhd$ is acyclic, α must be infinite; without restricting generality, $\alpha = \max \operatorname{Dom}\pi$. If α is a successor, $\alpha = \alpha' + 1$, we apply Claim 3.23.1, obtaining a finite tâtonnement path π^* of length m such that $\pi^*(0) = \pi(0)$ and $\pi^*(m) = \pi(\alpha' + 1) = \pi(\alpha) = \pi(0)$, which contradicts the acyclicity of $\rhd$.

If α is a limit, we obtain the same contradiction simply changing the origin of the cycle. Technically speaking, we first define a mapping $\sigma\colon \operatorname{Dom}\pi \to \operatorname{Dom}\pi$ by recursion: $\sigma(0) := 1$; whenever $\sigma(\beta)$ is defined and $(\beta + 1) \in \operatorname{Dom}\pi$, $\sigma(\beta + 1) := \sigma(\beta) + 1$; whenever $\beta \in \operatorname{Dom}\pi$ is a limit and $\sigma(\beta')$ is defined for all $\beta' < \beta$, $\sigma(\beta) := \sup_{\beta'<\beta} \sigma(\beta')$. It is immediately seen that $\sigma(\beta) = \beta + 1$ if β is finite, and $\sigma(\beta) = \beta$ otherwise. Then we define a tâtonnement path π' on the same $\operatorname{Dom}\pi$ by $\pi'(\beta) := \pi(\sigma(\beta))$ for all $\beta \in \operatorname{Dom}\pi$; thus we have $\pi'(\alpha) = \pi(\alpha) = \pi(0)$. Finally, we add one more point, $\alpha + 1$, to $\operatorname{Dom}\pi$ and extend π' by $\pi'(\alpha + 1) := \pi(1) = \pi'(0)$. Now we have, with π', the same situation as in the previous paragraph, hence obtain the same contradiction. □

Remark. Unlike Theorem 3.23, Theorem 3.5 cannot claim equivalence, only one-way implication. It is also worth noting that Ω-acyclicity does not generally imply the weak approximate FTP, nor even very weak FTP.

3.5 Endomorphisms and fixed points

With every mapping $f\colon X \to X$, we may associate the relation $\trianglerighteq^f$ by $y \trianglerighteq^f x \rightleftharpoons [y = f(x)]$ (it may be argued that the relation *is* the mapping). A slight modification, however, is much more convenient for our purposes here: $y \rhd^f x \rightleftharpoons [y = f(x) \neq x]$. The point is that maximizers of $\rhd^f$ are exactly fixed points of f. In the case of a correspondence $F\colon X \to 2^X \setminus \{\emptyset\}$, we define a binary relation $\rhd^F$ by $y \rhd^F x \rightleftharpoons [x \notin F(x) \ni y]$; again, maximizers of $\rhd^F$ are fixed points of F and vice versa.

Tâtonnement paths of $\rhd^F$ ($\rhd^f$) combine iterating F (or f) and picking limit points, so they may also be called iteration paths. We call a mapping F, or f, (strictly, weakly, Ω-, etc.) acyclic if so is $\rhd^F$ ($\rhd^f$).

Proposition 3.24. *A mapping $f\colon X \to X$ is strictly acyclic if and only if it is weakly acyclic.*

Given a mapping $f\colon X \to X$, a subset $Y \subseteq X$ is *f-invariant* if $f(Y) \subseteq Y$. There are two natural ways to extend the notion to correspondences $F\colon X \to 2^X \setminus \{\emptyset\}$: a subset $Y \subseteq X$ may be called *F-invariant* if $F(y) \subseteq Y$ for every $y \in Y$ or if $F(y) \cap Y \neq \emptyset$ for every $y \in Y$. The second version proves more convenient here, so we adopt it.

Proposition 3.25. *A correspondence $F\colon X \to 2^X \setminus \{\emptyset\}$ is strictly acyclic if and only if every F-invariant subset of X contains a fixed point of F. A correspondence $F\colon X \to 2^X \setminus \{\emptyset\}$ is acyclic if and only if every finite F-invariant subset of X contains a fixed point of F. A correspondence $F\colon X \to 2^X \setminus \{\emptyset\}$ is very weakly acyclic if and only if the* *closure of every F-invariant subset of X contains a fixed point of F. If a correspondence $F\colon X \to 2^X \setminus \{\emptyset\}$ is Ω-acyclic, then every compact F-invariant subset of X contains a fixed point of F.*

Example 3.26. Let X be a circle parameterized with a real number $0 \le \varphi < 2\pi$. We fix φ^0 incommensurable with 2π and define a mapping $f\colon X \to X$ by

$$f(\varphi) := \begin{cases} 0, & \varphi = 0; \\ \varphi \oplus \varphi^0, & \text{otherwise}, \end{cases}$$

where $\oplus$ denotes addition modulo 2π. Clearly, $\varphi = 0$ is a unique fixed point. f is acyclic and very weakly acyclic, but not Ω-acyclic. The only compact f-invariant subset of X is X itself. Therefore, the converse to the last statement in Proposition 3.25 is wrong.

The only well-known fixed point theorem that ensures universal convergence of iterations is that of Banach's.

Proposition 3.27. *Let X be a complete metric space and f be a mapping $X \to X$ such that $d(f(x), f(y)) \le \delta \cdot d(x,y)$ for some $\delta \in]0,1[$ and all $x, y \in X$. Then $\rhd^F$ has the approximate FTP.*

The assumptions of the Brouwer, Lefschetz, or Kakutani theorems do not guarantee anything of this kind. It is sufficient to consider the rotation of a circle around its center. As to the (Knaster-)Tarski theorem, roughly speaking, it ensures (Ω-)acyclicity on a chain, but not otherwise.

Example 3.28. Let $X := \{(0,0), (0,1), (1,0), (1,1)\} \subset \mathbb{R}^2$ and $f\colon X \to X$ be defined by $f(0,0) = (0,0)$, $f(1,0) = (0,1)$, $f(0,1) = (1,0)$, and $f(1,1) = (1,1)$. X is a finite lattice, f is increasing, there are even two fixed points, but iterations of f started at $(1,0)$ will never reach a fixed point.

Let X be a chain. A correspondence $F\colon X \to 2^X \setminus \{\emptyset\}$ is *ascending* if $\min\{y', y''\} \in F(x')$ and $\max\{y', y''\} \in F(x'')$ whenever $x', x'' \in X$, $x'' > x'$, $y' \in F(x')$, and $y'' \in F(x'')$.

Let X be simultaneously a metric space and a chain. We say that the order on X is *quasicontinuous* if both relations $>$ and $<$ are ω-transitive, and there are no two sequences $\langle x^k \rangle_{k \in \mathbb{N}}$ and $\langle y^h \rangle_{h \in \mathbb{N}}$ such that $x^k \to x^\omega$, $y^h \to y^\omega$, and

$$\forall k\ \forall h\ \big[x^\omega > y^h > y^{h+1} > x^{k+1} > x^k > y^\omega\big]. \tag{3.6}$$

Clearly, a continuous order is quasicontinuous as well. The lexicographic order on $\mathbb{R}^m$ is quasicontinuous, but not continuous (provided $m > 1$).

Theorem 3.29 (Kukushkin, 2000, Theorem 4.2). *Let X be simultaneously a metric space and a chain such that the order on X is quasicontinuous. Let a correspondence $F\colon X \to 2^X \setminus \{\emptyset\}$ be ascending. Then F is Ω-acyclic.*

Proof. Let π be a tâtonnement path of $\rhd^F$. We partition $\operatorname{Dom}\pi \setminus \{\max \operatorname{Dom}\pi\}$ into $\mathrm{B}^{\uparrow} := \{\beta \in \operatorname{Dom}\pi \setminus \{\max \operatorname{Dom}\pi\} \mid \pi(\beta+1) > \pi(\beta)\}$ and $\mathrm{B}^{\downarrow} := \{\beta \in \operatorname{Dom}\pi \setminus \{\max \operatorname{Dom}\pi\} \mid \pi(\beta+1) < \pi(\beta)\}$ ($\pi(\beta+1) = \pi(\beta)$ is obviously impossible).

Claim 3.29.1. *Let $\beta \in \mathrm{B}^{\uparrow}$ and $\beta+2 \in \operatorname{Dom}\pi$; then $(\beta+1) \in \mathrm{B}^{\uparrow}$ too.*

Proof. We have $\pi(\beta+1) > \pi(\beta)$, $\pi(\beta+1) \in F(\pi(\beta))$, and $\pi(\beta+1) \notin F(\pi(\beta+1)) \ni \pi(\beta+2)$. Since F is ascending, an assumption $\pi(\beta+1) \geq \pi(\beta+2)$ would imply $\pi(\beta+1) \in F(\pi(\beta+1))$ hence $\pi(\beta+2)$ could not have been defined. □

Claim 3.29.2. *Let $\beta \in \mathrm{B}^{\downarrow}$ and $\beta+2 \in \operatorname{Dom}\pi$; then $(\beta+1) \in \mathrm{B}^{\downarrow}$ too.*

The proof is dual to that of Claim 3.29.1.

The key role is played by the following statement.

Claim 3.29.3. *Let $\alpha \in \operatorname{Dom}\pi$ and $\beta < \alpha$. Then $\pi(\alpha) > \pi(\beta)$ if $\beta \in \mathrm{B}^{\uparrow}$ whereas $\pi(\alpha) < \pi(\beta)$ if $\beta \in \mathrm{B}^{\downarrow}$.*

Proof. The proof goes by transfinite induction in α. For $\alpha = 0$, the statement holds by default. Assuming it true for α, let us prove it for $\alpha + 1$.

Let us assume that $\alpha \in \mathrm{B}^{\uparrow}$. Then $\pi(\alpha+1) > \pi(\alpha) > \pi(\beta)$ whenever $\beta < \alpha$ and $\beta \in \mathrm{B}^{\uparrow}$. Let $\beta < \alpha$ and $\beta \in \mathrm{B}^{\downarrow}$; we have to prove that $\pi(\beta) > \pi(\alpha+1)$. By Claim 3.29.2, we have $(\beta+1) \in \mathrm{B}^{\downarrow}$, hence $\beta+1 < \alpha$. Now the induction hypothesis implies $\pi(\beta) > \pi(\beta+1) > \pi(\alpha)$. Then we argue similarly to the proof of Claim 3.29.1: we have $\pi(\alpha+1) \in F(\pi(\alpha))$ and $\pi(\beta) \notin F(\pi(\beta)) \ni \pi(\beta+1)$; since F is ascending and $\pi(\beta) > \pi(\alpha)$, an assumption $\pi(\alpha+1) \geq \pi(\beta)$ would imply $\pi(\beta) \in F(\pi(\alpha))$; but then $\pi(\beta) \in F(\pi(\beta))$ because F is ascending and $\pi(\beta) > \pi(\beta+1) \in F(\pi(\beta))$.

The case of $\alpha \in \mathrm{B}^{\downarrow}$ is treated dually.

To complete the induction step, let us assume that α is a limit and the statement of the lemma holds for all $\alpha' < \alpha$. We pick a sequence $\langle \beta^k \rangle_{k\in\mathbb{N}}$ in $\operatorname{Dom}\pi$ such that $\beta^{k+1} > \beta^k$ for all k, $\alpha = \sup_{k\in\mathbb{N}} \beta^k$, and $\pi(\beta^k) \to \pi(\alpha)$. Without restricting generality, we may assume $\beta^k \in \mathrm{B}^{\uparrow}$ for all k – the case of $\beta^k \in \mathrm{B}^{\downarrow}$ for all k is treated dually. The induction hypothesis implies $\pi(\beta^{k+1}) > \pi(\beta^k)$ for each $k \in \mathbb{N}$, hence $\pi(\alpha) > \pi(\beta^k)$ for each k by the ω-transitivity of the order. Whenever $\beta < \alpha$ and $\beta \in \mathrm{B}^{\uparrow}$, there is $k \in \mathbb{N}$ such that $\beta > \beta^k$, hence $\pi(\beta^k) > \pi(\beta)$ by the induction hypothesis.

Finally, let us assume, to the contrary, that there is $\beta^* < \alpha$ such that $\beta^* \in \mathrm{B}^{\downarrow}$ and $\pi(\beta^*) \leq \pi(\alpha)$. Without restricting generality, $\beta^k > \beta^*$ for all k. We define $\gamma^* := \min\{\gamma \in \mathrm{B}^{\uparrow} \mid \gamma > \beta^*\}$; clearly, $\beta^* < \gamma^* \leq \beta^0$, hence $\pi(\beta^0) > \pi(\gamma^*)$ by the induction hypothesis. Further, γ^* must be a limit: if $\gamma^* = \gamma+1$, then $\gamma \in \mathrm{B}^{\uparrow}$ by Claim 3.29.2, hence $\gamma > \beta^*$ as well, which contradicts the definition of γ^*. Therefore, we can pick a sequence $\langle \gamma^h \rangle_{h\in\mathbb{N}}$ in $\operatorname{Dom}\pi$ such that $\gamma^{h+1} > \gamma^h > \beta^*$ for all h, $\gamma^* = \sup_{h\in\mathbb{N}} \gamma^h$, and $\pi(\gamma^h) \to \pi(\gamma^*)$. Then $\gamma^h \in \mathrm{B}^{\downarrow}$ for each $h \in \mathbb{N}$ by the definition of γ^*, hence $\pi(\beta^*) > \pi(\gamma^h) > \pi(\beta^k)$ for each h and k by the induction hypothesis. Denoting $x^k := \pi(\beta^k)$, $x^\omega := \pi(\alpha)$, $y^h := \pi(\gamma^h)$, and $y^\omega := \pi(\gamma^*)$, we obtain a configuration (3.6) prohibited in the definition of a quasicontinuous order. □

The end of the proof is straightforward. If $\alpha \in \operatorname{Dom} \pi$ and $\alpha > 0$, then either $\pi(\alpha) > \pi(0)$ or $\pi(\alpha) < \pi(0)$ by Claim 3.29.3, hence no cycle is possible. □

Remark. If the order is continuous, then, for every ascending correspondence F and every tâtonnement path π of $\rhd^F$, one of the sets $\mathrm{B}^{\uparrow}$ and $\mathrm{B}^{\downarrow}$ must be empty. If X is, say, $\mathbb{R}^m$ ($m > 1$) with the lexicographic order, then, indeed, both may be non-empty simultaneously.

Concerning quasicontinuous orders, Kukushkin (2003) contains a couple of funny characterization results.

Theorem 3.30 (Kukushkin, 2003, Theorem 6). *Let X be simultaneously a metric space and a poset. Then the following conditions are equivalent:*

1. *The order on X is a quasicontinuous linear order.*
2. *Every ascending correspondence $F: X \to 2^X \setminus \{\emptyset\}$ is Ω-acyclic.*
3. *Every increasing mapping $f: X \to X$ is Ω-acyclic.*
4. *Every ascending correspondence $F: X \to 2^X \setminus \{\emptyset\}$ is quasi-Ω-acyclic.*
5. *Every compact subset $Y \subseteq X$ has the fixed point property.*

Theorem 3.31 (Kukushkin, 2003, Theorem 3). *A linear order on a separable metric space X is quasicontinuous if and only if for every nonempty compact $Y \subseteq X$ and every $Z \subseteq Y$ there exists a supremum of Z in Y.*

Taking into account the well-known characterization of chains compact in their intrinsic topology (Birkhoff, 1967), we immediately obtain the following

Corollary (Kukushkin, 2003). *A linear order on a separable metric space X is quasicontinuous if and only if every compact subset $Y \subseteq X$ is compact in its intrinsic topology.*

Theorem 3.29 implies that iterations of an increasing mapping (or ascending correspondence) from a chain to itself eventually lead to a fixed point. The convergence may take quite some time though.

Proposition 3.32 (Kukushkin, 2000, Theorem 4.3). *For every $\alpha \in \Omega$, there exist an increasing mapping $f^\alpha: [0,1] \to [0,1]$ such that $\operatorname{Dom} \pi \supseteq [0, \alpha]$ for every iteration path of f^α starting at 0 and ending at a fixed point of f^α.*

Proof. We define increasing mappings $f^\alpha: [0,1] \to [0,1]$ ($\alpha \in \Omega$) by transfinite recursion in such a way that $f^\alpha(x) > x$ whenever $0 \leq x < 1$, $f^\alpha(1) = 1$, and there is a unique tâtonnement path π^α of $\rhd^{f^\alpha}$ starting at $\pi^\alpha(0) = 0$ and ending at $\pi^\alpha(\alpha_*) = 1$ with $\alpha_* \geq \alpha$. The uniqueness of the path immediately follows from the inequality $f^\alpha(x) > x$ since all limit points in (3.3b) must be actually limits. We start with $f^0(x) := 1$ for all $x \in [0,1]$; then $\operatorname{Dom} \pi^0 = \{0,1\}$, $\pi^0(0) = 0$, and $\pi^0(1) = 1$.

Assuming f^α already defined, we define $f^{\alpha+1}$ by

$$f^{\alpha+1}(x) := \begin{cases} f^\alpha(2x)/2, & 0 \le x < 1/2; \\ 1, & 1/2 \le x \le 1. \end{cases}$$

Its monotonicity is obvious. For every $\beta \le \alpha$, we have $\pi^{\alpha+1}(\beta) = \pi^\alpha(\beta)/2$; in particular, $\pi^{\alpha+1}(\alpha_*) = 1/2$, hence $\pi^{\alpha+1}(\alpha_* + 1) = 1$.

Let α be a limit and f^β and π^β be already defined for all $\beta < \alpha$. We pick a sequence $\langle \beta^k \rangle_{k\in\mathbb{N}}$ in Ω such that $\beta^{k+1} > \beta^k$ for all k and $\alpha = \sup_{k\in\mathbb{N}} \beta^k$. Then we define $I_k := [1 - 1/2^k, 1 - 1/2^{k+1}[$ for each $k \in \mathbb{N}$ and

$$f^\alpha(x) := \begin{cases} f^{\beta^k}\big(2^{k+1}(x - 1 + 1/2^k)\big)/2^{k+1} + (1 - 1/2^k), & x \in I_k; \\ 1, & x = 1. \end{cases}$$

On every I_k, f^α is increasing as a monotone transformation of f^{β^k}; besides, $1 - 1/2^k \le f^\alpha(x) < 1 - 1/2^{k+1}$ whenever $x \in I_k$. Therefore, f^α is increasing on $[0, 1]$.

For each $k \in \mathbb{N}$, we define an iteration path of f^α by $\pi^k(\beta) := 1 - 1/2^k + \pi^{\beta^k}(\beta)/2^{k+1}$. Clearly, π^k starts at $1 - 1/2^k$ and ends at $\pi^k(\alpha_*^{\beta^k}) = 1 - 1/2^{k+1}$. Finally, we define π^α as a closed concatenation of all π^k with $\pi^\alpha(\alpha_*) = 1$. Applying Lemma 3.6, we obtain $\alpha_* \ge \alpha$. □

Transfinite iteration paths become redundant, in a sense, under a closed graph assumption. However, the assumption does not have as strong implications as upper semicontinuity of an abstract binary relation, cf. Theorems 3.5 and 3.23.

Example 3.33. Let X be a circle in the plane with polar coordinates, $\{(\rho, \varphi) \mid \rho = 1\}$ $(0 \le \varphi_i < 2\pi)$, and let $f : X \to X$ be defined by $f(\rho, \varphi) := (\rho, \varphi \oplus \varphi^0)$, where $\oplus$ denotes addition modulo 2π, and φ^0 is incommensurable with 2π. Clearly, f is continuous and acyclic; however, there is no fixed point.

