Cournot tâtonnement in aggregative games with monotone best responses

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Abstract

This paper establishes the acyclicity of Cournot tâtonnement in a strategic game with aggregation and monotone best responses, under the broadest assumptions on aggregation rules allowing the Huang–Dubey–Haimanko–Zapechelnyuk–Jensen trick to work and with minimal topological restrictions. *MSC2010* Classification: 91A10. *JEL* Classification Number: C 72.

Key words: Cournot tâtonnement; Cournot potential; aggregative game; monotone best responses

1 Introduction

The importance of aggregation for the existence of Nash equilibrium was first noticed by Novshek (1985), see also Kukushkin (1994). Kukushkin (2004) showed that monotonicity conditions in games with additive aggregation ensure the acyclicity of Cournot tâtonnement rather than the mere existence of an equilibrium. Dubey et al. (2006), having modified a trick invented by Huang (2002) for different purposes, developed a tool applicable to a broader class of aggregation rules. Kukushkin (2005) and, especially, Jensen (2010) extended its sphere of applicability much further. It looks plausible that the latter paper describes the most general class of aggregation rules for which this approach can still work.

This paper strives to establish the acyclicity of Cournot tâtonnement under the same assumptions on aggregation, but with minimal topological restrictions. The point is that the best response correspondences in the main results of Jensen (2010) were assumed upper hemicontinuous. Although one can plausibly argue that the upper hemiconinuity of the best responses holds in "most" of important economic models, "most" cannot be replaced with "all." Even more importantly, our main theorem implies the existence of an equilibrium where each player uses an arbitrarily fixed monotone selection from the best response correspondence; nothing like that could be derived from the previous literature.

There is an additional reason to look for the weakest possible topological conditions. When viewed as a fixed point theorem, our result occupies a position intermediate between Brouwer's and Tarski's theorems: the former is purely topological; the latter, order-theoretical. In our case, some combination

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of conditions of both kinds seems indispensable; so it is highly desirable to understand exactly what is needed from either side.

In Section 2, the most basic definitions are given; in Section 3, our (i.e., mostly Jensen's) assumptions, as well as the main theorem, are formulated. Section 4 contains a review of conditions ensuring appropriate monotonicity in strategic games. In Section 5, we briefly discuss alternative aggregation rules for which our main findings do, or do not, hold. Section 6 contains the proof of our main theorem and an example of Cournot dynamics when the best responses are not upper hemicontinuous.

2 Preliminaries

In the main theorem, we consider exogenously given best response correspondences rather than games as such. An *abstract game* is defined by a finite set of players N and, for each $i \in N$, a strategy set X_i and the *best response correspondence* $\mathcal{R}_i \colon X_{-i} \to 2^{X_i} \setminus \{\emptyset\}$, where $X_{-i} := \prod_{j \in N \setminus \{i\}} X_j$. We also denote $X_N := \prod_{i \in N} X_i$, the set of *strategy profiles*.

Remark. The definition of abstract games belongs to Vives (1990). Under his approach, however, the collection of responses $\mathcal{R}_i$ was immediately replaced with a single correspondence $\mathcal{R} \colon X_N \to X_N$, $\mathcal{R}(x_N) := \prod_{i \in N} \mathcal{R}_i(x_{-i})$, to which Tarski's fixed point theorem could be applied. In our case, the structure of a Cartesian product plays a crucial role and is retained to the end.

An *equilibrium* of an abstract game is $x^0_N \in X_N$ such that $x^0_i \in \mathcal{R}_i(x^0_{-i})$ for each $i \in N$. The first basic question about a particular model is whether it admits an equilibrium. In the case of a positive answer, the next question is whether iteration of the best responses leads to equilibria. In this paper, we derive a positive answer to the first question from the same answer to the second one.

We introduce the *best response improvement* relation on X_N ($i \in N$, $y_N, x_N \in X_N$):

$$y_N \rhd_i x_N \rightleftharpoons [y_{-i} = x_{-i} \;\&\; x_i \notin \mathcal{R}_i(x_{-i}) \ni y_i]; \tag{1a}$$

$$y_N \rhd x_N \rightleftharpoons \exists i \in N \, [y_N \rhd_i x_N]. \tag{1b}$$

Every equilibrium is a maximizer of $\rhd$. Since $\mathcal{R}_i(x_{-i}) \neq \emptyset$ for all $i \in N$ and $x_{-i} \in X_{-i}$, every maximizer of $\rhd$ is an equilibrium.

A *Cournot path* is a finite or infinite sequence $\langle x^k_N \rangle_{k=0,1,\ldots}$ such that $x^{k+1}_N \rhd x^k_N$ whenever x^{k+1}_N is defined. A *Cournot potential* is an irreflexive and transitive binary relation $\succ$ on X_N such that

$$\forall x_N, y_N \in X_N \, \big[y_N \rhd x_N \Rightarrow y_N \succ x_N \big]. \tag{2}$$

The existence of a Cournot potential is equivalent to the absence of *Cournot cycles*, i.e., Cournot paths $\langle x^0_N, x^1_N, \ldots, x^m_N \rangle$ such that $m > 0$ and $x^0_N = x^m_N$. If X_N is finite, this fact implies that every Cournot path, if continued whenever possible, reaches an equilibrium in a finite number of steps. Otherwise, the acyclicity of the best responses does not imply very much by itself. However, if a topological structure is assumed on the strategies and the notion of a Cournot potential is strengthened, conclusions can be drawn, not dissimilar from those in the finite case.

Let X be a metric space. A binary relation $\succ$ on X is *ω-transitive* if it is transitive and

$$\left[x^{\omega} = \lim_{k\to\infty} x^k \;\&\; \forall k \in \mathbb{N}[x^{k+1} \succ x^k]\right] \Rightarrow x^{\omega} \succ x^0. \tag{3}$$

It is essential that (3) implies $x^{\omega} \succ x^k$ for all $k = 1, \ldots$ as well.

Remark. The property seems to have been considered first by Gillies (1959), and then by Smith (1974), for orderings. The term "ω-transitivity" first appeared in Kukushkin (2003).

Theorem 1 from Kukushkin (2008) implies that an irreflexive and ω-transitive binary relation on a compact set always admits a maximizer. Therefore, the existence of an ω-transitive Cournot potential ensures the existence of an equilibrium. It also ensures the "transfinite convergence" (Kukushkin, 2003, 2010) of all iterations of the best responses to equilibria.

3 Main result

To derive the existence of an ω-transitive Cournot potential in a class of abstract games, we impose a set of assumptions combining topological and order-theoretical requirements. Quite a number of notations and definitions are needed.

