Cournot tatonnement and dominance solvability in finite games

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Abstract

If a finite strategic game is strictly dominance solvable, then every simultaneous best response adjustment path, as well as every non-discriminatory individual best response improvement path, ends at a Nash equilibrium after a finite number of steps. If a game is weakly dominance solvable, then every strategy profile can be connected to a Nash equilibrium with a simultaneous best response path and with an individual best response path (if there are more than two players, switches from one best response to another may be needed). Both statements remain valid if dominance solvability in the usual sense is replaced with "BR-dominance solvability," where a strategy can be eliminated if it is not among the best responses to anything, or if it is not indispensable for providing the best responses to all contingencies. For a two person game, some implications in the opposite direction are obtained.

Key words: Dominance solvability; Best response dynamics; Potential game

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1 Introduction

Moulin (1984) demonstrated connections between dominance solvability and nice behavior of best response dynamics, although he worked in a rather narrow context. Here we strive to produce a complete picture of "what depends on what." For technical convenience, we only consider finite games, where we can essentially restrict ourselves to finite improvement (or adjustment) paths; in a continuous game, this would be insufficient. Similarly, iterative elimination of dominated strategies in an infinite game raises quite a few complicated questions (Dufwenberg and Stegeman, 2002); in particular, very much depends on topological assumptions.

An apparently new notion of BR-dominance solvability is introduced; to be more precise, two versions of the notion. We assume that a strategy can be eliminated if it is not among the best responses to any profile of strategies of the partners/rivals, or if it is not indispensable for providing the best responses to all contingencies. This novelty allows us to formulate the weakest conditions for nice behavior of both sequential and simultaneous tâtonnement processes based on dominance solvability; in particular, *weak* dominance solvability has the same implications as the strict one if all best responses are unique. It also makes possible implications in the opposite direction and even equivalence results. One result of the type was obtained by Moulin (1984, Corollary of Lemmas 1 and 2), but, again, in a very special case.

Our basic model is a *strategic game with ordinal preferences.* It is defined by a finite set of players N, and strategy sets X_i and preference relations on $X_N = \prod_{i \in N} X_i$ for all $i \in N$. We always assume that each X_i is finite and preferences are described with ordinal utility functions $u_i \colon X_N \to \mathbb{R}$. For each $i \in N$, we denote $X_{-i} = \prod_{j \in N \setminus \{i\}} X_j$ and

$$R_i(x_{-i}) = \operatorname*{Argmax}_{x_i \in X_i} u_i(x_i, x_{-i})$$

for every $x_{-i} \in X_{-i}$ (the *best response correspondence*); if $\#N = 2$, then $-i$ denotes the partner/rival of player i.

2 Improvement paths

Given a strategic game Γ, we introduce the *individual improvement* relation $\rhd^{\mathrm{Ind}}$ and *best response improvement* relation $\rhd^{\mathrm{BR}}$ on X_N ($i \in N$, $y_N, x_N \in X_N$):

$$\begin{aligned}
y_N \rhd_i^{\mathrm{Ind}} x_N &\rightleftharpoons [y_{-i} = x_{-i} \ \&\ u_i(y_N) > u_i(x_N)],\\
y_N \rhd^{\mathrm{Ind}} x_N &\rightleftharpoons \exists i \in N\, [y_N \rhd_i^{\mathrm{Ind}} x_N];\\
y_N \rhd_i^{\mathrm{BR}} x_N &\rightleftharpoons [y_{-i} = x_{-i} \ \&\ x_i \notin R_i(x_{-i}) \ni y_i],\\
y_N \rhd^{\mathrm{BR}} x_N &\rightleftharpoons \exists i \in N\, [y_N \rhd_i^{\mathrm{BR}} x_N].
\end{aligned}$$

By definition, a strategy profile $x_N \in X_N$ is a Nash equilibrium if and only if x_N is a *maximizer* of $\rhd^{\mathrm{Ind}}$, i.e., if $y_N \rhd^{\mathrm{Ind}} x_N$ is impossible for any $y_N \in X_N$. In a finite game, $x_N \in X_N$ is a Nash equilibrium if and only if x_N is a maximizer of $\rhd^{\mathrm{BR}}$.

A (*best response*) *improvement path* is a finite or infinite sequence $\langle x_N^k \rangle_{k\in\mathbb{N}}$ such that $x_N^{k+1} \rhd^{\mathrm{Ind}} x_N^k$ ($x_N^{k+1} \rhd^{\mathrm{BR}} x_N^k$) whenever $k \geq 0$ and x_N^{k+1} is defined.

As in Kukushkin et al. (2005), we combine the terminology of Monderer and Shapley (1996), Milchtaich (1996), and Friedman and Mezzetti (2001). A game has the *finite improvement property* (*FIP*) if it admits no infinite improvement path. A game has the *finite best response improvement property* (*FBRP*) if it admits no infinite best response improvement path. The FIP (FBRP) implies that every (best response) improvement path reaches a Nash equilibrium in a finite number of steps. A game has the *weak FIP* (*weak FBRP*) if, for every $x_N \in X_N$, there exists a finite (best response) improvement path $\langle x_N^0, \ldots, x_N^m \rangle$ such that $x_N^0 = x_N$ and x_N^m is a Nash equilibrium. Clearly, FIP $\Rightarrow$ FBRP $\Rightarrow$ weak FBRP $\Rightarrow$ weak FIP.

A *Cournot potential* is an irreflexive and transitive binary relation $\succ$ on X_N such that $y_N \succ x_N$ whenever $y_N \rhd^{\mathrm{BR}} x_N$; a *weak Cournot potential* is an irreflexive and transitive binary relation $\succ$ on X_N such that, whenever x_N is not a Nash equilibrium, there is $y_N \in X_N$ such that $y_N \rhd^{\mathrm{BR}} x_N$ and $y_N \succ x_N$. By Propositions 6.1 and 6.2 from Kukushkin (2004), a finite game has the (weak) FBRP if and only if it admits a (weak) Cournot potential. Henceforth, best response improvement paths will be called just *Cournot paths*; clearly, the FBRP is equivalent to the absence of *Cournot cycles*, i.e., Cournot paths $\langle x_N^0, x_N^1, \ldots, x_N^m \rangle$ such that $m > 0$ and $x_N^0 = x_N^m$.

A property intermediate between the FBRP and weak FBRP deserves attention. We call an infinite Cournot path $\langle x_N^k \rangle_{k\in\mathbb{N}}$ *inclusive* if for each player $i \in N$ and each $m \in \mathbb{N}$, there is a $k \geq m$ such that $x_i^k \in R_i(x_{-i}^k)$. A game has the *finite inclusive best response improvement property* (*FIBRP*) if it admits no infinite inclusive Cournot path. It is immediately clear that the FIBRP implies, in particular, the convergence of the sequential tatonnement process as defined by Moulin (1984, p. 87) in a finite number of steps. A Cournot cycle $\langle x_N^0, x_N^1, \ldots, x_N^m = x_N^0 \rangle$ is *complete* if for each player $i \in N$ there is $k < m$ such that $x_i^k \in R_i(x_{-i}^k)$.