Theorem 3.34 (Kukushkin, 2000, Theorem 4.4). *Let X be a compact metric space. Let a correspondence $F : X \to 2^X \setminus \{\emptyset\}$ be upper hemicontinuous and Ω-acyclic. Let $\langle x^k \rangle_{k\in\mathbb{N}}$ be a tâtonnement path of $\trianglerighteq^F$, i.e., $x^{k+1} \in F(x^k)$ for each $k \in \mathbb{N}$. Then there is a fixed point of F among limit points of the sequence.*

Remark. The statement of the theorem implies that $\triangleright^F$ has the very weak FTP, but it is more than that. On the other hand, it is weaker than the weak approximate FTP, see Example 3.35 below.

Proof. We denote $Y \subseteq X$ the set of limit points of $\langle x^k \rangle_{k\in\mathbb{N}}$. Y is compact, hence $M(Y, \triangleright^F) \neq \emptyset$ by Theorem 3.21 [(3.5a) $\Rightarrow$ (3.5f)]. Let $x^\omega \in M(Y, \triangleright^F)$. If $x^\omega \in F(x^\omega)$, we are home; let $x^\omega \notin F(x^\omega)$. Since $x^\omega \in Y$, there is a strictly increasing sequence $\langle k_h \rangle_{h\in\mathbb{N}}$ such that $x^{k_h} \to x^\omega$. We denote $y^h := x^{k_h+1}$ $(h \in \mathbb{N})$; without restricting generality, $y^h \to y^\omega \in Y$. By our assumption, $y^h \in F(x^{k_h})$; since F is upper hemicontinuous, $y^\omega \in F(x^\omega)$. Thus, $y^\omega \triangleright^F x^\omega$, contradicting the choice of x^ω. □

Example 3.35 (Kukushkin, 2000, Example 4.3). We start with the definition of a compact subset in a plane with polar coordinates (ρ, φ) ($\rho \geq 0$, $0 \leq \varphi < 2\pi$):

$$X := \big\{(\rho,\varphi) \mid \rho \in \{1\} \cup \{1+1/(m+1)\}_{m\in\mathbb{N}} \ \&\ \varphi \in \{0\} \cup \{\pi/(m+1), 2\pi - \pi/(m+1)\}_{m\in\mathbb{N}} \ \& \\ \rho \leq \min\{\varphi, 2\pi - \varphi\}/\pi + 1\big\}$$

Then we define a mapping $f\colon X \to X$ by

$$f(\rho,\varphi) := \begin{cases} (1,0), & \rho = 1 \ \&\ \varphi = 0; \\ (\rho, \pi/(m+2)), & \rho \leq 1 + 1/(m+2) \ \&\ \varphi = \pi/(m+1); \\ (1+1/(m+2), 2\pi - \pi/(m+2)), & \rho = 1 + 1/(m+1) \ \&\ \varphi = \pi/(m+1); \\ (\rho, 2\pi - \pi/(m+1)), & \rho \leq 1 + 1/(m+2) \ \&\ \varphi = 2\pi - \pi/(m+2). \end{cases}$$

Clearly, f is continuous and $(1,0)$ is its unique fixed point. f is Ω-acyclic: if $(\rho,\varphi) \neq (1,0)$, then $f(\rho,\varphi)$ either has a lesser ρ, or the same ρ and a lesser φ; the appropriate lexicographic order on the plane is ω-transitive. An iteration path started on the circle $\rho = 1$ remains on the circle and converges to the fixed point $(1,0)$. An iteration path started outside the circle $\rho = 1$ converges to it so that every point on the circle is a limit point of the path. Thus, $\rhd^f$ has the very weak FTP, but not the weak approximate FTP.

Proposition 3.36. *Let X be simultaneously a compact metric space and a chain such that the order on X is continuous; let a correspondence $F\colon X \to 2^X \setminus \{\emptyset\}$ be upper hemicontinuous and ascending. Then $\rhd^F$ has the approximate FTP.*

Proof. Let π be a tâtonnement path of $\rhd^F$ with $\operatorname{Dom}\pi = \mathbb{N}$. Since Claims 3.29.1 and 3.29.2 are obviously applicable, we have either $\pi(k+1) > \pi(k)$ for all k, or $\pi(k+1) < \pi(k)$ for all k. Since X is compact and the order is continuous, we have $\pi(k) \to x^*$ as $k \to \infty$. Since the graph of F is closed and $\pi(k+1) \in F(\pi(k))$, we have $x^* \in F(x^*)$, i.e., $x^* \in M(X, \rhd^F)$. □

Remark. Most likely, Proposition 3.36 is wrong if the order is only quasicontinuous; however, I have not elaborated a complete counterexample. The mapping in Example 3.35 is increasing w.r.t. a lexicographic order which is ω-transitive, but not quasicontinuous.

4 Improvement dynamics in strategic games

4.1 A review of basic properties

Applying the notions developed in Sections 2.3 and 3 to improvement relations from Section 2.2, we obtain plenty of properties of strategic games. Artificial examples of games with or without the properties are easy to produce. Much more interesting is the fact that those properties are exhibited by natural classes of games, many of which had attracted attention for independent reasons.

Monderer and Shapley (1996a) called the FTP of the individual improvement relation $\rhd^{\mathrm{Ind}}$ (2.6) in a strategic game Γ the *finite individual improvement property* (*FIP*) of Γ. They defined a *generalized ordinal potential* of a finite game as a function $P\colon X_N \to \mathbb{R}$ satisfying (2.12). Their Lemma 2.5 proves that a finite game has the FIP if and only if it admits such a potential (cf. our Proposition 2.5 and the following remark). The best-known class of games having the FIP are Rosenthal's (1973) congestion games, see also Kukushkin (2007).

Friedman and Mezzetti (2001) called the weak FTP of the individual improvement relation $\rhd^{\mathrm{Ind}}$ (2.6) in a strategic game Γ the *weak FIP* of Γ. That property of a finite game with strategic complementarities was established by Kukushkin et al. (2005).

When each X_i is a metric space with a metric d_i, we define a distance function on X_N by

$$d(x_N, y_N) := \min_i d_i(x_i, y_i).$$

Then we will use, by analogy, the terms *very weak FIP* and (*weak*) *approximate FIP*, as well as *countable improvement property* (*CIP*) and *weak CIP*.

Continuing the analogy, we will use the terms *finite coalition improvement property* (*FCP*) and *countable coalition improvement property* (*CCP*), as well as (*very*) *weak FCP*, (*weak*) *approximate FCP*, and *weak CCP*, when referring to the strong coalition improvement relation $\rhd^{\mathrm{sCo}}$ defined by (2.7). The weak FCP of a certain subclass of congestion games was established by Holzman and Law-Yone (1997); the same property of a modification of congestion games was established by Konishi et al. (1997). When referring to the weak coalition improvement relation $\rhd^{\mathrm{wCo}}$ defined by (2.8), we will use the term *FC⁺P*, as well as *weak, very weak, approximate FC⁺P* etc.

When it comes to the best response improvements, some degree of caution is required. Suppose a game Γ is such that $\mathcal{R}_i(x_{-i}) = \emptyset$ for all $i \in N$ and $x_{-i} \in X_{-i}$. The FTP of the best response improvement relation $\rhd^{\mathrm{BR}}$ defined by (2.9) is obvious; however, there is no Nash equilibrium, so it cannot be reached after a finite number of best response improvements. There may be different ways to cope with that problem. We adopt the simplest of them: we only consider the relations $\rhd^{\mathrm{BR}}$ or $\rhd^{\mathrm{sBR}}$ when the game is BR-consistent. And then we use terminology quite similar to that introduced above: the (*weak*) *finite best response improvement property* ((*weak*) *FBRP*) is the (weak) FTP of $\rhd^{\mathrm{BR}}$, while the (*weak*) *finite simultaneous best response property* ((*weak*) *FSBRP*) is the (weak) FTP of the finite simultaneous best response relation $\rhd^{\mathrm{sBR}}$ defined by (2.10). The weak FBRP of a modification of congestion games was established by Milchtaich (1996). The meaning of the terms *very weak,* (*weak*) *approximate FBRP*, *very weak,* (*weak*) *approximate FSBRP*, (*weak*) *CBRP*, and (*weak*) *CSBRP* should be clear now.

Proposition 4.1. *Let Γ' be a subgame of Γ and let Γ have the FIP, or FC($^+$)P; then Γ' possesses a Nash equilibrium, or a (very) strong equilibrium. The same conclusion holds if Γ has the CIP, or CC($^+$)P, while each X'_i is compact.*

Proof. Immediately follows from Proposition 2.1. □

Remark. If every subgame of Γ possesses a Nash equilibrium, then Γ need not have the FIP (Takahashi and Yamamori, 2002).

It is easy to see that the following implications hold (the fourth and fifth rows assume BR-consistency):

$$
\begin{array}{ccccc}
\text{FC}^{+}\text{P} & \Rightarrow & \text{appr FC}^{+}\text{P} & \Rightarrow & \text{CC}^{+}\text{P} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{FCP} & \Rightarrow & \text{appr FCP} & \Rightarrow & \text{CCP} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{FIP} & \Rightarrow & \text{appr FIP} & \Rightarrow & \text{CIP} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{FBRP} & \Rightarrow & \text{appr FBRP} & \Rightarrow & \text{CBRP} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{weak FBRP} & \Rightarrow & \text{weak appr FBRP} & \Rightarrow & \text{weak CBRP} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{weak FIP} & \Rightarrow & \text{weak appr FIP} & \Rightarrow & \text{weak CIP}
\end{array}.
$$

These properties admit straightforward dynamic interpretations in the same style as the abstract (weak, etc.) FTP. They are also conducive to the convergence of more complicated scenarios (Young, 1993; Kandori and Rob, 1995).

Since an analog of Proposition 2.1 for the relations $\triangleright^{\text{BR}}$ and $\triangleright^{\text{sBR}}$ is obviously wrong, we obtain a partition of our improvement relations into two classes: "hereditary" ones ($\triangleright^{\text{Ind}}$, $\triangleright^{\text{sCo}}$, and $\triangleright^{\text{wCo}}$) and "non-hereditary" ones ($\triangleright^{\text{BR}}$ and $\triangleright^{\text{sBR}}$). The first class is considered in the next subsection; the second, in Subsection 4.3. Actually, $\triangleright^{\text{wCo}}$ is not considered here at all because our continuity assumptions do not make it much nicer. Instead, we consider a common generalization of individual and strong coalition improvements. Its use here is purely technical, but it may deserve attention for its own sake under proper circumstances.

Let $\mathcal{I} \subseteq \mathcal{N}$; we define a relation $\triangleright^{\mathcal{I}}$ on X_N by an analog of (2.7):

$$y_N \triangleright^{\mathcal{I}} x_N \rightleftharpoons \exists I \in \mathcal{I}\, [y_N \triangleright_I^{\text{sCo}} x_N]. \tag{4.1}$$

A tâtonnement path of $\triangleright^{\mathcal{I}}$ is called an $\mathcal{I}$-improvement path; a maximizer of $\triangleright^{\mathcal{I}}$, an $\mathcal{I}$-equilibrium. Clearly, $\triangleright^{\text{sCo}}$ is $\triangleright^{\mathcal{N}}$ while $\triangleright^{\text{Ind}}$ is $\triangleright^{\mathcal{I}}$ with $\mathcal{I}$ consisting of singleton subsets of $\mathcal{N}$; accordingly, the notions of Nash and strong equilibrium are particular cases of $\mathcal{I}$-equilibrium. The relations $\triangleright^{\mathcal{I}}$ are hereditary in the sense of Proposition 2.1, hence an analog of Proposition 4.1 is valid for them too.

Whenever π is an $\mathcal{I}$-improvement path and $(\alpha + 1) \in \operatorname{Dom} \pi$, we denote $\iota(\alpha) \in \mathcal{I}$ the coalition who move at the step α, i.e., $\pi(\alpha + 1) \triangleright_{\iota(\alpha)}^{\text{sCo}} \pi(\alpha)$.

4.2 "Quasi-continuous" preferences

Throughout this subsection, we make the following "quasi-continuity" assumption about the preference relations:

$$\forall i \in N \, \forall y_N, x_N \in X_N \Big[y_N \not\succ_i x_N \Rightarrow \\ \exists \delta \in \mathbb{R}_{++} \big[\forall y'_N, x'_N \in X_N \, [d(x_N, x'_N) < \delta \;\&\; d(y_N, y'_N) < \delta \Rightarrow (y_i, y'_{-i}) \not\succ_i x'_N] \big] \Big]. \quad (4.2)$$

The assumption has an immediate corollary for (individual or coalition) improvements:

$$\forall I \in \mathcal{N} \, \forall y_N, x_N \in X_N \Big[y_N \rhd_I^{\mathrm{sCo}} x_N \Rightarrow \\ \exists \delta \in \mathbb{R}_{++} \big[\forall x'_N \in X_N \, [d(x_N, x'_N) < \delta \Rightarrow (y_I, x'_{-I}) \rhd_I^{\mathrm{sCo}} x'_N] \big] \Big]. \quad (4.3)$$

A sufficient condition for (4.2) is the continuity of each preference relation. It is not necessary; e.g., when preferences are described by utility functions, (4.2) holds if each u_i is upper semicontinuous in x_N and continuous in x_{-i}.

Proposition 4.2. *Let a strategic game Γ satisfy condition (4.2), $\mathcal{I} \subseteq \mathcal{N}$, π be an $\mathcal{I}$-improvement path, $\alpha \in \operatorname{Dom} \pi$, and $\varepsilon > 0$. Then there exists a finite $\mathcal{I}$-improvement path π^* of length m such that $\pi^*(0) = \pi(0)$ and $d(\pi(\alpha), \pi^*(m)) < \varepsilon$.*

Proof. The proof goes by transfinite induction; if α is finite, the restriction of π to $[0, \alpha]$ will do. Assuming the statement valid for all $\alpha < \alpha'$, let us prove it for α'. If α' is a limit, there is $\alpha < \alpha'$ such that $d(\pi(\alpha'), \pi(\alpha)) < \varepsilon/2$. Applying the induction hypothesis to α, we obtain a finite $\mathcal{I}$-improvement path π^* of length m such that $\pi^*(0) = \pi(0)$ and $d(\pi(\alpha), \pi^*(m)) < \varepsilon/2$. Therefore, $d(\pi(\alpha'), \pi^*(m)) < \varepsilon$ and we are home.

Let α' be a successor, $\alpha' = \alpha + 1$. We invoke (4.3) with $I := \iota(\alpha)$, $y_N := \pi(\alpha')$ and $x_N := \pi(\alpha)$, and obtain $\delta > 0$ such that $(\pi_{\iota(\alpha)}(\alpha'), x'_{-\iota(\alpha)}) \rhd_{\iota(\alpha)}^{\mathrm{sCo}} x'_N$ whenever $d(\pi(\alpha), x'_N) < \delta$. Then we apply the induction hypothesis to α, obtaining a finite $\mathcal{I}$-improvement path π^* of length m such that $\pi^*(0) = \pi(0)$ and $d(\pi(\alpha), \pi^*(m)) < \min\{\varepsilon, \delta\}$. Finally, we extend π^* to $\{0, \ldots, m+1\}$ defining $\pi^*(m+1) := (\pi_{\iota(\alpha)}(\alpha'), \pi^*_{-\iota(\alpha)}(m))$. □

Corollary. *Let a strategic game Γ satisfy condition (4.2). Then the weak CIP implies the very weak FIP, and the weak CCP implies the very weak FCP.*

Theorem 4.3. *Let a strategic game Γ satisfy condition (4.2), $\#N = 2$, and $\mathcal{I} \subseteq \mathcal{N}$. Then the relation $\rhd^{\mathcal{I}}$ is Ω-acyclic if and only if it is acyclic.*

Proof. Suppose the contrary: $\rhd^{\mathcal{I}}$ is acyclic, but there is an $\mathcal{I}$-improvement path π and $\alpha \in \operatorname{Dom} \pi$ such that $\pi(\alpha) = \pi(0)$ and $\alpha > 0$. Since $\rhd^{\mathcal{I}}$ is acyclic, α must be infinite; without restricting generality, $\alpha = \max \operatorname{Dom} \pi$. Applying (4.3) to the relations $\pi(1) \rhd_{\iota(0)}^{\mathrm{sCo}} \pi(0)$ and $\pi(2) \rhd_{\iota(1)}^{\mathrm{sCo}} \pi(1)$, we obtain appropriate δ_1 and δ_2. Then we define $\varepsilon := \min\{\delta_1, \delta_2\}$ and apply

Proposition 4.2 with that ε to the restriction of π to $\operatorname{Dom}\pi \setminus \{0,1\}$, obtaining a finite $\mathcal{I}$-improvement path π^* of length m such that $\pi^*(0) = \pi(2)$ and $d(\pi(\alpha), \pi^*(m)) < \varepsilon$. Then we extend π^* to $\{0, \ldots, m+2\}$ defining $\pi^*(m+1) := (\pi_{\iota(0)}(1), \pi^*_{-\iota(0)}(m))$ and $\pi^*(m+2) := (\pi_{\iota(1)}(2), \pi^*_{-\iota(1)}(m+1))$; recalling the definition of ε and that $\pi(\alpha) = \pi(0)$, we see that the extended π^* remains a finite $\mathcal{I}$-improvement path. Finally, $\pi^*_{\iota(0)\cup\iota(1)}(m+2) = \pi_{\iota(0)\cup\iota(1)}(2) = \pi^*_{\iota(0)\cup\iota(1)}(0)$; since $\iota(0) \cup \iota(1) = N$, we have $\pi^*(m+2) = \pi^*(0)$, i.e., a contradiction with the supposed acyclicity. □

Theorem 4.4. *Let a strategic game Γ satisfy condition (4.2); let each X_i be compact; let $\mathcal{I} \subseteq \mathcal{N}$ and $\triangleright^{\mathcal{I}}$ be acyclic. Then $\triangleright^{\mathcal{I}}$ has the very weak FTP.*

Proof. Given $x^0_N \in X_N$, we denote $Y \subseteq X_N$ the set of strategy profiles that can be reached from x^0_N with finite $\mathcal{I}$-improvement paths. Then we define $Z := \operatorname{cl} Y$; clearly, Z is compact. We have to prove that Z contains an $\mathcal{I}$-equilibrium, i.e., a maximizer of $\triangleright^{\mathcal{I}}$ on X_N.

Claim 4.4.1. *If $z_N \in Z$ and $y_N \triangleright^{\mathcal{I}} z_N$, then $y_N \in Z$.*

Proof. Let $y_N \triangleright^{\text{sCo}}_I z_N$; by (4.3), there is $\delta > 0$ such that $(y_I, x_{-I}) \triangleright^{\text{sCo}}_I x_N$ whenever $d(z_N, x_N) < \delta$. Given $\varepsilon > 0$, there is a finite $\mathcal{I}$-improvement path π of length m such that $\pi(0) = x^0_N$ and $d(\pi(m), z_N) < \min\{\delta, \varepsilon\}$. We extend π, defining $\pi(m+1) := (y_I, \pi_{-I}(m))$; the extended π remains a finite $\mathcal{I}$-improvement path, hence $\pi(m+1) \in Y$. Since $d(\pi(m+1), y_N) < \varepsilon$ and ε was arbitrary, we have $y_N \in Z$ indeed. □

Now we may forget about $X_N \setminus Z$, and prove the existence of a maximizer of $\triangleright^{\mathcal{I}}$ on Z. Supposing the contrary, we fix, for every $x_N \in Z$, a $y_N(x_N) \in Z$ and an $I(x_N) \in \mathcal{I}$ such that $y_N(x_N) \triangleright^{\text{sCo}}_{I(x_N)} x_N$, and denote $U(x_N)$ the open ball around x_N of radius δ from (4.3). Since Z is compact, it is covered by a finite number of $U(x_N)$; we pick a finite set $X^* \subseteq Z$ accordingly, and define $X'_i := \{y_i(x_N) \mid x_N \in X^* \;\&\; i \in I(x_N)\} \cup \{x^0_i\}$ and $X'_N := \prod_{i\in M} X'_i$.

