Congestion games revisited*

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Abstract

Strategic games are considered where the players derive their utilities from participation in certain "processes." Two subclasses consisting exclusively of potential games are singled out. In the first, players choose where to participate, but there is a unique way of participation, the same for all players. In the second, the participation structure is fixed, but each player may have an arbitrary set of strategies. In both cases, the players sum up the intermediate utilities; thus the first class essentially coincides with that of congestion games. The necessity of additivity in each case is proven.

Key words: Nash equilibrium existence, potential game, congestion game, additive aggregation

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1 Introduction

Congestion games were introduced by Rosenthal (1973) as a class of strategic games where Nash equilibria exist without a tint of convexity. Later they played a central role in Monderer and Shapley's (1996) theory of potential games. Theorems 3.1 and 3.2 from that paper showed that a finite game admits an exact (i.e., cardinal) potential if and only if it can be represented as a congestion game (the sufficiency part was implicit in Rosenthal's reasoning). A simpler and more intuitive proof was given in Voorneveld et al. (1999, Theorem 3.3).

This paper strives to deepen the understanding of the "internal mechanism" of congestion games and to provide a stereoscopic view on their role in the theory of potential games.

First, a tentative concept of a *generalized congestion game* is introduced, where the sum of intermediate utilities is replaced with a list of arbitrary (continuous and strictly increasing) aggregation functions, one for each player and each number of arguments. It turns out that the existence of a Nash equilibrium regardless of other characteristics of the game is ensured if and only if every aggregation function is additive up to monotonic transformations satisfying certain restrictions (Theorem 1 below). The sufficiency part is an extension of Rosenthal's result; necessity is closely related to the famous Debreu–Gorman Theorem on additive representation of separable orderings (Debreu, 1960; Gorman, 1968). The presence of a cardinal potential under every circumstances requires more restrictions on the transformations (Theorem 2).

Second, a class of *games with structured utilities*, in a sense, "dual" to congestion games, is introduced. Here the players do not choose *which* facilities to use, but rather *how* to use facilities from a list fixed for each player. As in the case of congestion games, each facility generates an intermediate utility, which enters into the "ultimate" utility of each participant; as in the case of congestion games, additivity up to monotonic transformations is the only way (assuming continuity and strict monotonicity) to ensure the existence of a Nash equilibrium regardless of other characteristics of the game (Theorem 3). If a cardinal potential is desirable, the class of permissible transformations shrinks considerably (Theorem 4).

It turns out that games with structured utilities and additive aggregation of intermediate utilities are even better suited for the role of archetypal potential games than congestion games: A strategic game admits a cardinal potential if and only if it can be represented in the form (Theorem 5 below).

It is instructive to look on games from both classes as local public good (bad) models. Congestion games can be used to analyze phenomena like people "voting with their feet" in the style of Tiebout; however, the assumption that all players affect all public goods in the same way, inherent in Rosenthal's model, is rather restrictive. In a game with structured utilities, relocation is not considered at all, but the author of a model is absolutely free in the specification of how the players affect each locality. To achieve the existence of a potential, additivity of aggregation must be imposed in either case. From an economic viewpoint, it is not at all clear why exactly these two extreme cases should produce nice games; nonetheless, no comparable result has been obtained outside them.

The term "structured utilities" was borrowed from Kukushkin et al. (1985), where the aggregation of intermediate utilities was done with the minimum function, as in Germeier and Vatel' (1974). Additive aggregation in this context first appeared in Kukushkin (1994); the main theorem of that paper resembles our Theorem 3, but they are logically independent. Actually, two games from the class were present in Monderer and Shapley (1996): Cournot oligopoly with identical linear costs (cf. Kukushkin, 1994, Section 3) and the Stag Hunt game of Rousseau.

A broader view on "games with common intermediate objectives," a common generalization of both types of models considered in this paper, is presented in Kukushkin (2004). Theorems 5 and 7 from that paper are generalized by Theorems 1 and 3, respectively, below; Proposition 6.2 is virtually equivalent to our Theorem 5.

The next section reproduces the basic notions about potential games. Section 3 introduces the concept of a generalized congestion game and contains Theorems 1 and 2, which establish the crucial importance of additive aggregation; more complicated proofs are deferred to the Appendix. In Section 4, it is demonstrated that congestion games usually underlie the existence of Nash equilibrium in coalition formation games with additively separable utilities; the fact seems to be underestimated in the literature. Section 5 introduces the concept of a game with structured utilities and contains Theorems 3 and 4, which establish the crucial importance of additive aggregation (more complicated necessity proofs are again deferred to the Appendix), as well as Theorem 5 showing the special role of the class in the theory of potential games.

2 Basic notions

A *strategic game* Γ is defined by a finite set of players N (we denote $n = \#N$), and strategy sets X_i and utility functions u_i on $X = \prod_{i \in N} X_i$ for each $i \in N$. We introduce the (individual) *improvement relation* on X ($y, x \in X$, $i \in N$):

$$y \rhd_i x \iff [y_{-i} = x_{-i} \;\&\; u_i(y) > u_i(x)];$$

$$y \rhd x \iff \exists i \in N \, [y \rhd_i x].$$

A maximizer for $\rhd$, i.e., a strategy profile $x \in X$ such that $y \rhd x$ does not hold for any $y \in X$, is a *Nash equilibrium*.

An (individual) *improvement path* is a finite or infinite sequence $\{x^k\}_{k=0,1,\dots}$ such that $x^{k+1} \rhd x^k$ whenever $k \geq 0$ and x^{k+1} is defined. A game has the *finite improvement property* (*FIP*) if there exists no infinite improvement path. The property implies that every improvement path, if continued whenever possible, reaches a Nash equilibrium in a finite number of steps. For a finite game, it is equivalent to the absence of improvement cycles. For an infinite game, the property is rather exotic: it implies, e.g., that the set of utility levels available to a player when the choices of all partners are fixed cannot contain a non-degenerate interval.

A function $P : X \to \mathbb{R}$ is an *exact potential* of the game if $u_i(y) - u_i(x) = P(y) - P(x)$ whenever $i \in N$, $y, x \in X$, and $y_{-i} = x_{-i}$; equivalently, P is an exact potential if $u_i(x) = P(x) + Q_i(x_{-i})$ for every $i \in N$ and $x \in X$ (pick $x^0 \in X$ and define $Q_i(x_{-i}) = u_i(x_i^0, x_{-i}) - P(x_i^0, x_{-i})$). A function $P : X \to \mathbb{R}$ is an *ordinal potential* of the game if $\operatorname{sign}(u_i(y) - u_i(x)) = \operatorname{sign}(P(y) - P(x))$ whenever $i \in N$, $y, x \in X$, and $y_{-i} = x_{-i}$. A function $P : X \to \mathbb{R}$ is a *generalized ordinal potential* of the game if $P(y) > P(x)$ whenever $y, x \in X$ and $y \rhd x$. Clearly, an exact potential is an ordinal potential, and an ordinal potential is a generalized ordinal potential.

For a finite game, the existence of a generalized ordinal potential is equivalent to the FIP (Monderer and Shapley 1996, Lemma 2.5). If the strategy sets are compact, the presence of a *continuous* generalized ordinal potential ensures the existence of a Nash equilibrium; moreover, all improvement dynamics, in a sense, lead towards the set of Nash equilibria although one cannot, generally, be sure of reaching a Nash equilibrium either in a finite number of steps or even as a limit point. Continuity is by no means necessary for the conclusion, but we do not need more complicated conditions here.

3 Generalized congestion games

A *generalized congestion game* is defined by a finite set of players N, a set A of *processes* (Rosenthal called them "factors"; Monderer and Shapley, "facilities"), an *intermediate utility function* $\varphi_\alpha : \mathbb{N} \to \mathbb{R}$ for each $\alpha \in \mathrm{A}$, a finite set of strategies X_i for each $i \in N$, where each strategy $x_i \in X_i$ is a finite cortege $x_i = \langle \alpha_1, \dots, \alpha_{\#x_i} \rangle$ ($\#x_i > 0$) of members of A without repetitions, and *aggregation function* $V_i^{x_i} : \mathbb{R}^{\#x_i} \to \mathbb{R}$ for each $i \in N$ and $x_i \in X_i$, which must be monotonic in its arguments. The utility functions are defined in this way. Given a strategy profile $x \in X = \prod_{i \in N} X_i$ and $\alpha \in \mathrm{A}$, we define $N(\alpha, x)$ as the set of $i \in N$ for which α enters into x_i (at any position), i.e., the set of players participating in α at x. Now the utility of player i at x is

$$u_i(x) = V_i^{x_i}\big(\langle \varphi_\alpha(\#N(\alpha, x)) \rangle_{\alpha \in x_i}\big). \tag{1}$$

Remark. In principle, φ_α need not be defined on the whole $\mathbb{N}$, nor $V_i^{x_i}$ on the whole

$\mathbb{R}^{\# x_i}$, but we ignore such subtleties.

If each $V_i^{x_i}$ is just the sum of all its arguments, we obtain congestion games as a special case of this scheme. Since the order of arguments does not matter in this case, the corteges can be replaced with subsets.

Remark. The finiteness of each X_i is essential. One might think that an infinite A makes no sense since all the strategies of all players together can only contain a finite number of processes, but it is technically more convenient. For instance, in the sufficiency proof for Theorem 2 below, we even consider congestion games with infinite sets of facilities.

The concept of a *universal aggregation rule* will be used; it is perceived as an infinite sequence of functions $U^{(m)} : \mathbb{R}^m \to \mathbb{R}$, $m = 1, 2, \dots$, each of which is assumed continuous and *strictly increasing* in the sense of

$$\big[\forall s[v'_s \geq v_s] \ \&\ \exists s[v'_s > v_s]\big] \Rightarrow U^{(m)}(v') > U^{(m)}(v). \tag{2}$$

We say that a player $i \in N$ in a generalized congestion game *uses a universal aggregation rule* U_i if, for each $x_i \in X_i$, $V_i^{x_i}$ coincides with $U_i^{(\# x_i)}$; then (1) transforms into

$$u_i(x) = U_i^{(\# x_i)}\big(\langle \varphi_\alpha(\# N(\alpha, x))\rangle_{\alpha \in x_i}\big)$$

for every $x \in X$.

Theorem 1. *Let N be a finite set with $\# N \geq 2$; let $\langle U_i \rangle_{i \in N}$ be a list of universal aggregation rules such that every function $U_i^{(m)}$ is continuous and strictly increasing in the sense of* (2)*. Then the following conditions are equivalent.*

1.1. *Every generalized congestion game where N is the set of players and each player i uses the aggregation rule U_i has the FIP.*

1.2. *Every generalized congestion game where N is the set of players, each player i uses the aggregation rule U_i, two players have two strategies each and all other strategy sets are singletons possesses a Nash equilibrium.*

1.3. *All functions $U_i^{(m)}$ satisfy the following requirements:*

1. *there is a continuous and strictly increasing mapping $\nu : \mathbb{R} \to \mathbb{R}$ and a continuous and strictly increasing mapping $\lambda_i^m : m \cdot \nu(\mathbb{R}) \to \mathbb{R}$ for every $i \in N$ and $m \geq 1$ such that*
$$U_i^{(m)}(v_1, \dots, v_m) = \lambda_i^m\Big(\sum_{s=1}^{m} \nu(v_s)\Big) \tag{3}$$
for all $v_1, \dots, v_m \in \mathbb{R}$;

2. *for every $i \in N$ and $m, m' \geq 1$, there is a constant $\bar{u}_i^{mm'} \in \mathbb{R} \cup \{-\infty, +\infty\}$ such that*

$$\operatorname{sign}\big(\lambda_i^{m'}(u') - \lambda_i^m(u)\big) = \operatorname{sign}(u' - u - \bar{u}_i^{mm'}) \tag{4}$$

for all $u' \in m' \cdot \nu(\mathbb{R})$ and $u \in m \cdot \nu(\mathbb{R})$.

The implication [1.1] $\Rightarrow$ [1.2] is trivial. The proofs of [1.3] $\Rightarrow$ [1.1] and [1.2] $\Rightarrow$ [1.3] are deferred to the Appendix (Sections A and B, respectively).

Remark. The proof of [1.2] $\Rightarrow$ [1.3] remains valid, virtually without any modification, if each $U_i^{(m)}$ is assumed defined on R^m, where R is an open interval (bounded or not) in $\mathbb{R}$; e.g., $R = \mathbb{R}_{++}$. This is also true for the necessity parts of Theorems 2, 3, and 4. If R is not connected (e.g., if only integer-valued φ_α are considered), the proofs collapse; it is not yet known whether the theorems themselves remain valid in this case.

The monotonic transformations can change the meaning of the additive representation (3). For instance, if $\nu(\mathbb{R}) \subseteq \mathbb{R}_{++}$, then [1.3] holds for $U_i^{(m)}(v_1, \ldots, v_m) = \prod_{s=1}^m \nu(v_s)$. On the other hand, taking the average, $U_i^{(m)}(v_1, \ldots, v_m) = \frac{1}{m} \sum_{s=1}^m \nu(v_s)$, satisfies (3), but not (4).