A *preorder* is a reflexive and transitive binary relation; with every preorder $\succeq$, its asymmetric component $\succ$ and an equivalence relation $\sim$ are naturally associated. A *Cournot quasipotential* is a preorder $\succeq$ on X_N such that for every $x_N \in X_N$ there exists a subset $M(x_N) \subseteq N$ satisfying

$$y_N \rhd^{\mathrm{BR}} x_N \Rightarrow \big[y_N \succ x_N \ \text{ or } \ [y_N \sim x_N \ \&\ M(y_N) \subseteq M(x_N) \neq \emptyset] \big]; \tag{1a}$$

$$i \in M(x_N) \Rightarrow x_i \notin R_i(x_{-i}). \tag{1b}$$

If $\succ$ is a Cournot potential, then its reflexive closure $\succeq$ is a Cournot quasipotential with $M(x_N) = \emptyset$ for all $x_N \in X_N$. If $\succeq$ is a Cournot quasipotential, then we may extend its asymmetric component in this way:

$$y_N \gtrsim x_N \rightleftharpoons \big[y_N \succ x_N \text{ or } [y_N \sim x_N \,\&\, M(y_N) \subset M(x_N)]\big].$$

Clearly, $y_N \gtrsim x_N$ whenever $y_N \rhd_i^{\mathrm{BR}} x_N$ and $i \in M(x_N)$; therefore, $\gtrsim$ is a weak Cournot potential.

Proposition 2.1. *For every finite strategic game* Γ*, the following statements are equivalent:*

1. Γ *has the FIBRP;*
2. Γ *admits no complete Cournot cycle;*
3. Γ *admits a Cournot quasipotential.*

Proof. Infinite repetition of a complete Cournot cycle generates an infinite inclusive Cournot path, hence Statement 1 implies Statement 2.

Let Statement 2 hold. To verify Statement 3, we denote $\succeq$ the reflexive and transitive closure of $\rhd^{\mathrm{BR}}$: $y_N \succeq x_N$ if and only if there is a finite Cournot path $\langle x_N^0, x_N^1, \dots, x_N^m\rangle$ such that $x_N^0 = x_N$ and $x_N^m = y_N$ ($m \geq 0$). Let $Y \subseteq X_N$ be an equivalence class of its symmetric component $\sim$, and let $\#Y > 1$; we denote $D(Y) = \{i \in N \mid \forall x_N \in Y\, [x_i \notin R_i(x_{-i})]\}$. Since all $x_N \in Y$ can be arranged into a single Cournot cycle and that cycle cannot be complete, $D(Y) \neq \emptyset$. Now we define $M(x_N) = D(Y)$ if x_N belongs to a non-singleton equivalence class Y, and $M(x_N) = \emptyset$ otherwise. The conditions (1) are checked easily.

Finally, let $\succeq$ be a Cournot quasipotential and $\langle x_N^k\rangle_{k\in\mathbb{N}}$ be an infinite Cournot path; we have to show that the path is not inclusive. Since X_N is finite, at least one strategy profile $\bar{x}_N$ must enter into the path an infinite number of times. Let $x_N^m = \bar{x}_N$ for the first time; clearly, we must have $x_N^{k+1} \sim x_N^k$ for all $k \geq m$. By (1a), $M(x_N^{k+1}) = M(x_N^k) = M^0 \neq \emptyset$ for all $k \geq m$. By (1b), we have $x_i^k \notin R_i(x_{-i}^k)$ for all $i \in M^0$ and $k \geq m$. □

Remark. In the proof of Theorem 3 of Kukushkin (2004), the FBRP was derived from the presence of a "quasipotential" in an even weaker sense than (1). The point is that whenever a game satisfies the conditions of that theorem, so do all its reduced games. Generally, we only obtain FIBRP. In particular, dominance solvability (in any sense) need not be inherited by the reduced games, hence Theorem 4.3 below also asserts only FIBRP.

We introduce the *simultaneous best response adjustment* relation $\rhd^{\mathrm{sBR}}$ on X_N ($y_N, x_N \in X_N$):

$$y_N \rhd^{\mathrm{sBR}} x_N \rightleftharpoons \big[\forall i \in N\, [y_i = x_i \in R_i(x_{-i}) \text{ or } x_i \notin R_i(x_{-i}) \ni y_i] \,\&\, y_N \neq x_N\big].$$

In a finite game, $x_N \in X_N$ is a Nash equilibrium if and only if x_N is a maximizer of $\rhd^{\mathrm{sBR}}$. A *simultaneous Cournot path* is a finite or infinite sequence $\langle x_N^k\rangle_{k=0,1,\dots}$ such that $x_N^{k+1} \rhd^{\mathrm{sBR}} x_N^k$ whenever $k \geq 0$ and x_N^{k+1} is defined.

Remark. We do not use the term "improvement" here because $y_N \rhd^{\mathrm{sBR}} x_N$ is compatible with $u_i(y_N) < u_i(x_N)$ for all $i \in N$.

A game has the *finite simultaneous best response adjustment property* (*FSBRP*) if there exists no infinite simultaneous Cournot path. The FSBRP implies that every simultaneous Cournot path eventually leads to a Nash equilibrium. A game has the *weak FSBRP* if, for every $x_N \in X_N$, there exists a finite simultaneous Cournot path $\langle x_N^0, \dots, x_N^m \rangle$ such that $x_N^0 = x_N$ and x_N^m is a Nash equilibrium.

A *simultaneous Cournot potential* is an irreflexive and transitive binary relation $\succ$ on X_N such that $y_N \succ x_N$ whenever $y_N \rhd^{\mathrm{sBR}} x_N$; a *weak simultaneous Cournot potential* is an irreflexive and transitive binary relation $\succ$ on X_N such that, whenever x_N is not a Nash equilibrium, there is $y_N \in X_N$ such that $y_N \rhd^{\mathrm{sBR}} x_N$ and $y_N \succ x_N$. By Propositions 6.1 and 6.2 from Kukushkin (2004), a finite game has the (weak) FSBRP if and only if it admits a (weak) simultaneous Cournot potential.

Generally, there seems to be no relation between the convergence of Cournot paths and simultaneous Cournot paths (see Moulin, 1986). An exception is the case of two players, see Section 5 below.

A *pseudo-Cournot path* is a finite or infinite sequence $\langle x_N^k \rangle_{k=0,1,\dots}$ such that, whenever x_N^{k+1} is defined, there is $i \in N$ for which $x_{-i}^{k+1} = x_{-i}^k$, $x_i^{k+1} \neq x_i^k$, and $x_i^{k+1} \in R_i(x_{-i}^k)$. A *simultaneous pseudo-Cournot path* is a finite or infinite sequence $\langle x_N^k \rangle_{k=0,1,\dots}$ such that $x_N^{k+1} \neq x_N^k$ and $x_i^{k+1} \in R_i(x_{-i}^k)$ for all $i \in N$ whenever x_N^{k+1} is defined. A game has the *pseudo-FBRP* (*pseudo-FSBRP*) if, for every $x_N \in X_N$, there exists a finite (simultaneous) pseudo-Cournot path $\langle x_N^0, \dots, x_N^m \rangle$ such that $x_N^0 = x_N$ and x_N^m is a Nash equilibrium.

3 Elimination of dominated strategies

The term "dominance solvability" is due to Moulin (1979) although the origins of the notion itself can be traced back to Luce and Raiffa (1957). The elimination of strictly dominated strategies does not change, say, the set of Nash equilibria. The elimination of weakly dominated strategies is not at all innocuous (Samuelson, 1992), but, nonetheless, is often regarded as legitimate.