A *chain* is a linearly ordered set. A partially ordered set (*poset*) X is *chain-complete* if every chain $\emptyset \neq C \subseteq X$ admits both $\sup C$ and $\inf C$ in X. A poset X is *chain-complete downwards* (*upwards*) if only the existence of $\inf C$ ($\sup C$) is ensured for every nonempty chain in X. A *well ordered set* is a chain Δ such that every subset $\Delta' \subseteq \Delta$ contains its minimum. Dually, a *$*$-well ordered set* is a chain every subset of which contains its maximum.

Given a poset A and $b \in B \subseteq A$, we denote $B^{\rightarrow}(b) := \{a \in B \mid a > b\}$ and $B^{\leftarrow}(b) := \{a \in B \mid a < b\}$. Given a metric space A and $B \subseteq A$, $\operatorname{cl} B$ denotes the topological closure of B in A while $\operatorname{Int} B$ denotes its interior.

We assume throughout that a "universal" separable metric space A is given, which is simultaneously a poset. The order on A is consistent with the topology in the sense that:

$$\forall x \in A\left[\{y \in A \mid y \geq x\} = \operatorname{cl}(\operatorname{Int} A^{\rightarrow}(x))\right]; \tag{4a}$$

$$\forall x \in A\left[\{y \in A \mid y \leq x\} = \operatorname{cl}(\operatorname{Int} A^{\leftarrow}(x))\right]. \tag{4b}$$

These conditions imply that all upper sets $\{y \in A \mid y \geq x\}$ and all lower sets $\{y \in A \mid y \leq x\}$ ($x \in A$) are closed. A good example of such A is $\mathbb{R}^m$; a more general example is the space of continuous functions on a compact space C with point-wise order and the metric $d(f,g) := \max_{x\in C} |f(x) - g(x)|$. Both examples are also *lattices*, which fact comes in handy in Section 4.

Lemma 3.1. *If $x^k \to x^{\omega}$ and $x^{k+1} \geq x^k$ for all $k \in \mathbb{N}$, then $x^{\omega} = \sup_k x^k$.*

Proof. First, we denote $X^k := \{y \in A \mid y \geq x^k\}$ for each $k \in \mathbb{N}$; clearly, $x^h \in X^k$ whenever $h \geq k$. Since X^k is closed by (4a), we have $x^{\omega} \in X^k$, i.e., $x^{\omega} \geq x^k$ for each k. Assuming $y \geq x^k$ for each k, we denote $Y := \{x \in A \mid y \geq x\}$; Y is closed by (4b). Since $x^k \in Y$ for each k, we have $x^{\omega} \in Y$ as well; hence $x^{\omega} \leq y$. Thus, $x^{\omega} = \sup_k x^k$ indeed. $\square$

Lemma 3.2. *If $x^k \to x^\omega$ and $x^{k+1} \leq x^k$ for all $k \in \mathbb{N}$, then $x^\omega = \inf_k x^k$.*

The proof is dual to that of Lemma 3.1.

Henceforth, we assume that each X_i is a compact subset of the universal set A, and endow X_N with, say, the maximum metric. By Lemmas 3.1 and 3.2, each X_i is also chain-complete.

Further, we assume that there are continuous mappings $\sigma_i \colon X_{-i} \to \mathbb{R}$ ($i \in N$), *aggregation rules*. Denoting $S_i := \sigma_i(X_{-i}) \subset \mathbb{R}$, $s_i^- := \min S_i$ and $s_i^+ := \max S_i$, we assume the existence of correspondences $R_i \colon S_i \to 2^{X_i} \setminus \{\emptyset\}$ and continuous mappings $g \colon X_N \to \mathbb{R}$, $F_i \colon [s_i^-, s_i^+] \times X_i \to \mathbb{R}$ and $v_i \colon X_{-i} \to \mathbb{R}$ ($i \in N$) such that

$$\mathcal{R}_i(x_{-i}) = R_i(\sigma_i(x_{-i})) \tag{5}$$

and

$$g(x_N) = F_i(\sigma_i(x_{-i}), x_i) + v_i(x_{-i}) \tag{6}$$

for all $i \in N$, $x_{-i} \in X_{-i}$, and $x_N \in X_N$. Additionally, we assume that each F_i has a continuous derivative w.r.t. its first argument,

$$D_i(s_i, x_i) := \frac{\partial F_i}{\partial s_i}(s_i, x_i), \tag{7}$$

on $[s_i^-, s_i^+] \times X_i$.

Finally, we impose monotonicity assumptions:

$$\forall i \in N \, \forall s_i', s_i \in S_i \left[[s_i' > s_i \;\&\; x_i' \in R_i(s_i') \;\&\; x_i \in R_i(s_i)] \Rightarrow x_i' \geq x_i \right]; \tag{8}$$

$$\forall i \in N \, \forall s_i \in S_i \, \forall x_i', x_i \in X_i \left[x_i' > x_i \Rightarrow D_i(s_i, x_i') > D_i(s_i, x_i) \right]. \tag{9}$$

Theorem 1. *An abstract game satisfying all the above assumptions admits an ω-transitive Cournot potential (hence admits an equilibrium as well).*

The proof is deferred to Section 6.

Our conditions (6), (7), and (9) are exactly the same as in Jensen (2010). There was no need for explicit conditions like (4) there since all strategy sets were assumed to be subsets of $\mathbb{R}^m$. As to (8), Jensen preferred to consider *decreasing* best responses; the difference is of no significance since one can always replace σ_i with $-\sigma_i$.

Thus, if we assumed that each $\mathcal{R}_i$ is upper hemicontinuous, the difference between our Theorem 1 and the main result of Jensen (2010) would be quite minor. However, we do *not* impose that assumption.

The implications of Theorem 1 for the existence of equilibria in strategic games are considered in the next section. We follow the same logic as in Kukushkin (2005) and Jensen (2010): each $\mathcal{R}_i$ may be perceived either as the total best response correspondence or as an increasing selection from it.

4 Monotonicity conditions in strategic games

The difference between a *strategic game* and an abstract game is that the best responses in the former model are not exogenous, but generated by the maximization of (ordinal) utility functions $u_i\colon X_N \to \mathbb{R}$,

$$\mathcal{R}_i(x_{-i}) := \operatorname*{Argmax}_{x_i \in X_i} u_i(x_i, x_{-i}) \tag{10}$$

for each player $i \in N$ and every $x_{-i} \in X_{-i}$.