Theorem 2. *Let N be a finite set with $\#N \geq 2$; let $\langle U_i \rangle_{i \in N}$ be a list of universal aggregation rules such that every function $U_i^{(m)}$ is continuous and strictly increasing in the sense of* (2). *Then every generalized congestion game where N is the set of players and each player i uses the aggregation rule U_i admits an exact potential if and only if there is a continuous and strictly increasing mapping $\mu : \mathbb{R} \to \mathbb{R}$ and a constant $C_i^m \in \mathbb{R}$ for every $i \in N$ and $m \geq 1$ such that*

$$U_i^{(m)}(v_1, \ldots, v_m) = \sum_{s=1}^{m} \mu(v_s) + C_i^m \tag{5}$$

for all $v_1, \ldots, v_m \in \mathbb{R}$, $i \in N$ and $m \geq 1$.

The necessity proof is deferred to the Appendix (Section C).

Sufficiency proof. Having a generalized congestion game Γ where N is the set of players and each player i uses an aggregation rule U_i satisfying (5), we consider a congestion game Γ^* with the same set of players N, the set of facilities $M = \mathrm{A} \cup (N \times \mathbb{N})$, strategy sets $\Sigma^i = \{x_i \cup \{(i, \#x_i)\}\}_{x_i \in X_i}$, and cost functions $c_\alpha(h) = \mu(\varphi_\alpha(h))$ for $\alpha \in \mathrm{A}$ and $c_{(i,m)}(h) = C_i^m$. For natural bijections $g^i : X_i \to \Sigma^i$, $i \in N$, we obviously have

$$u_i^*(g(x)) = \sum_{\alpha \in x_i} \mu(\varphi_\alpha(\#N(\alpha, x))) + C_i^m = u_i(x)$$

for every $i \in N$ and $x \in X$, i.e., Γ and Γ^* are isomorphic. Since Γ^* is an exact potential game, so is Γ. $\square$

4 Coalition formation

Rosenthal (1973) is often cited in the literature on coalition formation games, but the technical power of his approach is usually underestimated. Actually, one should look for congestion games whenever the existence of a Nash equilibrium in the context is derived from additive separability in utility functions.

For instance, the group formation games considered by Hollard (2000) are congestion games. Each player there chooses an "action" $a_i \in A$ and the utility is

$$u_i(a_1, \ldots, a_n) = v^i(a_i) + I_{a_i}(n(a_i)) \; + \sum_{z \in A \setminus \{a_i\}} E_z(n(z)), \tag{6}$$

where $n(a)$ is the number of players having chosen a at the given strategy profile, and $v^i(\cdot)$, $I_a(\cdot)$, and $E_a(\cdot)$ are given functions. Let us define the set of facilities as the union of $A \times N$ and two copies of A: $M = (A \times N) \cup \{a^{\mathrm{Int}}\}_{a \in A} \cup \{a^{\mathrm{Ext}}\}_{a \in A}$ with $c_{(a,i)}(k) = v^i(a)$, $c_{a^{\mathrm{Int}}}(k) = I_a(k)$, and $c_{a^{\mathrm{Ext}}}(k) = E_a(n-k)$; the revised strategy sets will be $\Sigma^i = \big\{\{(a,i)\} \cup \{a^{\mathrm{Int}}\} \cup \{b^{\mathrm{Ext}}\}_{b \in A \setminus \{a\}}\big\}_{a \in A}$. Obviously, we have (6) for the utilities in the congestion game; in other words, Theorem 1 of Hollard (2000) is a special case of Rosenthal's (1973) theorem, so there was no need to prove it again.

Similarly, the additive utilities in Section 5 of Konishi et al. (1997b) are a particular case of (6), so a reference to Rosenthal (1973) would have been also sufficient to prove their implication Lemma 4.2 $\Rightarrow$ Proposition 4.1 (Lemma 4.2 itself belongs to a quite different set of ideas).

Bogomolnaia and Jackson (2002) considered "hedonic" coalition formation games and showed, among other things, the existence of a Nash stable coalition partition if the utilities are additively separable and symmetric. Strictly speaking, their basic model was not a strategic game, and a Nash stable partition was not explicitly defined as a Nash equilibrium. Nonetheless, their Proposition 2 can be derived from Rosenthal's theorem.

A hedonic game with additively separable and symmetric utilities is defined by the set of players N and $n \cdot (n-1)/2$ numbers v_{ij} ($i, j \in N$, $i \neq j$, $v_{ij} = v_{ji}$); whenever N is partitioned into disjoint coalitions S_k, each player i receives the utility equal to the sum of v_{ij} over all other members of the same element of the partition. A partition is Nash stable if no player can increase his utility by switching to another element of the partition or by forming a new, singleton, coalition.

To formalize the model as a congestion game, we introduce a set A of "rallying points"; the only condition on A is $\#A \geq n$. Then we denote $N^* = \{I \subseteq N \mid \#I = 2\}$ and $N_i^* = \{I \in N^* \mid i \in I\}$ for $i \in N$. Now we define a congestion game Γ^* by $M = A \times N^*$, $\Sigma_i = \{\{a\} \times N_i^*\}_{a \in A}$, and $c_{(a,\{i,j\})}(k) = 0$ if $k = 1$ and $c_{(a,\{i,j\})}(k) = v_{ij}$ if $k = 2$.

For every $i \in N$, we have a natural mapping $g_i : \Sigma_i \to A$, which is surjective. Every strategy profile s in Γ^* defines a partition of N into $S_a = \{i \in N \mid g_i(s_i) = a\}$ (perhaps empty S_a should be deleted). It is easily checked that each player's utility in Γ^* is again the sum of v_{ij} over all other players who have chosen the same a. Strictly speaking, we do not obtain an isomorphism between the original hedonic game and Γ^* because every partition is generated by several different strategy profiles in the congestion game, but this does not matter. The sum of v_{ij}, the maximization of which was suggested by Bogomolnaia and Jackson, coincides with Rosenthal's potential.

Admittedly, a reference to congestion games would hardly lead to a shorter proof in the last case, unlike the first two. Still, it is essential for deeper understanding of interrelationships between various models.

When the stability to coalition deviations is under investigation, Rosenthal (1973) is of no help. There are some positive results of the kind for congestion games (Holzman and Law-Yone, 1997) or their modifications (Milchtaich, 1996; Konishi et al., 1997a), to say nothing of a broader coalition formation context. However, there seems to be no single driving force behind the results.

5 Games with structured utilities

A *game with structured utilities* Γ may have an arbitrary finite set of players N and arbitrary sets of strategies whereas the utility functions satisfy certain structural requirements. There is a set A of *processes* and a finite cortege $\Upsilon^i = \langle \alpha_1, \ldots \alpha_{m_i} \rangle$ ($\alpha_s \in \mathrm{A}$, no repetition) of processes where each player $i \in N$ participates (given exogenously). With every $\alpha \in \mathrm{A}$, an *intermediate utility function* is associated, $\varphi_\alpha : X_{N(\alpha)} \to \mathbb{R}$, where $N(\alpha) = \{i \in N | \ \alpha \in \Upsilon^i\}$. The "ultimate" utility functions of the players are built of the intermediate utilities:

$$u_i(x) = V_i\big(\langle \varphi_\alpha(x_{N(\alpha)}) \rangle_{\alpha \in \Upsilon^i}\big), \tag{7}$$

where $i \in N$, $x \in X$, and V_i is a monotonic function defined on the appropriate subset of $\mathbb{R}^{m_i}$. We call Γ a *continuous* game with structured utilities if each X_i is a topological space, while all functions φ_α and V_i are continuous (in the appropriate product topologies); then each utility function u_i is continuous too.

Let us consider a couple of illustrations. The version of Stag Hunt game of Rousseau considered in Monderer and Shapley (1996, Section 5) is characterized by strategy sets $X_i = \{1, 2, \ldots, 7\}$ and utilities of the form $u_i(x) = a \cdot \min_{j \in N} x_j - b \cdot x_i + c$. Introducing $n+1$ processes, α_N and α_i, $i \in N$, assuming $\Upsilon^i = \langle \alpha_N, \alpha_i \rangle$, and defining $\varphi_{\alpha_N}(x) = a \cdot \min_{i \in N} x_i$ and $\varphi_{\alpha_i}(x_i) = -b \cdot x_i + c$, we see that our game belongs to the class with $V_i(\upsilon_N, \upsilon_i) = \upsilon_N + \upsilon_i$.

A Cournot oligopoly is characterized by $X_i \subseteq \mathbb{R}$ and utility functions $u_i(x) = x_i \cdot P(\sum_{j \in N} x_j) - C_i(x_i)$. To include the model into the class, we need $2n+1$ processes, α_N and α_i, β_i for $i \in N$, $\Upsilon^i = \langle \alpha_N, \alpha_i, \beta_i \rangle$, $\varphi_{\alpha_N}(x) = P(\sum_{j \in N} x_j)$, $\varphi_{\alpha_i}(x_i) = x_i$, $\varphi_{\beta_i}(x_i) = -C_i(x_i)$, and $V_i(\upsilon_N, \upsilon_i^\alpha, \upsilon_i^\beta) = \upsilon_N \cdot \upsilon_i^\alpha + \upsilon_i^\beta$. The aggregation function is not symmetric, so the order of the processes in Υ^i matters.

The same concept of a universal aggregation rule will be used. We say that a player $i \in N$ in a game Γ *uses a universal aggregation rule* U_i if the appropriate $U_i^{(m)}$ is substituted into (7):

$$u_i(x) = U_i^{(\#\Upsilon^i)}\big(\langle \varphi_\alpha(x_{N(\alpha)})\rangle_{\alpha\in\Upsilon^i}\big)$$

for every $x \in X$.

Theorem 3. *Let N be a finite set with $\#N \geq 2$; let $\langle U_i\rangle_{i\in N}$ be a list of universal aggregation rules such that every function $U_i^{(m)}$ is continuous and strictly increasing in the sense of* (2). *Then the following conditions are equivalent.*

3.1. *Every continuous game with structured utilities where N is the set of players and each player i uses the aggregation rule U_i admits a continuous ordinal potential.*

3.2. *Every game with structured utilities where N is the set of players, each player i uses the aggregation rule U_i, two players have two strategies each and all other strategy sets are singletons possesses a Nash equilibrium.*

3.3. *There is a continuous and strictly increasing mapping $\nu : \mathbb{R} \to \mathbb{R}$ and a continuous and strictly increasing mapping $\lambda_i^m : m \cdot \nu(\mathbb{R}) \to \mathbb{R}$ for every $i \in N$ and $m \geq 1$ such that* (3) *holds for all $v_1, \ldots, v_m \in \mathbb{R}$.*

The implication [3.1] $\Rightarrow$ [3.2] is trivial. The necessity proof, [3.2] $\Rightarrow$ [3.3], is deferred to the Appendix (Section D).

Sufficiency proof. ([3.3] $\Rightarrow$ [3.1]) If [3.3] holds, then $u_i(x) = \lambda_i^{\#\Upsilon^i}\big(\sum_{\alpha\in\Upsilon^i} \nu(\varphi_\alpha(x_{N(\alpha)}))\big)$ for every $i \in N$ and $x \in X$. Let us define $P : X \to \mathbb{R}$ by $P(x) = \sum_{\alpha\in \mathrm{A}} \nu(\varphi_\alpha(x_{N(\alpha)})) = \sum_{\alpha\in\Upsilon^i} \nu(\varphi_\alpha(x_{N(\alpha)})) + \sum_{\alpha\in \mathrm{A}\setminus\Upsilon^i} \nu(\varphi_\alpha(x_{N(\alpha)}))$; clearly, P is continuous. For every $i \in N$ and $x \in X$, we have $P(x) = (\lambda_i^{\#\Upsilon^i})^{-1}\big(u_i(x)\big) + Q_i(x_{-i})$; therefore, P is an ordinal potential indeed. $\square$

As in the case of Theorem 1, the aggregation rule $U_i^{(m)}(v_1, \ldots, v_m) = \prod_{s=1}^m \nu(v_s)$ is allowed, provided $\nu(\mathbb{R}) \subseteq \mathbb{R}_{++}$; Kukushkin (1997) used this representation for the voluntary provision of a public good with Cobb–Douglas utilities. This time, taking the average is an acceptable aggregation rule because every strategy of a given player in a given game involves the same processes, so the sum and the average define the same ordering.

Theorem 4. *Let N be a finite set with $\#N \geq 2$; let $\langle U_i\rangle_{i\in N}$ be a list of universal aggregation rules such that every function $U_i^{(m)}$ is continuous and strictly increasing in the sense of* (2). *Then every game with structured utilities where N is the set of players and* *each player i uses the aggregation rule U_i admits an exact potential if and only if there is a continuous and strictly increasing mapping $\mu : \mathbb{R} \to \mathbb{R}$ and a constant $C_i^m \in \mathbb{R}$ for every $i \in N$ and $m \geq 1$ such that* (5) *holds for all $v_1, \ldots, v_m \in \mathbb{R}$, $i \in N$ and $m \geq 1$.*

The necessity proof is deferred to the Appendix (Section E).