Then we recursively construct an infinite sequence $\langle x^k_N \rangle_{k\in\mathbb{N}}$, starting with x^0_N already given. Having $x^k_N \in Z$ defined, we pick $x_N \in X^*$ such that $x^k_N \in U(x_N)$ and define $x^{k+1}_N := (y_{I(x_N)}(x_N), x^k_{-I(x_N)})$. By (4.3), we have $x^{k+1}_N \triangleright^{\mathcal{I}} x^k_N$, hence $x^{k+1}_N \in Z$ by Claim 4.4.1. Therefore, $\langle x^k_N \rangle_{k\in\mathbb{N}}$ is an infinite $\mathcal{I}$-improvement path; moreover, $x^k_N \in X'_N$ for each $k \in \mathbb{N}$ by definition. Now the existence of an infinite $\mathcal{I}$-improvement path in a finite subgame contradicts the supposed acyclicity of $\triangleright^{\mathcal{I}}$. □

Corollary. *If a game with compact X_i's and utilities u_i upper semicontinuous in x_N and continuous in x_{-i} admits a generalized ordinal potential as defined by Monderer and Shapley (1996a), then it has the very weak FIP.*

Open Problem 4.5. *Let the preferences of the players in a strategic game Γ satisfy condition (4.2); let each X_i be compact; let $\mathcal{I} \subseteq \mathcal{N}$ and $\triangleright^{\mathcal{I}}$ be acyclic. Must $\triangleright^{\mathcal{I}}$ then have the weak approximate FTP? If the answer is negative, would the assumption of continuous preferences or restriction to two person games or to individual improvements help?*

What we can be sure of is that the assumptions of Problem 4.5 do not imply the CIP, nor even CBRP, if $n > 2$.

Example 4.6. Let us consider a strategic game Γ with $N := \{1, 2, 3\}$, $X_i := \{-1 + 1/2^k, 1 - 1/2^k\}_{k \in \mathbb{N}} \cup \{-1, 1\}$ for each $i \in \mathbb{N}$, and utility functions $u_i \colon X_N \to \mathbb{R}$ defined by the following constructions. First, we define a mapping $N \to N$ by $1' := 2$, $2' := 3$, and $3' := 1$; a mapping $\eta \colon X_i \to \mathbb{N} \cup \{\infty\}$ by $\eta(\pm 1) := \infty$ and $\eta(\pm 1 \mp 1/2^k) := k$; a mapping $\eta^+ \colon X_i \to \mathbb{N} \cup \{\infty\}$ by $\eta^+(x_N) := \max_i \eta(x_i)$; mappings $\varkappa, \nu \colon (\mathbb{N} \cup \{\infty\}) \times (\mathbb{N} \cup \{\infty\}) \to \mathbb{N} \cup \{\infty\}$ by $\varkappa(k, h) := \min\{k, h+1, \max\{k, h, 1\} - 1\}$ and $\nu(k, h) := \min\{k, h, \max\{k, h, 1\} - 1\}$. Then, for each $i \in N$, we define a mapping $r_i \colon X_{-i} \to X_i$ by

$$r_i(x_{-i}) := \begin{cases} -1 + 1/2^{\varkappa(\eta(x_{i''}), \eta(x_{i'}))}, & 0 \le x_{i'} \le 1,\ 0 \le x_{i''} \le 1; \\ -1 + 1/2^{\nu(\eta(x_{i'}), \eta(x_{i''}))}, & 0 \le x_{i'} \le 1,\ -1 \le x_{i''} \le 0; \\ 0, & -1 \le x_{i'} < 0,\ 0 < x_{i''} \le 1; \\ 1 - 1/2^{\varkappa(\eta(x_{i'}), \eta(x_{i''}))}, & -1 \le x_{i'} \le 0,\ -1 \le x_{i''} \le 0. \end{cases}$$

Finally we define

$$u_i(x_N) := \begin{cases} 1/2^{\eta(x_i)}, & x_i = r_i(x_{-i}); \\ 0, & \text{otherwise}. \end{cases}$$

It is easily checked that both r_i and u_i are continuous. The best response correspondence is $\mathcal{R}_i(x_{-i}) = \{r_i(x_{-i})\}$ if $\eta(r_i(x_{-i})) < \infty$ and $\mathcal{R}_i(x_{-i}) = X_i$ otherwise. There are three Nash equilibria: $(-1, -1, -1)$, $(0, 0, 0)$, and $(1, 1, 1)$.

The relation $\triangleright^{\text{Ind}}$, even $\triangleright^{\text{BR}}$, is not Ω-acyclic. For instance, let us consider the following mapping $[0, \omega + \omega] \to X_N$: $\pi(0) := (1, -1, -1)$; $\pi(1) := (1, 0, -1)$; $\pi(2k+2) := (1, -1 + 1/2^k, 1 - 1/2^k)$; $\pi(2k+3) := (1, -1 + 1/2^{k+1}, 1 - 1/2^k)$; $\pi(\omega) := (1, -1, 1)$; $\pi(\omega + 1) := (0, -1, 1)$; $\pi(\omega + 2k + 2) := (1 - 1/2^k, -1, -1 + 1/2^k)$; $\pi(\omega + 2k + 3) := (1 - 1/2^{k+1}, -1, -1 + 1/2^k)$; $\pi(\omega + \omega) := (1, -1, -1)$. It is easy to check that $\pi(2k+1) \triangleright_2^{\text{BR}} \pi(2k)$, $\pi(2k+2) \triangleright_3^{\text{BR}} \pi(2k+1)$, $\pi(\omega + 2k + 1) \triangleright_1^{\text{BR}} \pi(\omega + 2k)$, and $\pi(\omega + 2k + 2) \triangleright_3^{\text{BR}} \pi(\omega + 2k + 1)$ for all $k \in \mathbb{N}$. Moreover, $\pi(k) \to \pi(\omega)$ and $\pi(\omega + k) \to \pi(\omega + \omega) = \pi(0)$. Therefore, π is a best response improvement cycle.

On the other hand, whenever $y_N \triangleright_i^{\text{Ind}} x_N$, we must have $y_i = r_i(x_{-i})$ and $\eta(y_i) < \infty$; furthermore, $\eta(y_i) < \eta^+(y_N)$. It is easy to see now that $\triangleright^{\text{Ind}}$ is acyclic.

4.3 Cournot tâtonnement

Throughout this subsection, we restrict attention to BR-consistent strategic games; sufficient conditions for the property are given by Corollary to Proposition 2.9 and Corollary to Theorem 3.12. Henceforth, tâtonnement paths of $\triangleright^{\text{BR}}$ are called just *Cournot paths*; tâtonnement paths of $\triangleright^{\text{sBR}}$, *simultaneous Cournot paths.* A strict/weak order potential of $\triangleright^{\text{BR}}$ is called a *strict/weak Cournot potential*; a strict/weak order potential of $\triangleright^{\text{sBR}}$, a *strict/ weak simultaneous Cournot potential.* By Proposition 2.5, a game has the FBRP (FSBRP) if and only if it admits a strict (simultaneous) Cournot potential; by Proposition 2.7, a game has the weak FBRP (FSBRP) if and only if it admits a weak (simultaneous) Cournot potential.

A (weak) ω-potential of $\rhd^{\mathrm{BR}}$ is called a (*weak*) *Cournot ω-potential*; A (weak) ω-potential of $\rhd^{\mathrm{sBR}}$, a (*weak*) *simultaneous Cournot ω-potential.* By Theorem 3.21 [(3.5b) $\iff$ (3.5c)], a game has the CBRP (CSBRP) if and only if it admits a (simultaneous) Cournot ω-potential; by Proposition 3.22, a game has the weak CBRP (CSBRP) if it admits a weak (simultaneous) Cournot potential.

Proposition 4.7. *If a two person game Γ has the (approximate) FSBRP, then it has the (approximate) FBRP.*

Proof. By definition, $y_N \rhd^{\mathrm{sBR}} x_N$ whenever $y_N \rhd^{\mathrm{BR}} x_N$ and $x_i \in \mathcal{R}_i(x_{-i})$ for an $i \in N$. Therefore, every Cournot path becomes a simultaneous Cournot path after the first step. □

Besides best response *improvements*, it sometimes makes sense to consider best response "pseudo-improvements." We define

$$y_N \trianglerighteq^{\mathrm{BR}} x_N \rightleftharpoons \exists i \in N\,[x_i \neq y_i \in \mathcal{R}_i(x_{-i})\ \&\ y_{-i} = x_{-i}]; \tag{4.4}$$

$$y_N \trianglerighteq^{\mathrm{sBR}} x_N \rightleftharpoons \big[y_N \neq x_N\ \&\ \forall i \in N\,[y_i \in \mathcal{R}_i(x_{-i})]\big]. \tag{4.5}$$

A (*simultaneous*) *pseudo-Cournot path* is a tâtonnement path of $\trianglerighteq^{\mathrm{BR}}$ ($\trianglerighteq^{\mathrm{sBR}}$). A game has the *pseudo-FBRP* (*pseudo-FSBRP*) if, for every $x_N \in X_N$, there exists a finite (simultaneous) pseudo-Cournot path $\langle x_N^0, \dots, x_N^m \rangle$ such that $x_N^0 = x_N$ and x_N^m is a Nash equilibrium. A game has the *pseudo-CBRP* (*pseudo-CSBRP*) if, for every $x_N \in X_N$, there exists a (simultaneous) pseudo-Cournot path π such that $\pi(0) = x_N$, $\operatorname{Dom}\pi = [0,\alpha] \subset \Omega$, and $\pi(\alpha)$ is a Nash equilibrium.

Remark. A Nash equilibrium need not be a maximizer of $\trianglerighteq^{\mathrm{BR}}$, nor $\trianglerighteq^{\mathrm{sBR}}$; therefore, we cannot define pseudo-FBRP, pseudo-FSBRP, etc. as the weak FTP of $\trianglerighteq^{\mathrm{BR}}$, etc.

A (*simultaneous*) *pseudo-Cournot potential* of a game Γ is a strictly acyclic and transitive binary relation $\succ$ on X_N such that, whenever x_N is not a Nash equilibrium, there is $y_N \in X_N$ such that $y_N \succ x_N$ and $y_N \trianglerighteq^{\mathrm{BR}} x_N$ (respectively, $y_N \trianglerighteq^{\mathrm{sBR}} x_N$). A (*simultaneous*) *$\omega$-pseudo-potential* of Γ is an irreflexive and ω-transitive relation $\succ$ on X_N such that, whenever x_N is not a Nash equilibrium, there is $y_N \in X_N$ such that $y_N \succ x_N$ and $y_N \trianglerighteq^{\mathrm{BR}} x_N$ (respectively, $y_N \trianglerighteq^{\mathrm{sBR}} x_N$). Arguing quite similarly to the proofs of Propositions 2.7 and 3.22, we can show that Γ has the pseudo-FBRP (pseudo-FSBRP) if and only if it admits a (simultaneous) Cournot pseudo-potential, and that Γ has the pseudo-CBRP (pseudo-CSBRP) if it admits a (simultaneous) Cournot ω-pseudo-potential.

Defining a correspondence $\mathcal{R}_N \colon X_N \to 2^{X_N} \setminus \{\emptyset\}$ by $\mathcal{R}_N(x_N) := \prod_{i \in N} \mathcal{R}_i(x_{-i})$, we immediately see that every simultaneous pseudo-Cournot path is an iteration paths of $\mathcal{R}_N$. If we agree never to extend a simultaneous pseudo-Cournot path beyond a Nash equilibrium, the converse becomes true as well.

Proposition 4.8. *If a two person game Γ is strongly BR-consistent and has the pseudo-FSBRP, then it has the weak FBRP and weak FSBRP.*

Proof. Let Γ have the pseudo-FSBRP and $x_N^0 \in X_N$; then there is a simultaneous pseudo-Cournot path $\langle x_N^0, \ldots, x_N^m \rangle$ such that x_N^m is a Nash equilibrium. We define a sequence $\langle y_N^0, y_N^1, \ldots, y_N^{m+1} \rangle$ in this way: $y_N^0 := x_N^0$; $y_1^{2k+1} := x_1^{2k+1}$; $y_2^{2k+1} := x_2^{2k}$; $y_1^{2k+2} := x_1^{2k+1}$; $y_2^{2k+1} := x_2^{2k+2}$; if $2k = m$, we set $y_1^{m+1} := x_1^m$; if $2k+1 = m$, we set $y_2^{m+1} := x_2^m$. Thus, $y_N^{m+1} = x_N^m$ in either case.

By our construction, for each $k = 0, 1, \ldots, m$ we have $y_i^{k+1} \in \mathcal{R}_i(y_{-i}^k)$ and $y_{-i}^{k+1} = y_{-i}^k$ for an $i \in N$; therefore, for each $k = 1, \ldots, m$ we have $y_i^k \in \mathcal{R}_i(y_{-i}^k)$ for at least one $i \in N$. If $y_i^k \in \mathcal{R}_i(y_{-i}^k)$ for both $i \in N$, which inevitably occurs when $k = m+1$, then y_N^k is a Nash equilibrium. Otherwise, $y_N^{k+1} \rhd^{\mathrm{BR}} y_N^k$. We see that $\langle y_N^0, y_N^1, \ldots, y_N^{\bar{k}} \rangle$ ($\bar{k} \leq m$) is a Cournot path starting at $x_N^0 = y_N^0$ and ending at a Nash equilibrium. Since $x_N^0 \in X_N$ was arbitrary, Γ has the weak FBRP.

Let us show that Γ has the weak FSBRP as well. Given $x_N^0 \in X_N$, there is again a simultaneous pseudo-Cournot path $\langle x_N^0, \ldots, x_N^m \rangle$ such that x_N^m is a Nash equilibrium. If $x_N^{k+1} \rhd^{\mathrm{sBR}} x_N^k$ for each $k = 1, \ldots, m$, "pseudo" can be dropped, and we are home. Otherwise, let $\bar{k}$ be the first moment when "pseudo" was essential, i.e., $x_i^k \in \mathcal{R}_i(x_{-i}^k)$, but $x_i^{k+1} \neq x_i^k$ for an $i \in N$; then $x_{-i}^k \notin \mathcal{R}_i(x_i^k)$ because x_N^k would be a Nash equilibrium otherwise. Denoting $y_N^0 := (x_i^k, x_{-i}^{k+1})$, we have $y_N^0 \rhd^{\mathrm{sBR}} x_N^k$. Since Γ has the weak FBRP, there is a Cournot path starting at y_N^0 and ending at a Nash equilibrium. Since $y_{-i}^0 \in \mathcal{R}_{-i}(y_i^0)$, the path is a simultaneous Cournot path as well, exactly as in the proof of Proposition 4.7. □

Remark. The Battle of Sexes has the FBRP (even the FIP), but not the pseudo-FSBRP.

It is unclear whether any analog of Proposition 4.8 could be obtained without strong BR-consistency; on the other hand, the very notion of a pseudo-Cournot path becomes especially dubious in this case. When there are more than two players, there seems to be no relation between the convergence of Cournot paths and simultaneous Cournot paths even if the preferences are described by utility functions (see Moulin, 1986).

Two sorts of properties intermediate between the FBRP (CBRP) and weak FBRP (CBRP) deserve attention. The first of them is meaningful for $n > 2$; the second, closely related to the sequential tâtonnement process as defined by Moulin (1984, p. 87), for $n \geq 2$.

We call a Cournot path π *inclusive* if for each player $i \in N$ and each $\alpha \in \mathrm{Dom}\,\pi$, there is $\alpha' \geq \alpha$ such that $\pi_i(\alpha') \in \mathcal{R}_i(\pi_{-i}(\alpha'))$. A game has the *finite inclusive best response improvement property* (*FIBRP*) if it admits no infinite inclusive Cournot path. A game has the *approximate FIBRP* if the set of limit points of every inclusive Cournot path π such that $\mathrm{Dom}\,\pi = \mathbb{N}$ is a nonempty subset of the set of Nash equilibria. A game has the *countable inclusive best response improvement property* (*CIBRP*) if it admits no inclusive Cournot path π such that $\mathrm{Dom}\,\pi = \Omega$. Clearly,

$$\begin{array}{ccccc} \text{FBRP} & \Rightarrow & \text{approximate FBRP} & \Rightarrow & \text{CBRP} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{FIBRP} & \Rightarrow & \text{approximate FIBRP} & \Rightarrow & \text{CIBRP.} \end{array}$$

Proposition 4.9. *If a two person game has the FIBRP, then it has the FBRP.*

Proof. In a two person game, every infinite Cournot path is inclusive. □

A Cournot cycle π ($\pi(0) = \pi(\alpha)$, $\alpha > 0$) is *complete* if for each player $i \in N$ there is $\beta \in [0, \alpha]$ such that $\pi_i(\beta) \in \mathcal{R}_i(\pi_{-i}(\beta))$.

A *Cournot quasipotential* is a preorder (i.e., a reflexive and transitive binary relation) $\succeq$ on X_N such that its asymmetric component $\succ$ is strictly acyclic and for every $x_N \in X_N$ there exists a subset $M(x_N) \subseteq N$ satisfying

$$y_N \rhd^{\mathrm{BR}} x_N \Rightarrow \big[y_N \succ x_N \;\text{ or }\; [y_N \sim x_N \;\&\; M(y_N) \subseteq M(x_N) \neq \emptyset]\big]; \tag{4.6a}$$

$$i \in M(x_N) \Rightarrow x_i \notin \mathcal{R}_i(x_{-i}). \tag{4.6b}$$

If $\succ$ is a strict Cournot potential, then its reflexive closure $\succeq$ is a Cournot quasipotential with $M(x_N) = \emptyset$ for all $x_N \in X_N$. If $\succeq$ is a Cournot quasipotential, then its asymmetric component $\succ$ is a weak Cournot potential.

Proposition 4.10. *Given a BR-consistent strategic game* Γ*, let us consider these statements:*

$$\Gamma \textit{ admits a Cournot quasipotential}; \tag{4.7a}$$

$$\Gamma \textit{ has the FIBRP}; \tag{4.7b}$$

$$\Gamma \textit{ admits no finite complete Cournot cycle.} \tag{4.7c}$$

Then (4.7a) $\Rightarrow$ (4.7b) $\Rightarrow$ (4.7c)*. If* Γ *is finite,* (4.7c) $\Rightarrow$ (4.7a)*.*

Proof. Let $\succeq$ be a Cournot quasipotential and $\langle x_N^k \rangle_{k \in \mathbb{N}}$ be an infinite Cournot path; we have to show that the path is not inclusive. Since $\succ$ is strictly acyclic, (4.6a) implies that $x_N^{k+1} \sim x_N^k$ and $M(x_N^{k+1}) \subseteq M(x_N^k) \neq \emptyset$ for each $k \geq \bar{k}$. Since N is finite, we have $M(x_N^{k+1}) = M(x_N^k) \neq \emptyset$ for all $k \geq m\, [\geq \bar{k}]$. By (4.6b), $x_i^k \notin \mathcal{R}_i(x_{-i}^k)$ for all $i \in M(x_N^m)$ and $k \geq m$. Thus, $\langle x_N^k \rangle_{k \in \mathbb{N}}$ is not inclusive indeed.