Our Theorem 1 can be applied directly to the best response correspondences in a strategic game if each X_i is a compact subset of an appropriate universal set A, while $\mathcal{R}_i$'s defined by (10) satisfy all those assumptions. Let there be continuous aggregation rules $\sigma_i\colon X_{-i} \to \mathbb{R}$ ($i \in N$) such that

$$u_i(x_N) = U_i(\sigma_i(x_{-i}), x_i) \tag{11}$$

for all $i \in N$ and $x_N \in X_N$. For each $i \in N$, we denote $S_i := \sigma_i(X_{-i}) \subset \mathbb{R}$, and redefine the best response correspondence:

$$R_i(s_i) := \operatorname*{Argmax}_{x_i \in X_i} U_i(s_i, x_i).$$

Our assumption $\mathcal{R}_i(x_{-i}) \neq \emptyset$ is equivalent to $R_i(s_i) \neq \emptyset$ for each $s_i \in S_i$.

Remark. In principle, equality (5) may hold for all $i \in N$ and $x_{-i} \in X_{-i}$ without (11) holding for all $i \in N$ and $x_N \in X_N$. However, no natural class of strategic games where this happens is known.

Our assumptions on σ_i, i.e., (6), (7), and (9), do not depend on whether it is about an abstract game or a strategic game. Jensen (2010) provides quite a list of game models where the assumptions imposed on aggregation rules in Section 3 are satisfied.

As to assumption (8), the well-known studies of monotone comparative statics (Topkis, 1978; Vives, 1990; Veinott, 1992; Milgrom and Shannon, 1994) allow us to derive it from assumptions imposed directly on utility functions. We have to start with definitions.

A utility function exhibits the *single crossing* property (Milgrom and Shannon, 1994) if these conditions hold for all $i \in N$, $y_i, x_i \in X_i$, and $s'_i, s_i \in S_i$:

$$[y_i > x_i \;\&\; s'_i > s_i \;\&\; U_i(s_i, y_i) > U_i(s_i, x_i)] \Rightarrow U_i(s'_i, y_i) > U_i(s'_i, x_i); \tag{12a}$$

$$[y_i > x_i \;\&\; s'_i > s_i \;\&\; U_i(s_i, y_i) \geq U_i(s_i, x_i)] \Rightarrow U_i(s'_i, y_i) \geq U_i(s'_i, x_i). \tag{12b}$$

Also important are the *strict single crossing* property (Milgrom and Shannon, 1994):

$$[y_i > x_i \;\&\; s'_i > s_i \;\&\; U_i(s_i, y_i) \geq U_i(s_i, x_i)] \Rightarrow U_i(s'_i, y_i) > U_i(s'_i, x_i); \tag{13}$$

and the *weak single crossing* property (Shannon, 1995):

$$[y_i > x_i \;\&\; s'_i > s_i \;\&\; U_i(s_i, y_i) > U_i(s_i, x_i)] \Rightarrow U_i(s'_i, y_i) \geq U_i(s'_i, x_i). \tag{14}$$

When strategies are scalar, conditions like (12), (13) or (14) are sufficient for monotone comparative statics results. When X_i's are lattices, some versions of *quasisupermodularity* (Milgrom and Shannon, 1994; see also LiCalzi and Veinott, 1992) are needed too. We reproduce a few of them here.

First, four "quarters" of quasisupermodularity proper:

$$U_i(s_i, x_i) \geq U_i(s_i, y_i \wedge x_i) \Rightarrow [U_i(s_i, y_i \vee x_i) \geq U_i(s_i, x_i) \text{ or } U_i(s_i, y_i \vee x_i) \geq U_i(s_i, y_i)]; \tag{15a}$$
$$U_i(s_i, y_i) \geq U_i(s_i, y_i \vee x_i) \Rightarrow [U_i(s_i, y_i \wedge x_i) \geq U_i(s_i, x_i) \text{ or } U_i(s_i, y_i \wedge x_i) \geq U_i(s_i, y_i)]; \tag{15b}$$
$$U_i(s_i, x_i) > U_i(s_i, y_i \wedge x_i) \Rightarrow [U_i(s_i, y_i \vee x_i) > U_i(s_i, x_i) \text{ or } U_i(s_i, y_i \vee x_i) > U_i(s_i, y_i)]; \tag{15c}$$
$$U_i(s_i, y_i) > U_i(s_i, y_i \vee x_i) \Rightarrow [U_i(s_i, y_i \wedge x_i) > U_i(s_i, x_i) \text{ or } U_i(s_i, y_i \wedge x_i) > U_i(s_i, y_i)]. \tag{15d}$$

Then, two "halves" of *weak quasisupermodularity*:

$$U_i(s_i, x_i) > U_i(s_i, y_i \wedge x_i) \Rightarrow [U_i(s_i, y_i \vee x_i) \geq U_i(s_i, x_i) \text{ or } U_i(s_i, y_i \vee x_i) \geq U_i(s_i, y_i)]; \tag{16a}$$
$$U_i(s_i, y_i) > U_i(s_i, y_i \vee x_i) \Rightarrow [U_i(s_i, y_i \wedge x_i) \geq U_i(s_i, x_i) \text{ or } U_i(s_i, y_i \wedge x_i) \geq U_i(s_i, y_i)]. \tag{16b}$$

Remark. Each of conditions (15) and (16) holds trivially when $x_i \geq y_i$ or $x_i \leq y_i$.

Proposition 22 from Kukushkin (2013b) immediately implies this result:

Proposition 1. *Let, in a strategic game satisfying (11), X_i be a lattice and the utility function $U_i(s_i, x_i)$ satisfy these assumptions: the strict single crossing condition (13) holds for every $x_i, y_i \in X_i$ and $s_i, s'_i \in S_i$; there is $s^*_i \in S_i$ such that (15a) holds for all $x_i, y_i \in X_i$ and $s_i < s^*_i$ while (15b) holds for all $x_i, y_i \in X_i$ and $s_i > s^*_i$. Then the best response correspondence R_i is increasing in the sense of (8).*

If X_i is a *semilattice*, i.e., only the meet $y_i \wedge x_i$ is guaranteed to exist for every $x_i, y_i \in X_i$ (e.g., a budget set), then conditions (15) and (16) make no sense. However, a similar role is played by a condition that could be called *semiquasisupermodularity*:

$$U_i(s_i, y_i) > U_i(s_i, y_i \wedge x_i) \Rightarrow U_i(s_i, y_i \wedge x_i) \geq U_i(s_i, x_i). \tag{17}$$

Proposition 2. *Let, in a strategic game satisfying (11), X_i be a semilattice, the strict single crossing condition (13) hold for every $x_i, y_i \in X_i$ and $s_i, s'_i \in S_i$, and (17) hold for every $x_i, y_i \in X_i$ and $s_i \in S_i$. Then the best response correspondence R_i is increasing in the sense of (8).*

Proof. Let $s'_i > s_i$, $x_i \in R_i(s_i)$ and $y_i \in R_i(s'_i)$; we have to show $y_i \geq x_i$. Otherwise, we would have $x_i > y_i \wedge x_i$. Since $U_i(s_i, x_i) \geq U_i(s_i, y_i \wedge x_i)$, we have $U_i(s'_i, x_i) > U_i(s'_i, y_i \wedge x_i)$ by (13); hence $U_i(s'_i, y_i \wedge x_i) \geq U_i(s'_i, y_i)$ by (17). Thus, $U_i(s'_i, x_i) > U_i(s'_i, y_i)$, which contradicts $y_i \in R_i(s'_i)$. $\square$

Remark. Undoubtedly, condition (17) is much more demanding than (15) or (16); however, it may hold in a non-trivial way. Essentially, it shows that U_i may increase in x_i only along a "fixed path."