Sufficiency proof. If (5) holds, then an obvious modification of the proof of [3.3] $\Rightarrow$ [3.1] above shows that the same function P becomes an exact potential in this case. $\square$

Now it is easy to see that each player in the Stag Hunt game can be assumed using a universal aggregation rule (8); the presence of an exact potential in such a game accords with Theorem 4. As to the Cournot oligopoly, the above aggregation function V_i cannot be represented even in the form (3); the fact that a Cournot model need not generally have an equilibrium is well known. On the other hand, if we assume that the costs are identical and linear, $C_i(x_i) = c \cdot x_i$, we can, somewhat stretching our basic concepts, represent the model as one with an aggregation function satisfying (3): $\mathrm{A} = \{\alpha_N\} \cup \{\alpha_i\}_{i \in N}$; $\Upsilon^i = \langle \alpha_N, \alpha_i \rangle$; $\varphi_{\alpha_N}(x) = P(\sum_{j \in N} x_j) - c$; $\varphi_{\alpha_i}(x_i) = x_i$; $V_i(\upsilon_N, \upsilon_i) = \upsilon_i \cdot \upsilon_N$; $\nu(\upsilon) = \log(\upsilon)$; $\lambda(u) = \exp(u)$. The representation does not work where $\varphi_{\alpha_N}(x) \leq 0$, but such strategy profiles do not pose any problem.

Remark. McManus (1964) proved that identical *convex* costs are sufficient to ensure the existence of Cournot equilibrium; this fact appears to have nothing to do with Theorem 3. Actually, it only holds for *decreasing* price functions, while identical linear costs ensure the existence without that assumption.

Theorem 5. *A strategic game Γ admits an exact potential if and only if it can be represented as a game with structured utilities where each player uses the ("exact") additive aggregation rule*

$$U^{(m)}(v_1, \ldots, v_m) = \sum_{s=1}^{m} v_s. \tag{8}$$

Proof. If (8) holds, the sufficiency part of Theorem 4 applies.

Let $P : X \to \mathbb{R}$ be an exact potential of Γ. By Theorem 2.1 of Voorneveld et al. (1999), there are functions $Q_{-i} : X_{-i} \to \mathbb{R}$ ($i \in N$) such that $u_i(x) = P(x) + Q_{-i}(x_{-i})$ for all $i \in N$ and $x \in X$. We define $\mathrm{A} = N \cup \{N\}$, $\Upsilon^i = \mathrm{A} \setminus \{i\}$ (i.e., there are $n + 1$ processes; each player participates in n of them; one process is shared by all players; each of the other processes is shared by $n - 1$ players), $\varphi_N(x) = P(x) + \sum_{j \in N} Q_{-j}(x_{-j})$, and $\varphi_i(x_{-i}) = -Q_{-i}(x_{-i})$. Denoting $u_i^*(x)$ the structured utilities, we have

$$u_i^*(x) = \sum_{\alpha \in \Upsilon^i} \varphi_\alpha(x_{N(\alpha)}) = \varphi_N(x) + \sum_{j \neq i} \varphi_j(x_{-j}) =$$
$$P(x) + \sum_{j \in N} Q_{-j}(x_{-j}) - \sum_{j \neq i} Q_{-j}(x_{-j}) = P(x) + Q_{-i}(x_{-i}) = u_i(x)$$

for all $i \in N$ and $x \in X$. $\square$

If Γ is continuous, then, as was noted in Monderer and Shapley (1996), its exact potential must be continuous too, hence so are all Q_{-i}, hence all φ_α. In other words, a continuous game admits an exact potential if and only if it can be represented as a continuous game with structured utilities where each player uses the aggregation rule (8).

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Appendix

First of all, let us note that we refer to the sufficiency proof for Theorem 2 (Section 3) in the sufficiency proof for Theorem 1 (Section A), whereas the necessity proof for the former theorem (Section C) is built on the necessity statement of the latter theorem (Section B). Clearly, there is no circular reasoning here.

A Proof of sufficiency in Theorem 1

Let us prove the implication [1.3] $\Rightarrow$ [1.1].

Suppose to the contrary that a generalized congestion game Γ admits an improvement cycle $x^0, \dots, x^{\bar{m}} = x^0$, while the set of players in Γ is N and each player i uses the aggregation rule U_i. We define $N^* = \{i \in N | \ \exists k \in \{0, \dots, \bar{m}-1\}[x^{k+1} \rhd_i x^k]\}$ and $M_i = \{\#x_i^k\}_{k \in \{0,\dots,\bar{m}\}}$ for each $i \in N^*$. When dealing with the supposed cycle, there is no need to consider $i \notin N^*$ or $m \notin M_i$.

Let us fix an $i \in N^*$. We say that m and m' *overlap* if $\lambda_i^m(m \cdot \nu(\mathbb{R})) \cap \lambda_i^{m'}(m' \cdot \nu(\mathbb{R})) \neq \emptyset$. In this case $\bar{u}_i^{mm'} \in \mathbb{R}$ satisfying (4) is unique; in particular, $\bar{u}_i^{mm} = 0$. An *overlap path* is a sequence $m_0, m_1, \dots, m_k$ such that $m_h \in M_i$ and m_h and m_{h+1} overlap for each $h \in \{0, \dots, k-1\}$. We call m and m' *contiguous* if there is an overlap path $m = m_0, m_1, \dots, m_k = m'$. Clearly, M_i is partitioned into equivalence classes. The union of $\lambda_i^m(m \cdot \nu(\mathbb{R}))$ for all m from a class is an open interval; such intervals defined by distinct classes cannot intersect. Therefore, all $m \in M_i$ form one equivalence class, i.e., are contiguous.

Lemma A.1. *Let* $m_0, m_1, \ldots, m_k$ *be an overlap path. Then* (4) *holds for* $m = m_0$, $m' = m_k$, *and* $\bar{u}_i^{mm'} = \sum_{h=0}^{k-1} \bar{u}_i^{m_h m_{h+1}}$, *where each* $\bar{u}_i^{m_h m_{h+1}}$ *is uniquely defined by* (4) *with* $m = m_h$ *and* $m' = m_{h+1}$.

Proof. We argue by induction. For $k = 1$, the statement is tautological. Let it hold for overlap paths of the "length" $k \geq 1$ or less; we have to prove it for any path of the length $k+1$. For each $s = 0, 1, \ldots, k+1$, we denote $W^s = \lambda_i^{m_s}(m_s \cdot \nu(\mathbb{R})) \subseteq \mathbb{R}$.

Supposing first that $W^{k+1} \cap W^0 = \emptyset$, we may assume that $w'' > w$ for all $w'' \in W^{k+1}$ and $w \in W^0$ (the case of opposite inequalities is treated dually). Since $\bigcup_{s=1}^{k} W^s$ is an open interval which intersects with both W^{k+1} and W^0, there are s and $w' \in W^s$ such that $1 \leq s \leq k$ and $w'' > w' > w$ for all $w'' \in W^{k+1}$ and $w \in W^0$. Let $w' = \lambda_i^{m_s}(u')$. By the induction hypothesis, we have $u' > u + \sum_{h=0}^{s-1} \bar{u}^{m_h m_{h+1}}$ for all $u \in m_0 \cdot \nu(\mathbb{R})$, and $u'' > u' + \sum_{h=s}^{k} \bar{u}^{m_h m_{h+1}}$ for all $u'' \in m_{k+1} \cdot \nu(\mathbb{R})$; therefore, $u'' > u + \sum_{h=0}^{k} \bar{u}^{m_h m_{h+1}}$, i.e., (4) holds.

Now let $W = W^{k+1} \cap W^0 \neq \emptyset$; then $\bar{u}^{m_0 m_{k+1}}$ satisfying (4) is unique. Since $\bigcup_{s=1}^{k} W^s$ is an open interval which intersects with both W^{k+1} and W^0, there is s ($1 \leq s \leq k$) such that $W^s \cap W \neq \emptyset$; therefore, there are $u \in m_0 \cdot \nu(\mathbb{R})$, $u' \in m_s \cdot \nu(\mathbb{R})$, and $u'' \in m_{k+1} \cdot \nu(\mathbb{R})$ such that $\lambda_i^{m_0}(u) = \lambda_i^{m_s}(u') = \lambda_i^{m_{k+1}}(u'')$. By the induction hypothesis, we have $u' = u + \sum_{h=0}^{s-1} \bar{u}^{m_h m_{h+1}}$ and $u'' = u' + \sum_{h=s}^{k} \bar{u}^{m_h m_{h+1}}$, hence $u'' = u + \sum_{h=0}^{k} \bar{u}^{m_h m_{h+1}}$, hence $\bar{u}^{m_0 m_{k+1}} = \sum_{h=0}^{k} \bar{u}^{m_h m_{h+1}}$.

Thus, the induction step is completed, hence the lemma is proven. □

Lemma A.1 immediately implies that, whenever $m_0, m_1, \ldots, m_k = m_0$ is an overlap cycle, we have $\sum_{h=0}^{k-1} \bar{u}_i^{m_h m_{h+1}} = 0$. Now, for each $i \in N^*$ and each $m, m' \in M_i$, we define $\bar{u}_i^{mm'} = \sum_{h=0}^{k-1} \bar{u}_i^{m_h m_{h+1}}$ for an overlap path $m = m_0, m_1, \ldots, m_k = m'$; the value does not depend on the choice of a particular path. Moreover, $\bar{u}_i^{mm''} = \bar{u}_i^{mm'} + \bar{u}_i^{m'm''}$ for all $m, m', m'' \in M_i$.

For each $i \in N^*$, we pick $\bar{m}_i \in M_i$, and define $C_i^m = \bar{u}_i^{m\bar{m}_i}$ whenever $i \in N^*$ and $m \in M_i$, while $C_i^m = 0$ otherwise. Denoting Γ^* the game with the same players, processes, intermediate utilities, and strategies, but with the aggregation functions $U_i^{(m)}(v_1, \ldots, v_m) = \sum_{s=1}^{m} \nu(v_s) + C_i^m$, we see that $x^0, \ldots, x^{\bar{m}} = x^0$ is an improvement cycle in Γ^* as well: $u_i(x^{k+1}) > u_i(x^k)$ implies $\lambda_i^{\#x_i^{k+1}}\Big(\sum_{\alpha \in x_i^{k+1}} \nu\big(\varphi_\alpha(\#N(\alpha, x^{k+1}))\big)\Big) > \lambda_i^{\#x_i^k}\Big(\sum_{\alpha \in x_i^k} \nu\big(\varphi_\alpha(\#N(\alpha, x^k))\big)\Big)$, hence, by condition (4), $\sum_{\alpha \in x_i^{k+1}} \nu\big(\varphi_\alpha(\#N(\alpha, x^{k+1}))\big) > \sum_{\alpha \in x_i^k} \nu\big(\varphi_\alpha(\#N(\alpha, x^k))\big) + \bar{u}_i^{\#x_i^k \#x_i^{k+1}} = \sum_{\alpha \in x_i^k} \nu\big(\varphi_\alpha(\#N(\alpha, x^k))\big) + \bar{u}_i^{\#x_i^k \bar{m}_i} - \bar{u}_i^{\#x_i^{k+1} \bar{m}_i}$, hence $u_i^*(x^{k+1}) > u_i^*(x^k)$. Now we have a contradiction with the sufficiency part of Theorem 2 (Section 3).

Remark. Replacing our hypothetical improvement cycle with a "weak improvement cycle" of Voorneveld and Norde (1996) and referring to the main theorem of that paper, we would prove the existence of an ordinal potential. However, the FIP appears a much more important property.

B Proof of necessity in Theorem 1

Let us prove the implication [1.2] $\Rightarrow$ [1.3].

B.1 Basic lemmas

Lemma B.1. *Let $i, j \in N$, $m \geq 2$, $v_1, v_1', v_2, v_2' \in \mathbb{R}$, $1 \leq s_1, s_2 \leq m$, $s_1 \neq s_2$, $w, w' \in \mathbb{R}^m$, $w_{s_1} = v_1$, $w_{s_2} = v_2$, $w'_{s_1} = v'_1$, $w'_{s_2} = v'_2$, $w_s = w'_s$ for all $s \neq s_1, s_2$, and*

$$U_i^{(m)}(w) \leq U_i^{(m)}(w'); \tag{9a}$$

let $m' \geq 2$, $1 \leq s'_1, s'_2 \leq m'$, $s'_1 \neq s'_2$, $w'', w''' \in \mathbb{R}^{m'}$, $w''_{s'_1} = v_1$, $w''_{s'_2} = v_2$, $w'''_{s'_1} = v'_1$, $w'''_{s'_2} = v'_2$, and $w''_s = w'''_s$ for all $s \neq s'_1, s'_2$. Then

$$U_j^{(m')}(w'') \leq U_j^{(m')}(w'''). \tag{9b}$$

The interpretation of the statement should be clear: if a simultaneous replacement of v_1 with v'_1, and of v_2 with v'_2 did not produce an increase of the utility $U_i^{(m)}$, then the same replacement must not increase any utility $U_j^{(m')}$ under any circumstances.