Infinite repetition of a finite complete Cournot cycle generates an infinite inclusive Cournot path, hence the FIBRP implies the absence of complete Cournot cycles.

Finally, let Γ be finite and there be no complete Cournot cycle. We denote $\succeq$ the reflexive and transitive closure of $\rhd^{\mathrm{BR}}$: $y_N \succeq x_N$ if and only if there is a finite Cournot path $x_N^0, x_N^1, \ldots, x_N^m$ such that $x_N^0 = x_N$ and $x_N^m = y_N$ ($m \geq 0$). Let $Y \subseteq X_N$ be an equivalence class of $\sim$ with $\#Y > 1$; we denote $D(Y) = \{i \in N \mid \forall x_N \in Y\, [x_i \notin \mathcal{R}_i(x_{-i})]\}$. Since all $x_N \in Y$ can be arranged into a single Cournot cycle and that cycle cannot be complete, $D(Y) \neq \emptyset$. Now we define $M(x_N) = D(Y)$ if x_N belongs to a non-singleton equivalence class Y, and $M(x_N) = \emptyset$ otherwise. The conditions (4.6) are checked easily. □

Remark. In the proof of Theorem 3 of Kukushkin (2004), the FBRP was derived from the presence of a "quasipotential" in an even weaker sense than (4.6). The point is that whenever a game satisfies the conditions of that theorem, so do all its subgames. Generally, we only obtain FIBRP. In particular, dominance solvability (in any sense) need not be inherited by the subgames, hence Theorem 5.3 below also asserts only FIBRP. It may also be noted that Proposition 4.10 is actually about an arbitrary binary relation with a *disjunctive structure*; its analogs are valid for $\rhd^{\text{Ind}}$ or even $\rhd^{\text{sCo}}$ for that matter. The only difference is that no interesting applications of such analogs are known at the moment.

An *approximate Cournot quasipotential* of Γ is a preorder $\succeq$ on X_N such that its asymmetric component $\succ$ is strictly acyclic on every set

$$X(\delta) := \{x_N \in X_N \mid \forall x'_N \in X_N \left[d(x_N, x'_N) < \delta \Rightarrow \exists y_N \in X_N \left[y_N \rhd^{\text{BR}} x'_N\right]\right]\} \quad (\delta > 0), \quad (4.8)$$

while for every $x_N \in X_N$ there exists a subset $M(x_N) \subseteq N$ satisfying (4.6).

Proposition 4.11. *Let Γ be a BR-consistent game where each X_i is a compact metric space and the set of Nash equilibria is closed in X_N. Let Γ admit an approximate Cournot quasipotential. Then Γ has the approximate FIBRP.*

Proof. We employ a combination of the proofs of Propositions 3.3 and 4.10. Let $\succeq$ be an approximate Cournot quasipotential and $\langle x_N^k \rangle_{k\in\mathbb{N}}$ be an infinite inclusive Cournot path; (4.6a) implies that $x_N^{k+1} \succeq x_N^k$ for all k. Let y_N be a limit point of the path. An assumption that y_N does not belong to the closure of the set of Nash equilibria would imply the existence of $\delta > 0$ such that $x_N^k \in X(\delta)$ for infinitely many k. Since $\succ$ is strictly acyclic on $X(\delta)$, (4.6a) would imply that $x_N^{k+1} \sim x_N^k$ and $M(x_N^{k+1}) \subseteq M(x_N^k) \neq \emptyset$ for each $k \geq \bar{k}$. Since N is finite, we would have $M(x_N^{k+1}) = M(x_N^k) \neq \emptyset$ for all $k \geq m\,[\geq \bar{k}]$. By (4.6b), $x_i^k \notin \mathcal{R}_i(x_{-i}^k)$ for all $i \in M(x_N^m)$ and $k \geq m$, which contradicts the assumption that $\langle x_N^k \rangle_{k\in\mathbb{N}}$ is inclusive. $\square$

A *Cournot ω-quasipotential* is an ω-transitive preorder $\succeq$ on X_N such that for every $x_N \in X_N$ there exists a subset $M(x_N) \subseteq N$ satisfying (4.6) and this condition:

$$\left[x^\omega = \lim_{k\to\infty} x^k \;\&\; \forall k \in \mathbb{N}\,[x^{k+1} \sim x^k \;\&\; M(x^{k+1}) = M(x^k)\,]\right] \Rightarrow [x^\omega \succ x^0 \;\text{ or }\; M(x^\omega) = M(x^0)\,]. \quad (4.9)$$

If $\succ$ is a Cournot ω-potential, then its reflexive closure $\succeq$ is a Cournot ω-quasipotential with $M(x_N) = \emptyset$ for all $x_N \in X_N$. If $\succeq$ is a Cournot ω-quasipotential, then its asymmetric component $\succ$ is a weak Cournot ω-potential.

Proposition 4.12. *Let Γ be a BR-consistent strategic game where each X_i is a compact metric space. Then these statements are equivalent:*

Γ *admits a Cournot ω-quasipotential*; (4.10a)

Γ *has the CIBRP*; (4.10b)

Γ *admits no complete Cournot cycle.* (4.10c)

Proof. Let $\succeq$ be a Cournot ω-quasipotential and π be a Cournot path with $\operatorname{Dom}\pi = \Omega$. By (4.6a), ω-transitivity, and Lemma 3.9, $\pi(\alpha) \succeq \pi(\beta)$ whenever $\alpha > \beta$. We have to show that π is not inclusive. By Proposition 3.15, the asymmetric component $\succ$ of $\succeq$ is ω-transitive too, hence Proposition 3.11 implies the existence of $\alpha^* \in \Omega$ such that $\pi(\alpha) \sim \pi(\beta)$ whenever $\alpha, \beta \geq \alpha^*$. Now (4.6a) implies that $\emptyset \neq M(\pi(\alpha+1)) \subseteq M(\pi(\alpha))$ whenever $\alpha \geq \alpha^*$. The finiteness of N and (4.9) imply that $M(\pi(\alpha)) = M(\pi(\beta)) \neq \emptyset$ whenever $\alpha, \beta \geq \alpha^{**}[\geq \alpha^*]$. By (4.6b), $\pi_i(\alpha) \notin \mathcal{R}_i(\pi_{-i}(\alpha))$ for all $i \in M(\pi(\alpha^{**}))$ and $\alpha \geq \alpha^{**}$. Thus, π is not inclusive indeed.

Endless repetition of a complete Cournot cycle generates an inclusive Cournot path with $\operatorname{Dom}\pi = \Omega$; formalism is the same as in the proof of Theorem 3.21. Therefore, the CIBRP implies the absence of complete Cournot cycles.

Assuming (4.10c), we denote $\succeq$ the reflexive and ω-transitive closure of $\rhd^{\mathrm{BR}}$ and $\succ$ its asymmetric component. $\succeq$ is ω-transitive by definition. For every $x_N \in X_N$, we define $N(x_N) := \{i \in N \mid x_i \notin \mathcal{R}_i(x_{-i})\}$ and $M(x_N) := \bigcap_{y_N \sim x_N} N(y_N)$. Clearly, $N(x_N) = \emptyset$ if and only if x_N is a Nash equilibrium; let us show that the same holds for $M(x_N)$. Indeed, let $M(x_N) = \emptyset$ while $N(x_N) \neq \emptyset$. Since N is finite, there is a finite set $\{y_N^1, \ldots, y_N^m\} \subseteq X_N$ $(m \leq n)$ such that $y_N^k \sim x_N$ for each k and $\bigcap_k N(y_N^k) = \emptyset$. We denote π the concatenation of Cournot paths going from x_N to y_N^1, then back to x_N, then to y_N^2, then back to x_N, ..., then to y_N^m, then, finally, back to x_N. Clearly, π is a Cournot cycle; moreover, it is complete because for each $i \in N$ there is y_N^k such that $i \notin N(y_N^k)$. The contradiction with (4.10c) proves that $N(x_N) = \emptyset$. Now it is clear that $\succeq$ with $M(\cdot)$ is a Cournot ω-quasipotential. □

For each one-to-one mapping $\sigma\colon \{1, \ldots, n\} \to N$, we define a correspondence $\mathcal{R}^\sigma\colon X_N \to 2^{X_N} \setminus \{\emptyset\}$ by $\big[\, y_N \in \mathcal{R}^\sigma(x_N)$ if and only if there exists a mapping $\tau\colon \{0, 1, \ldots, n\} \to X_N$ such that: $\tau(0) = x_N$; for each $k \in \{0, 1, \ldots, n-1\}$, either $\tau(k+1) \rhd^{\mathrm{BR}}_{\sigma(k)} \tau(k)$, or $\tau_{\sigma(k)}(k) \in \mathcal{R}_{\sigma(k)}(\tau_{-\sigma(k)}(k))$ and $\tau(k+1) = \tau(k)$; $\tau(n) = y_N \,\big]$. Naturally, we call the [weak] FTP (CTP) of $\rhd^{\mathcal{R}^\sigma}$ the [weak] FσBRP (CσBRP) of the game. Since every iteration path of $\mathcal{R}^\sigma$ is a Cournot path, while every infinite iteration path of $\mathcal{R}^\sigma$ is inclusive, we have [FIBRP $\Rightarrow$ FσBRP]; [weak FσBRP $\Rightarrow$ weak FBRP]; [CBRP $\Rightarrow$ CσBRP]; [weak CσBRP $\Rightarrow$ weak CBRP]; etc. for any σ.

An obvious similarity between (simultaneous) Cournot paths and iteration paths of the best response correspondence(s) may inspire hope that a closed graph assumption could be used to dispense with transfinite (simultaneous) Cournot paths. It turns out, however, that the assumption is even less biting here than in Section 3.5, although *some* "positive" results can be derived nonetheless.

Proposition 4.13. *If a preference relation $\ggcurly_i$ satisfies condition* (4.2)*, then the best response correspondence $\mathcal{R}_i$ is upper hemicontinuous.*

Proof. We have to show that the complement of the graph of $\mathcal{R}_i$ is open. Let $x_N \in X_N$ and $x_i \notin \mathcal{R}_i(x_{-i})$. By definition, there is $y_i \in X_i$ such that $y_i \ggcurly_i^{x_{-i}} x_i$. By (4.2), there is an open neighborhood U of x_N such that $y_i \ggcurly_i^{x'_{-i}} x'_i$ whenever $x'_N \in U$. Therefore, $x'_i \notin \mathcal{R}_i(x'_{-i})$ for all $x'_N \in U$. □

Example 4.14. Let us consider an analog of Example 3.33. Let $N := \{1, 2\}$ and $X_1 = X_2$ be circles in the plane with polar coordinates, $\{(\rho_i, \varphi_i) \mid \rho_i = 1\}$ $(0 \le \varphi_i < 2\pi)$, while utility functions be $u_1(x_1, x_2) := -d(\varphi_1, \varphi_2)$ and $u_2(x_1, x_2) := -d(\varphi_1 \oplus \varphi^0, \varphi_2)$, where $d(\varphi, \psi)$ is the distance between points $(1, \varphi)$ and $(1, \psi)$ in the plane, $\oplus$ denotes addition modulo 2π, and φ^0 is incommensurable with 2π. Both utility functions are continuous; both relations $\triangleright^{\mathrm{BR}}$ and $\triangleright^{\mathrm{sBR}}$ are acyclic. However, there is no Nash equilibrium, to say nothing of stronger properties.

Theorem 4.15 (Kukushkin, 2000, Theorem 5.1). *Let Γ be a BR-consistent game where $\#N = 2$, each X_i is a compact metric space, and each $\mathcal{R}_i$ is upper hemicontinuous. Let Γ have the CBRP. Let $\langle x_N^k \rangle_{k\in\mathbb{N}}$ be an infinite pseudo-Cournot path. Then there is a Nash equilibrium among limit points of the path.*

Remark. Similarly to Theorem 3.34, the statement of the theorem describes a property of Γ intermediate between the very weak FBRP and the weak approximate FBRP, see Example 4.16 below.

Proof. There may be a way to derive our statement from Theorem 3.34, but it seems simpler just to argue similarly to that proof. Again we denote $Y \subseteq X_N$ the set of limit points of $\langle x_N^k \rangle_{k\in\mathbb{N}}$ and pick $x_N^\omega \in M(Y, \triangleright^{\mathrm{BR}})$. If $x_N^\omega \in \mathcal{R}_N(x_N^\omega)$, we are home; let $x_i^\omega \notin \mathcal{R}_i(x_{-i}^\omega)$. Since $x_N^\omega \in Y$, there is a strictly increasing sequence $\langle k_h \rangle_{h\in\mathbb{N}}$ such that $x_N^{k_h} \to x_N^\omega$. We denote $y_N^h := x_N^{k_h+1}$ $(h \in \mathbb{N})$; without restricting generality, $y_N^h \to y_N^\omega \in Y$. Since $\mathcal{R}_i$ is upper hemicontinuous, there holds $x_i^{k_h} \notin \mathcal{R}_i(x_{-i}^{k_h})$ for all h large enough; without restricting generality, for all h. Therefore, $x_N^{k_h} \triangleright_{-i}^{\mathrm{BR}} x_N^{k_h-1}$, hence $y_N^h \triangleright_i^{\mathrm{BR}} x_N^{k_h}$, hence $y_i^h \in \mathcal{R}_i(x_{-i}^{k_h})$. Since $\mathcal{R}_i$ is upper hemicontinuous, $y_i^\omega \in \mathcal{R}_i(x_{-i}^\omega)$. Thus, $y_N^\omega \triangleright_i^{\mathrm{BR}} x_N^\omega$, contradicting the choice of x_N^ω. □

Example 4.16. Let $N := \{1, 2\}$, each player's strategy set be the same compact subset X of the plane as in Example 3.35, and the utilities be $u_1(x_N) := -d(x_1, f(x_2))$ and $u_2(x_N) := -d(x_1, x_2)$, where d denotes distance in the plane and $f \colon X \to X$ is defined in Example 3.35. Clearly, both utilities are continuous; $\mathcal{R}_1(x_2) = \{f(x_2)\}$ and $\mathcal{R}_2(x_1) = \{x_1\}$. The strategy profile $((1, 0), (1, 0))$ is a unique Nash equilibrium. Since each $\mathcal{R}_i(x_{-i})$ is a singleton, at most two Cournot paths can be started from any strategy profile. In projection to either X_i, every Cournot path reproduces an iteration path from Example 3.35. Therefore, the analysis remains essentially the same: the game does not have the weak approximate FBRP although it has the very weak FBRP, in accordance with Theorem 4.15.

Example 4.17. Let us consider a game Γ where $N := \{1, 2\}$, $X_1 := X_2 := [-1, 1]$, and the preferences are defined by the utility functions $u_i(x_N) := -d(x_N, G_i) - s_i(x_N)$, where d denotes distance in the plane and the sets G_i and functions s_i are these:

$$G_1 := \{(x_1, x_2) \in X_N \mid \max\{2x_1 + x_2 + 1, 2x_1 - x_2 + 1\} \cdot \min\{2x_1 + x_2 - 1, 2x_1 - x_2 - 1\} = 0\};$$

$$G_2 := \{(x_1, x_2) \in X_N \mid \max\{2x_2 + x_1 + 1, 2x_2 - x_1 + 1\} \cdot \min\{2x_2 + x_1 - 1, 2x_2 - x_1 - 1\} = 0\};$$

$$s_1(x_1,x_2) := \begin{cases} \min\{x_1,x_2,1-x_1,1-x_2\}/2, & 0 \le x_1 \le 1,\ 0 \le x_2 \le 1; \\ 0, & -1 \le x_1 \le 0,\ 0 \le x_2 \le 1; \\ 0, & 0 \le x_1 \le 1,\ -1 \le x_2 \le 0; \\ -\max\{x_1,x_2\}/2, & -1 \le x_1 \le 0,\ -1 \le x_2 \le 0; \end{cases}$$

$$s_2(x_1,x_2) := \begin{cases} \min\{x_1,x_2\}/2, & 0 \le x_1 \le 1,\ 0 \le x_2 \le 1; \\ 0, & -1 \le x_1 \le 0,\ 0 \le x_2 \le 1; \\ 0, & 0 \le x_1 \le 1,\ -1 \le x_2 \le 0; \\ \min\{-x_1,-x_2,x_1+1,x_2+1\}/2, & -1 \le x_1 \le 0,\ -1 \le x_2 \le 0. \end{cases}$$

Each utility function u_i is continuous in x_N. The best responses are:

$$\mathcal{R}_1(x_2) = \begin{cases} \{-1,1\}, & x_2 = 1; \\ \{(-1-x_2)/2\}, & 0 < x_2 < 1; \\ \{-1/2,1/2\}, & x_2 = 0; \\ \{(1-x_2)/2\}, & -1 \le x_2 < 0; \end{cases} \qquad \mathcal{R}_2(x_1) = \begin{cases} \{(-1-x_1)/2\}, & 0 < x_1 \le 1; \\ \{-1/2,1/2\}, & x_1 = 0; \\ \{(1-x_1)/2\}, & -1 < x_1 < 0; \\ \{-1,1\}, & x_1 = -1. \end{cases}$$

There are two Nash equilibria: $(1,-1)$ and $(-1,1)$.

Whenever $x_1 \cdot x_2 \neq 0$, a unique simultaneous Cournot path can be started from x_N. If $x_1 \cdot x_2 < 0$, the path converges to a Nash equilibrium; otherwise, it does not converge. Let $\langle x_N^k \rangle_{k\in\mathbb{N}}$ be an infinite simultaneous Cournot path such that $x_1^0 > 0 < x_2^0$. We have $x_i^{2k+1} = -(x_{-i}^{2k}+1)/2$ and $x_i^{2k+2} = (1-x_{-i}^{2k+1})/2$ for both i and all k. Therefore, $x_i^{2k+1} = -1+(1-x_{-i}^0)/2^{2k+1}$ and $x_i^{2k} = 1-(1-x_i^0)/2^{2k}$ for both i and all k. We see that $x_N^{2k} \to (1,1)$, while $x_N^{2k+1} \to (-1,-1)$. None of the limit points is an equilibrium, hence the game does not have even the very weak FSBRP. (On the other hand, the game has the approximate FBRP.)

It is easy to see that the path inevitably reaches a Nash equilibrium after the first transfinite step, regardless of which limit point is chosen as x_N^ω: $(1,-1) \rhd^{\mathrm{sBR}} (1,1)$ and $(1,-1) \rhd^{\mathrm{sBR}} (-1,-1)$. Thus, the game has the CSBRP, which fact shows that the appropriate analog of Theorem 4.15 is just wrong.

Example 4.18. Let us consider a game Γ where $N := \{1,2,3\}$, $X_i := [-1,1]$, and the preferences are defined by the utility functions $u_i(x_N) := -d(x_N,G_i) - \min\{\delta, d(x_N,(1,1,1))\}$, where d denotes distance in $\mathbb{R}^3$, $\delta > 0$ is small enough, and the sets G_i are these:

$$G_1 := \{x_N \in X_N \mid 2x_1 + x_3 = 1 \ \&\ x_1 \ge 0\} \cup \{x_N \in X_N \mid 2x_1 + x_3 = -1 \ \&\ x_1 \le 0\} \cup \{(1,-1,1),(1,1,1)\};$$

$$G_2 := \{x_N \in X_N \mid x_1 + x_2 = 0\} \cup \{(1,1,-1),(1,1,1)\};$$

$$G_3 := \{x_N \in X_N \mid x_2 + x_3 = 0\} \cup \{(-1,1,1),(1,1,1)\};$$

Each utility function u_i is continuous in x_N. The best responses are easy to compute. There is a unique Nash equilibrium, $(1,1,1)$.

