

# ON THE GENERIC FINITENESS OF EQUILIBRIUM OUTCOME DISTRIBUTIONS IN BIMATRIX GAME FORMS

NIKOLAI S. KUKUSHKIN, CRISTIAN M. LITAN,  
AND FRANCISCO MARHUENDA

**ABSTRACT.** We provide an example of an outcome game form with two players for which there is in an open set of utilities for both players such that, in each of the associated games, the set of Nash equilibria induces a continuum of outcome distributions.

## 1. INTRODUCTION

Harsanyi [5] shows that, generically in the space of utilities, a finite normal form game has a finite (and odd) number of mixed equilibria. Kreps and Wilson [6] note that when the normal form is derived from an extensive form, Harsanyi's result has no immediate implications, because many strategies lead to the same final node. Even if the payoffs at the final nodes can be perturbed independently, the finiteness of the number of equilibria need not be a generic property; appropriate counterexamples are easy to construct. Thus, Harsanyi's result cannot be applied directly to many economic models of interest.

For games in extensive form, Kreps and Wilson [6] prove that for generic utility payoffs on the terminal nodes, the Nash equilibria induce finitely many probability distributions on the same set of terminal nodes. Based on techniques from the theory of semi-algebraic sets, Govindan and Wilson [4] provide a more direct proof of the results above, for normal as well as for extensive form games.

On the other hand, Mas-Colell [8] and later Govindan and McLennan [3] point out that, in many economic instances, different strategy profiles generate the same outcome; moreover, this may be true for different terminal nodes of an extensive game. Thus, the same criticism can be reiterated for the Kreps and Wilson [6] result.

A suitable general setup to address the above problem is obtained with the notion of an outcome game form, i.e., a mapping from strategy profiles to outcomes. We can now pose the question as follows: assuming that the utility functions on outcomes may be independently

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*Key words and phrases.* Generic finiteness, Nash equilibrium.

This problem was communicated to us by Francesco De Sinopoli. We thank Andreu Mas-Colell for providing us with the manuscript [8]. The authors acknowledge financial support from the project No. SEJ2004-00968/ECON from the Spanish Ministry of Education and Science. C. Litan acknowledges financial support from "Plan de Formación de Personal Investigador" of the Comunidad de Madrid.

perturbed, is it true that, generically, the equilibrium strategies of the associated game induce a finite number of probability distributions on the set of outcomes?

When there are at least three players, Govindan and McLennan [3] provide a negative answer to the above question. They show an example of a game form with six outcomes where, for an open set of utility functions, the Nash equilibria induce a continuum of outcome distributions. On the positive side and for games with any number of players, Govindan and McLennan [3] prove that if there are only two outcomes, generic finiteness on equilibrium outcome distributions is restored.

So far, for two person game forms, partial evidence had been obtained that suggested a possible positive answer to the question above. Mas-Colell [8] shows that the equilibrium payoffs are generically finite for two player game forms. Govindan and McLennan [2] prove the generic finiteness of equilibrium distributions on outcomes for zero sum games and common interest games. González-Pimienta [1] argues that the same result holds for games with two players and three outcomes.

This note closes the last remaining gap in the case of two players, in a rather unexpected way. We provide an example of a game form (with four outcomes) and an open set of utility functions such that in each of the associated games there is a continuum of outcome distributions induced by Nash equilibria. By introducing dummy players, our example can be extended, in an obvious way, to any number of players and four outcomes. The results in González-Pimienta [1] and Govindan and McLennan [3] show that this is the minimum number of outcomes needed for a counterexample.

## 2. THE CONJECTURE

Let  $\Omega$  be a finite outcome space and let  $S_1, S_2$  denote the finite strategy spaces of each of the two players. An outcome game form is a mapping  $\theta : S_1 \times S_2 \rightarrow \Omega$ . The utilities of the players are defined by the functions  $u_i : \Omega \rightarrow \mathbb{R}$ ,  $i = 1, 2$ .

For each  $i = 1, 2$ , let  $\Delta_i = \{\mu \in \mathbb{R}_{++}^{S_i} : \sum_{x \in S_i} \mu(x) = 1\}$ . A pair of strategy vectors  $p_i \in \Delta_i$  of the players induces a probability distribution in  $\Omega$ .

An outcome game form  $\theta$  and the utilities of the players  $u_i$ ,  $i = 1, 2$  determine a bimatrix game. We denote by  $A(u_i)$  the associated matrices of utilities of the players in this game. That is, the entry  $a_{jl}$  of  $A(u_i)$  is  $u_i(\theta(s_j, s_l))$ .