Proof. Assuming first that $i \neq j$, we suppose to the contrary that $U_j^{(m')}(w'') > U_j^{(m')}(w''')$. Defining $w'''(\delta) \in \mathbb{R}^{m'}$ by $w'''(\delta)_{s'_1} = v'_1 + \delta$ and $w'''(\delta)_s = w'''_s$ for all $s \neq s'_1$, and $w'(\delta) \in \mathbb{R}^m$ by $w'(\delta)_{s_1} = v'_1 + \delta$ and $w'(\delta)_s = w'_s$ for all $s \neq s_1$, we can pick $\delta > 0$ such that $u_j^2 = U_j^{(m')}(w'') > U_j^{(m')}(w'''(\delta)) = u_j^1$; by monotonicity from (9a), $u_i^1 = U_i^{(m)}(w) < U_i^{(m)}(w'(\delta)) = u_i^2$.

Let us consider a generalized congestion game with $m + m' + 1$ processes where each player $k \in N$ uses the aggregation rule U_k: $\mathrm{A} = \{a, b, c, d, g\} \cup \{e_s\}_{s=1,\ldots,m,\, s_1 \neq s \neq s_2} \cup \{f_s\}_{s=1,\ldots,m',\, s'_1 \neq s \neq s'_2}$; $X_i = \{\alpha^1, \alpha^2\}$, where $\#\alpha^1 = \#\alpha^2 = m$, $\alpha^1_{s_1} = a$, $\alpha^1_{s_2} = b$, $\alpha^2_{s_1} = c$, $\alpha^2_{s_2} = d$, and $\alpha^k_s = e_s$ for $k = 1, 2$ and $s \neq s_1, s_2$; $X_j = \{\beta^1, \beta^2\}$, where $\#\beta^1 = \#\beta^2 = m'$, $\beta^1_{s'_1} = a$, $\beta^1_{s'_2} = b$, $\beta^2_{s'_1} = c$, $\beta^2_{s'_2} = d$, and $\beta^k_s = f_s$ for $k = 1, 2$ and $s \neq s'_1, s'_2$; $X_k = \{\langle g \rangle\}$ for $k \in N \setminus \{i, j\}$; $\varphi_a(1) = \varphi_c(1) = v'_1 + \delta$, $\varphi_a(2) = \varphi_c(2) = v_1$; $\varphi_b(1) = \varphi_d(1) = v'_2$, $\varphi_b(2) = \varphi_d(2) = v_2$; $\varphi_{e_s}(1) = w_s$ ($s = 1, \ldots, m$, $s_1 \neq s \neq s_2$), $\varphi_{f_s}(1) = w''_s$ ($s = 1, \ldots, m'$, $s'_1 \neq s \neq s'_2$). The 2×2 matrix of the essential part of the game looks as follows:

$$\begin{array}{ccc} & \beta^1 & \beta^2 \\ \alpha^1 & (u_i^1, u_j^2) & (u_i^2, u_j^1) \\ \alpha^2 & (u_i^2, u_j^1) & (u_i^1, u_j^2). \end{array}$$

Since $u_k^2 > u_k^1$ $(k = i, j)$, the game possesses no Nash equilibrium.

If $i = j$, we pick $k \neq i$ (we have assumed $n \geq 2\,!$) and obtain

$$U_k^{(m)}(w) \leq U_k^{(m)}(w')$$

first, and then (9b). □

The exact analogues of Lemma B.1 with equalities in (9) as well as strict inequalities of the same sign easily follow.

Lemma B.2. *Let $i \in N$, $m \geq 2$, $v_1, v_2 \in \mathbb{R}$, $1 \leq s_1, s_2 \leq m$, $s_1 \neq s_2$, $w, w' \in \mathbb{R}^m$, $w_{s_1} = v_1 = w'_{s_2}$, $w_{s_2} = v_2 = w'_{s_1}$, and $w_s = w'_s$ for all $s \neq s_1, s_2$; then*

$$U_i^{(m)}(w) = U_i^{(m)}(w'). \tag{10}$$

Proof. Applying Lemma B.1 with $i = j$, $m' = m$, $v'_2 = v_1$, $v'_1 = v_2$, $s'_1 = s_2$, and $s'_2 = s_1$, we obtain $U_i^{(m)}(w) \leq U_i^{(m)}(w') \Rightarrow U_i^{(m)}(w') \leq U_i^{(m)}(w)$, hence (10). □

Lemma B.3. *Let $i \in N$ and $v'_s, v''_s, v'''_s \in \mathbb{R}$ for $s = 1, 2$; let*

$$U_i^{(2)}(v'_1, v''_2) = U_i^{(2)}(v''_1, v'_2) \tag{11a}$$

and

$$U_i^{(2)}(v'_1, v'''_2) = U_i^{(2)}(v''_1, v''_2) = U_i^{(2)}(v'''_1, v'_2). \tag{11b}$$

Then

$$U_i^{(2)}(v''_1, v'''_2) = U_i^{(2)}(v'''_1, v''_2). \tag{12}$$

Proof. Supposing the contrary, we may, without restricting generality, assume $U_i(v'''_1, v''_2) > U_i(v''_1, v'''_2)$. By continuity, there exists $\delta_1 > 0$ such that

$$U_i(v'''_1 - \delta_1, v''_2) > U_i(v''_1, v'''_2). \tag{13a}$$

Pick $j \neq i$; by Lemma B.1, the equalities (11) are valid for $U_j^{(2)}$ as well. By monotonicity from (11b) for j, $U_j(v''_1, v''_2) > U_j(v'''_1 - \delta_1, v'_2)$; therefore, there is $\delta_2 > 0$ such that $U_j(v''_1, v''_2) > U_j(v'''_1 - \delta_1, v'_2 + \delta_2)$; by continuity, there is $\delta'_1 > 0$ such that

$$U_j(v''_1 - \delta'_1, v''_2) > U_j(v'''_1 - \delta_1, v'_2 + \delta_2). \tag{13b}$$

By monotonicity from (13a), $U_i(v'''_1 - \delta_1, v''_2) > U_i(v''_1 - \delta'_1, v'''_2)$, hence, by continuity, there is $\delta'_2 > 0$ such that

$$U_i(v'''_1 - \delta_1, v''_2 - \delta'_2) > U_i(v''_1 - \delta'_1, v'''_2). \tag{13c}$$

By monotonicity from (11a),

$$U_i(v''_1, v'_2 + \delta_2) > U_i(v'_1, v''_2); \tag{13d}$$

by monotonicity from (11b) for j,

$$U_j(v_1', v_2''') > U_j(v_1'', v_2'' - \delta_2'). \tag{13e}$$

Now we denote $u_i^1 = U_i(v_1', v_2'')$, $u_i^2 = U_i(v_1'', v_2' + \delta_2)$, $u_j^1 = U_j(v_1''' - \delta_1, v_2' + \delta_2)$, $u_j^2 = U_j(v_1'' - \delta_1', v_2'')$, $u_j^3 = U_j(v_1'', v_2'' - \delta_2')$, $u_j^4 = U_j(v_1', v_2''')$, $u_i^3 = U_i(v_1'' - \delta_1', v_2''')$, and $u_i^4 = U_i(v_1''' - \delta_1, v_2'' - \delta_2')$. We have $u_i^2 > u_i^1$ by (13d), $u_j^2 > u_j^1$ by (13b), $u_i^4 > u_i^3$ by (13c), and $u_j^4 > u_j^3$ by (13e).

Let us consider a generalized congestion game with the set of players N where each player k uses the aggregation rule U_k: $\mathrm{A} = \{a, b, c, d, e\}$; $X_i = \big\{\langle a, c\rangle, \langle b, d\rangle\big\}$; $X_j = \big\{\langle b, c\rangle, \langle a, d\rangle\big\}$; $X_k = \big\{\langle e\rangle\big\}$ for $k \in N \setminus \{i, j\}$; $\varphi_a(2) = v_1'$, $\varphi_a(1) = v_1''' - \delta_1$; $\varphi_b(2) = v_1'' - \delta_1'$, $\varphi_b(1) = v_1''$; $\varphi_c(2) = v_2'' - \delta_2'$, $\varphi_c(1) = v_2''$; $\varphi_d(2) = v_2' + \delta_2$, $\varphi_d(1) = v_2'''$. The 2×2 matrix of the essential part of the game looks as follows:

$$\begin{array}{ccc} & \text{bc} & \text{ad} \\ \text{ac} & (u_i^4, u_j^3) & (u_i^1, u_j^4) \\ \text{bd} & (u_i^3, u_j^2) & (u_i^2, u_j^1). \end{array}$$

There is no Nash equilibrium in the game. □

Those lemmas are sufficient for additive representation (3).

Lemma B.1 and its analogues imply that all two-dimensional sections of all functions $U_i^{(m)}$ ($i \in N$, $m = 2, \dots$) define the same ordering on $\mathbb{R}^2$, which is symmetric by Lemma B.2. In other words, the ordering defined by each function $U_i^{(m)}$ ($i \in N$, $m > 2$) on $\mathbb{R}^m$ admits a separable projection to every two-dimensional subspace; by the main result of Gorman (1968), it admits an additive representation. For $m = 2$, Lemma B.3 implies that the condition depicted in Figure 1(a) of Debreu (1960) holds; therefore, by the Blaschke–Thomsen results cited by Debreu, we have the additive representation for $m = 2$ as well. Unfortunately, those references give us no way to show that the same function ν can be used for all $i \in N$, all $m \geq 1$, and all coordinates.

Fishburn (1970; Chapter 5) provides almost complete proofs for a version of the Debreu–Gorman Theorem and a version of the Blaschke–Thomsen result. It is easily seen from the proofs that, under symmetry, the same function ν can be used everywhere. Unfortunately, in both cases the assumptions are stronger than what is obtained from the lemmas of Section B.1.

In order not to leave our main results without secure foundation, the next subsection contains a complete derivation of (3) from the lemmas. The scheme is essentially the same as in Kukushkin (1994), but symmetry simplifies everything considerably. By the way, the necessity of additivity in the latter paper, unlike the current situation, could not be derived from the Debreu–Gorman Theorem.

A straightforward corollary to Lemma B.3 is useful in the following.

Lemma B.4. *Let $i \in N$ and $v'_s, v''_s, v'''_s \in \mathbb{R}$ for $s = 1, 2$; let*

$$U_i^{(2)}(v'_1, v''_2) = U_i^{(2)}(v''_1, v'_2), \tag{14a}$$

$$U_i^{(2)}(v''_1, v''_2) = U_i^{(2)}(v'''_1, v'_2), \tag{14b}$$

and

$$U_i^{(2)}(v''_1, v'''_2) = U_i^{(2)}(v'''_1, v''_2). \tag{14c}$$

Then

$$U_i^{(2)}(v'_1, v'''_2) = U_i^{(2)}(v''_1, v''_2). \tag{15}$$

Proof. Without restricting generality, we may assume that $v'_s < v''_s < v'''_s$ for both s.

If the left hand side in (15) is greater than the right hand side, then there is $v^*_2 \in]v''_2, v'''_2[$ such that $U_i^{(2)}(v'_1, v^*_2) = U_i^{(2)}(v''_1, v''_2)$. Now we apply Lemma B.3, replacing v'''_2 with v^*_2, and obtain $U_i^{(2)}(v''_1, v^*_2) = U_i^{(2)}(v'''_1, v''_2)[= U_i^{(2)}(v''_1, v'''_2)]$. Since $v^*_2 < v'''_2$, this contradicts strict monotonicity (2).

Similarly, if the left hand side in (15) is less than the right hand side, then there is $v^*_1 \in]v'_1, v''_1[$ such that $U_i^{(2)}(v^*_1, v'''_2) = U_i^{(2)}(v''_1, v''_2)$. Now we apply Lemma B.3 to $\langle v'''_1, v''_1, v^*_1 \rangle$ and $\langle v'''_2, v''_2, v'_2 \rangle$ (i.e., reversing the order of v_s and replacing v'_1 with v^*_1), obtaining $U_i^{(2)}(v^*_1, v''_2) = U_i^{(2)}(v''_1, v'_2)[= U_i^{(2)}(v'_1, v''_2)]$. Since $v'_1 < v^*_1$, this contradicts strict monotonicity again. □

B.2 Additive representation

First of all, we fix a player; without restricting generality, $1 \in N$. An *integer net* is a strictly increasing mapping $\psi : H \to \mathbb{R}$, where H is the set of integers h satisfying $h^- < h < h^+$ ($h^- \in \{-\infty, \dots, -2, -1\}$, $h^+ \in \{2, 3, \dots, +\infty\}$), such that:

$$\psi(0) = 0; \tag{16a}$$

$$U_1^{(2)}\big(\psi(h+1), \psi(k)\big) = U_1^{(2)}\big(\psi(h), \psi(k+1)\big) \tag{16b}$$

for all $h^- < h, k < h^+ - 1$; if $h^+ < +\infty$, then

$$U_1^{(2)}\big(\psi(h^+ - 1), \psi(1)\big) > U_1^{(2)}(v, 0) \tag{16c}$$

for all $v \in \mathbb{R}$; if $h^- > -\infty$, then

$$U_1^{(2)}\big(v, \psi(1)\big) > U_1^{(2)}\big(\psi(h^- + 1), 0\big) \tag{16d}$$

for all $v \in \mathbb{R}$.