If we choose the identity mapping as σ, we easily see that every infinite iteration path of $\mathcal{R}^\sigma$, if started far enough from the equilibrium, has six limit points: $(1,-1,1)$, $(-1,1,-1)$, $(1,1,-1)$, $(-1,-1,1)$, $(-1,1,1)$, and $(1,-1,-1)$, none of which is an equilibrium. Therefore, Γ does not even have the very weak FσBRP, although it has CσBRP. In other words, Theorem 4.15 cannot be extended to $n > 2$ even if we agree to restrict attention to sequential tâtonnement.

5 Dominance solvability

Moulin (1984) demonstrated connections between dominance solvability and nice behavior of best response dynamics. However, he worked in a rather restricted framework. Here we depict a broader picture. On the other hand, we consider only implications in one direction.

5.1 Strict dominance and strong BR-dominance

Let Γ be a strategic game, $i \in N$, and $x_i, y_i \in X_i$. We say that y_i *strictly dominates* x_i, $y_i \ggg_i x_i$, if for every $x_{-i} \in X_{-i}$, there holds $y_i \succ_i^{x_{-i}} x_i$. A strategy $y_i \in X_i$ is *strictly dominant* if $y_i \ggg_i x_i$ for every $x_i \neq y_i$. A strategy $x_i \in X_i$ is *strictly dominated* if there exists $y_i \in X_i$ such that $y_i \ggg_i x_i$.

Given a strategic game Γ, we denote $\mathfrak{G}_\Gamma$ the set of all subgames of Γ. If $\Gamma', \Gamma'' \in \mathfrak{G}_\Gamma$ and $X'_i \subseteq X''_i$ for each $i \in N$, then we write $\Gamma' \leq \Gamma''$. Thus, $\mathfrak{G}_\Gamma$ becomes a poset; actually, it would be a lattice if we allowed empty strategy sets in subgames, but it is more convenient not to do that.

An *elimination scheme* in Γ is a decreasing mapping $\lambda\colon \Lambda \to \mathfrak{G}_\Gamma$ such that:

$$\Lambda \text{ is a well ordered set;} \tag{5.1a}$$

$$\lambda(0) = \Gamma; \tag{5.1b}$$

$$\lambda(\alpha) = \bigcap_{\beta<\alpha} \lambda(\beta) \text{ whenever } \alpha \in \Lambda \text{ is a limit.} \tag{5.1c}$$

Given an elimination scheme λ, we use notation $X_i^{\lambda(\alpha)}$ for the strategy set of player i in $\lambda(\alpha)$. If there exists $\max \Lambda$, we may define an "inverse" mapping $\mu_i\colon X_i \to \Lambda$ by

$$\mu_i(x_i) = \max\{\alpha \in \Lambda \mid x_i \in X_i^{\lambda(\alpha)}\} \tag{5.2}$$

(the maximum is attained because of (5.1c)). We also define $\mu^-\colon X_N \to \Lambda$ by

$$\mu^-(x_N) = \min_{i \in N} \mu_i(x_i). \tag{5.3}$$

An elimination scheme is *perfect* if there exists $\max \Lambda$ and every strategy profile in $\lambda(\max \Lambda)$ is a Nash equilibrium there. An *SD-scheme* is an elimination scheme such that every deleted strategy $x_i \in X_i^{\lambda(\alpha)} \setminus X_i^{\lambda(\alpha+1)}$ is strictly dominated in $\lambda(\alpha)$ whenever $(\alpha+1) \in \Lambda$. It is worth noting that the relation $y_i \ggg_i x_i$ in Γ implies the same relation in every subgame Γ' of Γ such that $x_i, y_i \in X'_i$.

A game Γ is *strictly dominance solvable* if it admits a perfect SD-scheme.

Remark. Two, at least, stronger definitions of strict dominance solvability should be mentioned. First, in the definition of a perfect elimination scheme we may demand that $y_N \not\gg_i x_N$ does not hold for any $i \in N$ and $y_N, x_N \in X_N^{\lambda(\max \Lambda)}$. Second, in the definition of an SD-scheme we may demand that every deleted strategy $x_i \in X_i^{\lambda(\alpha)} \setminus X_i^{\lambda(\alpha+1)}$ be strictly dominated in $\lambda(\alpha)$ by $y_i \in X_i^{\lambda(\alpha+1)}$. However, none of the stronger requirements seems to have stronger implications for improvement dynamics.

Obviously, a BR-consistent two-person game is strictly dominance solvable if one player has a strictly dominant strategy x_i^+; however, the behavior of improvement paths with $x_i \neq x_i^+$ may be arbitrary, hence we cannot hope to derive too much from dominance solvability (say, the FIP). It turns out that some implications for *Cournot* dynamics can be derived nonetheless. Moreover, a distinctly weaker property of “strong BR-dominance solvability” is sufficient for everything.

To the end of this section, we restrict attention to BR-consistent games; the definition is in Subsection 2.2; sufficient conditions for the property are given by Corollary to Proposition 2.9 and Corollary to Theorem 3.12. Given $i \in N$ and $x_i \in X_i$, we denote $\mathcal{R}_i^{-1}(x_i) := \{x_{-i} \in X_{-i} \mid x_i \in \mathcal{R}_i(x_{-i})\}$. A strategy $x_i \in X_i$ is *strongly BR-dominated* if $\mathcal{R}_i^{-1}(x_i) = \emptyset$.

An *SBR-scheme* is an elimination scheme such that every deleted strategy $x_i \in X_i^{\lambda(\alpha)} \setminus X_i^{\lambda(\alpha+1)}$ is strongly BR-dominated in $\lambda(\alpha)$ whenever $(\alpha+1) \in \Lambda$. A game Γ is *strongly BR-dominance solvable* (*SBRDS*) if it admits a perfect SBR-scheme. It is immediately clear that a strictly dominated strategy is strongly BR-dominated, hence every SD-scheme is SBR-scheme, hence every strictly dominance solvable game is SBRDS.

Proposition 5.1. *Let Γ be a BR-consistent game and $\lambda\colon \Lambda \to \mathfrak{G}_\Gamma$ be an SBR-scheme. Then, for every $\alpha \in \Lambda$:*

$$\forall i \in N\, \forall x_{-i} \in X_{-i}^{\lambda(\alpha)}\, [\mathcal{R}_i(x_{-i}) \subseteq X_i^{\lambda(\alpha)}]; \tag{5.4a}$$

$$\textit{if } x_N^0 \textit{ is a Nash equilibrium in } \Gamma, \textit{ then } x_N^0 \in X_N^{\lambda(\alpha)}; \tag{5.4b}$$

$$\textit{if } x_N^0 \textit{ is a Nash equilibrium in } \lambda(\alpha), \textit{ then } x_N^0 \textit{ is a Nash equilibrium in } \Gamma. \tag{5.4c}$$

Proof. Supposing (5.4a) wrong, we pick the minimal $\alpha \in \Lambda$ for which there exist $i \in N$, $x_{-i} \in X_{-i}^{\lambda(\alpha)}$, and $y_i \in \mathcal{R}_i(x_{-i}) \setminus X_i^{\lambda(\alpha)}$. Clearly, $\alpha = \alpha'+1$, hence $y_i \in X_i^{\lambda(\alpha')} \setminus X_i^{\lambda(\alpha)}$ because α is minimal. However, y_i cannot be strongly BR-dominated in $\lambda(\alpha')$, which contradicts the definition of SBR-scheme.

Supposing (5.4b) wrong, we pick the minimal $\alpha \in \Lambda$ such that $x_N^0 \notin X_N^{\lambda(\alpha)}$ and immediately obtain a contradiction with (5.4a).

Supposing (5.4c) wrong, we pick $i \in N$ and $y_i \in X_i$ such that $y_i \ggcurly_i^{x^0_{-i}} x^0_i$. Since Γ is BR-consistent, there is $z_i \in \mathcal{R}_i(x^0_{-i})$ such that $z_i \ggcurly_i^{x^0_{-i}} x^0_i$. Now $z_i \in X_i^{\lambda(\alpha)}$ by (5.4a), hence x^0_N cannot be a Nash equilibrium in $\lambda(\alpha)$. □

Corollary. *Let Γ be a BR-consistent game and $\lambda\colon \Lambda \to \mathfrak{G}_\Gamma$ be a perfect SBR-scheme. Then $X_N^{\lambda(\max \Lambda)}$ is the set of Nash equilibria in Γ.*

Lemma 5.2. *If λ is a perfect SBR-scheme and $x_N \in X_N$ is not a Nash equilibrium, then for every $i \in N$ and $y_i \in \mathcal{R}_i(x_{-i})$, there holds $\mu_i(y_i) > \mu^-(x_N)$.*

Proof. Since $x_{-i} \in X_{-i}^{\lambda(\mu^-(x_N))}$, we have $y_i \in X_i^{\lambda(\mu^-(x_N))}$ by (5.4a). Since x_N is not a Nash equilibrium, we have $\mu^-(x_N) < \max \Lambda$. Therefore, $y_i \in X_i^{\lambda(\mu^-(x_N)+1)}$ because y_i cannot be BR-dominated in $\lambda(\mu^-(x_N))$. □

Theorem 5.3. *If a BR-consistent game Γ is SBRDS with a finite set Λ, then it has the FSBRP and the FIBRP.*

Proof. Fixing a perfect SBR-scheme with a finite Λ, we consider the functions μ_i and μ^- defined by (5.2) and (5.3). We also denote $m := \max \Lambda$.

Let us consider the total preorder represented by μ^- and its asymmetric component:

$$y_N \succeq x_N \rightleftharpoons \mu^-(y_N) \geq \mu^-(x_N); \tag{5.5a}$$

$$y_N \succ x_N \rightleftharpoons \mu^-(y_N) > \mu^-(x_N). \tag{5.5b}$$

First, we show that $\succ$ is a strict simultaneous Cournot potential. If $y_N \rhd^{\text{sBR}} x_N$, then $\mu_i(y_i) > \mu^-(x_N)$ for every $i \in N$ by Lemma 5.2, hence $\mu^-(y_N) > \mu^-(x_N)$ as well.

Second, we show that $\succeq$ is a Cournot quasipotential with $M(x_N) = \operatorname{Argmin}_{i \in N} \mu_i(x_i)$ when $\mu^-(x_N) < m$ and $M(x_N) = \emptyset$ otherwise. If $\mu^-(x_N) = m$, then $x_N \in X_N^{\lambda(m)}$, hence x_N is a Nash equilibrium in Γ by Corollary to Proposition 5.1.

Let $y_N \rhd_i^{\text{BR}} x_N$; then Lemma 5.2 is applicable. If $i \notin M(x_N)$, then $\mu^-(y_N) = \mu^-(x_N)$ and $M(y_N) = M(x_N)$. Let $i \in M(x_N)$; then $\mu_i(y_i) > \mu^-(x_N)$, hence either $\mu^-(y_N) > \mu^-(x_N)$ or $\mu^-(y_N) = \mu^-(x_N)$ and $M(y_N) = M(x_N) \setminus \{i\}$. We see that condition (4.6a) holds. Finally, if $x_N \in X_N$ and $i \in M(x_N)$, then $\mu_i(x_i) = \mu^-(x_N) < m$; if $x_i \in R_i(x_{-i})$, then Lemma 5.2 would imply $\mu_i(x_i) > \mu_i(x_i)$. Thus, (4.6b) holds as well. □

Corollary. *If a two person BR-consistent game Γ is SBRDS with a finite set Λ, then it has the FBRP.*

Proof. The statement immediately follows from Theorem 5.3 and Proposition 4.9. □

If there are more than two players, the FIBRP in the formulation of Theorem 5.3 cannot be replaced with the FBRP: if one player has a strictly dominant strategy x_i^+, then any behavior of the best responses of the other players is compatible even with strict dominance solvability. Without the finiteness of Λ, Theorem 5.3 is simply wrong.

Example 5.4. Let us consider a game Γ where $N := \{1,2\}$, $X_1 := X_2 := [0,1] \cup \{2\}$, and the preferences are defined by these utility functions:

$$u_1(x_1,x_2) := \begin{cases} -x_1, & 0 \le x_1 \le 1,\ x_2 = 2; \\ 1, & x_1 = 2,\ x_2 = 2; \\ -|2x_1 - x_2|, & 0 \le x_1 \le 1,\ 0 < x_2 \le 1; \\ -x_2/4, & x_1 = 2,\ 0 < x_2 \le 1; \\ x_1, & 0 \le x_1 \le 1,\ x_2 = 0; \\ 1, & x_1 = 2,\ x_2 = 0; \end{cases}$$

$$u_2(x_1,x_2) := u_1(x_2,x_1).$$

Each utility function u_i is upper semicontinuous in x_N and continuous in x_i (given x_{-i}). The best responses are:

$$\mathcal{R}_1(x_2) = \begin{cases} \{2\}, & x_2 = 2; \\ \{x_2/2\}, & 0 < x_2 \le 1; \\ \{1\}, & x_2 = 0; \end{cases} \quad \mathcal{R}_2(x_1) = \begin{cases} \{2\}, & x_1 = 2; \\ \{x_1/2\}, & 0 < x_1 \le 1; \\ \{1\}, & x_1 = 0. \end{cases}$$

The strategy profile $(2,2)$ is a unique Nash equilibrium.

If $x_i \in \mathcal{R}_i(x_{-i})$ and $x_{-i} \le 1$, then $x_i \le 1$ too. Denoting $X_N^- := [0,1] \times [0,1]$, we see that X_N^- is an undominated set w.r.t. either $\rhd^{\mathrm{BR}}$ or $\rhd^{\mathrm{sBR}}$; besides, X_N^- is closed. Since it contains no Nash equilibrium, Γ has neither weak CSBRP nor weak CBRP, to say nothing of the FSBRP or FBRP.

On the other hand, Γ is SBRDS, even strictly dominance solvable. Let us define $\lambda\colon (\mathbb{N} \cup \{\omega\}) \to \mathfrak{G}_\Gamma$ by $\lambda(0) := \Gamma$, $X_i^{\lambda(1)} := X_i \setminus \{0\}$, $X_i^{\lambda(k)} :=]0, 1/2^{k-1}] \cup \{2\}$ for all $k > 1$, and $X_i^{\lambda(\omega)} := \{2\}$. We have $2 \ggg_i 0$ for either i in Γ. Once 0 is deleted from both X_i, we have $1/2 \ggg_i x_i$ whenever $1/2 < x_i \le 1$; by induction, $1/2^{k-1} \ggg_i x_i$ in $\lambda(k)$ whenever $1/2^{k-1} < x_i \le 1/2^k$. Finally, $X_i^{\lambda(\omega)} = \bigcap_{k \in \mathbb{N}} X_i^{\lambda(k)}$. Thus, λ is a perfect SD-scheme.

The convergence of Cournot dynamics to Nash equilibria can be derived from "infinite" or "transfinite" strong BR-dominance solvability (in particular, strict dominance solvability) under an additional assumption that each best response correspondence is upper hemicontinuous. By Proposition 4.13, (4.2) is sufficient for that. In the case of preferences described by utility functions, we may assume that each u_i is upper semicontinuous in x_N and continuous in x_{-i}; it is the last condition that is lacking in Example 5.4.

Given a BR-consistent game Γ, we define the *most radical strong BR-dominance elimination scheme* $\lambda_{\mathrm{SBRD}}\colon \mathbb{N} \to \mathfrak{G}_\Gamma$, setting $\lambda_{\mathrm{SBRD}}(0) := \Gamma$ and then, recursively,

$$X_i^{\lambda_{\mathrm{SBRD}}(k+1)} := \{x_i \in X_i^{\lambda_{\mathrm{SBRD}}(k)} \mid \left(\mathcal{R}_i^{\lambda_{\mathrm{SBRD}}(k)}\right)^{-1}(x_i) \ne \emptyset\}$$

for each $k \in \mathbb{N}$ and $i \in N$. In other words, $X_i^{\lambda_{\mathrm{SBRD}}(k+1)}$ consists of all $x_i \in X_i^{\lambda_{\mathrm{SBRD}}(k)}$ that are not strongly BR-dominated in $\lambda_{\mathrm{SBRD}}(k)$. Clearly, λ_{SBRD} is an SBR-scheme.

Proposition 5.5. *Let a game Γ be BR-consistent, each X_i be compact, and the graph of each best response correspondence $\mathcal{R}_i$ be closed. Then the following statements hold:*

λ_{SBRD} *can be extended to an SBR-scheme* $\mathbb{N} \cup \{\omega\} \to \mathfrak{G}_\Gamma$,

$$\textit{i.e.}, \forall i \in N \left[X_i^{\lambda_{\mathrm{SBRD}}(\omega)} := \bigcap_{k\in\mathbb{N}} X_i^{\lambda_{\mathrm{SBRD}}(k)} \neq \emptyset\right]; \quad (5.6a)$$

$$\textit{there is no strongly BR-dominated strategy in } \lambda_{\mathrm{SBRD}}(\omega); \quad (5.6b)$$

for every SBR-scheme $\lambda\colon \Lambda \to \mathfrak{G}_\Gamma$, *every* $\alpha \in \Lambda$ *and* $i \in N$,

$$\textit{there holds } X_i^{\lambda_{\mathrm{SBRD}}(\omega)} \subseteq X_i^{\lambda(\alpha)}; \quad (5.6c)$$

Γ *is SBRDS if and only if every strategy profile*

$$\textit{in } \lambda_{\mathrm{SBRD}}(\omega) \textit{ is a Nash equilibrium.} \quad (5.6d)$$

Proof. First, let us show that the set of strongly BR-dominated strategies of each player i is open in X_i. Indeed, if $\mathcal{R}_i^{-1}(x_i) = \emptyset$, then the compact subset $\{x_i\} \times X_{-i} \subseteq X_N$ does not intersect the (compact) graph of $\mathcal{R}_i$; denoting δ the distance between them, we see that $\delta > 0$. Therefore, whenever $d(x_i, x_i') < \delta$, x_i' is strongly BR-dominated as well. (Recall that d was defined as $\min_i d_i$.)

Since $X_i \setminus X_i^{\lambda_{\mathrm{SBRD}}(1)}$ is open, $X_i^{\lambda_{\mathrm{SBRD}}(1)}$ is closed in X_i, hence compact; it cannot be empty since Γ is BR-consistent. Thus, $\lambda_{\mathrm{SBRD}}(1)$ satisfies all assumptions imposed on Γ, and we can continue by induction, obtaining that $X_i^{\lambda_{\mathrm{SBRD}}(k)}$ is nonempty and compact for each $k \in \mathbb{N}$. (5.6a) immediately follows.

To prove (5.6b), we pick $i \in N$ and $x_i \in X_i^{\lambda_{\mathrm{SBRD}}(\omega)}$. For each $k \in \mathbb{N}$, there is $x_{-i}^k \in X_{-i}^{\lambda_{\mathrm{SBRD}}(k)}$ such that $x_i \in \mathcal{R}_i(x_{-i}^k)$. Without restricting generality, $x_{-i}^k \to x_{-i}^\omega \in X_{-i}$; since the graph of $\mathcal{R}_i$ is closed, $x_i \in \mathcal{R}_i(x_{-i}^\omega)$. Since each $X_{-i}^{\lambda_{\mathrm{SBRD}}(k)}$ is closed in X_{-i}, we have $x_{-i}^\omega \in X_{-i}^{\lambda_{\mathrm{SBRD}}(\omega)}$. Thus, x_i is not strongly BR-dominated in $\lambda_{\mathrm{SBRD}}(\omega)$.