When the best response correspondences in a strategic game with appropriate aggregation do *not* satisfy (8), our Theorem 1 may still be applicable in a more sophisticated way. If there are increasing selections from the best response correspondences, we may consider the abstract game defined by those selections rather than total best responses. We cannot obtain any statement about *all* Cournot dynamics in this way, but the existence of an equilibrium, at least, will be ensured. In the case when those selections are "natural" in one sense or another, the acyclicity of the corresponding dynamics may also be of interest. Finally, the existence of an equilibrium where each player uses a fixed selection from the best response correspondence may be important in some contexts.

Combining Proposition 26 from Kukushkin (2013b) and Theorem 2.2 from Kukushkin (2013a), we obtain this result.

Proposition 3. *Let, in a strategic game satisfying* (11), X_i *be a lattice and the utility function* $U_i(s_i, x_i)$ *satisfy these conditions: the weak single crossing condition* (14) *holds for every* $x_i, y_i \in X_i$ *and* $s_i, s'_i \in S_i$*; there is* $s^*_i \in S_i$ *such that* (15c) *holds for all* $x_i, y_i \in X_i$ *and* $s_i < s^*_i$ *while* (15d) *holds for all* $x_i, y_i \in X_i$ *and* $s_i > s^*_i$*; every* $R_i(s_i)$ *is nonempty and chain-complete. Then there exists an increasing selection* r_i *from the best response correspondence* R_i*.*

Invoking Proposition 28 from Kukushkin (2013b) and the same Theorem 2.2 from Kukushkin (2013a), we obtain this result.

Proposition 4. *Let, in a strategic game satisfying* (11), X_i *be a lattice and the utility function* $U_i(s_i, x_i)$ *satisfy these conditions: the single crossing conditions* (12) *hold for every* $x_i, y_i \in X_i$ *and* $s_i, s'_i \in S_i$*; there is* $s^*_i \in S_i$ *such that* (16a) *holds for all* $x_i, y_i \in X_i$ *and* $s_i < s^*_i$ *while* (16b) *holds for all* $x_i, y_i \in X_i$ *and* $s_i > s^*_i$*; every* $R_i(s_i)$ *is nonempty and chain-complete. Then there exists an increasing selection* r_i *from the best response correspondence* R_i*.*

A similar statement is valid for semilattices too.

Proposition 5. *Let, in a strategic game satisfying* (11), X_i *be a semilattice and the utility function* $U_i(s_i, x_i)$ *satisfy conditions* (12a) *and* (17) *for every* $x_i, y_i \in X_i$ *and* $s_i, s'_i \in S_i$*, while every* $R_i(s_i)$ *be nonempty and chain-complete downwards. Then there exists an increasing selection* r_i *from the best response correspondence* R_i*.*

Proof. For every $s_i \in S_i$, we denote $R^-_i(s_i)$ the "lower frontier" of $R_i(s_i)$, i.e., $R^-_i(s_i) \rightleftharpoons \{x_i \in R_i(s_i) \mid \nexists y_i \in R_i(s_i)\,[y_i < x_i]\}$. Since $R_i(s_i)$ is nonempty and chain-complete downwards, Zorn's Lemma implies that $R^-_i(s_i) \neq \emptyset$. Let us show that an arbitrary selection $r_i \colon S_i \to X_i$ from $R^-_i(s_i)$ is increasing.

Let $s'_i > s_i$, $x_i = r_i(s_i)$ and $y_i = r_i(s'_i)$; we have to show $y_i \geq x_i$. Otherwise, we would have $x_i > y_i \wedge x_i$; hence $U_i(s_i, x_i) > U_i(s_i, y_i \wedge x_i)$ because $x_i \in R^-_i(s_i)$. Now we have $U_i(s'_i, x_i) > U_i(s'_i, y_i \wedge x_i)$ by (12a); hence $U_i(s'_i, y_i \wedge x_i) \geq U_i(s'_i, y_i)$ by (17). Thus, $U_i(s'_i, x_i) > U_i(s'_i, y_i)$, which contradicts $y_i \in R_i(s'_i)$. $\square$

5 On other aggregation rules

It is important to note that Theorem 1, i.e., the Huang–Dubey–Haimanko–Zapechelnyuk–Jensen trick, does not cover *all* nice aggregation rules. For instance, abstract games satisfying (5) and (8) with

$X_i \subset \mathbb{R}$ and $\sigma_i(x_{-i}) = \min_{j\neq i} x_j$ for all $i \in N$ or $\sigma_i(x_{-i}) = -\min_{j\neq i} x_j$ for all $i \in N$ also admit ω-transitive Cournot potentials. Moreover, given a subset $I(i) \subseteq N \setminus \{i\}$ for each $i \in N$ such that $j \in I(i) \iff i \in I(j)$, the aggregation rules $\sigma_i(x_{-i}) = \min_{j\in I(i)} x_j$ or $\sigma_i(x_{-i}) = -\min_{j\in I(i)} x_j$ are also acceptable (Kukushkin, 2003, Theorems 7 and 8). It goes without saying that the minimum can be replaced with the maximum. Apparently, functions g and F_i satisfying (6) and (9) cannot exist in this case.

Aggregation rules mapping X_{-i} into chains "longer" than $\mathbb{R}$ (lexicographies) may also be nice although there are very few established facts so far. On the other hand, *partially* ordered aggregates seem hopeless. For instance, additive aggregation makes sense for $X_i \subset \mathbb{R}^m$ with $m > 1$ as well, in which case $S_i \subset \mathbb{R}^m$ too. Moreover, in the case of $\sigma_i(x_{-i}) = \sum_{j\neq i} x_j$, condition (8) implies that the best responses are increasing, and hence the existence of equilibria is looked after by the Tarski Theorem (provided the strategy sets remain lattices). Nonetheless, there may be Cournot cycles in such games. When $\sigma_i(x_{-i}) = -\sum_{j\neq i} x_j$, even the mere existence of an equilibrium is not guaranteed.