Lemma B.5. *Let ψ be an integer net. Then whenever $j \in N$, $\bar{m} \geq 2$, and $h'_k, h_k \in H$ for all $k = 1, \ldots, \bar{m}$, we have*

$$U_j^{(\bar{m})}\big(\langle\psi(h'_k)\rangle_k\big) \geq U_j^{(\bar{m})}\big(\langle\psi(h_k)\rangle_k\big) \iff \sum_{k=1}^{\bar{m}} h'_k \geq \sum_{k=1}^{\bar{m}} h_k. \tag{17}$$

Proof. First, we show that an equality in the right hand side of (17) implies an equality in the left hand side. Taking into account the symmetry of $U_j^{(\bar{m})}$, it is obviously sufficient to show

$$\begin{multline} U_j^{(\bar{m})}\big(\psi(h_1+1), \psi(h_2), \psi(h_3), \ldots, \psi(h_{\bar{m}})\big) = \\ U_j^{(\bar{m})}\big(\psi(h_1), \psi(h_2+1), \psi(h_3), \ldots, \psi(h_{\bar{m}})\big) \end{multline} \tag{18}$$

whenever $h^- < h_k, h_1+1, h_2+1 < h^+$ $(k = 1, \ldots, \bar{m})$. By (16b), we have $U_1^{(2)}\big(\psi(h_1+1), \psi(h_2)\big) = U_1^{(2)}\big(\psi(h_1), \psi(h_2+1)\big)$. Now we can apply the analogue of Lemma B.1 with equalities in (9), setting $i = 1$, $m = 2$, $w = \big(\psi(h_1+1), \psi(h_2)\big)$, $w' = \big(\psi(h_1), \psi(h_2+1)\big)$, $m' = \bar{m}$, $w'' = \big(\psi(h_1+1), \psi(h_2), \psi(h_3), \ldots, \psi(h_{\bar{m}})\big)$, and $w''' = \big(\psi(h_1), \psi(h_2+1), \psi(h_3), \ldots, \psi(h_{\bar{m}})\big)$. Then (9b) with an equality becomes just (18).

If there is a strong inequality in the right hand side of (17), we can find $\langle h''_k\rangle_{k=1,\ldots,\bar{m}}$ such that $\sum_{k=1}^{\bar{m}} h''_k = \sum_{k=1}^{\bar{m}} h_k$ while $\langle h'_k\rangle_{k=1,\ldots,\bar{m}}$ Pareto dominates $\langle h''_k\rangle_{k=1,\ldots,\bar{m}}$. Now $U_j^{(\bar{m})}\big(\langle\psi(h'_k)\rangle_k\big) > U_j^{(\bar{m})}\big(\langle\psi(h''_k)\rangle_k\big) = U_j^{(\bar{m})}\big(\langle\psi(h_k)\rangle_k\big)$ by the strict monotonicity of $U_j^{(\bar{m})}$ and the findings of the previous paragraph. Finally, an (in)equality in the left hand side of (17) implies the same (in)equality in the right hand side because $\sum_{k=1}^{\bar{m}} h'_k$ and $\sum_{k=1}^{\bar{m}} h_k$ are always comparable. □

Lemma B.6. *There exists an integer net.*

Proof. First, we define $\psi(0) = 0$ and pick $\psi(1) > 0$ arbitrarily. Then we define $\psi(h+1)$ for integer $h \geq 1$ inductively, by the equalities

$$U_1^{(2)}\big(0, \psi(h+1)\big) = U_1^{(2)}\big(\psi(1), \psi(h)\big). \tag{19a}$$

There are two alternatives: either no solution $\psi(h+1)$ to (19a) can be found at a stage $h \geq 1$, in which case we stop the process and define $h^+ = h+1$; or $\psi(h)$ will be defined for all $h \geq 0$, in which case we set $h^+ = +\infty$.

For $h \leq 0$, $\psi(h-1)$ is also defined inductively, by the equalities

$$U_1^{(2)}\big(\psi(1), \psi(h-1)\big) = U_1^{(2)}\big(0, \psi(h)\big). \tag{19b}$$

Note that (19a) and (19b) only differ in their viewpoint. Again, if no solution $\psi(h-1)$ to (19b) can be found at a stage $h \leq 0$, we stop the process and define $h^- = h-1$; if $\psi(h)$ is defined for all $h \leq 0$, we set $h^- = -\infty$.

Turning to the definition (16), we notice that the condition (16a) is satisfied automatically. Let us check (16b); note that we already have it for all k if $h = 0$ (and for all h if $k = 0$). We organize two inductive processes in h, upwards and downwards; inside each step of each process, we organize two inductive processes in k. Each step consists in an application of Lemma B.3 or Lemma B.4 with $i = 1$.

On a "double upward" step ($h, k > 0$), we assume $v_1' = \psi(h-1)$, $v_1'' = \psi(h)$, $v_1''' = \psi(h+1)$, $v_2' = \psi(k-1)$, $v_2'' = \psi(k)$, and $v_2''' = \psi(k+1)$. Now we have (11a) and "a half" of (11b) from the induction hypothesis of the h-process, and the "second half" of (11b) from the induction hypothesis of the k-process. The statement of Lemma B.3, (12), gives us (16b).

Similarly, on a "double downward" step ($h, k < 0$), we assume $v_1' = \psi(h+2)$, $v_1'' = \psi(h+1)$, $v_1''' = \psi(h)$, $v_2' = \psi(k+2)$, $v_2'' = \psi(k+1)$, and $v_2''' = \psi(k)$. Again, we have (11a) and "a half" of (11b) from the induction hypothesis of the h-process, and the "other half" of (11b) from the induction hypothesis of the k-process. The statement of Lemma B.3 gives us (16b).

Because of the symmetry of $U_1^{(2)}$, we may only consider one "upward-downward" step; let $h < 0$ and $k > 0$. From the induction hypothesis of the k-process, we have $U_1^{(2)}\big(\psi(h), \psi(k)\big) = U_1^{(2)}\big(\psi(h+1), \psi(k-1)\big)$; from the induction hypothesis of the h-process, $U_1^{(2)}\big(\psi(h+1), \psi(k)\big) = U_1^{(2)}\big(\psi(h+2), \psi(k-1)\big)$ and $U_1^{(2)}\big(\psi(h+1), \psi(k+1)\big) = U_1^{(2)}\big(\psi(h+2), \psi(k)\big)$. Assuming $v_1' = \psi(h)$, $v_1'' = \psi(h+1)$, $v_1''' = \psi(h+2)$, $v_2' = \psi(k-1)$, $v_2'' = \psi(k)$, and $v_2''' = \psi(k+1)$, we apply Lemma B.4. The statement of the lemma, (15), gives us (16b).

Finally, let us turn to (16c) and (16d). If $h^+ < +\infty$, but (16c) does not hold, i.e., there is $v \in \mathbb{R}$ such that $U_1^{(2)}(0, v) \geq U_1^{(2)}(\psi(1), \psi(h^+ - 1))$, then a solution to (19a) with $h = h^+ - 1$ exists, so our inductive process could not have stopped here. Quite similarly, the "downward" process (19b) can only stop at a finite h if (16d) is satisfied. $\square$

Let ψ and $\bar{\psi}$ be two integer nets; we call $\bar{\psi}$ a *doubling* of ψ if, whenever $2h \in \bar{H}$, we have $h \in H$ and $\psi(h) = \bar{\psi}(2h)$. Clearly, $\bar{h}^\pm = \pm\infty$ only if $h^\pm = \pm\infty$. A *binary net* is an infinite sequence of integer nets $\psi^0, \psi^1, \ldots$ such that each ψ^{d+1} is a doubling of ψ^d.

Lemma B.7. *There exists a binary net.*

Proof. It is obviously sufficient to prove that every integer net admits a doubling. From $U_1^{(2)}(\psi(1), \psi(1)) > U_1^{(2)}(\psi(1), 0) > U_1^{(2)}(0, 0)$ and continuity, we immediately derive the existence of $t^* \in]0, \psi(1)[$ such that $U_1^{(2)}(t^*, t^*) = U_1^{(2)}(\psi(1), 0)$. We define $\bar{\psi}(1) = t^*$ and $\bar{\psi}(2h) = \psi(h)$ for every $h \in H$. The definition of $\bar{\psi}(h)$ for odd h depends on the sign.

For $h \geq 1$, we try to define $\bar{\psi}(2h+1)$ by the equality

$$U_1^{(2)}\big(\bar{\psi}(2h+1), 0\big) = U_1^{(2)}\big(\bar{\psi}(2h), \bar{\psi}(1)\big); \tag{20a}$$

if $(h+1) \in H$, then we have $U_1^{(2)}(\psi(h+1),0) = U_1^{(2)}(\psi(h),\psi(1)) > U_1^{(2)}(\bar\psi(2h),\bar\psi(1)) > U_1^{(2)}(\bar\psi(2h),0) = U_1^{(2)}(\psi(h),0)$, hence there is a unique solution to (20a), which satisfies $\psi(h) < \bar\psi(2h+1) < \psi(h+1)$. In particular, if $h^+ = +\infty$, then $\bar\psi(h)$ is defined for all $h \ge 0$, i.e., $\bar h^+ = +\infty$ as well. If $h^+ < +\infty$, then (20a) for $h = h^+ - 1$ may, or may not, admit a solution; in the first case, we set $\bar h^+ = 2h^+$; in the second, $\bar h^+ = 2h^+ - 1$.

For $h \le 0$, we try to define $\bar\psi(2h-1)$ by the equality

$$U_1^{(2)}\big(\bar\psi(2h-1),\psi(1)\big) = U_1^{(2)}\big(\bar\psi(2h),\bar\psi(1)\big); \tag{20b}$$

if $(h-1) \in H$, then we have $U_1^{(2)}(\psi(h),\psi(1)) > U_1^{(2)}(\bar\psi(2h),\bar\psi(1)) > U_1^{(2)}(\bar\psi(2h),0) = U_1^{(2)}(\psi(h-1),\psi(1))$, hence there is a unique solution to (20b), which satisfies $\psi(h-1) < \bar\psi(2h-1) < \psi(h)$. In particular, if $h^- = -\infty$, then $\bar\psi(h)$ is defined for all $h \le 0$, i.e., $\bar h^- = -\infty$ as well. If $h^- > -\infty$, then (20b) for $h = h^- + 1$ may, or may not, admit a solution; in the first case, we set $\bar h^- = 2h^-$; in the second, $\bar h^- = 2h^- + 1$.

Let us check (16b). By the definition of $\bar\psi(1)$, we have $U_1^{(2)}(\bar\psi(1),\bar\psi(1)) = U_1^{(2)}(\psi(1),0) = U_1^{(2)}(\bar\psi(2),0)$. By symmetry, we have $U_1^{(2)}(\bar\psi(1),\bar\psi(2)) = U_1^{(2)}(\bar\psi(2),\bar\psi(1))$, while the right hand side, by the definition of $\bar\psi(3)$, equals $U_1^{(2)}(\bar\psi(3),0)$. Applying Lemma B.3 with $v_1' = \bar\psi(1)$, $v_1'' = \bar\psi(2)$, $v_1''' = \bar\psi(3)$, $v_2' = 0$, $v_2'' = \bar\psi(1)$, and $v_2''' = \bar\psi(2)$, we obtain $U_1^{(2)}(\bar\psi(3),\bar\psi(1)) = U_1^{(2)}(\bar\psi(2),\bar\psi(2))$. Then we notice that $U_1^{(2)}(\bar\psi(2),\bar\psi(2)) = U_1^{(2)}(\psi(1),\psi(1)) = U_1^{(2)}(\psi(2),0) = U_1^{(2)}(\bar\psi(4),0)$; therefore, we can apply Lemma B.3 with $v_1' = \bar\psi(2)$, $v_1'' = \bar\psi(3)$, $v_1''' = \bar\psi(4)$, $v_2' = 0$, $v_2'' = \bar\psi(1)$, and $v_2''' = \bar\psi(2)$, obtaining $U_1^{(2)}(\bar\psi(3),\bar\psi(2)) = U_1^{(2)}(\bar\psi(4),\bar\psi(1))$. By the definition of $\bar\psi(5)$, the right hand side equals $U_1^{(2)}(\bar\psi(5),0)$, so we can again apply Lemma B.3 and so on.

The downward movement is executed in a similar way. By the definition of $\bar\psi(-1)$, we have $U_1^{(2)}(\bar\psi(-1),\bar\psi(2)) = U_1^{(2)}(0,\bar\psi(1))$. Applying Lemma B.3 with $v_1' = \bar\psi(1)$, $v_1'' = 0$, $v_1''' = \bar\psi(-1)$, $v_2' = \bar\psi(2)$, $v_2'' = \bar\psi(1)$, and $v_2''' = 0$, we obtain $U_1^{(2)}(\bar\psi(-1),\bar\psi(1)) = U_1^{(2)}(0,0)$. Now (16b) for ψ implies $U_1^{(2)}(0,0) = U_1^{(2)}(\psi(-1),\psi(1)) = U_1^{(2)}(\bar\psi(-2),\bar\psi(2))$; therefore, we can apply Lemma B.3 with $v_1' = 0$, $v_1'' = \bar\psi(-1)$, $v_1''' = \bar\psi(-2)$, $v_2' = \bar\psi(2)$, $v_2'' = \bar\psi(1)$, and $v_2''' = 0$, obtaining $U_1^{(2)}(\bar\psi(-1),0) = U_1^{(2)}(\bar\psi(-2),\bar\psi(1))$. The definition of $\bar\psi(-3)$ implies that the right hand side equals $U_1^{(2)}(\bar\psi(-3),\bar\psi(2))$, so Lemma B.3 can be applied again and so on.