Supposing that (5.6c) is violated, let there be an SBR-scheme $\lambda\colon \Lambda \to \mathfrak{G}_\Gamma$, $\alpha \in \Lambda$, $i \in N$, and $x_i \in X_i$ such that $x_i \in X_i^{\lambda_{\mathrm{SBRD}}(\omega)} \setminus X_i^{\lambda(\alpha)}$ $[\neq \emptyset]$. Without restricting generality, we may assume that $X_j^{\lambda_{\mathrm{SBRD}}(\omega)} \subseteq X_j^{\lambda(\beta)}$ for all $j \in N$ and $\beta < \alpha$. By (5.6b), we have $x_i \in \mathcal{R}_i(x_{-i})$ for some $x_{-i} \in X_{-i}^{\lambda_{\mathrm{SBRD}}(\omega)} \subseteq X_{-i}^{\lambda(\beta)}$ for all $\beta < \alpha$. Therefore, x_i is not strongly BR-dominated in any $\lambda(\beta)$ $(\beta < \alpha)$, hence $x_i \in X_i^{\lambda(\alpha)}$ because λ is an SBR-scheme, contradicting our original assumption.

(5.6d) immediately follows from (5.6c) and Proposition 5.1. □

Theorem 5.6. *Let a game Γ be BR-consistent and SBRDS, each X_i be compact, and the graph of each best response correspondence $\mathcal{R}_i$ be closed. Then Γ has the approximate FSBRP and approximate FIBRP.*

Proof. By (5.6d) and (5.4b), $M(X_N, \rhd^{\mathrm{sBR}})$ coincides with $X_N^{\lambda_{\mathrm{SBRD}}(\omega)}$, which is closed by Proposition 5.5. In the light of Proposition 3.3, the approximate FSBRP will be established if we produce an approximate potential of $\rhd^{\mathrm{sBR}}$. Let us show that $\succ$ defined by (5.5b) fits the role if we define μ_i and μ^- by (5.2) and (5.3) with $\lambda := \lambda_{\mathrm{SBRD}}$, understood as a mapping $\mathbb{N} \cup \{\omega\} \to \mathfrak{G}_\Gamma$.

Condition (2.2) immediately follows from Lemma 5.2 exactly as in Theorem 5.3. Let $\delta > 0$; $X(\delta)$ defined in (3.1) coincides with $\{x_N \in X_N \mid d(x_N, X_N^{\lambda_{\mathrm{SBRD}}(\omega)}) \geq \delta\}$. An assumption that $X(\delta) \cap X_N^{\lambda_{\mathrm{SBRD}}(k)} \neq \emptyset$ for all $k \in \mathbb{N}$ would lead to $X(\delta) \cap X_N^{\lambda_{\mathrm{SBRD}}(\omega)} \neq \emptyset$ as well because $X(\delta)$ is closed, hence compact. Therefore, there is $m \in \mathbb{N}$ such that $X(\delta) \cap X_N^{\lambda_{\mathrm{SBRD}}(k)} = \emptyset$ for all $k \geq m$, hence $X(\delta) \subseteq \{x_N \in X_N \mid \mu^-(x_N) \leq m\}$. Strict acyclicity of $\succ$ on $X(\delta)$ is now obvious.

Turning to the approximate FIBRP, we show that $\succeq$ defined by (5.5a) with $\lambda := \lambda_{\mathrm{SBRD}}$ is an approximate Cournot quasipotential with $M(x_N) := \mathrm{Argmin}_{i \in N} \mu_i(x_i)$ when $\mu^-(x_N) < \omega$ and $M(x_N) := \emptyset$ otherwise. First, $\succ$ is strictly acyclic on every $X(\delta)$ ($\delta > 0$) as shown above; second, both conditions (4.6) hold for the same reasons as in Theorem 5.3. Now Proposition 4.11 applies. □

Corollary. *If a two person game Γ is BR-consistent and SBRDS, each X_i is compact, and the graph of each best response correspondence $\mathcal{R}_i$ is closed, then Γ has the approximate FBRP.*

Upper hemicontinuity of the best responses can be replaced with a restriction on the elimination scheme, viz., that strategies eliminated at every particular step form open sets. However, we only obtain CIBRP and CSBRP in this case.

Theorem 5.7. *Let Γ be a BR-consistent game where each X_i is compact. Let Γ admit a perfect SBR-scheme λ such that every $X_i^{\lambda(\alpha)}$ ($i \in N$, $\alpha \in \Lambda$) is closed in X_i. Then Γ has the CSBRP and CIBRP.*

Proof. We again consider the functions μ_i and μ^- defined by (5.2) and (5.3). Since every $X_i^{\lambda(\alpha)}$ is closed, both $\succ$ and $\succeq$ defined by (5.5) are ω-transitive. Therefore, Lemma 5.2 implies that $\succ$ is a simultaneous Cournot ω-potential. Defining $M(x_N) := \mathrm{Argmin}_{i \in N} \mu_i(x_i)$ when $\mu^-(x_N) < \max \Lambda$ and $M(x_N) := \emptyset$ otherwise, we see that $\succeq$ is a Cournot ω-quasipotential, exactly as in the proof of Theorem 5.3; (4.9) holds because every $X_i^{\lambda(\alpha)}$ is closed. Thus, Γ has the CSBRP and CIBRP. □

Corollary. *Let Γ be a BR-consistent game where $\#N = 2$ and each X_i is compact. Let Γ admit a perfect SBR-scheme λ such that every $X_i^{\lambda(\alpha)}$ ($i \in N$, $\alpha \in \Lambda$) is closed in X_i. Then Γ has the CBRP.*

Example 5.8. Let us consider a game Γ where $N := \{1, 2\}$, $X_i := [-1, 1]$, and the preferences are defined by these utility functions: $u_i(x_N) := \min\{2x_i - x_{-i},\ x_{-i} - 2x_i\}$ if $x_{-i} < 0$; $u_i(x_N) := \min\{2x_i - x_{-i},\ x_{-i} - 2x_i + 2\}$ if $x_{-i} \geq 0$ ($i \in N$). Each u_i is upper semicontinuous in x_N, but not continuous in x_{-i}. The best response correspondences are single-valued, but not upper hemicontinuous: $\mathcal{R}_i(x_{-i}) = \{x_{-i}/2\}$ if $x_{-i} < 0$; $\mathcal{R}_i(x_{-i}) = \{(x_{-i}+1)/2\}$ if $x_{-i} \geq 0$. The strategy profile $(1,1)$ is a unique Nash equilibrium.

Γ is SBRDS, even strictly dominance solvable. Let us define $\lambda\colon [0, \omega+\omega] \to \mathfrak{G}_\Gamma$ by $\lambda(0) := \Gamma$, $X_i^{\lambda(k)} := [-1/2^k, 1]$ for each $k \in \mathbb{N}$, $X_i^{\lambda(\omega)} := [0,1]$, $X_i^{\lambda(\omega+k)} := [1-1/2^k, 1]$ for each $k \in \mathbb{N}$, and $X_i^{\lambda(\omega+\omega)} := \{1\}$. We have $-1/2^{k+1} \ggg_i x_i$ in $\lambda(k)$ whenever $-1/2^k \leq x_i < -1/2^{k+1}$, and $1-1/2^{k+1} \ggg_i x_i$ in $\lambda(\omega+k)$ whenever $1-1/2^k \leq x_i < 1-1/2^{k+1}$. Thus, λ is a perfect SD-scheme.

By Theorem 5.7, Γ has the CBRP and CSBRP. On the other hand, it does not have the very weak FBRP or FSBRP: If we start a (simultaneous) Cournot path from, say, $(-1,-1)$, then it will remain in $[-1,0[\times[-1,0[$ after any finite number of steps. The equilibrium $(1,1)$ will only be reached at the step $\omega+\omega$.

Remark. Similarly to Proposition 3.32, the example can be modified so that the equilibrium $(1,1)$ will only be reached at an arbitrarily chosen step $\alpha \in \Omega$.

If a game is strictly dominance solvable, then it is SBRDS, hence Theorem 5.6 or its corollary apply (provided other assumptions are satisfied). However, λ_{SBRD} may eliminate strategies that are *not* dominated. Since the elimination of strictly dominated strategies in an infinite game, in particular, the problem of whether "maximal reduction" is well defined, has attracted considerable attention, we delve into the matter a bit. Dufwenberg and Stegeman (2002, Theorem 1(a)) showed the existence and uniqueness of maximal reduction (with nonempty strategy sets) if the strategy sets are compact and the payoff functions continuous. Our Proposition 5.9 shows that (4.2) is sufficient. Actually, Dufwenberg and Stegeman assumed that strategy sets are topological, rather than metric, spaces; however, distance functions seem superfluous here as well. On the other hand, we do not assume that the preferences are described by utility functions.

Given Γ, we define the *most radical strict dominance elimination scheme* $\lambda_{\text{SD}}\colon \mathbb{N} \to \mathfrak{G}_\Gamma$ ("*maximal reduction*"), setting $\lambda_{\text{SD}}(0) := \Gamma$ and then, recursively, $X_i^{\lambda_{\text{SD}}(k+1)} := \{x_i \in X_i^{\lambda_{\text{SD}}(k)} \mid \nexists y_i \in X_i^{\lambda_{\text{SD}}(k)}\,[y_i \ggg x_i]\}$ for each $k \in \mathbb{N}$ and $i \in N$. In other words, $X_i^{\lambda_{\text{SD}}(k+1)}$ consists of all $x_i \in X_i^{\lambda_{\text{SD}}(k)}$ that are not strictly dominated in $\lambda_{\text{SD}}(k)$.

Proposition 5.9. *Let each preference relation $\succ\!\!\succ_i$ in a BR-consistent game Γ satisfy* (4.2)*; let each X_i be compact. Then the following statements hold:*

λ_{SD} *can be extended to an SD-scheme* $\mathbb{N} \cup \{\omega\} \to \mathfrak{G}_\Gamma$,

$$\textit{i.e., } \forall i \in N\,[X_i^{\lambda_{\text{SD}}(\omega)} := \bigcap_{k\in\mathbb{N}} X_i^{\lambda_{\text{SD}}(k)} \neq \emptyset]; \quad (5.7a)$$

$$\textit{there is no strictly dominated strategy in } \lambda_{\text{SD}}(\omega); \quad (5.7b)$$

for every SD-scheme $\lambda\colon \Lambda \to \mathfrak{G}_\Gamma$, *every* $\alpha \in \Lambda$ *and* $i \in N$,

$$\textit{there holds } X_i^{\lambda_{\text{SD}}(\omega)} \subseteq X_i^{\lambda(\alpha)}; \quad (5.7c)$$

Γ *is strictly dominance solvable if and only if*

every strategy profile in $\lambda_{\mathrm{SD}}(\omega)$ *is a Nash equilibrium.* (5.7d)

Proof. First, let us show that the set of strictly dominated strategies of each player i is open in X_i. Let $y_i \ggg x_i$. For every $x_{-i} \in X_{-i}$, (4.2) implies the existence of $\delta_{x_{-i}} > 0$ such that $y_i \succ_i^{x'_{-i}} x'_i$ whenever $d(x_{-i}, x'_{-i}) < \delta_{-i}$ and $d(x_i, x'_i) < \delta_{-i}$. Since X_{-i} is compact, it is covered by a finite number of open balls of radius $\delta_{x_{-i}}$ with the center at x_{-i}; we denote $\delta > 0$ the minimum of those $\delta_{x_{-i}}$. Now we have $y_i \ggg x_i$ whenever $d(x_i, x'_i) < \delta$.

(5.7a) is proven in exactly the same way as in Proposition 5.5: If $x_i \in \mathcal{R}(x_{-i})$, then x_i cannot be strictly dominated; hence BR-consistency implies that $X_i^{\lambda_{\mathrm{SD}}(1)} \neq \emptyset$; hence each $X_i^{\lambda_{\mathrm{SD}}(k)}$ is nonempty and compact; hence $X_i^{\lambda_{\mathrm{SD}}(\omega)}$ is nonempty and compact as well.

To prove (5.7b), we suppose the contrary: there are $i \in N$ and $y_i, x_i \in X_i^{\lambda_{\mathrm{SD}}(\omega)}$ such that $y_i \ggg x_i$ (in $\lambda_{\mathrm{SD}}(\omega)$). For each $k \in \mathbb{N}$, we have $y_i, x_i \in X_i^{\lambda_{\mathrm{SD}}(k+1)}$, hence there is $x_{-i}^k \in X_{-i}^{\lambda_{\mathrm{SD}}(k)}$ such that $y_i \succ_i^{x_{-i}^k} x_i$ does not hold. Without restricting generality, $x_{-i}^k \to x_{-i}^\omega \in X_{-i}$; since each $X_{-i}^{\lambda_{\mathrm{SD}}(k)}$ is closed in X_{-i}, we have $x_{-i}^\omega \in X_{-i}^{\lambda_{\mathrm{SD}}(\omega)}$. Therefore, $y_i \succ_i^{x_{-i}^\omega} x_i$; but then $y_i \succ_i^{x_{-i}^k} x_i$ for some $k \in \mathbb{N}$ by (4.2).

(5.7c) is proven in virtually the same way as (5.6c) in Proposition 5.5. (5.7d) immediately follows from (5.7c). □

5.2 Weak (BR-)dominance

Let us suppose for awhile that all preference relations are orderings (hence each player's weak preference relation $\succcurlyeq_i$ is transitive and total). Besides strict dominance defined at the beginning of Subsection 5.1, we may consider *weak dominance* in this case:

$$y_i \gg_i x_i \rightleftharpoons \forall x_{-i} \in X_{-i}\, [y_i \succcurlyeq_i^{x_{-i}} x_i] \;\&\; \exists x_{-i} \in X_{-i}\, [y_i \succ_i^{x_{-i}} x_i].$$

A strategy $x_i \in X_i$ is *weakly dominated* if there exists $y_i \in X_i$ such that $y_i \gg_i x_i$. A *WD-scheme* is an elimination scheme such that for every deleted strategy $x_i \in X_i^{\lambda(\alpha)} \setminus X_i^{\lambda(\alpha+1)}$ ($(\alpha+1) \in \Lambda$), there is $y_i \in X_i^{\lambda(\alpha+1)}$ such that $y_i \gg_i x_i$. A game Γ is *weakly dominance solvable* if it admits a perfect WD-scheme.

Weak dominance solvability also has some implications for best response dynamics; not surprisingly, they are much weaker than in the previous subsection. We again introduce an even weaker property expressed in terms of the best responses, which is sufficient for everything. Strictly speaking, preference relations need not be orderings, but we do need strong BR-consistency.

Given $X'_i \subseteq X_i$, we denote $\mathcal{R}_i^{-1}(X'_i) := \{x_{-i} \in X_{-i} \mid \mathcal{R}_i(x_{-i}) \cap X'_i \neq \emptyset\}$. A subset $X'_i \subseteq X_i$ is *BR-sufficient* (in Γ) if $R_i^{-1}(X'_i) = X_{-i}$, i.e., a best response for every $x_{-i} \in X_{-i}$ can be found in X'_i. A *WBR-scheme* is an elimination scheme λ such that $X_i^{\lambda(\alpha+1)}$ is BR-sufficient in $\lambda(\alpha)$ whenever $(\alpha+1) \in \Lambda$. We call Γ *weakly BR-dominance solvable* (*WBRDS*) if it admits a perfect WBR-scheme.

Lemma 5.10. *Given a strongly BR-consistent game* Γ *and a WBR-scheme* λ*, there holds* $\mathcal{R}_i^{\lambda(k)}(x_{-i}) = \mathcal{R}_i(x_{-i}) \cap X_i^{\lambda(k)} \neq \emptyset$ *for every* $i \in N$*, finite ordinal* $k \in \Lambda$*, and* $x_{-i} \in X_{-i}^{\lambda(k)}$*.*

Proof. Straightforward induction based on the definition of a WBR-scheme shows $\mathcal{R}_i(x_{-i}) \cap X_i^{\lambda(k)} \neq \emptyset$; the equality immediately follows from strong BR-consistency. □

Corollary. *Let* Γ *be strongly BR-consistent and WBRDS with a finite set* Λ*; let* $x_N^0 \in X_N^{\max \Lambda}$*. Then* x_N^0 *is a Nash equilibrium in* Γ*.*

Theorem 5.11. *If a game* Γ *is strongly BR-consistent and WBRDS with a finite set* Λ*, then it has the pseudo-FSBRP and pseudo-FBRP.*

Proof. Fixing a perfect WBR-scheme with a finite Λ, we consider the functions μ_i and μ^- defined by (5.2) and (5.3), and again denote $m := \max \Lambda = \max_{x_N \in X_N} \mu^-(x_N)$.

Claim 5.11.1. *If* $\mu^-(x_N) < m$*, then for each* $i \in N$ *there is* $y_i \in \mathcal{R}_i(x_{-i})$ *such that* $\mu_i(y_i) > \mu^-(x_N)$*.*

Proof. Given $i \in N$, we have $x_{-i} \in X_{-i}^{\lambda(\mu^-(x_N))}$. By Lemma 5.10, $\mathcal{R}_i(x_{-i}) \cap X_i^{\lambda(\mu^-(x_N)+1)} \neq \emptyset$. For every y_i from the intersection, we have $\mu_i(y_i) \geq \mu^-(x_N) + 1$. □

Claim 5.11.1 immediately implies that the order (5.5b) defined by the function μ^- is a simultaneous pseudo-Cournot potential, hence Γ has the pseudo-FSBRP.

To prove the second statement, we define $\mathrm{M}(x_N) := \mathrm{Argmin}_{i \in N} \mu_i(x_i)$ for every $x_N \in X_N$, and

$$y_N \succ x_N \rightleftharpoons \bigl[\mu^-(y_N) > \mu^-(x_N) \ \text{ or } \ [\mu^-(x_N) = \mu^-(y_N) \ \& \ \mathrm{M}(y_N) \subset \mathrm{M}(x_N)]\bigr]. \tag{5.8}$$

Let us show that $\succ$ is a pseudo-Cournot potential. Assuming that x_N is not a Nash equilibrium, we pick $i \in \mathrm{M}(x_N)$ and, invoking Claim 5.11.1, $y_i \in \mathcal{R}_i(x_{-i})$ such that $\mu_i(y_i) > \mu^-(x_N) = \mu_i(x_i)$. Denoting $y_N := (y_i, x_{-i})$, we immediately see that either $\mu^-(y_N) > \mu^-(x_N)$ [if $\mathrm{M}(x_N) = \{i\}$] or $\mu^-(y_N) = \mu^-(x_N)$ and $\mathrm{M}(y_N) \subset \mathrm{M}(x_N)$; therefore, $y_N \succ x_N$. □

Theorem 5.12. *If a two person game is strongly BR-consistent and WBRDS with a finite set* Λ*, then it has the weak FSBRP and weak FBRP.*

Proof. Immediately follows from Theorem 5.11 and Proposition 4.8. □

Theorem 5.12 is wrong for more than two players.

Example 5.13. Let us consider a three person $2 \times 3 \times 2$ game (where player 1 chooses rows, player 2 columns, and player 3 matrices):

$$\begin{bmatrix} (3,3,3) & (2,1,1) & (1,2,2) \\ (3,3,3) & (1,2,2) & (2,1,1) \end{bmatrix} \quad \begin{bmatrix} (0,0,0) & \underline{(2,1,1)} & \underline{(1,2,2)} \\ (0,0,0) & \underline{(1,2,2)} & \underline{(2,1,1)} \end{bmatrix}.$$

The game is weakly dominance solvable: the choice of the left matrix weakly dominates the choice of the right matrix; when the latter is deleted, the left column becomes strictly dominant. Both strategy profiles in that column are Nash equilibria; however, none of the underlined strategy profiles could be connected to any equilibrium with an individual improvement path or with a simultaneous Cournot path. Thus, the game has neither weak FIP nor weak FSBRP.