Example 1. Let us consider an abstract game where $N := \{1, 2, 3\}$, $X_1 := \{(0, 0, 0), (1, 0, 0)\}$, $X_2 := \{(0, 0, 0), (0, 1, 0)\}$, $X_3 := \{(0, 0, 0), (0, 0, 1)\}$, $\sigma_i(x_{-i}) := -\sum_{j\neq i} x_j$, and

$$R_1(s_1) := \begin{cases} \{(0,0,0)\} & \text{if } s_1 \leq (0,-1,0); \\ \{(1,0,0)\} & \text{otherwise;} \end{cases} \quad R_2(s_2) := \begin{cases} \{(0,0,0)\} & \text{if } s_2 \leq (0,0,-1); \\ \{(0,1,0)\} & \text{otherwise;} \end{cases}$$

$$R_3(s_3) := \begin{cases} \{(0,0,0)\} & \text{if } s_3 \leq (-1,0,0); \\ \{(0,0,1)\} & \text{otherwise.} \end{cases}$$

Condition (8) is obvious; nonetheless, there is no equilibrium.

6 Proof

For every $s_i \in S_i$, we define $\bar{R}_i(s_i) := \{x_i \in X_i \mid (s_i, x_i) \in \mathrm{cl}(\mathrm{graph}\, R_i)\}$; clearly, (8) holds for $\bar{R}_i$ as well. For each $i \in N$, we define $X_i^0 := \bigcup_{s_i\in S_i} \bar{R}_i(s_i)$. The compactness of S_i and upper hemicontinuity of $\bar{R}_i$ imply that X_i^0 is closed in X_i; hence it is compact too. For every $x_N \in X_N$, we set $N^0(x_N) := \{i \in N \mid x_i \in X_i^0\}$.

For each $i \in N$, we pick an arbitrary selection $r_i \colon S_i \to X_i$ from R_i; by (8), r_i is increasing in the sense of $s'_i \geq s_i \Rightarrow r_i(s'_i) \geq r_i(s_i)$. For every $s_i \in S_i$, we denote $S_i^{\to}(s_i) := \{s'_i \in S_i \mid s'_i > s_i\}$ and $S_i^{\leftarrow}(s_i) := \{s'_i \in S_i \mid s'_i < s_i\}$, and then $S_i^{\to} := \{s_i \in S_i \mid s_i = \inf S_i^{\to}(s_i)\}$ and $S_i^{\leftarrow} := \{s_i \in S_i \mid s_i = \sup S_i^{\leftarrow}(s_i)\}$. For $s_i \in S_i^{\to}$, we set $r_i^+(s_i) := \inf r_i(S_i^{\to}(s_i))$; for $s_i \in S_i^{\leftarrow}$, $r_i^-(s_i) := \sup r_i(S_i^{\leftarrow}(s_i))$. Assumption (8) immediately implies that

$$\forall s_i \in S_i^{\to} \cap S_i^{\leftarrow}\, [r_i^+(s_i) \geq r_i^-(s_i)];$$
$$\forall s_i \in S_i^{\to}\, \forall s'_i > s_i\, \forall x_i \in \bar{R}_i(s'_i)\, [x_i \geq r_i^+(s_i)];$$
$$\forall s_i \in S_i^{\leftarrow}\, \forall s'_i < s_i\, \forall x_i \in \bar{R}_i(s'_i)\, [x_i \leq r_i^-(s_i)];$$

in the following equality, if $r_i^+(s_i)$ and/or $r_i^-(s_i)$ are not defined, the corresponding term(s) should be ignored:

$$\forall s_i \in S_i\, [\bar{R}_i(s_i) = \mathrm{cl}(R_i(s_i)) \cup \{r_i^-(s_i)\} \cup \{r_i^+(s_i)\}];$$
$$\forall s_i \in S_i^{\rightarrow} \cap S_i^{\leftarrow}\, [r_i^+(s_i) = r_i^-(s_i) \Rightarrow \bar{R}_i(s_i) = R_i(s_i) = \{r_i(s_i)\}].$$

Step 6.1. *If $s_i \in S_i^{\leftarrow}$ and $r_i(s_i) = r_i^-(s_i)$, then r_i is left continuous at s_i.*

Proof. Let $\langle s_i^k \rangle_k$ be a strictly increasing sequence such that $s_i^k \to s_i$; then $s_i = \sup_k s_i^k$, and hence $r_i(s_i) = r_i^-(s_i) = \sup_k r_i(s_i^k)$. Since X_i is compact, we may assume $r_i(s_i^k) \to x_i^* \in X_i$; by Lemma 3.1, we have $x_i^* = \sup_k r_i(s_i^k) = r_i(s_i)$; hence $r_i(s_i^k) \to r_i(s_i)$. □

Step 6.2. *If $s_i \in S_i^{\rightarrow}$ and $r_i(s_i) = r_i^+(s_i)$, then r_i is right continuous at s_i.*

The proof is dual to that of Step 6.1.

Now we extend r_i to the whole $[s_i^-, s_i^+]$ with the following construction. For every $s_i \in [s_i^-, s_i^+]$ we define $\xi_i^+(s_i) = \min\{\xi_i \in S_i \mid \xi_i \geq s_i\}$ and $\xi_i^-(s_i) = \max\{\xi_i \in S_i \mid \xi_i \leq s_i\}$. Obviously, $\xi_i^+(s_i) = \xi_i^-(s_i) = s_i$ if and only if $s_i \in S_i$; otherwise, $\xi_i^-(s_i) < s_i < \xi_i^+(s_i)$. Now for every $s_i \in [s_i^-, s_i^+] \setminus S_i$ we define $r_i(s_i) = r_i(\xi_i^-(s_i))$ if $s_i - \xi_i^-(s_i) \leq \xi_i^+(s_i) - s_i$, and $r_i(s_i) = r_i(\xi_i^+(s_i))$ otherwise.

Step 6.3. *r_i is continuous at every $s_i \in [s_i^-, s_i^+]$ except for a countable subset.*

Proof. S_i being compact, its complement, $[s_i^-, s_i^+] \setminus S_i$, consists of a countable number of disjoint open intervals. $S_i \setminus (S_i^{\rightarrow} \cap S_i^{\leftarrow})$ consists of the end points of those same intervals. The way we extended r_i beyond S_i ensures continuity everywhere with the possible exception of the end points and the middle of each interval. By Steps 6.1 and 6.2, r_i is continuous wherever $r_i^-(s_i) = r_i^+(s_i)$.