Let Γ be a strategic game, $i \in N$, and $x_i, y_i \in X_i$. We say that y_i *strictly dominates* x_i if for every $x_{-i} \in X_{-i}$, there holds $u_i(y_i, x_{-i}) > u_i(x_i, x_{-i})$. We say that y_i *weakly dominates* x_i if $u_i(y_i, x_{-i}) \geq u_i(x_i, x_{-i})$ for every $x_{-i} \in X_{-i}$, while $u_i(y_i, x_{-i}) > u_i(x_i, x_{-i})$ for some $x_{-i} \in X_{-i}$.

Given a strategic game Γ, an *elimination scheme* of *length* $m \geq 0$ is a collection of sequences X_i^k for $i \in N$ and $k = 0, 1, \dots, m$ such that $X_i^0 = X_i$ and $\emptyset \neq X_i^{k+1} \subseteq X_i^k$ for each i and k. Naturally, a sequence of *subgames* Γ^k of Γ is associated with such a scheme: the set of players remains the same; the strategy sets are X_i^k; the preferences are defined by the restrictions of the same utility functions to $X_N^k = \prod_{i \in N} X_i^k$. An elimination scheme of length $m \geq 0$ is *perfect* if every $x_N \in X_N^m$ is a Nash equilibrium in Γ^m.

A game Γ is *strictly/weakly dominance solvable* if it admits a perfect elimination scheme such that, for each $k \in \{0, \dots, m-1\}$, every deleted strategy $x_i \in X_i^k \setminus X_i^{k+1}$ is strictly/ weakly dominated in Γ^k.

Remark. A more usual requirement is that each player should become indifferent between all *outcomes* when the elimination process is completed; our perfect schemes do not ensure that. However, our weaker condition is sufficient for all "nice" conclusions.

Given $X_i' \subseteq X_i$, we denote $R_i^{-1}(X_i') = \{x_{-i} \in X_{-i} \mid R_i(x_{-i}) \cap X_i' \neq \emptyset\}$. When $X_i' = \{x_i\}$, we write $R_i^{-1}(x_i)$ rather than $R_i^{-1}(\{x_i\})$. A strategy $x_i \in X_i$ is *strongly BR-dominated* if $R_i^{-1}(x_i) = \emptyset$. It is immediately clear that a strictly dominated strategy is strongly BR-dominated. A subset $X_i' \subseteq X_i$ is *BR-sufficient* if $R_i^{-1}(X_i') = X_{-i}$.

An S-scheme is an elimination scheme of length m such that every $x_i \in X_i^k \setminus X_i^{k+1}$ is strongly BR-dominated in Γ^k ($k \in \{0, \dots, m-1\}$). A W-scheme is an elimination scheme of length m such that every X_i^{k+1} is BR-sufficient in Γ^k ($k \in \{0, \dots, m-1\}$). We call Γ *strongly/weakly BR-dominance solvable* (*SBRDS/WBRDS*) if it admits a perfect S-scheme/W-scheme.

The idea of iterative elimination of strongly BR-dominated strategies was implicit in Lemma 2 of Moulin (1984), where it was shown to lead to the same result as the elimination of strictly dominated strategies (under rather strong assumptions, naturally). Generally, it can be viewed as an ordinal analogue of the rationalizability concept (Bernheim, 1984; Pearce, 1984). Admittedly, there is a serious difference between the two situations: If a pure strategy is not a best response to any probability distribution on the strategies of the other players, then it is dominated by a mixed strategy, hence the latter provides a justification for the elimination of the former. When only pure strategies are allowed, the fact that a strategy is not a best response to any profile of strategies of the partners does not make it inferior to any other strategy.

An ordinal version of rationalizability was developed by Borges (1993), but its departure from conventional notions of dominance was less radical than here. Actually, the question of which strategies are not needed by a player can only be resolved with a particular scenario (or a list of scenarios) in view; e.g., the Stackelberg solution of a two person game may well include the choice of a strictly dominated strategy by the leader. And it is easy to see that the elimination of strongly BR-dominated strategies does not change the set of Nash equilibria.

Since BR-dominance solvability seems to have never been studied in the literature, we provide exact formulations and proofs of familiar properties in the new context. Three implications are obvious: a strictly (weakly) dominance solvable game is strongly (weakly) BR-dominance solvable with the same elimination scheme; an SBRDS game is WBRDS.

Proposition 3.1. *If x_N^0 is a Nash equilibrium in Γ, then $x_N^0 \in X_N^k$ for every S-scheme of length $m \geq k$.*

Proof. Supposing the contrary, let k be the first step when $x_N^0 \notin X_N^k$; then $x_i^0 \in X_i^{k-1} \setminus X_i^k$ for some $i \in N$, hence x_i^0 is strongly BR-dominated in Γ^{k-1}. On the other hand, $x_i^0 \in R_i(x_{-i}^0)$ in Γ and $x_{-i}^0 \in X_{-i}^{k-1}$: a contradiction. □

Lemma 3.2. *Given a W-scheme of length m, each best response correspondence R_i^k ($i \in N$ and $k \in \{0, \ldots, m\}$) in Γ^k satisfies $R_i^k(x_{-i}) = R_i(x_{-i}) \cap X_i^k$ $[\neq \emptyset]$ for every $x_{-i} \in X_{-i}^k$.*

Proof. Straightforward induction based on the definition of a W-scheme shows $R_i(x_{-i}) \cap X_i^k \neq \emptyset$; the rest is obvious. □

Proposition 3.3. *If Γ is WBRDS and $x_N \in X_N^m$, then x_N is a Nash equilibrium in Γ.*

Proof. For each $i \in N$, we apply Lemma 3.2 to $x_{-i} \in X_{-i}^m$ and pick $y_i \in R_i(x_{-i}) \cap X_i^m$. By the definition of a perfect scheme, $u_i(y_i, x_{-i}) = u_i(x_N)$, hence $x_i \in R_i(x_{-i})$ as well. □

Propositions 3.1 and 3.3 immediately imply that the set of Nash equilibria in a strongly BR-dominance solvable game is rectangular, and all perfect S-schemes eliminate the strategies not participating in the equilibria. As to perfect W-schemes, every such scheme can be extended until X_N^m becomes a singleton; however, which Nash equilibrium of the original game will be selected may depend on the particular elimination scheme. This dependence remains possible in the case of weak dominance solvability, but can be ruled out under reasonable assumptions (Gilboa et al., 1990; Marx and Swinkels, 1997).

4 Implications of BR-dominance solvability

Given an elimination scheme of length m, we define $\mu_i \colon X_i \to \{0, \ldots, m\}$ by

$$\mu_i(x_i) = \max \{k \in \{0, \ldots, m\} \mid x_i \in X_i^k\}. \tag{2}$$

Then we define $\mu^- \colon X_N \to \{0, \ldots, m\}$ by

$$\mu^-(x_N) = \min_{i \in N} \mu_i(x_i). \tag{3}$$

Lemma 4.1. *Let there be an S-scheme of length m and $x_N \in X_N$ such that $\mu^-(x_N) < m$; then for every $i \in N$ and $y_i \in R_i(x_{-i})$, there holds $\mu_i(y_i) > \mu^-(x_N)$.*

Proof. We have $x_{-i} \in X_{-i}^{\mu^-(x_N)}$, hence $y_i \in X_i^{\mu^-(x_N)+1}$ since $y_i \in R_i(x_{-i})$; therefore, $\mu_i(y_i) \geq \mu^-(x_N) + 1$. □