Example 5.13 also shows that Theorem 5.3 becomes wrong if Γ is only weakly dominance solvable. Example 5.14 shows the same for Corollary to Theorem 5.3.

Example 5.14. Let us consider the following bimatrix game:

$$\begin{matrix} (0,1) & \underline{(1,0)} & \underline{(0,1)} \\ (0,1) & \underline{(0,1)} & \underline{(1,0)}. \\ (2,2) & (1,0) & (1,0) \end{matrix}$$

The bottom row and the left column are weakly dominant; the southwestern corner of the matrix is a unique Nash equilibrium. The underlined subgame is a Cournot cycle (hence a simultaneous Cournot cycle as well).

Theorems 5.11 and 5.12 become wrong without the finiteness assumption as Example 5.4 shows. Unlike the situation with strong BR-dominance solvability, even the restriction to preferences defined with continuous utilities is not of much help here.

Example 5.15. Let us consider a game Γ where $N := \{1,2\}$, $X_1 := X_2 := [-3,-1] \cup \{0\} \cup [1,3]$, and the preferences are defined by these utility functions:

$$u_1(x_1,x_2) := \begin{cases} -1, & 1 \le x_1 \le 3 \ \& \ 0 \le x_2 \le 3; \\ \min\{3 - x_1, (x_1-1)(x_2+3)/(1-x_2)\}, & 1 \le x_1 \le 3 \ \& \ -3 \le x_2 \le -1; \\ -1, & x_1 = 0 \ \& \ x_2 \ne 0; \\ 0, & x_1 = 0 \ \& \ x_2 = 0; \\ \min\{x_1 + 3, (x_1+1)(x_2-3)/(x_2+1)\}, & -3 \le x_1 \le -1 \ \& \ 1 \le x_2 \le 3; \\ -1, & -3 \le x_1 \le -1 \ \& \ -3 \le x_2 \le 0; \end{cases}$$

$$u_2(x_1,x_2) := \begin{cases} \min\{3 - x_2, (3-x_1)(x_2-1)/(1+x_1)\}, & 1 \le x_1 \le 3 \ \& \ 1 \le x_2 \le 3; \\ -1, & 1 \le x_1 \le 3 \ \& \ -3 \le x_2 \le 0; \\ -1, & x_1 = 0 \ \& \ x_2 \ne 0; \\ 0, & x_1 = 0 \ \& \ x_2 = 0; \\ -1, & -3 \le x_1 \le -1 \ \& \ 0 \le x_2 \le 3; \\ \min\{x_2 + 3, (x_1+3)(x_2+1)/(x_1-1)\}, & -3 \le x_1 \le -1 \ \& \ -3 \le x_2 \le -1. \end{cases}$$

Each utility function u_i is continuous in x_N. The best responses are:

$$\mathcal{R}_1(x_2) = \begin{cases} [-3,-1], & x_2 = 3; \\ \{-(3+x_2)/2\}, & 1 \le x_2 < 3; \\ \{0\}, & x_2 = 0; \\ \{(3-x_2)/2\}, & -3 < x_2 \le -1; \\ [1,3], & x_2 = -3; \end{cases}$$

$$\mathcal{R}_2(x_1) = \begin{cases} [1,3], & x_1 = 3; \\ \{(x_1+3)/2\}, & 1 \le x_1 < 3; \\ \{0\}, & x_1 = 0; \\ \{(x_1-3)/2\}, & -3 < x_1 \le -1; \\ [-3,-1], & x_1 = -3. \end{cases}$$

A unique Nash equilibrium is $(0,0)$.

Let us denote $X'_i := X_i \setminus \{0\}$ and $X'_N := X'_1 \times X'_2$. Whenever $x_N \in X'_N$ and $y_i \in \mathcal{R}_i(x_{-i})$ for both $i \in N$, we have both $y_N \in X'_N$ and $(y_i, x_{-i}) \in X'_N$. Since X'_N is closed and contains no Nash equilibrium, no (simultaneous) pseudo-Cournot path originating there will ever reach an equilibrium, hence the game does not even have weak CBRP or CSBRP, to say nothing of stronger properties.

On the other hand, Γ is WDS: For each $i \in N$ and $k \in \mathbb{N}$, we denote $y_i^k := 3 - 1/2^k \in X_i$ and $z_i^k := -3 + 1/2^k \in X_i$. Then we define an infinite sequence of subgames $\Gamma^0, \Gamma^1, \dots$ of Γ by $\Gamma^0 := \Gamma$ and $X_i^k :=]-3, z_i^{k-1}] \cup \{0\} \cup [y_i^{k-1}, 3[$ for both i and all $k > 0$. It is easily checked that $y_i^0 = 2 \gg x_i$ in Γ for every $x_i \in [1,2[$ as well as for $x_i = 3$; similarly, $z_i^0 = -2 \gg x_i$ in Γ for every $x_i \in \{-3\} \cup]-2,1]$. Furthermore, $y_i^k \gg x_i$ in Γ^k $(k > 0)$ for every $x_i \in [y_i^{k-1}, y_i^k[$, while $z_i^k \gg x_i$ in Γ^k $(k > 0)$ for every $x_i \in]z_i^k, z_i^{k-1}]$. Therefore, the sequence $\langle \Gamma^k \rangle$ is a WD-scheme. Defining Γ^ω by $X_i^\omega := \{0\}$, we obtain a perfect WD-scheme.

Remark. The perfect WD-scheme in Example 5.15 can be called "maximal reduction": at each step, *all* weakly dominated strategies are eliminated. It can also be noted that if we change each utility function at one point, setting $u_i(0,0) := -2$, then the game will have no Nash equilibrium, but remain WDS: the same sequence of subgames will remain a perfect WD-scheme; however, it will no longer be "maximal reduction".

Weaker analogs of Theorems 5.11 and 5.12 can be obtained under the same restriction on the elimination scheme as in Theorem 5.7, viz., when strategies eliminated at every particular step form open sets. Unlike that theorem, we still need a closedness assumption here.

Lemma 5.16. *Let Γ be a strongly BR-consistent game where every X_i is compact and every $\mathcal{R}_i(x_{-i})$ is closed in X_i. Let λ be a WBR-scheme such that every $X_i^{\lambda(\alpha)}$ $(i \in N,\ \alpha \in \Lambda)$ is closed in X_i. Then there holds $\mathcal{R}_i^{\lambda(\alpha)}(x_{-i}) = \mathcal{R}_i(x_{-i}) \cap X_i^{\lambda(\alpha)} \neq \emptyset$ for every $i \in N$, $\alpha \in \Lambda$, and $x_{-i} \in X_{-i}^{\lambda(\alpha)}$.*

Proof. We argue similarly to Lemma 5.10. Strong BR-consistency implies that it is enough to show

$$\mathcal{R}_i(x_{-i}) \cap X_i^{\lambda(\alpha)} \neq \emptyset. \tag{5.9}$$

And this is done by induction: if (5.9) holds for α, it holds for $\alpha+1$ by the definition of a WBR-scheme; if α is a limit and (5.9) holds for all $\beta < \alpha$, then we have $\mathcal{R}_i(x_{-i}) \cap X_i^{\lambda(\alpha)} = \mathcal{R}_i(x_{-i}) \cap \bigcap_{\beta<\alpha} X_i^{\lambda(\beta)} = \bigcap_{\beta<\alpha}[\mathcal{R}_i(x_{-i}) \cap X_i^{\lambda(\beta)}] \neq \emptyset$ since $\mathcal{R}_i(x_{-i}) \cap X_i^{\lambda(\beta)}$ form a chain of nonempty compact subsets of X_i. □

Corollary. *Let Γ be a strongly BR-consistent game where every X_i is compact and every $\mathcal{R}_i(x_{-i})$ is closed in X_i. Let Γ admit a perfect WBR-scheme λ such that every $X_i^{\lambda(\alpha)}$ ($i \in N,\ \alpha \in \Lambda$) is closed in X_i. Let $x_N^0 \in X_N^{\max\Lambda}$. Then x_N^0 is a Nash equilibrium in Γ.*

Theorem 5.17. *Let Γ be a strongly BR-consistent game where every X_i is compact and every $\mathcal{R}_i(x_{-i})$ is closed in X_i. Let Γ admit a perfect WBR-scheme λ such that every $X_i^{\lambda(\alpha)}$ ($i \in N,\ \alpha \in \Lambda$) is closed in X_i. Then Γ has the pseudo-CSBRP and pseudo-CBRP. If, additionally, $\#N = 2$, then Γ has the weak CSBRP and weak CBRP.*

Proof. We again consider the functions μ_i and μ^- defined by (5.2) and (5.3), and again denote $\mathrm{M}(x_N) := \mathrm{Argmin}_{i\in N}\, \mu_i(x_i)$ for every $x_N \in X_N$.

Claim 5.17.1. *If $\mu^-(x_N) < \max\Lambda$, then for each $i \in N$ there is $y_i \in \mathcal{R}_i(x_{-i})$ such that $\mu_i(y_i) > \mu^-(x_N)$.*

Proof. We argue exactly as in the proof of Claim 5.11.1, only replacing the reference to Lemma 5.10 with that to Lemma 5.16. □

Both orders defined by (5.5b) and (5.8) are ω-transitive. Therefore, the first is a simultaneous pseudo-Cournot ω-potential, while the second is a pseudo-Cournot ω-potential, exactly as in the proof of Theorem 5.11. Accordingly, Γ has the pseudo-CSBRP and pseudo-CBRP.

Proving the second statement, we no longer can rely on Proposition 4.8. In order to present explicit potentials, we introduce a modification of relation (5.8) on X_N:

$$\begin{aligned} y_N \succ x_N \rightleftharpoons \big[\,\mu^-(y_N) > \mu^-(x_N) \ \text{ or } \ [\mu^-(x_N) = \mu^-(y_N)\ \&\ \mathrm{M}(y_N) \subset \mathrm{M}(x_N)] \ \text{ or} \\ \exists i \in N\, [y_i = x_i \in \mathcal{R}_i(x_{-i})\ \&\ \mathrm{M}(y_N) = \mathrm{M}(x_N) = \{i\}\ \&\ x_{-i} \notin \mathcal{R}_{-i}(x_i) \ni y_{-i}]\big]. \end{aligned} \tag{5.10}$$

The relation is obviously irreflexive; the transitivity is obvious as long as the first or second disjunctive terms in (5.10) are applicable. Let $y_N \succ x_N$ by the third term; then the appropriate i is unique and $x_i = y_i$, hence $\mu^-(x_N) = \mu_i(x_i) = \mu^-(y_N)$. Now if $z_N \succ y_N$, then the second disjunctive term in (5.10) cannot be valid because $\mathrm{M}(y_N) = \{i\}$ while the third term cannot be valid because $y_{-i} \in \mathcal{R}_{-i}(y_i)$, hence $\mu^-(z_N) > \mu^-(y_N) = \mu^-(x_N)$, hence $z_N \succ x_N$ by the first term in (5.10). Similarly, if $x_N \succ z_N$, then the third term cannot be valid because $x_{-i} \notin \mathcal{R}_{-i}(x_i)$, hence either the first or the second term must be applicable; therefore, $y_N \succ z_N$ by the same term.

Since there may be only a finite number of consecutive domination by the second or third term in (5.10), $\succ$ is ω-transitive.

Let $x_N \in X_N$; if $\mu^-(x_N) = \max \Lambda$, then x_N is a Nash equilibrium by Corollary to Lemma 5.16. Assuming $\mu^-(x_N) < \max \Lambda$, we may apply Claim 5.17.1. For each $i \in N$, we set $y_i := x_i$ if $x_i \in \mathcal{R}_i(x_{-i})$, and pick y_i such that $\mu_i(y_i) > \mu^-(x_N)$ otherwise. If $y_N = x_N$, then x_N is again a Nash equilibrium. Otherwise, $y_N \rhd^{\mathrm{sBR}} x_N$. We also set $y'_N := (y_i, x_{-i})$ where i minimizes μ_i under condition $x_i \notin \mathcal{R}_i(x_{-i})$.

Let us show $y_N \succ x_N$, hence $\succ$ is a weak simultaneous Cournot potential. If there is $i \in \mathrm{M}(x_N)$ such that $y_i \neq x_i$, we have $y_N \succ x_N$ by the first or second term in (5.10). Otherwise, we must have $\mathrm{M}(x_N) = \{i\}$ and $y_i = x_i$, hence $\mathrm{M}(y_N) = \{i\}$ as well. It is easily seen now that $y_N \succ x_N$ by the third disjunctive term in (5.10).

To show that $\succ$ is a weak Cournot potential, we combine arguments from the preceding paragraph and from the last paragraph of the proof of Theorem 5.11. If $i \in \mathrm{M}(x_N)$, then $y'_N \succ x_N$ by the first or second term in (5.10). Otherwise, $\mathrm{M}(x_N) = \{-i\}$, hence $y'_N \succ x_N$ by the third disjunctive term in (5.10). □

Remark. The problem in Example 5.15 is just that the strategy sets eliminated at the first step are not open, hence Theorem 5.17 does not apply.

Example 5.18 (Example 4.17, continued). Let us consider the same game Γ as in Example 4.17. Just recall that $N = \{1, 2\}$, $X_1 = X_2 = [-1, 1]$, and the preferences are defined by continuous utility functions, which are piece-meal monotone in own choice. The game has the approximate FBRP as well as the CSBRP, but not the very weak FSBRP.

Let us define $\lambda \colon (\mathbb{N} \cup \{\omega\}) \to \mathfrak{G}_\Gamma$ by $X_i^{\lambda(k)} := [-1, -1+1/2^k] \cup [1-1/2^k, 1]$ for all $k \in \mathbb{N}$, and $X_i^{\lambda(\omega)} := \{-1, 1\}$. It is easy to check that λ is an SD-scheme, although not perfect because $(1, -1) \not\succ_1 (-1, -1)$ and $(1, -1) \not\succ_2 (1, 1)$. However, defining $X_1^{\lambda(\omega+1)} := \{1\}$ and $X_2^{\lambda(\omega+1)} := \{-1\}$, we obtain a perfect WD-scheme.

Thus, the assumptions of Theorem 5.17, even when supplemented with continuous preferences and weak dominance solvability ("almost" strict dominance solvability!), do not imply the very weak FSBRP. The very weak FBRP in Example 4.17 is not quite "accidental," cf. Theorem 4.15, but it cannot be derived from a general theorem at the moment: Theorem 4.15 assumes CBRP, while Theorem 5.17 only gives weak CBRP.

Example 5.19. Let $N := \{1, 2\}$, $X_1 := X_2 := [0, 1]$, and the preferences be defined by these utility functions: $u_1(x_N) := 1$ if $2x_1 = x_2$; $u_1(x_N) := 0$ otherwise; $u_2(x_N) := 1$ if $\big[x_2 \neq 0 \ \& \ [x_1 = 0 \ \text{ or } \ x_1 = 2x_2]\big]$; $u_2(x_N) := 0$ otherwise. Clearly, $\mathcal{R}_1(x_2) = \{x_2/2\}$ for all x_2 whereas $\mathcal{R}_2(0) =]0, 1]$ and $\mathcal{R}_2(x_1) = \{x_1/2\}$ for $x_1 > 0$. Defining $X_i^k := [0, 1/2^k]$ ($i \in N$, $k \in \mathbb{N}$) and $X_i^\omega := \{0\}$, we obtain a perfect WD-scheme: $y_i^1 := 1/2$ weakly dominates every $x_i \in]y_i^1, 1]$ in Γ; similarly, for each $k \in \mathbb{N}$, $y_i^{k+1} := 1/2^{k+1}$ weakly dominates every $x_i \in]y_i^{k+1}, y_i^k]$ in Γ^k. Therefore, Γ is WDS. On the other hand, it possesses no Nash equilibrium, hence cannot have any stronger property. All assumptions of Theorem 5.17 are satisfied, except $\mathcal{R}_2(0)$ is not closed in X_2.

6 Strategic complementarity

Assuming a (partial) order on each X_i, we say that a preference relation $\succ\!\!\succ_i$ has the *single crossing* property if these conditions hold:

$$\forall x_i, y_i \in X_i \, \forall x_{-i}, y_{-i} \in X_{-i} \left[[y_i \succ\!\!\succ_i^{x_{-i}} x_i \;\&\; y_i > x_i \;\&\; y_{-i} > x_{-i}] \Rightarrow y_i \succ\!\!\succ_i^{y_{-i}} x_i \right]; \quad (6.1a)$$

$$\forall x_i, y_i \in X_i \, \forall x_{-i}, y_{-i} \in X_{-i} \left[[y_i \succ\!\!\succ_i^{x_{-i}} x_i \;\&\; x_i > y_i \;\&\; x_{-i} > y_{-i}] \Rightarrow y_i \succ\!\!\succ_i^{y_{-i}} x_i \right]. \quad (6.1b)$$

This definition is equivalent to Milgrom and Shannon's (1994) if every $\succ\!\!\succ_i$ is an ordering represented by a numeric function.

Theorem 6.1. *Let Γ be a two player BR-consistent game; let each X_i be a chain; let one of them contain its maximum and minimum. Let each preference relation $\succ\!\!\succ_i$ satisfy* (6.1) *and every $\succ\!\!\succ_i^{x_{-i}}$ be strictly acyclic. Then Γ has the FBRP.*

Proof. Without restricting generality, we may assume $N = \{1, 2\}$ and $\max X_1$ and $\min X_1$ exist. Suppose to the contrary that $\langle x_N^k \rangle_{k \in \mathbb{N}}$ is an infinite Cournot path. Since we could start the path anyplace, we may assume

$$x_1^{2k} \notin \mathcal{R}_1(x_2^{2k}) \ni x_1^{2k+1} = x_1^{2k+2}; \quad \mathcal{R}_2(x_1^{2k}) \ni x_2^{2k} = x_2^{2k+1} \notin \mathcal{R}_2(x_1^{2k+1}).$$

Again without restricting generality, we may assume $x_1^1 > x_1^0$. Now if we assume $x_2^2 < x_2^0$, then the relation $x_2^2 \succ\!\!\succ_2^{x_1^1} x_2^0$ and condition (6.1b) would imply $x_2^2 \succ\!\!\succ_2^{x_1^0} x_2^0$, contradicting our assumption $x_2^0 \in \mathcal{R}_2(x_1^0)$. A straightforward inductive argument shows that $x_2^{2k+2} > x_2^{2k+1}$ and $x_1^{2k+1} > x_1^{2k}$ for all $k \in \mathbb{N}$. Now the relation $x_2^{2k+2} \succ\!\!\succ_2^{x_1^{2k+1}} x_2^{2k}$ and condition (6.1a) imply $x_2^{2k+2} \succ\!\!\succ_2^{\max X_1} x_2^{2k}$, hence $\langle x_2^{2k} \rangle_{k \in \mathbb{N}}$ is an infinite tâtonnement path of $\succ\!\!\succ_2^{\max X_1}$, which fact contradicts the latter's strict acyclicity. □

Remark. A bit more detailed analysis shows that it is enough to assume in Theorem 6.1 that one X_i contains $\min X_i$ and one X_i contains $\max X_i$. If, say, $\max X_i$ does not exist for both i, there may be no Nash equilibrium (Kukushkin, 2010, Example 4.1). The theorem becomes wrong if $n > 2$ (Kukushkin et al., 2005, Example 4) or X_i are not chains (Kukushkin et al., 2005, Example 2), even if the preferences are defined with utility functions and all X_i are finite.