Therefore, we only have to prove that the set $\{s_i \in S_i^{\rightarrow} \cap S_i^{\leftarrow} \mid r_i^+(s_i) > r_i^-(s_i)\}$ is countable. We pick a countable and dense subset of A and denote it Z. Given $s_i \in S_i^{\rightarrow} \cap S_i^{\leftarrow}$ such that $r_i^+(s_i) > r_i^-(s_i)$, we denote $U' := \mathrm{Int}\, A^{\leftarrow}(r_i^-(s_i))$ and $U'' := \mathrm{Int}\, A^{\leftarrow}(r_i^+(s_i))$. By (4b), we have $r_i^+(s_i) \notin \mathrm{cl}\, U'$; hence there is an open set $U \subset A$ such that $r_i^+(s_i) \in U$ and $U \cap \mathrm{cl}\, U' = \emptyset$. Since $r_i^+(s_i) \in \mathrm{cl}\, U''$, we have $U \cap U'' \neq \emptyset$. Now we pick $z(s_i) \in Z \cap U \cap U''$ arbitrarily; this is possible since Z is dense in A. Since $z(s_i) \notin \mathrm{cl}\, U'$, we have $r_i^-(s_i) \not\geq z(s_i)$; since $z(s_i) \in U''$, we have $r_i^+(s_i) > z(s_i)$. Finally, $z(s_i') \neq z(s_i)$ whenever $s_i' > s_i$, because $r_i^-(s_i') \geq r_i^+(s_i) > z(s_i)$ while $r_i^-(s_i') \not\geq z(s_i')$. □

For every $x_N \in X_N$, we define a function

$$H(x_N) := g(x_N) + \sum_{i \in N} \Big[-F_i(s_i^+, x_i) + \int_{s_i^-}^{s_i^+} \min\{D_i(s_i, x_i), D_i(s_i, r_i(s_i))\}\, ds_i \Big]. \tag{18}$$

In light of Step 6.3, the integral in (18) exists even in the Riemann sense. Let $i \in N$, $x_N \in X_N$, and $x_i \in \bar{R}_i(s_i^*)$ for $s_i^* \in S_i$. The key role in the Huang–Dubey–Haimanko–Zapechelnyuk–Jensen trick is played by this equality, easily following from (6) and the monotonicity (9) of $D_i(s_i, x_i)$:

$$H(x_N) = F_i(\sigma_i(x_{-i}), x_i) - F_i(s_i^*, x_i) + \int_{s_i^-}^{s_i^*} D_i(s_i, r_i(s_i))\, ds_i + C(x_{-i}). \tag{19}$$

It will be easily derived from (19) that $H(y_N) = H(x_N)$ whenever $x_{-i} = y_{-i}$ and $x_i \in \bar{R}_i(\sigma_i(x_{-i})) \ni y_i$. Not so easy, but also straightforward is the derivation from (19) that $H(y_N) > H(x_N)$ whenever $x_{-i} = y_{-i}$ and $x_i \notin \bar{R}_i(\sigma_i(x_{-i})) \ni y_i$. The most cumbersome part of the whole construction is the definition of the order in the sense of which the current strategy profile goes upwards when $x_i \in \bar{R}_i(\sigma_i(x_{-i})) \setminus R_i(\sigma_i(x_{-i}))$ is replaced with $y_i \in R_i(\sigma_i(x_{-i}))$. We generally follow Kukushkin (2005), but additional subtleties are needed because X_i's are no longer chains. This whole part becomes superfluous if every R_i is upper hemicontinuous, so $R_i = \bar{R}_i$.

For every $i \in N$, we define binary relations on X_i:

$$y_i \rhd\!\!\rhd_i x_i \rightleftharpoons \exists \bar{s}_i \in S_i \, [y_i \in R_i(\bar{s}_i) \ \& \ x_i \in \bar{R}_i(\bar{s}_i) \setminus R_i(\bar{s}_i)] \tag{20}$$

(in the following, we say "$y_i \rhd\!\!\rhd_i x_i$ holds at $\bar{s}_i$");

$$y_i \rhd\!\!\rhd_i^+ x_i \rightleftharpoons [y_i \rhd\!\!\rhd_i x_i \ \& \ y_i > x_i];$$
$$y_i \rhd\!\!\rhd_i^- x_i \rightleftharpoons [y_i \rhd\!\!\rhd_i x_i \ \& \ y_i < x_i].$$

An *i-singular upward chain* is a countable well ordered subset $\Delta \subseteq X_i$ such that (i) $y_i \rhd\!\!\rhd_i^+ x_i$ whenever $y_i \in \Delta$ and $x_i = \max \Delta^{\leftarrow}(y_i)$ [in which case $y_i = \min \Delta^{\rightarrow}(x_i)$], and (ii) $y_i = \sup \Delta^{\leftarrow}(y_i)$ whenever $y_i \in \Delta$ and $\max \Delta^{\leftarrow}(y_i)$ does not exist. We set $y_i \rightarrowtail_i^+ x_i$ iff $y_i > x_i$ and there is an i-singular upward chain $\Delta \subseteq X_i$ such that $y_i = \max \Delta$ and $x_i = \min \Delta$.

An *i-singular downward chain* is defined dually as a countable $*$-well ordered subset $\Delta \subseteq X_i$ such that (i) $y_i \rhd\!\!\rhd_i^- x_i$ whenever $y_i \in \Delta$ and $x_i = \min \Delta^{\rightarrow}(y_i)$ [in which case $y_i = \max \Delta^{\leftarrow}(x_i)$], and (ii) $y_i = \inf \Delta^{\rightarrow}(y_i)$ whenever $y_i \in \Delta$ and $\min \Delta^{\rightarrow}(y_i)$ does not exist. We set $y_i \rightarrowtail_i^- x_i$ iff $y_i < x_i$ and there is an i-singular downward chain $\Delta \subseteq X_i$ such that $y_i = \min \Delta$ and $x_i = \max \Delta$.

Then, we define

$$y_i \rightarrowtail_i x_i \rightleftharpoons [y_i \rhd\!\!\rhd_i x_i \ \text{ or } \ y_i \rightarrowtail_i^+ x_i \ \text{ or } \ y_i \rightarrowtail_i^- x_i]. \tag{21}$$

Remark. Obviously, $y_i \rightarrowtail_i^+ x_i$ if $y_i \rhd\!\!\rhd_i^+ x_i$, while $y_i \rightarrowtail_i^- x_i$ if $y_i \rhd\!\!\rhd_i^- x_i$. However, $y_i \rhd\!\!\rhd_i x_i$ may hold when y_i and x_i are incomparable in the order on X_i.

Now, we are ready to define our potential, a binary relation on X_N:

$$\begin{aligned} y_N \succ x_N \rightleftharpoons \big[N^0(y_N) \supset N^0(x_N) \ \text{ or } \ [N^0(y_N) = N^0(x_N) \ \& \ H(y_N) > H(x_N)] \ \text{ or } \\ \big(N^0(y_N) = N^0(x_N) \ \& \ H(y_N) = H(x_N) \ \& \\ \forall i \in N \, [y_i = x_i \ \text{ or } \ y_i \rightarrowtail_i x_i] \ \& \ \exists i \in N \, [y_i \rightarrowtail_i x_i]\big)\big]. \end{aligned} \tag{22}$$

Obviously, $\succ$ is irreflexive. Checking its ω-transitivity and (2) needs quite some effort.