Thus, (16b) for $\bar\psi$ holds whenever $k = 0$ or $k = 1$ (hence when $h = 0$ or $h = 1$ as well); the derivation of (16b) for all k and h is done by the same double induction as in the proof of Lemma B.6.

Checking (16c) for $\bar\psi$ is straightforward: If (20a) admits no solution for $h = h^+ - 1$, then (16c) must hold for $\bar h^+ = 2h^+ - 1$. If a solution $\bar\psi(2h^+ - 1)$ exists, then $U_1^{(2)}(\psi(h^+ - 1),\psi(1)) = U_1^{(2)}(\bar\psi(2h^+ - 2),\bar\psi(2)) = U_1^{(2)}(\bar\psi(2h^+ - 1),\bar\psi(1))$ by (16b) for $\bar\psi$, so (16c) for ψ implies (16c) for $\bar\psi$.

Finally, let us suppose that (16d) for $\bar\psi$ is violated; then $U_1^{(2)}\big(v_1',\bar\psi(1)\big) = U_1^{(2)}\big(\bar\psi(\bar h^- +$

$1), 0\big)$ for a $v_1' < \bar\psi(\bar h^- + 1)$. If (20b) admits no solution for $h = h^- + 1$, then $\bar h^- + 1 = 2h^- + 2$. Denoting $v_1'' = \bar\psi(2h^- + 2) = \psi(h^- + 1)$, $v_1''' = \bar\psi(2h^- + 3)$, $v_2' = 0$, $v_2'' = \bar\psi(1)$, and $v_2''' = \bar\psi(2) = \psi(1)$, we have (14a) from the definition of v_1', and (14b) and (14c) from (16b) for $\bar\psi$. Now Lemma B.4 renders $U_1^{(2)}(v_1', \psi(1)) = U_1^{(2)}(\bar\psi(2h^- + 2), \bar\psi(1))$, i.e., v_1' is a solution of (20b) with $h = h^- + 1$.

Similarly, if (20b) with $h = h^- + 1$ was satisfied by $\bar\psi(2h^- + 1)$, then $\bar h^- + 1 = 2h^- + 1$. We denote $v_1'' = \bar\psi(2h^- + 1)$, $v_1''' = \bar\psi(2h^- + 2) = \psi(h^- + 1)$, $v_2' = 0$, $v_2'' = \bar\psi(1)$, and $v_2''' = \bar\psi(2) = \psi(1)$; again, all equalities (14) follow from the definition of v_1' and (16b) for $\bar\psi$. Lemma B.4 renders $U_1^{(2)}(v_1', \psi(1)) = U_1^{(2)}(\bar\psi(2h^- + 1), \bar\psi(1))[= U_1^{(2)}(\psi(h^- + 1), 0)]$. Now we see that (16d) is violated for ψ itself. □

Let us fix a binary net $\psi^0, \psi^1, \ldots$ and denote $\Psi = \{\psi^d(h)\}_{d\in\mathbb{N},\, h\in H^d}$.

Lemma B.8. *The set Ψ is dense in $\mathbb{R}$.*

Proof. Suppose the contrary: there are $u_- < u_+$ such that $\Psi \cap [u_-, u_+] = \emptyset$. We denote $\Psi_- = \{t \in \Psi \mid t \le u_-\}$ and $\Psi_+ = \{t \in \Psi \mid t \ge u_+\}$. There are three alternatives:

1. $\Psi_- \ne \emptyset \ne \Psi_+$;
2. $\Psi_+ = \emptyset$;
3. $\Psi_- = \emptyset$.

Step B.8.1. *The first alternative cannot hold.*

Proof. Denoting $\tau_- = \sup \Psi_-$ and $\tau_+ = \inf \Psi_+$, we have $\tau_- \le u_- < u_+ \le \tau_+$ and $\Psi \cap]\tau_-, \tau_+[= \emptyset$. By monotonicity,

$$U_1^{(2)}(\tau_+, \tau_+) > U_1^{(2)}(\tau_-, \tau_+) > U_1^{(2)}(\tau_-, \tau_-);$$

by continuity, there are open intervals V_+ and V_- containing τ_+ and τ_-, respectively, and such that the strict inequalities are retained on $V_+ \times V_+$, $V_- \times V_+$, and $V_- \times V_-$ (hence $V_- \cap V_+ = \emptyset$).

Clearly, there are $d_-, d_+ \in \mathbb{N}$, $h_- \in H^{d_-}$ and $h_+ \in H^{d_+}$ such that $\psi^{d_-}(h_-) \in V_-$ and $\psi^{d_+}(h_+) \in V_+$. Defining $d = \max\{d_-, d_+\}$, we see that $\{\psi^d(h)\}_{h\in H^d}$ intersects both V_- and V_+; therefore, $\psi^d(h) \in V_-$ and $\psi^d(h+1) \in V_+$ for some $h \in H^d \ni (h+1)$. Let us note that $\psi^d(h) = \psi^{d+1}(2h) < \psi^{d+1}(2h+1) < \psi^{d+1}(2h+2) = \psi^d(h+1)$ and $\psi^{d+1}(2h+1) \notin]\tau_-, \tau_+[$.

Let $\psi^{d+1}(2h+1) \le \tau_-$; then $\psi^{d+1}(2h+1) \in V_-$. Therefore, $U_1^{(2)}\big(\psi^{d+1}(2h), \psi^{d+1}(2h+2)\big) > U_1^{(2)}\big(\psi^{d+1}(2h+1), \psi^{d+1}(2h+1)\big)$, but this contradicts (16b).

The assumption $\psi^{d+1}(2h+1) \ge \tau_+$ is refuted dually. □

Step B.8.2. *The second alternative cannot hold.*

Proof. We denote $\tau_+ = \sup \Psi < +\infty$; by the continuity and strict monotonicity of $U_1^{(2)}$, there are $\Delta, \Delta' > 0$ such that

$$U_1^{(2)}(\tau_+ + \Delta, 0) > U_1^{(2)}(\tau_+, \Delta').$$

By Step B.8.1, there is $d \in \mathbb{N}$ such that $\psi^d(1) < \Delta'$. We consider two alternatives.

If $h^{+d} < +\infty$, then $U_1^{(2)}(\tau_+ + \Delta, 0) > U_1^{(2)}(\tau_+, \psi^d(1)) \geq U_1^{(2)}(\psi^d(h^{+d} - 1), \psi^d(1))$, contradicting (16c).

Let $h^{+d} = +\infty$; then $\tau_+ = \sup\{\psi^d(h)\}_{h\in\mathbb{N}}$. By continuity from $U_1^{(2)}(\tau_+, \psi^d(1)) > U_1^{(2)}(\tau_+, 0)$, there is $\tau_* < \tau_+$ such that $U_1^{(2)}(t, \psi^d(1)) > U_1^{(2)}(t', 0)$ whenever $t > \tau_*$ and $t' < \tau_+$. On the other hand, if $\psi^d(h) > \tau_*$, then $\psi^d(h+1) < \tau_+$, but $U_1^{(2)}(\psi^d(h), \psi^d(1)) = U_1^{(2)}(\psi^d(h+1), 0)$. ☐

Step B.8.3. *The third alternative cannot hold.*

The proof is dual to that of Step B.8.2. The lemma is proved. ☐

Lemma B.9. *For all $d, d' \in \mathbb{N}$, $h \in H^d$, and $h' \in H^{d'}$, there holds $\psi^{d'}(h') \geq \psi^d(h) \iff h'/2^{d'} \geq h/2^d$.*

Proof. When $d = d'$, this is just the monotonicity of ψ^d. Then a straightforward inductive process in $\max\{d', d\} - \min\{d', d\}$ based on the definition of a doubling works. ☐

For every $d \in \mathbb{N}$, we denote $q_d^+ = (h^{+d} - 1)/2^d$ and $q_d^- = (h^{-d} + 1)/2^d$; then we define $Q = \bigcup_{d\in\mathbb{N}}[q_d^-, q_d^+] \setminus \{+\infty, -\infty\}$. Clearly, Q is a non-degenerate interval (actually, open). For every $v \in \mathbb{R}$, we define $\nu(v) = \sup\{h/2^d |\ d \in \mathbb{N}\ \&\ h \in H^d\ \&\ \psi^d(h) \leq v\}$. By Lemmas B.8 and B.9, ν is strictly increasing and $\nu(v) \in Q$ for every $v \in \mathbb{R}$. Conversely, if $w \in Q$, we define $v = \sup\{\psi^d(h) |\ d \in \mathbb{N}\ \&\ h \in H^d\ \&\ h/2^d \leq w\}$ and easily derive from Lemma B.8 that $w = \nu(v)$. Thus, we have a strictly increasing mapping onto a non-degenerate interval; therefore, both ν and its inverse are continuous.

Now let us turn to (3). It is sufficient to prove

$$U_i^{(m)}(v'_1, \ldots, v'_m) \geq U_i^{(m)}(v_1, \ldots, v_m) \iff \sum_{s=1}^{m} \nu(v'_s) \geq \sum_{s=1}^{m} \nu(v_s) \tag{21}$$

for all $i \in N$, $m \in \mathbb{N}$, and $v'_1, v_1, \ldots, v'_m, v_m \in \mathbb{R}$.

Suppose $\Delta = \sum_{s=1}^{m} \nu(v'_s) - \sum_{s=1}^{m} \nu(v_s) > 0$. By Lemma B.8, for every $s = 1, \ldots, m$, there is $d'_s \in \mathbb{N}$ such that $\psi^{d'_s}(h) \leq v'_s$ for some $h \in H^{d'_s}$ and $d_s \in \mathbb{N}$ such that $\psi^{d_s}(h) > v_s$ for some $h \in H^{d_s}$. Let us pick $d \in \mathbb{N}$ such that $d \geq \max_s \max\{d'_s, d_s\}$ and $2^{d-1} \geq m/\Delta$. For every $s = 1, \ldots, m$, we denote $h'_s = \max\{h \in H^d |\ \psi^d(h) \leq v'_s\}$ and $h_s = \min\{h \in H^d |\ \psi^d(h) > v_s\}$. Clearly, we have $h'_s/2^d \leq \nu(v'_s) < (h'_s + 1)/2^d$ and $(h_s -$

$1)/2^d \leq \nu(v_s) < h_s/2^d$ for all s. Therefore, $\sum_{s=1}^{m}(h'_s/2^d) > \sum_{s=1}^{m}\nu(v'_s) - m/2^d$ and $\sum_{s=1}^{m}(h_s/2^d) \leq \sum_{s=1}^{m}\nu(v_s) + m/2^d$, hence $\sum_{s=1}^{m}(h'_s/2^d) - \sum_{s=1}^{m}(h_s/2^d) > \Delta - 2m/2^d \geq 0$, hence $\sum_{s=1}^{m} h'_s > \sum_{s=1}^{m} h_s$. Now Lemma B.5 (for ψ^d) and strict monotonicity of $U_i^{(m)}$ imply a strict inequality in the left hand side of (21).

An equality in the right hand side of (21) means that we have both strict inequalities in any open neighborhood, hence the same inequalities in the left hand side of (21), hence an equality.

The opposite implication is proven exactly as in Lemma B.5. Now (21) is proven, hence so is (3).

B.3 Consistency

Now let us turn to the second statement in [1.3]. If $m' = m$, then $\bar{u}^{mm} = 0$ obviously satisfies (4). If $\lambda_i^m(m \cdot \nu(\mathbb{R})) \cap \lambda_i^{m'}(m' \cdot \nu(\mathbb{R})) = \emptyset$, then either $\lambda_i^{m'}(u') > \lambda_i^m(u)$ for all $u' \in m' \cdot \nu(\mathbb{R})$ and $u \in m \cdot \nu(\mathbb{R})$, or vice versa. In the first case, we define $\bar{u}^{mm'} = -\infty$; in the second, $\bar{u}^{mm'} = +\infty$. The condition (4) obviously holds.

Let us fix $i \in N$ and $m' > m \geq 1$ such that $V = \lambda_i^m(m \cdot \nu(\mathbb{R})) \cap \lambda_i^{m'}(m' \cdot \nu(\mathbb{R})) \neq \emptyset$. Obviously, V is an open interval (bounded or not), hence $W' = (\lambda_i^{m'})^{-1}(V)$ and $W = (\lambda_i^m)^{-1}(V)$ are open intervals too. We denote $\nu(+\infty)$ (respectively, $\nu(-\infty)$) the supremum (infimum) of $\nu(\mathbb{R})$. If $\nu(\mathbb{R}) = \mathbb{R}$, much of the following becomes superfluous, but there is no ground for such a simplifying assumption. In any case, $\nu(\mathbb{R}) =]\nu(-\infty), \nu(+\infty)[$.