Theorem 4.2. *If a finite game Γ is SBRDS, then it has the FSBRP.*

Proof. Fixing a perfect S-scheme, we consider the functions μ and μ^- defined by (2) and (3). Let us show that the strict ordering represented by μ^-, i.e.,

$$y_N \succ x_N \rightleftharpoons \mu^-(y_N) > \mu^-(x_N),$$

is a simultaneous Cournot potential. Let $y_N \rhd^{\text{sBR}} x_N$; then $\mu^-(x_N) < m$. By Lemma 4.1, $\mu_i(y_i) > \mu^-(x_N)$ for every $i \in N$, hence $\mu^-(y_N) > \mu^-(x_N)$ as well. $\square$

Theorem 4.3. *If a finite game Γ is SBRDS, then it has the FIBRP.*

Proof. Fixing a perfect S-scheme, we again consider the functions μ and μ^- defined by (2) and (3). Let us show that the total preorder represented by μ^-, i.e.,

$$y_N \succeq x_N \rightleftharpoons \mu^-(y_N) \geq \mu^-(x_N),$$

is a Cournot quasipotential with $M(x_N) = \text{Argmin}_{i \in N}\, \mu_i(x_i)$ when $\mu^-(x_N) < m$ and $M(x_N) = \emptyset$ otherwise. If $\mu^-(x_N) = m$, then $x_N \in X_N^m$, hence x_N is a Nash equilibrium in Γ by Proposition 3.3.

Let $y_N \rhd_i^{\text{BR}} x_N$; then $\mu^-(x_N) < m$, hence Lemma 4.1 is applicable. If $i \notin M(x_N)$, then $\mu^-(y_N) = \mu^-(x_N)$ and $M(y_N) = M(x_N)$. Let $i \in M(x_N)$; then $\mu_i(y_i) > \mu^-(x_N)$, hence either $\mu^-(y_N) > \mu^-(x_N)$ or $\mu^-(y_N) = \mu^-(x_N)$ and $M(y_N) = M(x_N) \setminus \{i\}$. We see that condition (1a) holds. Finally, if $i \in M(x_N)$, then $\mu_i(x_i) = \mu^-(x_N) < m$; if $x_i \in R_i(x_{-i})$, then Lemma 4.1 would imply $\mu_i(x_i) > \mu_i(x_i)$. Thus, (1b) holds as well. $\square$

The FIBRP in the formulation of Theorem 4.3 cannot be replaced with the FBRP if there are more than two players: if one player has a strictly dominant strategy x_i^+, then any behavior of improvement paths with $x_i^k \neq x_i^+$ is compatible with strict dominance solvability. When $\#N = 2$, Theorem 5.2 below asserts the FBRP.

Lemma 4.4. *Let there be a W-scheme of length m and $x_N \in X_N$ such that $\mu^-(x_N) < m$; then for each $i \in N$ there is $y_i \in R_i(x_{-i})$ such that $\mu_i(y_i) > \mu^-(x_N)$.*

Proof. For each $i \in N$, we pick y_i maximizing μ_i over $R_i(x_{-i})$. The definition of a W-scheme implies that $\mu_i(y_i) \geq \mu^-(x_N) + 1$ because $x_{-i} \in X_{-i}^{\mu^-(x_N)}$. $\square$

Theorem 4.5. *If a finite game is WBRDS, then it has the pseudo-FSBRP and pseudo-FBRP.*

Proof. Fixing a perfect W-scheme, we consider the functions μ and μ^- defined by (2) and (3). As above, if $\mu^-(x_N) = m$, then x_N is already a Nash equilibrium. Otherwise, we pick y_i maximizing μ over $R_i(x_{-i})$ for each $i \in N$.

To prove the first statement, we notice that $\langle x_N, y_N\rangle$ is a simultaneous pseudo-Cournot path. By Lemma 4.4, $\mu^-(y_N) > \mu^-(x_N)$. If y_N is not a Nash equilibrium, we make a similar step, and so on. Thus we obtain a simultaneous pseudo-Cournot path along which μ^- strictly increases until a Nash equilibrium is reached.

To prove the second statement, we pick $i \in \mathrm{Argmin}_{i\in N}\, \mu_i(x_i)$. This time, $\langle x_N, (y_i, x_{-i})\rangle$ is a pseudo-Cournot path; by Lemma 4.4, we have either $\mu^-(y_i, x_{-i}) > \mu^-(x_N)$ or $\mu^-(y_i, x_{-i}) = \mu^-(x_N)$ and $M(y_i, x_{-i}) \subset M(x_N)$. The final argument is virtually the same as in the previous paragraph. □

5 Two-person games

Proposition 5.1. *If a finite two person game Γ has the FIBRP or FSBRP, then it has the FBRP.*

Proof. In the first case, Γ admits no complete Cournot cycle by Proposition 2.1; on the other hand, best response improvements by one player cannot form a cycle in any game. In the second case, we notice that $y_N \rhd^{\mathrm{sBR}} x_N$ whenever $y_N \rhd^{\mathrm{BR}} x_N$ and $x_i \in R_i(x_{-i})$ for an $i \in N$. Therefore, every Cournot path becomes a simultaneous Cournot path after the first step. □

Theorem 5.2. *If a finite two person game Γ is SBRDS, then it has the FBRP.*

Proof. The statement immediately follows from Theorem 4.3 and Proposition 5.1. □

The FBRP in the formulation of Theorem 5.2 cannot be replaced with the FIP: if one player has a strictly dominant strategy x_i^+, then any behavior of improvement paths with $x_i^k \neq x_i^+$ is compatible with strict dominance solvability.

Proposition 5.3. *If a finite two person game Γ has the pseudo-FBRP, then it has the weak FBRP.*

Proof. Let Γ have the pseudo-FBRP and $x_N^0 \in X_N$. If x_N^0 is a Nash equilibrium, there is nothing to prove. Otherwise, there is $x_N^1 \in X_N$ such that $x_N^1 \rhd^{\mathrm{BR}} x_N^0$. If x_N^1 is a Nash equilibrium, we are home again. Otherwise, there is a pseudo-Cournot path $\langle x_N^1, \ldots, x_N^m\rangle$ such that x_N^m is a Nash equilibrium; without restricting generality, we assume that no shorter pseudo-Cournot path from x_N^1 to an equilibrium exists. If $\langle x_N^1, \ldots, x_N^m\rangle$ happens to be a Cournot path, we are home once again. Otherwise, let k ($1 \le k < m$) be the least where $x_N^{k+1} \rhd^{\mathrm{BR}} x_N^k$ does not hold, i.e., $x_{-i}^{k+1} = x_{-i}^k$ and $x_i^k \in R_i(x_{-i}^k) \ni x_i^{k+1}$ for an $i \in N$. On the other hand, we have $x_N^k \rhd^{\mathrm{BR}} x_N^{k-1}$, hence $x_j^{k-1} \notin R_j(x_{-j}^k) \ni x_j^k$ for a $j \in N$. Now if $i \neq j$, then x_N^k is a Nash equilibrium, hence the path $\langle x_N^1, \ldots, x_N^m\rangle$ is not the shortest. If $i = j$, then we have $x_N^{k+1} \rhd_i^{\mathrm{BR}} x_N^{k-1}$, hence the path $\langle x_N^1, \ldots, x_N^{k-1}, x_N^{k+1}, \ldots, x_N^m\rangle$ is a shorter pseudo-Cournot path from x_N^1 to x_N^m. □

Proposition 5.4. *If a finite two person game Γ has the pseudo-FSBRP, then it has the weak FBRP and weak FSBRP.*

Proof. Let Γ have the pseudo-FSBRP and $x_N^0 \in X_N$; then there is a simultaneous pseudo-Cournot path $\langle x_N^0, \ldots, x_N^m \rangle$ such that x_N^m is a Nash equilibrium. We define a sequence $\langle y_N^0, y_N^1, \ldots, y_N^{m+1} \rangle$ in this way: $y_N^0 = x_N^0$; $y_1^{2k+1} = x_1^{2k+1}$; $y_2^{2k+1} = x_2^{2k}$; $y_1^{2k+2} = x_1^{2k+1}$; $y_2^{2k+2} = x_2^{2k+2}$; if $2k = m$, we set $y_1^{m+1} = x_1^m$; if $2k+1 = m$, we set $y_2^{m+1} = x_2^m$. Thus, $y_N^{m+1} = x_N^m$ in either case.