Step 6.4. *Both relations $\rightarrowtail_i^+$ and $\rightarrowtail_i^-$ are ω-transitive.*

Proof. It is sufficient to consider one of the relations, say, $\rightarrowtail_i^+$. Let $z_i \rightarrowtail_i^+ y_i \rightarrowtail_i^+ x_i$. By definition, there are two i-singular upward chains, Δ' and Δ'', such that $\min \Delta' = x_i$, $\max \Delta' = y_i = \min \Delta''$, and $\max \Delta'' = z_i$. Defining $\Delta = \Delta' \cup \Delta''$, we see that Δ is an i-singular upward chain – when checking each condition in the definition, we will find ourselves either totally inside Δ' or totally inside Δ''. Since $x_i = \min\Delta$ and $z_i = \max\Delta$, we have $z_i \succ\!\!\succ_i^+ x_i$.

The proof of (3) is quite similar. Let $x_i^k \to x_i^\omega$ and $x_i^{k+1} \succ\!\!\succ_i^+ x_i^k$ for all k; let Δ^k $(k = 0, 1, \dots)$ be an i-singular upward chain such that $x_i^k = \min\Delta^k$ and $x_i^{k+1} = \max\Delta^k$. Denoting $\Delta = \{x_i^\omega\} \cup \bigcup_{k\in\mathbb{N}} \Delta^k$, we again obtain that Δ is an i-singular upward chain (the condition $x_i^\omega = \sup_{k\in\mathbb{N}} x_i^k$ is essential here), $x_i^0 = \min\Delta$ and $x_i^\omega = \max\Delta$. $\square$

Step 6.5. *If $z_i \rhd\!\!\rhd_i y_i \rhd\!\!\rhd_i x_i$, then either $z_i \rhd\!\!\rhd_i^+ y_i \rhd\!\!\rhd_i^+ x_i$ or $z_i \rhd\!\!\rhd_i^- y_i \rhd\!\!\rhd_i^- x_i$.*

Proof. Let $y_i \rhd\!\!\rhd_i x_i$ at $s_i' \in S_i$ and $z_i \rhd\!\!\rhd_i y_i$ at $s_i'' \in S_i$. Since $y_i \in R_i(s_i') \setminus R_i(s_i'')$, $s_i'' \neq s_i'$.

Suppose that $s_i'' > s_i'$. If $y_i < x_i$ or y_i and x_i are incomparable in the order on X_i, then $y_i' > y_i$ for every $y_i' \in \bar{R}_i(s_i'')$; hence $y_i \in \bar{R}_i(s_i'')$ is impossible, and hence $z_i \rhd\!\!\rhd_i y_i$ at s_i'' is impossible too. If $y_i > x_i$, then $y_i \rhd\!\!\rhd_i^+ x_i$. Since $y_i' \geq y_i$ for every $y_i' \in R_i(s_i'')$, $z_i \rhd\!\!\rhd_i^+ y_i$ too.

The case of $s_i'' < s_i'$ is treated dually. $\square$

Step 6.6. *If $z_i \succ\!\!\succ_i y_i \succ\!\!\succ_i x_i$, then either $z_i \succ\!\!\succ_i^+ y_i \succ\!\!\succ_i^+ x_i$ or $z_i \succ\!\!\succ_i^- y_i \succ\!\!\succ_i^- x_i$.*

Proof. First, if we suppose that $z_i \rhd\!\!\rhd_i y_i$ does not hold, then either $z_i \succ\!\!\succ_i^+ y_i$ or $z_i \succ\!\!\succ_i^- y_i$. In the first case, there is an i-singular upward chain Δ such that $z_i = \max\Delta$ and $y_i = \min\Delta$. Setting $z_i' := \min(\Delta \setminus \{y_i\})$, we obtain $z_i' \rhd\!\!\rhd_i^+ y_i$. Dually, if $z_i \succ\!\!\succ_i^- y_i$, we pick an i-singular downward chain Δ such that $z_i = \min\Delta$ and $y_i = \max\Delta$. Setting $z_i' := \max(\Delta \setminus \{y_i\})$, we obtain $z_i' \rhd\!\!\rhd_i^- y_i$. Obviously, if $z_i \rhd\!\!\rhd_i y_i$ does hold, then we may set $z_i' := z_i$ and have $z_i' \rhd\!\!\rhd_i y_i$ again. Let $z_i' \rhd\!\!\rhd_i y_i$ at $\bar{s}_i \in S_i$.

Turning to the second relation, we see that Step 6.5 applies if $y_i \rhd\!\!\rhd_i x_i$; hence either $z_i' \rhd\!\!\rhd_i^+ y_i \rhd\!\!\rhd_i^+ x_i$ or $z_i' \rhd\!\!\rhd_i^- y_i \rhd\!\!\rhd_i^- x_i$; hence either $z_i \succ\!\!\succ_i^+ y_i \succ\!\!\succ_i^+ x_i$ or $z_i \succ\!\!\succ_i^- y_i \succ\!\!\succ_i^- x_i$ indeed.

Let $y_i \succ\!\!\succ_i^+ x_i$ while $y_i \not\rhd\!\!\rhd_i x_i$. Then there is an i-singular upward chain Δ such that $x_i = \min\Delta$ and $y_i = \sup(\Delta \setminus \{y_i\})$. By definition, for every $x_i' \in \Delta \setminus \{y_i\}$, there is $y_i' \in \Delta \setminus \{y_i\}$ such that $y_i' \rhd\!\!\rhd_i^+ x_i'$; hence $y_i', x_i' \in \bar{R}(s_i)$ for some $s_i \in S_i$. Since $y_i', x_i' < y_i$, we must have $s_i < \bar{s}_i$; hence $y_i', x_i' \leq z_i'$. Clearly, $y_i = \sup(\Delta \setminus \{y_i\})$ is only possible if $z_i' \geq y_i$; hence $z_i' \rhd\!\!\rhd_i^+ y_i$, and hence $z_i \succ\!\!\succ_i^+ y_i \succ\!\!\succ_i^+ x_i$.