A completely mixed Nash equilibrium in this game consists of two strategy vectors  $p_1 \in \Delta_1$  and  $p_2 \in \Delta_2$  such that

$$\begin{aligned} A(u_2) p_1 &= \beta e && \text{for some } \beta \in \mathbb{R}. \\ p_2 A(u_1) &= \alpha e && \text{for some } \alpha \in \mathbb{R}. \end{aligned}$$

where  $e$  denotes the vector (in the appropriate Euclidean space) with all of its entries equal to 1.

**Conjecture 1.** For every bimatrix game form  $\theta$ , there is a generic set of utilities in  $\mathbb{R}^\Omega \times \mathbb{R}^\Omega$  for which the set of distributions induced on outcomes by the **completely mixed Nash equilibria** is finite.

For the case of two person games, this conjecture corresponds to Conjecture T in Govindan and McLennan [3]. They disprove the conjecture for games with at least three players.

### 3. THE EXAMPLE

We now give an example that shows that Conjecture 1 does not hold. Let there be four outcomes, denoted by  $\Omega = \{a, b, c, d\}$ . And consider the outcome matrix

$$A(a, b, c, d) = \begin{pmatrix} c & a & b & b \\ d & a & a & b \\ c & d & b & c \end{pmatrix}$$

Let us use the notation  $a_i = u_i(a)$ ,  $b_i = u_i(b)$ ,  $c_i = u_i(c)$ ,  $d_i = u_i(d)$ , and  $u^i = (a_i, b_i, c_i, d_i) \in \mathbb{R}^4$  for the utilities of agent  $i = 1, 2$ . We denote  $G = \{(u^1, u^2) \in \mathbb{R}^8 \mid d_1, b_1 < a_1, c_1 \text{ and } d_2 < b_2 < a_2, c_2\}$ , an open subset in the space of utilities. For each  $(u^1, u^2) \in G$  and  $t \in \mathbb{R}$ , we define

$$p_1(u^2) = \frac{1}{a_2 - b_2 + c_2 - d_2} (b_2 - d_2, c_2 - b_2, a_2 - b_2) \in \mathbb{R}^3$$

and

$$p_2(u^1; t) = \left( \frac{a_1 - b_1}{a_1 - b_1 + c_1 - d_1} - \frac{(a_1 - b_1)t}{a_1 - d_1}, \frac{(c_1 - b_1)t}{a_1 - d_1}, \frac{c_1 - d_1}{a_1 - b_1 + c_1 - d_1} - \frac{(c_1 - d_1)t}{a_1 - d_1}, t \right) \in \mathbb{R}^4.$$

One checks immediately that the pair  $\langle p_1(u^2), p_2(u^1; t) \rangle$  is a completely mixed Nash equilibrium provided  $t$  is positive and small enough. The probability of outcome  $a$  induced by the equilibrium is computed easily as

$$p_a = \frac{(b_2 - c_2)(c_1 - d_1)}{(a_1 - b_1 + c_1 - d_1)(-a_2 + b_2 - c_2 + d_2)} + \frac{b_2(d_1 - c_1) + b_1(c_2 - d_2) + c_1d_2 - c_2d_1}{(a_1 - d_1)(-a_2 + b_2 - c_2 + d_2)} t$$

Therefore, there is a continuum of equilibrium probability distributions (i.e. for  $t$  positive and small enough) on the set of outcomes as long as  $(u^1, u^2) \in G$  and  $b_2(d_1 - c_1) + b_1(c_2 - d_2) + c_1d_2 - c_2d_1 \neq 0$ ; such values of  $u^1$  and  $u^2$  form an open subset in the space of utilities. Remark that when  $u^1 = \pm u^2$ , the coefficient of  $t$  in the above expression for  $p_a$  (as well as for the other outcomes) disappears and there is a unique probability distribution on outcomes, as proved in Govindan and McLennan [2] and Litan [7].

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RUSSIAN ACADEMY OF SCIENCES, DORODNICYN COMPUTING CENTER, MOSCOW.  
*E-mail address:* ququns@pochta.ru

UNIVERSITY CARLOS III DE MADRID, DEPARTMENT OF ECONOMICS, MADRID.  
*E-mail address:* clitan@eco.uc3m.es

UNIVERSITY CARLOS III DE MADRID, DEPARTMENT OF ECONOMICS, MADRID.  
*E-mail address:* marhuend@eco.uc3m.es