By our construction, for each $k = 0, 1, \ldots, m$ we have $y_i^{k+1} \in R_i(y_{-i}^k)$ and $y_{-i}^{k+1} = y_{-i}^k$ for an $i \in N$; therefore, $\langle y_N^0, y_N^1, \ldots, y_N^{m+1} \rangle$ is a pseudo-Cournot path ending at a Nash equilibrium. Since $x_N^0 \in X_N$ was arbitrary, Γ has the pseudo-FBRP. Now Proposition 5.3 implies the weak FBRP.

Let us show that Γ has the weak FSBRP as well. Given $x_N^0 \in X_N$, there is again a simultaneous pseudo-Cournot path $\langle x_N^0, \ldots, x_N^m \rangle$ such that x_N^m is a Nash equilibrium. If $x_N^{k+1} \rhd^{\mathrm{sBR}} x_N^k$ for each $k = 1, \ldots, m$, "pseudo" can be dropped, and we are home. Otherwise, let $\bar{k}$ be the first moment when $x_i^k \in R_i(x_{-i}^k)$, but $x_i^{k+1} \neq x_i^k$ for an $i \in N$. If $x_{-i}^k \in R_i(x_i^k)$, then x_N^k is a Nash equilibrium, and we are home again. Supposing $x_{-i}^k \notin R_i(x_i^k)$ and denoting $y_N^0 = (x_i^k, x_{-i}^{k+1})$, we have $y_N^0 \rhd^{\mathrm{sBR}} x_N^k$. Since Γ has the weak FBRP, there is a Cournot path starting at y_N^0 and ending at a Nash equilibrium. Since $y_{-i}^0 \in R_{-i}(y_i^0)$, the path is a simultaneous Cournot path as well, exactly as in the proof of Proposition 5.1. □

Theorem 5.5. *If a finite two person game is WBRDS, then it has the weak FSBRP and the weak FBRP.*

Proof. The statement immediately follows from Theorem 4.5 and Proposition 5.4. □

6 Main necessity results

A very interesting feature of Moulin (1984) is an equivalence result (Corollary of Lemmas 1 and 2), even though obtained in a rather special case. From our current viewpoint, that result is just a fortunate coincidence: when all best responses are unique, both levels of BR-dominance solvability become equivalent. Generally, it seems impossible to derive *strong* BR-dominance solvability from any nice property of best response dynamics. There also seems to be no necessity result whatsoever for games with more than two players.

The Battle of Sexes, which has the FIP but is not even WBRDS, sets limits to necessity results. An obvious way around the example is to notice that it does not have even the weak FSBRP. Another, unexpectedly helpful, observation is that the set of Nash equilibria in the Battle of Sexes is not rectangular.

Theorem 6.1. *If a finite two person game Γ has the weak FBRP and the set of Nash equilibria in Γ is rectangular, then Γ is WBRDS.*

Proof. We assume that the set of Nash equilibria in Γ is Y_N^0 and recursively define Y_N^k for all $k \in \mathbb{N}$ by $Y_i^{k+1} = R_{-i}^{-1}(Y_{-i}^k)$.

Claim 6.1.1. *$Y_N^k \subseteq Y_N^h$ whenever $k, h \in \mathbb{N}$ and $h \geq k$.*

Proof. Straightforward induction starting with $Y_i^0 \subseteq R_{-i}^{-1}(Y_{-i}^0) = Y_i^1$. □

Since X_N is finite, the sequence Y_N^k stabilizes at some stage $\bar{m} \in \mathbb{N}$.

Claim 6.1.2. *Let $x_N^0, \dots, x_N^m$ be a Cournot path such that $x_N^m \in Y_N^0$. Then $x_i^k \in Y_i^{\bar{m}}$ for each $i \in N$ and $k = 1, \dots, m$.*

Proof. We argue by backward induction along the path. There is no problem with $m = 0$ or $k = m$, so we are home immediately if $m \leq 1$. Let $m > 1$ and $x_N^m \rhd_i^{\mathrm{BR}} x_N^{m-1}$; then $x_{-i}^{m-1} = x_{-i}^m \in Y_{-i}^0$ and $x_{-i}^{m-1} \in R_{-i}(x_i^{m-1})$; therefore, $x_i^{m-1} \in R_i^{-1}(x_{-i}^{m-1}) \subseteq R_i^{-1}(Y_{-i}^0) = Y_i^1$. Iterating this argument, we come to $x_i^1 \in Y_i^{m-1}$ for both i; however, we cannot say anything about x_i^0 such that $x_N^1 \rhd_i^{\mathrm{BR}} x_N^0$. □

Claim 6.1.3. *There holds $Y_N^{\bar{m}} = X_N$.*

Proof. Let $i \in N$ and $x_i \in X_i \setminus Y_i^0$. We start with picking $x_{-i} \in Y_{-i}^0$. If $x_{-i} \in R_{-i}(x_i)$, then $x_i \in Y_i^1 \subseteq Y_i^{\bar{m}}$. Otherwise, we pick $x'_{-i} \in R_{-i}(x_i)$ and have $(x_i, x'_{-i}) \rhd_{-i}^{\mathrm{BR}} x_N$. Since Γ has the weak FBRP, there is a Cournot path from (x_i, x'_{-i}) to a Nash equilibrium. Since $x_i \notin Y_i^0$, the length of the path is strictly positive. Adding x_N to the path at the left and invoking Claim 6.1.2, we have $x_i \in Y_i^{\bar{m}}$. □

Now the sequences $X_i^k = Y_i^{\bar{m}-k}$ $(i \in N,\ k = 0, 1, \dots, \bar{m})$ form a perfect W-scheme. □

Corollary. *If a finite two person game Γ has the weak FBRP and the set of Nash equilibria in Γ is rectangular, then Γ has the weak FSBRP.*

Proof. By Theorem 6.1, Γ is WBRDS. Therefore, Γ has the weak FSBRP by Theorem 5.5. □

Theorem 6.2. *If a finite two person game Γ has the weak FSBRP, then Γ is WBRDS.*

Proof. Without restricting generality, $N = \{1, 2\}$. Given a Nash equilibrium $x_N \in X_N$, we define $Y_N^0(x_N) = \{x_N\}$, and then recursively define $Y_i^k(x_N)$ for both i and all $k \in \mathbb{N}$ in essentially the same way as in the proof of Theorem 6.1: $Y_i^{k+1}(x_N) = R_{-i}^{-1}(Y_{-i}^k(x_N))$.

Claim 6.2.1. *$Y_i^k(x_N) \subseteq Y_i^h(x_N)$ whenever $i \in N$, $k, h \in \mathbb{N}$, and $h \geq k$.*

The proof is the same as in Theorem 6.1. Exactly as in the same proof, the sequence $Y_N^k(x_N)$ stabilizes at some stage $\bar{m} \in \mathbb{N}$. Since X_N is finite, the same $\bar{m}$ will do for all Nash equilibria x_N.