Let $u^1 > u^2$ and $u^t \in V$ for $t = 1, 2$, i.e.,

$$u^t = \lambda_i^m(\sigma_t) = \lambda_i^{m'}(\sigma'_t) \;\&\; \sigma_t \in m \cdot \nu(\mathbb{R}) \;\&\; \sigma'_t \in m' \cdot \nu(\mathbb{R}) \text{ for } t = 1, 2. \tag{22a}$$

There exist $v^t \in \mathbb{R}$ such that $\sigma_t = m \cdot \nu(v^t)$ $(t = 1, 2)$. If u^1 and u^2 are close enough to each other, then

$$\sigma'_1 - \sigma'_2 < (m' - m) \cdot [\nu(+\infty) - \nu(-\infty)] \tag{22b}$$

(if $\nu(\mathbb{R}) = \mathbb{R}$, the inequality holds for all u^1, u^2). It can be rewritten as $\sigma'_2 - (m' - m) \cdot \nu(-\infty) > \sigma'_1 - (m' - m) \cdot \nu(+\infty)$. Since the left-hand side is greater than $m \cdot \nu(-\infty)$ whereas the right-hand side is less than $m \cdot \nu(+\infty)$, there is $\sigma_0 \in m \cdot \nu(\mathbb{R})$ such that $\sigma'_t \in \sigma_0 + (m' - m) \cdot \nu(\mathbb{R})$ for both t. Therefore, there are $v^0, \bar{v}^1, \bar{v}^2 \in \mathbb{R}$ such that $\sigma'_t = m \cdot \nu(v^0) + (m' - m) \cdot \nu(\bar{v}^t)$ for both t.

Lemma B.10. *If both conditions* (22) *hold, then* $\sigma_1 - \sigma_2 = \sigma'_1 - \sigma'_2$.

Proof. Let us suppose first that $\sigma_1 - \sigma_2 > \sigma'_1 - \sigma'_2$. We pick $\delta > 0$ such that

$$\sigma'_1 - \sigma'_2 < (m' - m) \cdot [\nu(\bar{v}^1 + \delta) - \nu(\bar{v}^2 - \delta)] < \sigma_1 - \sigma_2$$

(the first inequality holds automatically). Denoting $\sigma_j^2 = \sigma_2 + (m' - m) \cdot \nu(\bar{v}^1 + \delta)$ and $\sigma_j^1 = \sigma_1 + (m' - m) \cdot \nu(\bar{v}^2 - \delta)$, we see that $\sigma_j^1 > \sigma_j^2$; since both belong to $m' \cdot \nu(\mathbb{R})$, there is $\sigma_j^0 \in m' \cdot \nu(\mathbb{R})$ such that

$$\sigma_j^1 > \sigma_j^0 > \sigma_j^2; \tag{23a}$$

clearly, $\sigma_j^0 = m' \cdot \nu(\hat{v})$ for $\hat{v} \in \mathbb{R}$. We denote $u'' = \lambda_i^{m'}(m \cdot \nu(v^0) + (m' - m) \cdot \nu(\bar{v}^2 - \delta))$ and $u' = \lambda_i^{m'}(m \cdot \nu(v^0) + (m' - m) \cdot \nu(\bar{v}^1 + \delta))$; clearly,

$$u'' < u^2 < u^1 < u'. \tag{23b}$$

Let us pick $j \in N$, $j \neq i$, and consider a generalized congestion game where N is the set of players and each player k uses the aggregation rule U_k: $\mathrm{A} = \{a_1, \ldots, a_m, b_{m+1}, \ldots, b_{m'}, c_1, \ldots, c_m, d_1, \ldots, d_{m'}, e\}$; $X_i = \big\{\langle a_1, \ldots, a_m\rangle, \langle c_1, \ldots, c_m, b_{m+1}, \ldots, b_{m'}\rangle\big\}$; $X_j = \big\{\langle a_1, \ldots, a_m,\ b_{m+1}, \ldots, b_{m'}\rangle,\ \langle d_1, \ldots, d_{m'}\rangle\big\}$; $X_k = \big\{\langle e\rangle\big\}$ for $k \in N \setminus \{i, j\}$; $\varphi_{a_s}(t) = v^t$ $(t = 1, 2;\ s = 1, \ldots, m)$; $\varphi_{b_s}(2) = \bar{v}^2 - \delta$, $\varphi_{b_s}(1) = \bar{v}^1 + \delta$ $(s = m+1, \ldots, m')$; $\varphi_{c_s}(1) = v^0$ $(s = 1, \ldots, m)$; $\varphi_{d_s}(1) = \hat{v}$ $(s = 1, \ldots, m')$. The 2×2 matrix of the essential part of the game looks as follows:

$$\begin{array}{ccc} & \text{ab} & \text{d} \\ \text{a} & \big(u^2, \lambda_j^{m'}(\sigma_j^2)\big) & \big(u^1, \lambda_j^{m'}(\sigma_j^0)\big) \\ \text{bc} & \big(u'', \lambda_j^{m'}(\sigma_j^1)\big) & \big(u', \lambda_j^{m'}(\sigma_j^0)\big). \end{array}$$

The inequalities (23) imply that there is no Nash equilibrium in the game.

Now let $\sigma_1 - \sigma_2 < \sigma_1' - \sigma_2'$. We argue similarly to the previous case, but with some modifications. Pick $\delta > 0$ such that

$$\sigma_1 - \sigma_2 < (m' - m) \cdot [\nu(\bar{v}^1 - \delta) - \nu(\bar{v}^2 + \delta)] < \sigma_1' - \sigma_2'$$

(the second inequality holds automatically). Denoting $\sigma_j^1 = \sigma_2 + (m' - m) \cdot \nu(\bar{v}^1 - \delta)$ and $\sigma_j^2 = \sigma_1 + (m' - m) \cdot \nu(\bar{v}^2 + \delta)$, we see that $\sigma_j^1 > \sigma_j^2$; since both belong to $m' \cdot \nu(\mathbb{R})$, there is $\sigma_j^0 \in m' \cdot \nu(\mathbb{R})$ such that

$$\sigma_j^1 > \sigma_j^0 > \sigma_j^2; \tag{24a}$$

clearly, $\sigma_j^0 = m' \cdot \nu(\hat{v})$ for $\hat{v} \in \mathbb{R}$. We denote $u'' = \lambda_i^{m'}(m \cdot \nu(v^0) + (m' - m) \cdot \nu(\bar{v}^2 + \delta))$ and $u' = \lambda_i^{m'}(m \cdot \nu(v^0) + (m' - m) \cdot \nu(\bar{v}^1 - \delta))$; clearly,

$$u^2 < u'' < u' < u^1. \tag{24b}$$

Now we pick $j \in N$, $j \neq i$, and consider a generalized congestion game where N is the set of players and each player k uses the aggregation rule U_k: $\mathrm{A} = \{a_1, \ldots, a_m, b_{m+1}, \ldots, b_{m'}, c_1, \ldots, c_m, d_1, \ldots, d_{m'}, e\}$; $X_i = \big\{\langle a_1, \ldots, a_m\rangle, \langle c_1, \ldots, c_m, b_{m+1}, \ldots, b_{m'}\rangle\big\}$; $X_j = \big\{\langle a_1, \ldots, a_m, b_{m+1}, \ldots, b_{m'}\rangle,\ \langle d_1, \ldots, d_{m'}\rangle\big\}$; $X_k = \big\{\langle e\rangle\big\}$ for $k \in N \setminus \{i, j\}$; $\varphi_{a_s}(t) = v^t$ $(t = 1, 2;\ s = 1, \ldots, m)$; $\varphi_{b_s}(2) = \bar{v}^2 + \delta$, $\varphi_{b_s}(1) = \bar{v}^1 - \delta$ $(s = m+1, \ldots, m')$; $\varphi_{c_s}(1) = v^0$

$(s=1,\ldots,m)$; $\varphi_{d_s}(1)=\hat{v}$ $(s=1,\ldots,m')$. The 2×2 matrix of the essential part of the game looks as follows:

$$\begin{array}{ccc} & \text{ab} & \text{d} \\ \text{a} & \big(u^2, \lambda_j^{m'}(\sigma_j^1)\big) & \big(u^1, \lambda_j^{m'}(\sigma_j^0)\big) \\ \text{bc} & \big(u'', \lambda_j^{m'}(\sigma_j^2)\big) & \big(u', \lambda_j^{m'}(\sigma_j^0)\big). \end{array}$$

The inequalities (24) imply that there is no Nash equilibrium in the game. □

Lemma B.10 implies that the function $\Lambda(v)=(\lambda_i^{m'})^{-1}(v)-(\lambda_i^{m})^{-1}(v)$ is locally constant on V; therefore, it is a constant on V. Let us denote it $\bar{u}^{mm'}$ and show that (4) holds for all $u'\in m'\cdot\nu(\mathbb{R})$ and $u\in m\cdot\nu(\mathbb{R})$. Note that $W'=W+\bar{u}^{mm'}$ by the same Lemma B.10.

Let $u\in W$, i.e., $\lambda_i^m(u)=\lambda_i^{m'}(u+\bar{u}^{mm'})$; then, for every $u'\in m'\cdot\nu(\mathbb{R})$, we have $\operatorname{sign}\big(\lambda_i^{m'}(u')-\lambda_i^m(u)\big)=\operatorname{sign}\big(\lambda_i^{m'}(u')-\lambda_i^{m'}(u+\bar{u}^{mm'})\big)=\operatorname{sign}(u'-u-\bar{u}^{mm'})$ since $\lambda_i^{m'}$ is strictly increasing.

Let $u\notin W$, say, $u\geq\sup W$, hence $\lambda_i^m(u)>\lambda_i^{m'}(u')$ for all $u'\in m'\cdot\nu(\mathbb{R})$, hence the left hand side of (4) equals -1. Suppose there is $u'\in m'\cdot\nu(\mathbb{R})$ such that $u'\geq u+\bar{u}^{mm'}$, hence $u'\geq\sup W'$. We see that $(\sup W)\in m\cdot\nu(\mathbb{R})$ and $(\sup W')\in m'\cdot\nu(\mathbb{R})$; therefore, $\lambda_i^m(\sup W)=\lambda_i^{m'}(\sup W')$ by continuity, hence $(\sup W)\in W$, which is impossible for an open interval.

The case of $u\leq\inf W$ is treated dually.

C Proof of necessity in Theorem 2

If a finite game admits an exact potential, then it possesses a Nash equilibrium, so the implication [1.2] $\Rightarrow$ [1.3] from Theorem 1 applies. Therefore, there exist continuous and strictly increasing functions $\nu:\mathbb{R}\to\mathbb{R}$ and $\lambda_i^m: m\cdot\nu(\mathbb{R})\to\mathbb{R}$ $(i\in N,\ m\in\mathbb{N})$ for which (3) holds. In particular, the order of the processes does not matter, so we may assume that strategies are just finite subsets of A.

As in Section B.3, the matters would be simplified if $\nu(\mathbb{R})=\mathbb{R}$.

Lemma C.1. *Let $i,j\in N$, $m,m'\in\mathbb{N}$, $\sigma\in(m-1)\cdot\nu(\mathbb{R})$, $\sigma'\in(m'-1)\cdot\nu(\mathbb{R})$, and $v',v''\in\nu(\mathbb{R})$. Then*

$$\lambda_i^m(v''+\sigma)-\lambda_i^m(v'+\sigma)=\lambda_j^{m'}(v''+\sigma')-\lambda_j^{m'}(v'+\sigma'). \tag{25}$$

Proof. Let $w_2,\ldots,w_m\in\mathbb{R}$ be such that $\sigma=\sum_{s=2}^m\nu(w_s)$, $w'_2,\ldots,w'_{m'}\in\mathbb{R}$ such that $\sigma'=\sum_{s=2}^{m'}\nu(w'_s)$, and $w_1,w'_1\in\mathbb{R}$ be arbitrary; we denote $v_1=\nu(w_1)$ and $v'_1=\nu(w'_1)$.

Assuming first that $i\neq j$, we consider a generalized congestion game Γ with $m+m'+2$ processes where each player $k\in N$ uses the aggregation rule U_k: $\mathrm{A}=\{a,b_1,\ldots,b_m,c_1,\ldots,c_{m'},d\}$; $X_i=\big\{\langle a,b_2,\ldots,b_m\rangle,\langle b_1,b_2,\ldots,b_m\rangle\big\}$; $X_j=$

$\{\langle a, c_2, \ldots, c_{m'}\rangle, \langle c_1, c_2, \ldots, c_{m'}\rangle\}$; $X_k = \{\langle d\rangle\}$ for $k \in N \setminus \{i,j\}$; $\varphi_a(1) = \nu^{-1}(\upsilon')$, $\varphi_a(2) = \nu^{-1}(\upsilon'')$; $\varphi_{b_s}(1) = w_s$ $(s = 1, \ldots, m)$; $\varphi_{c_s}(1) = w'_s$ $(s = 1, \ldots, m')$. The 2×2 matrix of the essential part of the game looks as follows:

$$\begin{array}{ccc} & \text{ac} & \text{c} \\ \text{ab} & \big(\lambda_i^m(\upsilon''+\sigma), \lambda_j^{m'}(\upsilon''+\sigma')\big) & \big(\lambda_i^m(\upsilon'+\sigma), \lambda_j^{m'}(\upsilon'_1+\sigma')\big) \\ \text{b} & \big(\lambda_i^m(\upsilon_1+\sigma), \lambda_j^{m'}(\upsilon'+\sigma')\big) & \big(\lambda_i^m(\upsilon_1+\sigma), \lambda_j^{m'}(\upsilon'_1+\sigma')\big). \end{array}$$

By our assumption, Γ admits an exact potential. By Theorem 2.8 of Monderer and Shapley (1996), we obtain

$$\lambda_i^m(\upsilon''+\sigma) - \lambda_i^m(\upsilon_1+\sigma) + \lambda_j^{m'}(\upsilon'+\sigma') - \lambda_j^{m'}(\upsilon'_1+\sigma') = \\ \lambda_j^{m'}(\upsilon''+\sigma') - \lambda_j^{m'}(\upsilon'_1+\sigma') + \lambda_i^m(\upsilon'+\sigma) - \lambda_i^m(\upsilon_1+\sigma),$$

hence (25).