Theorem 6.2. *Let each X_i in a game Γ be a chain containing its maximum and minimum. Let each preference relation $\succ\!\!\succ_i$ satisfy* (6.1) *and every $\succ\!\!\succ_i^{x_{-i}}$ be strictly acyclic and transitive. Then Γ has the weak FBRP.*

Proof. We define

$$X^{\uparrow} := \{x_N \in X_N \mid \forall i \in N \, \forall y_N \in X_N \, [y_N \rhd_i^{\mathrm{BR}} x_N \Rightarrow y_i > x_i]\}. \quad (6.2)$$

Claim 6.2.1. *If $x_N \in X^{\uparrow}$ and $y_N \rhd^{\mathrm{BR}} x_N$, then $y_N \in X^{\uparrow}$ too.*

Proof. Let $y_N \rhd_i^{\mathrm{BR}} x_N$; then $y_i > x_i$ since $x_N \in X^{\uparrow}$. Suppose, to the contrary, that there are $z_N \in X_N$ and $j \in N$ such that $z_N \rhd_j^{\mathrm{BR}} y_N$ and $y_j > z_j$. Since $y_i \in \mathcal{R}_i(y_{-i})$, we have $j \neq i$, hence $z_{-j} = y_{-j} > x_{-j}$, hence $z_j \succ\!\!\succ_j^{x_{-j}} x_j$ by (6.1b), hence $x_j \notin \mathcal{R}_j(x_{-j})$. Now we have $z_j \notin \mathcal{R}_j(x_{-j})$ because we would have $(z_j, x_{-j}) \rhd_j^{\mathrm{BR}} x_N$ and $z_j < x_j [= y_j]$ otherwise, contradicting $x_N \in X^{\uparrow}$. Since Γ is BR-consistent, there is $z'_N \in X_N$ such that $z'_N \rhd_j^{\mathrm{BR}} (z_j, x_{-j})$. Since $\succ\!\!\succ_j$ is transitive, we have $z'_N \rhd_j^{\mathrm{BR}} x_N$ as well. Therefore, $z'_j > x_j$ because $x_N \in X^{\uparrow}$, hence $(z'_j, y_{-j}) \rhd_j^{\mathrm{BR}} z_N$ by (6.1a), hence $z_j \notin \mathcal{R}_j(y_{-j})$, contradicting our assumption $z_N \rhd_j^{\mathrm{BR}} y_N$. □

If $x_N^0 \in X^{\uparrow}$, but is not an equilibrium, we pick an arbitrary $x_N^1 \in X_N$ such that $x_N^1 \rhd^{\mathrm{BR}} x_N^0$; then $x_N^1 \in X^{\uparrow}$ by Claim 6.2.1. Iterating this rule, we obtain a Cournot path $\langle x_N^k \rangle_k$ such that $x_N^k \in X^{\uparrow}$ whenever x_N^k is defined. Besides, $x_i^{k+1} > x_i^k$ whenever $x_N^{k+1} \rhd_i^{\mathrm{BR}} x_N^k$; by (6.1a), we have $x_i^{k+1} \succ\!\!\succ_i^{\max X_{-i}} x_i^k$ for all such k. If the path is infinite, then we will have an infinite number of improvements for, at least, one i (actually, two), contradicting the assumed strict acyclicity. Therefore, it must stop at some stage, and that is only possible at an equilibrium.

If $x_N^0 \notin X^{\uparrow}$, we pick $i \in N$ and $x_N^1 \in X_N$ such that $x_N^1 \rhd_i^{\mathrm{BR}} x_N^0$ and $x_i^1 < x_i^0$; if $x_N^1 \notin X^{\uparrow}$, we behave similarly. Iterating this rule as long as $x_N^k \notin X^{\uparrow}$, we obtain a Cournot path $\langle x_N^k \rangle_k$ such that $x_i^{k+1} < x_i^k$ whenever $x_N^{k+1} \rhd_i^{\mathrm{BR}} x_N^k$. The path cannot be infinite for the same (or rather dual) reason as in the previous paragraph. Once $x_N^k \in X^{\uparrow}$, we already know that an infinite Cournot path is impossible. □

Remark. Theorem 6.2 remains valid if we allow one of X_i's not to contain $\min X_i$ and one of them not to contain $\max X_i$. It remains unclear whether the transitivity assumption could be weakened.

Theorem 6.3. *Let Γ be a two player strategic game. Let each X_i be a compact metric space and a complete chain such that the order is continuous. Let each preference relation $\succ\!\!\succ_i$ satisfy* (6.1) *and every relation $\succ\!\!\succ_i^{x_{-i}}$ be irreflexive and ω-transitive. Then Γ has the CBRP.*

Proof. First of all, Γ is BR-consistent by Corollary to Theorem 3.12. We invoke $X^{\uparrow}$ defined by (6.2). Since the proof of Claim 6.2.1 does not use the strict acyclicity assumption, it remains valid here.

Claim 6.3.1. *Let $x_N^k \in X^{\uparrow}$ and $x_N^{k+1} \geq x_N^k$ for all $k \in \mathbb{N}$. Let $x_i = \sup_{k \in \mathbb{N}} x_i^k$ and $x_i \succ\!\!\succ_i^{x_{-i}} x_i^k$ for both $i \in N$ and each $k \in \mathbb{N}$. Then $x_N \in X^{\uparrow}$ as well.*

Proof. Supposing, to the contrary, the possibility of $y_N \rhd_i^{\mathrm{BR}} x_N$ and $y_i < x_i$, we pick $k \in \mathbb{N}$ such that $y_i < x_i^k$. We have $y_i \succ\!\!\succ_i^{x_{-i}} x_i \succ\!\!\succ_i^{x_{-i}} x_i^k$, hence $y_i \succ\!\!\succ_i^{x_{-i}^k} x_i^k$ by (6.1b). By BR-consistency, there is $z_i \in \mathcal{R}_i(x_{-i}^k)$ such that either $z_i = y_i$ or $z_i \succ\!\!\succ_i^{x_{-i}^k} y_i$; by the transitivity of $\succ\!\!\succ_i^{x_{-i}^k}$, we have $z_i \succ\!\!\succ_i^{x_{-i}^k} x_i^k$ in either case. Since $x_N^k \in X^{\uparrow}$ and $(z_i, x_{-i}^k) \rhd_i^{\mathrm{BR}} x_N^k$, we have $z_i > x_i^k [> y_i]$. Now (6.1a) applies, producing $z_i \succ\!\!\succ_i^{x_{-i}} y_i$, which is incompatible with $y_i \in \mathcal{R}_i(x_{-i})$. □

Defining

$$X^{\downarrow} := \{x_N \in X_N \mid \forall i \in N\, \forall y_N \in X_N\, [y_N \rhd_i^{\mathrm{BR}} x_N \Rightarrow y_i < x_i]\}, \tag{6.3}$$

we easily see that "dual" versions of Claims 6.2.1 and 6.3.1 are valid too.

Claim 6.3.2. *Let π be a Cournot path. If $\pi(2)$ is defined, then $\pi(2) \in X^{\uparrow} \cup X^{\downarrow}$.*

Proof. Let $\pi(2) \rhd_i^{\mathrm{BR}} \pi(1)$; then $\pi_i(0) = \pi_i(1)$, hence $\pi_i(2) \neq \pi_i(0)$. Let us show that $\pi(2) \in X^{\uparrow}$ if $\pi_i(2) > \pi_i(0)$. Indeed, if $x_N \rhd^{\mathrm{BR}} \pi(2)$, then $x_N \rhd_{-i}^{\mathrm{BR}} \pi(2)$. An assumption that $x_{-i} < \pi_{-i}(2)\, [= \pi_{-i}(1)]$ would imply $x_{-i} \succ\!\!\succ_{-i}^{\pi_i(0)} \pi_{-i}(1)$ by (6.1b), which is incompatible with $\pi_{-i}(1) \in \mathcal{R}_{-i}(\pi_i(0))$.

Dually, if $\pi_i(2) < \pi_i(0)$, then $\pi(2) \in X^{\downarrow}$. □

Let π be a Cournot path such that $\pi(2)$ is defined. Straightforward induction based on Claims 6.2.1 and 6.3.1 or their "dual" versions shows that either $\pi(\alpha) \in X^{\uparrow}$ for all $\alpha \in \operatorname{Dom}\pi$, $\alpha \geq 1$, or $\pi(\alpha) \in X^{\downarrow}$ for all $\alpha \in \operatorname{Dom}\pi$, $\alpha \geq 1$. In either case, an assumption that $\operatorname{Dom}\pi = \Omega$ contradicts Proposition 3.11 applied to $\succ\!\!\succ_{-i}^{\max X_i}$ in the first case or to $\succ\!\!\succ_{-i}^{\min X_i}$ in the second. □

The continuity assumption in Theorem 6.3 cannot be replaced with quasicontinuity.

Example 6.4 (Kukushkin, 2000, Example 5.2). Let $N := \{1, 2\}$, $X_i := [0,1] \times [0,1]$ with the lexicographic order (first component, x_i^1, matters first), and the preferences be defined by these utility functions: $u_i(x_N) := -\left|\varphi_i(x_i) - \psi_{-i}(x_{-i})\right|$, where

$$\varphi_1(x_1) := \begin{cases} x_1^2 + 1, & x_1^1 = 1; \\ x_1^1, & x_1^1 < 1,\ x_1^2 = 1; \\ -1, & \text{otherwise}; \end{cases} \quad \varphi_2(x_2) := \begin{cases} 2x_2^1 + 1, & 0 < x_2^1 \leq 1/2,\ x_2^2 = 0; \\ 2x_2^2 - 1, & x_2^1 = 0,\ x_2^2 \geq 1/2; \\ -1, & \text{otherwise}; \end{cases}$$

$$\psi_1(x_1) := \begin{cases} x_1^2 + 1, & x_1^1 = 1; \\ x_1^1, & x_1^1 < 1; \end{cases} \quad \psi_2(x_2) := \begin{cases} x_2^1 + 1, & x_2^1 > 0; \\ x_2^2, & x_2^1 = 0. \end{cases}$$

Conditions (6.1) are checked easily; the best responses are single-valued and increasing. There is a unique Nash equilibrium: $((1,0),(0,1))$; there is even the weak CBRP. However, the relation $\rhd^{\mathrm{BR}}$ is not Ω-acyclic. Let us consider the following mapping $[0, \omega + \omega] \to X_N$: $\pi(0) := ((1,0),(0,0))$; $\pi(2k+1) := ((1 - 1/2^k, 1), (0, 1 - 1/2^k))$; $\pi(2k+2) := ((1 - 1/2^k, 1), (0, 1 - 1/2^{k+1}))$; $\pi(\omega) := ((1,1),(0,1))$; $\pi(\omega + 2k + 1) := ((1, 1/2^k), (1/2^{k+1}, 0))$; $\pi(\omega + 2k + 2) := ((1, 1/2^{k+1}), (1/2^{k+1}, 0))$; $\pi(\omega + \omega) := ((1,0),(0,0))$. It is easy to check that $\pi(2k+1) \rhd_1^{\mathrm{BR}} \pi(2k)$, $\pi(2k+2) \rhd_2^{\mathrm{BR}} \pi(2k+1)$, $\pi(\omega + 2k + 1) \rhd_2^{\mathrm{BR}} \pi(\omega + 2k)$, and $\pi(\omega + 2k + 2) \rhd_1^{\mathrm{BR}} \pi(\omega + 2k + 1)$ for all $k \in \mathbb{N}$. Moreover, $\pi(k) \to \pi(\omega)$ and $\pi(\omega + k) \to \pi(\omega + \omega) = \pi(0)$. Therefore, π is a Cournot cycle.

Conjecture 6.5. *Let Γ be a two player strategic game. Let each X_i be a compact metric space and a complete chain such that the order is quasicontinuous. Let each preference relation $\ggcurly_i$ satisfy* (6.1) *and every relation $\ggcurly_i^{x_{-i}}$ be irreflexive and ω-transitive. Then Γ has the CσBRP for both permutations σ.*

Theorem 6.6. *Let each X_i in a strategic game Γ be a compact metric space and a complete chain such that the order is continuous. Let each preference relation $\ggcurly_i$ satisfy* (6.1) *and every relation $\ggcurly_i^{x_{-i}}$ be irreflexive and ω-transitive. Then Γ has the weak CBRP.*

The proof is essentially a combination of those of Theorems 6.2 and 6.3.

Turning to games with "multi-dimensional" strategy sets, we have to assume that the preference relations are essentially orderings; we also need a rather weak version of Milgrom and Shannon's (1994) quasisupermodularity property. Given $i \in N$ and $x_{-i} \in X_{-i}$, we consider this condition:

$$\forall x_i, y_i \in X_i \left[x_i \ggcurly_i^{x_{-i}} y_i \wedge x_i \Rightarrow [(y_i \vee x_i \ggcurly_i^{x_{-i}} x_i) \text{ or } (y_i \vee x_i \ggcurly_i^{x_{-i}} y_i)]\right]. \quad (6.4)$$

Theorem 6.7. *Let each X_i in a strategic game Γ be a compact metric space and a complete lattice such that the order is continuous. Let each preference relation $\ggcurly_i$ satisfy* (6.1a)*. Let every $\ggcurly_i^{x_{-i}}$ be an upper semicontinuous ordering satisfying* (6.4)*. Then Γ has the weak CIP.*

Proof. We define

$$X^{\uparrow} := \{x_N \in X_N \mid \exists y_N \in X_N \, [y_N > x_N \;\&\; y_N \rhd^{\mathrm{Ind}} x_N]\}; \;\; X^{\downarrow} := X_N \setminus X^{\uparrow};$$

$$y_N \succ x_N \rightleftharpoons \big[\, [y_N \in X^{\downarrow} \;\&\; x_N \in X^{\uparrow}] \text{ or } [x_N, y_N \in X^{\uparrow} \;\&\; y_N > x_N] \text{ or } [x_N, y_N \in X^{\downarrow} \;\&\; y_N < x_N]\big]. \quad (6.5)$$

Clearly, $\succ$ is irreflexive and transitive; let us show that it is a weak ω-potential of $\rhd^{\mathrm{Ind}}$.

Claim 6.7.1. *If $x_N \in X_N$ is not a Nash equilibrium, then there exists $y_N \in X_N$ such that $y_N \rhd^{\mathrm{Ind}} x_N$ and $y_N \succ x_N$; in other words,* (2.13) *holds.*

Proof. If $x_N \in X^{\uparrow}$, then we pick $y_N \in X_N$ such that $y_N \rhd^{\mathrm{Ind}} x_N$ and $y_N > x_N$. If $y_N \in X^{\downarrow}$, then $y_N \succ x_N$ by the first disjunctive term in (6.5). If $y_N \in X^{\uparrow}$, then $y_N \succ x_N$ by the second disjunctive term in (6.5).

Let $x_N \in X^{\downarrow}$. We pick $i \in N$ and $y_N \in X_N$ such that $y_N \rhd_i^{\mathrm{Ind}} x_N$. Denoting $Y_i := \{z_i \in X_i \mid z_i \leq x_i\}$, we pick $z_i \in M(Y_i, \ggcurly_i^{x_{-i}})$ [$\neq \emptyset$ because Y_i is compact and $\ggcurly_i^{x_{-i}}$ upper semicontinuous]. Since $x_N \in X^{\downarrow}$, $y_i > x_i$ is impossible. If $y_i < x_i$, then $z_i \succcurlyeq_i^{x_{-i}} y_i$, hence $z_i \ggcurly_i^{x_{-i}} x_i$ and $z_i < x_i$. If y_i and x_i are incomparable in the order, then $y_i \vee x_i > x_i$ and $y_i \wedge x_i < x_i$. An assumption that $x_i \succcurlyeq_i^{x_{-i}} y_i \wedge x_i$ would imply $y_i \ggcurly_i^{x_{-i}} y_i \wedge x_i$, hence $y_i \vee x_i \ggcurly_i^{x_{-i}} x_i$ by (6.4), contradicting our assumption that $x_N \in X^{\downarrow}$. Therefore,

$y_i \wedge x_i \not\succ_i^{x_{-i}} x_i$, hence $z_i \not\succ_i^{x_{-i}} x_i$ and $z_i < x_i$ again. Denoting $z_N := (z_i, x_{-i})$, we see that $z_N \rhd^{\mathrm{Ind}} x_N$ and $z_N < x_N$. To show that $z_N \succ x_N$, we only have to show that $z_N \in X^{\downarrow}$.

Suppose the contrary: there are $j \in N$ and $y'_j > z_j$ such that

$$y'_j \not\succ_j^{z_{-j}} z_j. \tag{6.6}$$

Let us consider two alternatives.

If $j = i$ (hence $z_{-j} = x_{-i}$), $y'_i > x_i$ would contradict $x_N \in X^{\downarrow}$ while $y'_i < x_i$ would contradict the choice of z_i; therefore, we have to assume that y'_i and x_i are incomparable, hence $y'_i \vee x_i > x_i$. The choice of z_i implies $z_i \succcurlyeq_i^{x_{-i}} y'_i \wedge x_i$, hence, by (6.6), $y'_i \not\succ_i^{x_{-i}} y'_i \wedge x_i$, hence, by (6.4), $y'_i \vee x_i \not\succ_i^{x_{-i}} x_i$, contradicting the assumption $x_N \in X^{\downarrow}$.

Thus, we are led to $j \neq i$, hence $y'_j > z_j = x_j$ and $z_{-j} < x_{-j}$. Now (6.6) and (6.1a) imply $y'_j \not\succ_j^{x_{-j}} x_j$, again contradicting the assumption $x_N \in X^{\downarrow}$. □

Claim 6.7.2. *$\succ$ is ω-transitive.*

Proof. Let $x_N^k \to x_N^{\omega}$ and $x_N^{k+1} \succ x_N^k$ for all $k \in \mathbb{N}$. We have to show that $x_N^{\omega} \succ x_N^0$. The only point worth discussing is that the assumptions that $x_N^k \in X^{\downarrow}$ and $x_N^{k+1} < x_N^k$ for all k imply $x_N^{\omega} \in X^{\downarrow}$. Suppose the contrary: there exist $i \in N$ and $y_i \in X_i$ such that $y_i > x_i^{\omega}$ (hence $y_i > x_i^k$ for all k large enough) and $y_i \not\succ_i^{x_{-i}^{\omega}} x_i^{\omega}$. Since $\not\succ_i^{x_{-i}^{\omega}}$ is upper semicontinuous, we have

$$y_i \not\succ_i^{x_{-i}^{\omega}} x_i^k \tag{6.7}$$

for all k large enough. Obviously, $x_{-i}^{\omega} \leq x_{-i}^k$ for any k; therefore, (6.7) and (6.1a) imply $y_i \not\succ_i^{x_{-i}^k} x_i^k$, contradicting the assumption $x_N^k \in X^{\downarrow}$. □

In the light of Claims 6.7.1 and 6.7.2, a reference to Proposition 3.22 finishes the proof. □

Remark. Theorem 6.7 extends Theorem 1 of Kukushkin et al. (2005) to infinite games, simultaneously weakening its assumptions. The dual argument proves an analog of Theorem 6.7 where (6.1a) is replaced with (6.1b), while (6.4) with

$$\forall x_i, y_i \in X_i \left[x_i \not\succ_i^{x_{-i}} y_i \vee x_i \Rightarrow [(y_i \wedge x_i \not\succ_i^{x_{-i}} x_i) \text{ or } (y_i \wedge x_i \not\succ_i^{x_{-i}} y_i)] \right]. \tag{6.8}$$

Corollary. *If, under the assumptions of Theorem 6.7, each preference relation $\not\succ_i$ satisfies* (4.2)*, then Γ has the very weak FIP.*

Proof. Immediately follows from Theorem 6.7 and Corollary to Proposition 4.2. □

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