Claim 6.2.2. *Let $x_N, y_N \in X_N$, x_N be a Nash equilibrium, and there be a simultaneous Cournot path starting at y_N and ending at x_N. Then $y_N \in Y_N^{\bar{m}}(x_N)$.*

Proof. The same backward induction along the path as in the proof of Theorem 6.1. □

Claim 6.2.3. *If x_N and y_N are Nash equilibria and $x_N \in Y_N^{\bar{m}}(y_N)$, then $Y_N^{\bar{m}}(x_N) \subseteq Y_N^{\bar{m}}(y_N)$.*

Proof. Let $x_N \in Y^m(y_N)$. Straightforward induction along the definition of $Y^k(x_N)$ shows that $Y^k(x_N) \subseteq Y^{m+k}(y_N)$ for all $k \in \mathbb{N}$. □

Claim 6.2.4. *There exists a Nash equilibrium $x_N \in X_N$ such that $Y_N^{\bar{m}}(x_N) = X_N$.*

Proof. Let us pick a Nash equilibrium $x_N^0 \in X_N$ with a maximal $Y_N^{\bar{m}}(x_N^0)$. If $Y_N^{\bar{m}}(x_N^0) = X_N$, then we are home. Supposing the contrary, we may, w.r.g., assume the existence of $x_1 \in X_1 \setminus Y_1^{\bar{m}}(x_N^0)$ [actually, if $Y_N^{\bar{m}}(x_N^0) \neq X_N$, then $Y_i^{\bar{m}}(x_N^0) \neq X_i$ for both i]; then $(x_1, x_2^0) \notin Y_N^{\bar{m}}(x_N^0)$. By the weak FSBRP and Claim 6.2.2, there is a Nash equilibrium $y_N \in X_N$ such that $(x_1, x_2^0) \in Y_N^{\bar{m}}(y_N)$. Let $k \in \mathbb{N}$ be such that $x_2^0 \in Y_2^k(y_N)$; then $x_1^0 \in Y_1^{k+1}(y_N)$ since $x_2^0 \in R_1(x_1^0)$. Therefore, $x_N^0 \in Y_N^{\bar{m}}(y_N)$, hence $Y_N^{\bar{m}}(x_N^0) \subset Y_N^{\bar{m}}(y_N)$ by Claim 6.2.3, contradicting the choice of x_N^0. □

We pick a Nash equilibrium $x_N \in X_N$ as in Claim 6.2.4 and finish the proof in exactly the same way as in Theorem 6.1. □

Corollary. *A finite two person game has the weak FSBRP if and only if it is WBRDS.*

Remark. In the light of Propositions 5.3 and 5.4, there would be no point in distinguishing between the weak F(S)BRP and pseudo-F(S)BRP.

7 Intermediate BR-dominance

Although strong BR-dominance solvability does not follow from the FBRP or FSBRP, something stronger than weak BR-dominance solvability *can* be derived. Unfortunately, those intermediate versions are not sufficient for the FBRP or FSBRP, nor for any nicer properties of Cournot dynamics than those following from weak BR-dominance solvability.

Let $x_i, y_i \in X_i$; we say that y_i (*strictly*) *BR-dominates* x_i *in an intermediate sense*, denoting the fact $y_i \gtrsim_i x_i$ ($y_i \gg_i x_i$), if $y_i \neq x_i$ and $R_i^{-1}(x_i) \subseteq R_i^{-1}(y_i)$ ($R_i^{-1}(x_i) \subset R_i^{-1}(y_i)$); note that $\gg_i$ is the asymmetric component of $\gtrsim_i$. It is immediately clear that $y_i \gtrsim_i x_i$ whenever y_i weakly dominates x_i or x_i is strongly BR-dominated.

An I-scheme (I!-scheme) is an elimination scheme of length m such that, for every $k \in \{0, \ldots, m-1\}$ and every $x_i \in X_i^k \setminus X_i^{k+1}$, there is $y_i \in X_i^{k+1}$ such that $y_i \gtrsim_i x_i$ ($y_i \gg_i x_i$). We call Γ (*strictly*) *BR-dominance solvable in an intermediate sense* (*IBRDS/ I!BRDS*) if it admits a perfect I-scheme (I!-scheme). Obviously, SBRDS $\Rightarrow$ I!BRDS $\Rightarrow$ IBRDS $\Rightarrow$ WBRDS.

Theorem 7.1. *If a finite two person game* Γ *has the FSBRP, then it is IBRDS.*

Proof.

Claim 7.1.1. *Either every strategy profile* $x_N \in X_N$ *is a Nash equilibrium, or* Γ *contains a strategy BR-dominated in an intermediate sense.*

Proof. Let the first statement not hold: there is, at least, one pair of strategy profiles such that $y_N \rhd^{\text{sBR}} x_N$. Since there is no simultaneous Cournot cycle, we can pick an $x_N^* \in X_N$ which is not a Nash equilibrium and for which $x_N^* \rhd^{\text{sBR}} x_N$ is impossible for any $x_N \in X_N$.

For each $i \in N$, we denote $X_{-i}^* = R_i^{-1}(x_i^*) \subseteq X_{-i}$. If $X_{-i}^* = \emptyset$ for an $i \in N$, then x_i^* is even strongly BR-dominated and we are home. Let $X_N^* = X_1^* \times X_2^* \neq \emptyset$. Since x_N^* is not a Nash equilibrium, there must be $i \in N$ and $x_i^0 \in X_i^*$ such that $x_i^0 \neq x_i^*$. If $R_i^{-1}(x_i^0) \supseteq X_{-i}^*$, then $x_i^0 \gtrsim_i x_i^*$ and we are home again; otherwise, there is $x_{-i}^0 \in X_{-i}^*$ such that $x_i^0 \notin R_i(x_{-i}^0)$. Since $x_N^* \rhd^{\text{sBR}} x_N^0$ is assumed impossible, we must have $x_{-i}^* \neq x_{-i}^0 \in R_{-i}(x_i^0)$. Again, if $R_{-i}^{-1}(x_{-i}^0) \supseteq X_i^*$, then $x_{-i}^0 \gtrsim_{-i} x_{-i}^*$. Otherwise, there is $x_i^1 \in X_i^*$ such that $x_{-i}^0 \notin R_{-i}(x_i^1)$; we denote $x_N^1 = (x_i^1, x_{-i}^0) \in X_N^*$. Since $x_N^* \rhd^{\text{sBR}} x_N^1$ is assumed impossible, we must have $x_i^* \neq x_i^1 \in R_i(x_{-i}^0)$; therefore, $x_N^1 \rhd^{\text{sBR}} x_N^0$. Again, if $R_i^{-1}(x_i^1) \supseteq X_{-i}^*$, then $x_i^1 \gtrsim_i x_i^*$; otherwise, there is $x_{-i}^2 \in X_{-i}^*$ such that $x_i^1 \notin R_i(x_{-i}^2)$. We denote $x_N^2 = (x_i^1, x_{-i}^2) \in X_N^*$; again, $x_N^2 \rhd^{\text{sBR}} x_N^1 \rhd^{\text{sBR}} x_N^0$, and so on.