If $i = j$, we pick $k \neq i$ and obtain (25) with k as j, $m' = m$, and $\sigma' = \sigma$ first, and then (25) as it is. $\square$

The lemma immediately implies the existence of a function $\chi : \nu(\mathbb{R})^2 \to \mathbb{R}$ such that

$$\lambda_i^m(\upsilon''+\sigma) - \lambda_i^m(\upsilon'+\sigma) = \chi(\upsilon'', \upsilon')$$

for all $i \in N$, $m \in \mathbb{N}$, $\sigma \in (m-1)\cdot\nu(\mathbb{R})$, and $\upsilon', \upsilon'' \in \nu(\mathbb{R})$. Let $\upsilon'' > \upsilon'$. If $\delta > 0$ is such that $\upsilon''+\delta \in \nu(\mathbb{R})$, then $\upsilon'+\delta \in \nu(\mathbb{R})$ too. Pick $\sigma \in \nu(\mathbb{R})$ such that $\sigma < \upsilon''$; then $\sigma+\delta \in \nu(\mathbb{R})$ and $\chi(\upsilon'', \upsilon') = \lambda_i^2(\upsilon''+(\sigma+\delta)) - \lambda_i^2(\upsilon'+(\sigma+\delta)) = \lambda_i^2((\upsilon''+\delta)+\sigma) - \lambda_i^2((\upsilon'+\delta)+\sigma) = \chi(\upsilon''+\delta, \upsilon'+\delta)$. Quite similarly, $\chi(\upsilon'', \upsilon') = \chi(\upsilon''-\delta, \upsilon'-\delta)$ whenever $\upsilon'-\delta \in \nu(\mathbb{R})$.

Therefore, there is a function $\psi :]\nu(-\infty)-\nu(+\infty), \nu(+\infty)-\nu(-\infty)[\to \mathbb{R}$ such that $\chi(\upsilon'', \upsilon') = \psi(\upsilon''-\upsilon')$. Clearly, $\psi(-\Delta) = -\psi(\Delta)$; in particular, $\psi(0) = 0$. Since λ_i^m is continuous and strictly increasing, so is ψ.

For each $\Delta \in]\nu(-\infty)-\nu(+\infty), \nu(+\infty)-\nu(-\infty)[$, we have $\psi(\Delta) = \lambda_i^1(\upsilon+\Delta) - \lambda_i^1(\upsilon) = \lambda_i^1(\upsilon+\Delta) - \lambda_i^1(\upsilon+\Delta/2) + \lambda_i^1(\upsilon+\Delta/2) - \lambda_i^1(\upsilon) = 2\psi(\Delta/2)$. Similarly, $\psi(r\Delta) = r\psi(\Delta)$ for every rational $r \in [0,1]$. Since ψ is continuous, there is $B > 0$ such that $\psi(\Delta) = B\Delta$.

Finally, we define $\mu(w) = B\nu(w)$ and $C_i^m = \lambda_i^m(m\cdot\nu(0)) - m\cdot\mu(0)$. For $m = 1$, we have $U_i^{(1)}(w) = \lambda_i^1(\nu(w)) = B\cdot(\nu(w)-\nu(0)) + \lambda_i^1(\nu(0)) = \mu(w) + C_i^1$. A straightforward inductive argument shows that (5) holds for all $m \in \mathbb{N}$.

D Proof of necessity in Theorem 3

Let us prove the implication [3.2] $\Rightarrow$ [3.3]. The general scheme of the proof is the same as in Section B (and even simpler because we do not have to prove the second statement).

Lemma D.1. *Let $i, j \in N$, $m \geq 2$, $\upsilon_1, \upsilon_1', \upsilon_2, \upsilon_2' \in \mathbb{R}$, $1 \leq s_1, s_2 \leq m$, $s_1 \neq s_2$, $w, w' \in \mathbb{R}^m$, $w_{s_1} = \upsilon_1$, $w_{s_2} = \upsilon_2$, $w'_{s_1} = \upsilon_1'$, $w'_{s_2} = \upsilon_2'$, $w_s = w'_s$ for all $s \neq s_1, s_2$, and*

$$U_i^{(m)}(w) \leq U_i^{(m)}(w'); \tag{26a}$$

let $m' \geq 2$, $1 \leq s_1', s_2' \leq m'$, $s_1' \neq s_2'$, $w'', w''' \in \mathbb{R}^{m'}$, $w''_{s_1'} = \upsilon_1$, $w''_{s_2'} = \upsilon_2$, $w'''_{s_1'} = \upsilon_1'$, $w'''_{s_2'} = \upsilon_2'$, and $w''_s = w'''_s$ for all $s \neq s_1', s_2'$. Then

$$U_j^{(m')}(w'') \leq U_j^{(m')}(w'''). \tag{26b}$$

The interpretation is the same as in Lemma B.1.

Proof. Assuming first that $i \neq j$, we suppose to the contrary that $U_j^{(m')}(w'') > U_j^{(m')}(w''')$. Defining $w'''(\delta) \in \mathbb{R}^{m'}$ by $w'''(\delta)_{s_1'} = \upsilon_1' + \delta$ and $w'''(\delta)_s = w'''_s$ for all $s \neq s_1'$, and $w'(\delta) \in \mathbb{R}^m$ by $w'(\delta)_{s_1} = \upsilon_1' + \delta$ and $w'(\delta)_s = w'_s$ for all $s \neq s_1$, we can pick $\delta > 0$ such that $u_j^2 = U_j^{(m')}(w'') > U_j^{(m')}(w'''(\delta)) = u_j^1$; by monotonicity from (26a), $u_i^1 = U_i^{(m)}(w) < U_i^{(m)}(w'(\delta)) = u_i^2$.

Let us consider a game with structured utilities with $m + m' - 1$ processes where each player $k \in N$ uses the aggregation rule U_k: $\mathrm{A} = \{a, b, e\} \cup \{c_s\}_{s=1,\ldots,m,\, s_1 \neq s \neq s_2} \cup \{d_s\}_{s=1,\ldots,m',\, s_1' \neq s \neq s_2'}$; $\#\Upsilon^i = m$; $\#\Upsilon^j = m'$; $\Upsilon^i_{s_1} = a = \Upsilon^j_{s_1'}$; $\Upsilon^i_{s_2} = b = \Upsilon^j_{s_2'}$; $\Upsilon^i_s = c_s$ for all $s = 1, \ldots, m$, $s_1 \neq s \neq s_2$; $\Upsilon^j_s = d_s$ for all $s = 1, \ldots, m'$, $s_1' \neq s \neq s_2'$; $\Upsilon^k = \langle e \rangle$ for $k \in N \setminus \{i, j\}$; $X_i = X_j = \{1, 2\}$, $X_k = \{1\}$ for $k \neq i, j$; $\varphi_a(x_i, x_j) = \upsilon_1' + \delta$ if $x_i = x_j$, $\varphi_a(x_i, x_j) = \upsilon_1$ if $x_i \neq x_j$; $\varphi_b(x_i, x_j) = \upsilon_2'$ if $x_i = x_j$, $\varphi_b(x_i, x_j) = \upsilon_2$ if $x_i \neq x_j$; $\varphi_{c_s}(x_i) = w_s$ ($s = 1, \ldots, m$, $s_1 \neq s \neq s_2$); $\varphi_{d_s}(x_j) = w''_s$ ($s = 1, \ldots, m'$, $s_1' \neq s \neq s_2'$). The 2×2 matrix of the essential part of the game looks as follows:

$$\begin{matrix} (u_i^2, u_j^1) & (u_i^1, u_j^2) \\ (u_i^1, u_j^2) & (u_i^2, u_j^1). \end{matrix}$$

Since $u_k^2 > u_k^1$ ($k = i, j$), the game possesses no Nash equilibrium.

If $i = j$, we pick $k \neq i$ and obtain

$$U_k^{(m)}(w) \leq U_k^{(m)}(w')$$

first, and then (26b). □

The exact analogues of Lemma D.1 with equalities in (26) as well as strict inequalities of the same sign easily follow. Lemma B.2 in the current situation also follows from Lemma D.1.

Lemma D.2. *Let $i \in N$ and $v_s', v_s'', v_s''' \in \mathbb{R}$ for $s = 1, 2$; let*

$$U_i^{(2)}(v_1', v_2'') = U_i^{(2)}(v_1'', v_2')$$

and

$$U_i^{(2)}(v_1', v_2''') = U_i^{(2)}(v_1'', v_2'') = U_i^{(2)}(v_1''', v_2').$$

Then

$$U_i^{(2)}(v_1'', v_2''') = U_i^{(2)}(v_1''', v_2'').$$

The statement immediately follows from Lemma 2 of Kukushkin (1994).

Additive representation (3) now follows from Lemmas D.1 and D.2 exactly as in Section B.2.

E Proof of necessity in Theorem 4

As in Section C, we apply the implication [3.2] $\Rightarrow$ [3.3] from Theorem 3, obtaining the existence of continuous and strictly increasing functions $\nu : \mathbb{R} \to \mathbb{R}$ and $\lambda_i^m : m \cdot \nu(\mathbb{R}) \to \mathbb{R}$ ($i \in N$, $m \in \mathbb{N}$) for which (3) holds.

Lemma E.1. *Let $i, j \in N$, $m, m' \in \mathbb{N}$, $\sigma \in (m-1) \cdot \nu(\mathbb{R})$, $\sigma' \in (m'-1) \cdot \nu(\mathbb{R})$, and $v', v'' \in \nu(\mathbb{R})$. Then*

$$\lambda_i^m(v'' + \sigma) - \lambda_i^m(v' + \sigma) = \lambda_j^{m'}(v'' + \sigma') - \lambda_j^{m'}(v' + \sigma'). \tag{27}$$

Proof. Let $w_2, \ldots, w_m \in \mathbb{R}$ be such that $\sigma = \sum_{s=2}^{m} \nu(w_s)$, $w_2', \ldots, w_{m'}' \in \mathbb{R}$ such that $\sigma' = \sum_{s=2}^{m'} \nu(w_s')$, and $w_1, w_1' \in \mathbb{R}$ be arbitrary; we denote $v_1 = \nu(w_1)$ and $v_1' = \nu(w_1')$.

Assuming first that $i \neq j$, we consider a game with structured utilities Γ with $m + m'$ processes where each player $k \in N$ uses the aggregation rule U_k: $\mathrm{A} = \{a, b_2, \ldots, b_m, c_2, \ldots, c_{m'}, d\}$; $\Upsilon^i = \{a, b_2, \ldots, b_m\}$, $\Upsilon^j = \{a, c_2, \ldots, c_{m'}\}$, $\Upsilon^k = \{d\}$ for $k \neq i, j$; $X_i = X_j = \{1, 2\}$, $X_k = \{1\}$ for $k \neq i, j$; $\varphi_a(x_i, x_j) = \nu^{-1}(v'')$ if $x_i = x_j = 2$ and $\varphi_a(x_i, x_j) = \nu^{-1}(v')$ otherwise; $\varphi_{b_s}(x_i) = w_s$ ($s = 2, \ldots, m$, $x_i \in X_i$); $\varphi_{c_s}(x_j) = w_s'$ ($s = 2, \ldots, m'$, $x_j \in X_j$).

By our assumption, Γ admits an exact potential. By Theorem 2.8 of Monderer and Shapley (1996), we obtain

$$u_i(2,2) - u_i(1_i, 2_j) + u_j(1_i, 2_j) - u_j(1,1) = u_j(2,2) - u_j(2_i, 1_j) + u_i(2_i, 1_j) - u_i(1,1).$$

Taking into account that $u_i(2,2) = \lambda_i^m(v'' + \sigma)$, $u_j(2,2) = \lambda_j^{m'}(v'' + \sigma')$, $u_i(x_i, x_j) = \lambda_i^m(v' + \sigma)$ for all other (x_i, x_j), and $u_j(x_i, x_j) = \lambda_j^{m'}(v' + \sigma')$ for all other (x_i, x_j), we obtain (27).

If $i = j$, we pick $k \neq i$ and obtain (27) with k as j, $m' = m$, and $\sigma' = \sigma$ first, and then (27) as it is. $\square$

The rest of the proof is the same as in Section C.