The case of $y_i \succ\!\!\succ_i^- x_i$ is treated dually. $\square$

Step 6.7. *The relation $\succ\!\!\succ_i$ is ω-transitive.*

Proof. The statement immediately follows from Steps 6.6 and 6.4. $\square$

Step 6.8. *The relation $\succ$ is irreflexive and ω-transitive.*

Proof. The irreflexivity of $\succ$ is obvious; checking transitivity is very easy. Let us check (3). The situation $N^0(x_N^{k+1}) \supset N^0(x_N^k)$ can only happen for a finite number of k; without restricting generality, $N^0(x_N^{k+1}) = N^0(x_N^k)$ for all k, and hence $N^0(x_N^\omega) \supseteq N^0(x_N^0)$ since each X_i^0 is closed. If $H(x_N^{k+1}) > H(x_N^k)$ for a single k, then $H(x_N^\omega) > H(x_N^0)$ since H is continuous, and we are home. Finally, let $N^0(x_N^{k+1}) = N_N^0(x^k)$ and $H(x_N^{k+1}) = H(x_N^k)$ for all k. Then, for each $i \in N$, either $x_i^{k+1} \succ\!\!\succ_i x_i^k$ for some k, or $x_i^{k+1} = x_i^k$ for all k. In the first case, Step 6.7 applies, producing $x_i^\omega \succ\!\!\succ_i x_i^0$; in the second, $x_i^\omega = x_i^0$. In either case, we have $x_N^\omega \succ x_N^0$. $\square$

Step 6.9. *If $y_N \rhd x_N$, then $y_N \succ x_N$.*

Proof. Let $y_N \rhd_i x_N$ and $\bar{s}_i := \sigma_i(x_{-i})$. We have $y_i \in R_i(\bar{s}_i)$ by definition; hence $y_i \in X_i^0$ and $N^0(y_N) \supseteq N^0(x_N)$. If the inclusion is strict, we are home.

Let us assume $N^0(y_N) = N^0(x_N)$, i.e., $x_i \in \bar{R}_i(s_i^*)$ for $s_i^* \in S_i$. Invoking (19) separately for x_N and y_N, we obtain

$$H(y_N) - H(x_N) = \int_{s_i^-}^{\bar{s}_i} D_i(s_i, r_i(s_i))\, ds_i - F_i(\bar{s}_i, x_i) + F_i(s_i^*, x_i) - \int_{s_i^-}^{s_i^*} D_i(s_i, r_i(s_i))\, ds_i = \int_{\bar{s}_i}^{s_i^*} [D_i(s_i, x_i) - D_i(s_i, r_i(s_i))]\, ds_i. \quad (23)$$

If $s_i^* = \bar{s}_i$, i.e., $x_i \in \bar{R}_i(\bar{s}_i)$, then $H(y_N) = H(x_N)$. Since $x_i \in \bar{R}_i(\bar{s}_i) \setminus R_i(\bar{s}_i)$ and $y_i \in R_i(\bar{s}_i)$, we have $y_i \rhd\!\!\rhd_i x_i$ by (20) and hence $y_i \succ\!\!\succ_i x_i$ by (21). Therefore, $y_N \succ x_N$ by the third term in (22).

Finally, let $x_i \notin \bar{R}_i(\bar{s}_i)$; hence $s_i^* \neq \bar{s}_i$. If $s_i^* > \bar{s}_i$, then the integrand in (23) is nonnegative on the whole interval because $x_i \geq r_i(s_i)$; since $x_i \notin \bar{R}_i(\bar{s}_i)$ and the graph of $\bar{R}_i$ is closed, $x_i > r_i(s_i)$ in an open neighborhood of $\bar{s}_i$ and hence the integrand is strictly positive by (9). If $s_i^* < \bar{s}_i$, then the integrand is nonpositive on the whole interval and strictly negative in an open neighborhood of $\bar{s}_i$, but $ds_i < 0$ (the lower limit is greater than the upper one). In either case, $H(y_N) > H(x_N)$, hence $y_N \succ x_N$ by the second term in (22). □

Thus, $\succ$ is an ω-transitive Cournot potential and Theorem 1 is proved.

Example 2. Let us consider a game where $N := \{1, 2\}$, $X_i := [0, 2]$, and the utility functions are "isomorphic":

$$u_i(x_N) := \begin{cases} \min\{4x_i + 4x_{-i}, 7x_{-i} - 2x_i + 6\}, & \forall j \in N\, [x_j \geq 1]; \\ \min\{4x_i + 4x_{-i}, 7x_{-i} - 2x_i + 3\}, & \text{otherwise.} \end{cases}$$

Obviously, both u_i are upper semicontinuous, but not continuous. The unique best responses are easy to compute:

$$\mathcal{R}_i(x_{-i}) = \begin{cases} \{x_{-i}/2 + 1\}, & x_{-i} \geq 1; \\ \{x_{-i}/2 + 1/2\}, & x_{-i} < 1. \end{cases}$$

There is a unique Nash equilibrium, $(2, 2)$.

The monotonicity condition (8) being obvious, this game belongs to the class covered by Theorem 1 with $\sigma_i(x_{-i}) := x_{-i}$. The function $H(x_N)$ defined by (18) looks as follows:

$$H(x_N) = x_1 x_2 + \psi(x_1) + \psi(x_2),$$

where

$$\psi(x) := \begin{cases} 2x - x^2 - 3/2, & 3/2 \le x \le 2; \\ 3/4 - x, & 1 \le x \le 3/2; \\ x - x^2 - 1/4, & 1/2 \le x \le 1; \\ 0, & x \le 1/2. \end{cases}$$

Every Cournot path started from x_N^0 with $x_i^0 < 1$ for both i converges to $x_N^\omega = (1,1)$, which is not an equilibrium; $H(x_N^k)$ strictly increases at each step. When, say, player 1 makes a best response improvement by replacing $x_1^\omega = 1$ with $x_1^{\omega+1} = 3/2$, H remains the same, but $3/2 \not\succ_1^+ 1$; hence the strategy profile goes upwards in the sense of $\succ$ defined by (22). After that, a unique Cournot path converges to $(2,2)$, which *is* an equilibrium; $H(x_N^{\omega+k})$ again strictly increases at each step $k > 1$.

This example demonstrates why Jensen's construction (18) alone is insufficient when the best responses need not be upper hemiconinuous.

Actually, this game admits a simpler Cournot potential, represented by an upper semicontinuous real-valued function

$$P(x_N) := \min_i u_i(x_N).$$

It is easy to check that $P(y_N) > P(x_N)$ whenever $y_N \rhd x_N$. On the other hand, this simpler potential hinges on specifics of this particular example, whereas (18) and (22) work for every game from the class.

Remark. Kukushkin (2005) contains a more elaborate example where the function H remains constant along a "double infinite" Cournot path (i.e., up to the second limit). Similarly, an example can be produced where all assumptions of Theorem 1 are satisfied, while an arbitrary (countable) transfinite number of best response improvements may be needed to reach an equilibrium, with H being constant all the way. Thus, the behavior of Cournot dynamics when the best responses are not upper hemiconinuous may be much more complicated than in Theorem 2 of Jensen (2010).

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