Since Γ has the FSBRP, the simultaneous Cournot path $\langle x_N^0, x_N^1, \dots \rangle$ cannot be infinite. On the other hand, the next profile x_N^{k+1} cannot be defined only if $x_i^k \gtrsim_i x_i^*$ for an $i \in N$, hence x_i^* is BR-dominated in an intermediate sense. □

If every strategy profile $x_N \in X_N$ is a Nash equilibrium, we are home immediately. Otherwise, we delete (all or some) strategies BR-dominated in an intermediate sense, obtaining a subgame Γ^1 and an I-scheme of length 1. By Lemma 3.2, we have $R_i^1(x_{-i}) = R_i(x_{-i}) \cap X_i^1$ for all $i \in N$ and $x_{-i} \in X_{-i}^1$. Therefore, the relation $\rhd^{\text{sBR}}$ in Γ^1 is the restriction of $\rhd^{\text{sBR}}$ in Γ to X_N^1, hence Γ^1 also has the FSBRP, hence Claim 7.1.1 applies to Γ^1. The process only stops when every strategy profile in Γ^m is a Nash equilibrium; then the I-scheme will be perfect. □

The Battle of Sexes shows that the FSBRP in Theorem 7.1 cannot be replaced with the FBRP (or even FIP). This becomes possible under an additional assumption that the set of Nash equilibria is rectangular.

Theorem 7.2. *If a finite two person game* Γ *has the FBRP and the set of Nash equilibria in* Γ *is rectangular, then* Γ *is I!BRDS.*

Proof. Let the set of Nash equilibria of Γ be $X_N^0 = X_1^0 \times X_2^0$, where $X_i^0 \subseteq X_i$ for each $i \in N$.

Claim 7.2.1. *Either $X_N^0 = X_N$, or Γ contains a strategy strictly BR-dominated in an intermediate sense.*

Proof. Let $X_N^0 \subset X_N$. By definition, $R_i^{-1}(x_i^0) \supseteq X_{-i}^0$ for both $i \in N$ and all $x_i^0 \in X_i^0$. We pick an $x_N \in X_N \setminus X_N^0 \neq \emptyset$ and start a Cournot path from x_N; since Γ has the FBRP, the path must end at an $x_N^* \in X_N^0$; therefore, $R_i^{-1}(x_i^*) \supset X_{-i}^0$ for an $i \in N$.

Now we define a binary relation $\rhd$ on X_i:

$$y_i \rhd x_i \rightleftharpoons \exists x_{-i} \in X_{-i} \, [x_i \notin R_i(x_{-i}) \ni y_i \ \& \ x_{-i} \in R_{-i}(x_i) \ \& \ x_{-i} \notin R_{-i}(y_i)]. \qquad (4)$$

Let us show that $\rhd$ is acyclic. Supposing to the contrary that $x_i^0, x_i^1, \dots, x_i^m = x_i^0$ are such that $x_i^{k+1} \rhd x_i^k$ for each $k = 0, \dots, m-1$, we pick, for each k, an x_{-i}^k from (4). Then we define $x_N^{2k} = (x_i^k, x_{-i}^k)$ and $x_N^{2k+1} = (x_i^{k+1}, x_{-i}^k)$ for each $k = 0, \dots, m-1$. It follows immediately from (4) that $x_N^0, x_N^1, \dots, x_N^{2m} = x_N^0$ is a Cournot cycle in Γ, which contradicts the FBRP.

Since X_i is finite and $\rhd$ is acyclic, there is $y_i \in X_i$ such that $y_i \rhd x_i$ does not hold for any $x_i \in X_i$. If $R_i^{-1}(y_i) = \emptyset$, then y_i is even strongly BR-dominated, hence we are home immediately. For every $x_{-i} \in R_i^{-1}(y_i)$, we consider two alternatives: If $x_{-i} \in R_{-i}(y_i)$, then (y_i, x_{-i}) is a Nash equilibrium, hence $x_{-i} \in X_{-i}^0$. If $x_{-i} \notin R_{-i}(y_i)$, then we pick $x_i \in R_{-i}^{-1}(x_{-i}) \neq \emptyset$; then $x_i \in R_i(x_{-i})$ because we would have $y_i \rhd x_i$ otherwise; therefore, (x_i, x_{-i}) is a Nash equilibrium, hence $x_{-i} \in X_{-i}^0$ again. Thus, $R_i^{-1}(y_i) \subseteq X_{-i}^0 \subset R_i^{-1}(x_i^*)$, i.e., y_i is strictly BR-dominated in an intermediate sense. □

Now we apply Claim 7.2.1 in the same way as Claim 7.1.1 was applied in the proof of Theorem 7.1. □

8 Unique best responses

The relationship between BR-dominance solvability and nice best response dynamics becomes especially simple in the case of two person games with unique best responses, as in Moulin (1984). According to Propositions 8.1 and 8.3, there is then no need to distinguish between strong and weak versions of the properties. The set of Nash equilibria is rectangular if and only if it is a singleton.

Proposition 8.1. *For every finite game Γ where $R_i(x_{-i})$ is a singleton for every $i \in N$ and $x_{-i} \in X_{-i}$, the weak FSBRP implies the FSBRP. If $\#N = 2$, then the weak FBRP implies the FBRP.*

Proof. No more than one simultaneous Cournot path can be started from any x_N. Therefore, if there were a simultaneous Cournot cycle, no equilibrium could be reached from any strategy profile belonging to the cycle. Similarly, no more than one Cournot path can be started from x_N such that $x_i \in R_i(x_{-i})$ for at least one $i \in N$, and every Cournot cycle must consist of such profiles. $\square$

Lemma 8.2. *If $R_i(x_{-i})$ is a singleton for every $i \in N$ and $x_{-i} \in X_{-i}$, then every W-scheme is an S-scheme.*

Proof. Let $0 \leq k \leq m-1$, $i \in N$, and $x_i \in X_i^k \setminus X_i^{k+1}$. Since X_i^{k+1} is BR-sufficient in Γ^k and each $R_i(x_{-i})$ is a singleton, we have $R_i(X_{-i}^k) \subseteq X_i^{k+1}$, hence $x_i \notin R_i(X_{-i}^k)$, hence x_i is strongly BR-dominated. $\square$

Proposition 8.3. *If Γ is WBRDS and $R_i(x_{-i})$ is a singleton for every $i \in N$ and $x_{-i} \in X_{-i}$, then Γ is SBRDS.*

Proof. The statement immediately follows from Lemma 8.2. $\square$

Corollary to Theorems 4.2 and 6.2. *Let Γ be a finite two person game such that $R_i(x_{-i})$ is a singleton for every $i \in N$ and $x_{-i} \in X_{-i}$. Then Γ has the FSBRP if and only if it is SBRDS.*

Corollary to Theorems 4.3 and 6.1. *Let Γ be a finite two person game such that $R_i(x_{-i})$ is a singleton for every $i \in N$ and $x_{-i} \in X_{-i}$, and there is no more than one Nash equilibrium in Γ. Then Γ has the FBRP if and only if it is SBRDS.*

9 "Counterexamples"

This section consists of examples showing the impossibility of easy extensions of our results. It should be noted that the preferences of the players in every game are "generic on outcomes," i.e., whenever a player is indifferent between two strategy profiles, each other player is indifferent too.

Example 9.1 shows that Theorems 4.2 and 4.3 become wrong if Γ is only weakly dominance solvable (or strictly BR-dominance solvable in an intermediate sense); Example 9.2 shows the same for Theorem 5.2. Example 9.1 simultaneously shows that Theorem 5.5 is wrong for more than two players.

Example 9.1. Let us consider a three person $2 \times 3 \times 2$ game (where player 1 chooses rows, player 2 columns, and player 3 matrices):

$$\begin{bmatrix} (3,3,3) & (2,1,1) & (1,2,2) \\ (3,3,3) & (1,2,2) & (2,1,1) \end{bmatrix} \quad \begin{bmatrix} (0,0,0) & \underline{(2,1,1)} & \underline{(1,2,2)} \\ (0,0,0) & \underline{(1,2,2)} & \underline{(2,1,1)} \end{bmatrix}.$$

Nash equilibria fill the left column of the left matrix; however, none of the underlined strategy profiles could be connected to any equilibrium with an individual improvement path or with a simultaneous Cournot path. Thus, the game does not have even the weak FIP or the weak FSBRP. On the other hand, it is weakly dominance solvable: The choice of the left matrix weakly dominates the choice of the right matrix; there is also strict BR-dominance in an intermediate sense. When the right matrix is deleted, the left column becomes strictly dominant.

Example 9.2. Let us consider the following bimatrix game:

$$\begin{array}{ccc} (0,1) & \underline{(1,0)} & \underline{(0,1)} \\ (0,1) & \underline{(0,1)} & \underline{(1,0)} \\ (2,2) & (1,0) & (1,0) \end{array}.$$

The bottom row and the left column are weakly dominant as well as strictly BR-dominant in an intermediate sense. The southwestern corner of the matrix is a unique Nash equilibrium. The underlined fragment is a Cournot cycle (hence a simultaneous Cournot cycle as well).

The Battle of Sexes has the FIP, but is not even weakly BR-dominance solvable; therefore, the converse to Theorems 4.3 and 5.2 is wrong. Example 9.3 shows that no general necessity result would be possible without the idea of BR-dominance solvability. Example 9.4 shows the impossibility to reverse Theorem 5.2 even when the set of Nash equilibria is rectangular. Example 9.5 shows the impossibility to reverse Theorem 4.2, or assert strict BR-dominance solvability in an intermediate sense in Theorem 7.1.

Example 9.3. Let us consider the following bimatrix game:

$$\begin{array}{ccc} (0,6) & (4,8) & (8,7) \\ (1,5) & (5,4) & (7,3) \\ \underline{(2,2)} & (3,0) & (6,1) \end{array}.$$

The southwestern corner of the matrix is a unique Nash equilibrium. The game has even the FIP as well as FSBRP; it is also SBRDS. However, there is no weakly dominated strategy.

Example 9.4. Let us consider a two person 2×2 game:

$$\begin{array}{cc} (0,2) & (2,0) \\ (1,1) & (1,1) \end{array}.$$

The southwestern corner is a unique Nash equilibrium. The game obviously has the FIP as well as FBRP. On the other hand, each strategy of each player is a best response to a strategy of the partner; therefore, the game is not SBRDS.

Example 9.5. Let us consider a two person 2×3 game:

$$\begin{array}{ccc} (3,2) & (1,3) & (1,3) \\ (3,2) & (2,1) & (0,0) \end{array}.$$

There are two Nash equilibria: the northeastern and southwestern corners. The game has the FSBRP: the longest possible simultaneous Cournot path starts from the southeastern corner and then passes through all non-equilibrium strategy profiles. On the other hand, there is no BR-dominance of any kind between the strategies of player 1; among the strategies of player 2, there is only non-strict BR-dominance in an intermediate sense between the second and third columns. Therefore, the game is not SBRDS, nor even I!BRDS. (On the other hand, it *is* weakly dominance solvable)

Example 9.6 shows that both Theorem 6.1 and Theorem 7.2 are wrong for more than two players; Example 9.7 shows the same for Theorems 6.2 and 7.1.

Example 9.6. Let us consider a three person $2 \times 2 \times 2$ game (where player 1 chooses rows, player 2 columns, and player 3 matrices):

$$\begin{bmatrix} (3,4,3) & (0,0,0) \\ (5,5,5) & (4,3,4) \end{bmatrix} \quad \begin{bmatrix} (2,2,1) & (1,1,2) \\ (0,0,0) & (2,2,1) \end{bmatrix}.$$

The southwestern corner is a unique Nash equilibrium; the FBRP is easy to check. On the other hand, each strategy of each player is the unique best response to a strategy profile of the partners. Therefore, the game is not even WBRDS.

Example 9.7. Let us consider a three person $2 \times 2 \times 2$ game (where player 1 chooses rows, player 2 columns, and player 3 matrices):

$$\begin{bmatrix} \underline{(2,1,2)} & (4,4,4) \\ \underline{\underline{(0,0,0)}} & \underline{(1,3,3)} \end{bmatrix} \quad \begin{bmatrix} \underline{\underline{(0,0,0)}} & \underline{(3,2,1)} \\ (4,4,4) & \underline{\underline{(0,0,0)}} \end{bmatrix}.$$

The two Nash equilibria are not underlined. Each of the three strategy profiles underlined once is dominated in the sense of $\triangleright^{\mathrm{sBR}}$ only by a Nash equilibrium; each of the three strategy profiles underlined twice is dominated in the same sense only by a strategy profile underlined once. Thus, the game has the FSBRP. On the other hand, each strategy of each player is a unique best response to a strategy profile of the partners. Therefore, the game is not even WBRDS.

Example 9.8 shows that the adjectives "weak" in the corollary to Theorem 6.1 cannot be dropped.

Example 9.8. Let us consider the following bimatrix game:

$$\begin{matrix} (1,3) & (5,2) & (0,0) \\ (3,1) & (0,0) & (1,3). \\ (0,0) & (4,4) & \underline{(4,4)} \end{matrix}$$

The game has even the FIP; the southeastern corner of the matrix is a unique Nash equilibrium. In accordance with Theorem 7.2, it is WBRDS. However, it does not have the FSBRP: the profiles on the diagonal with utilities $(0,0)$ form a simultaneous Cournot cycle.

Example 9.9 shows that BR-dominance solvability in an intermediate sense cannot be asserted in Theorems 6.1 or 6.2. Example 9.10 shows that weak dominance solvability does not imply strict BR-dominance solvability in an intermediate sense (although implying the "non-strict" version of the property).

Example 9.9. Let us consider a two person 6×6 game defined by the left matrix:

$$\begin{array}{cccccc} (3,3) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (2,1) & (1,2) & (2,1) & (1,2) & (0,0) \\ (0,0) & (0,0) & (2,1) & (1,2) & (2,1) & (1,2) \\ (0,0) & (1,2) & (0,0) & (2,1) & (1,2) & (2,1) \\ (0,0) & (2,1) & (1,2) & (0,0) & (2,1) & (1,2) \\ (1,2) & (1,2) & (2,1) & (1,2) & (0,0) & (2,1) \end{array} \qquad \begin{bmatrix} 0 & 4 & 2 & 4 & 4 & 2 \\ 3 & 4 & 3 & 4 & 5 & 3 \\ 3 & 4 & 4 & 5 & 4 & 3 \\ 5 & 5 & 5 & 6 & 5 & 6 \\ 3 & 4 & 3 & 4 & 4 & 3 \\ 1 & 5 & 2 & 5 & 4 & 2 \end{bmatrix}.$$

The northwestern corner is a unique Nash equilibrium. The weak FSBRP is easy to check: the right matrix shows the length of the shortest simultaneous Cournot path leading to the equilibrium from every strategy profile. By Proposition 5.4, the game has the weak FBRP as well. On the other hand, none of the sets $R_i^{-1}(x_i)$ include each other for either $i \in N$, even if non-strict inclusion is taken into account. Therefore, no strategy is BR-dominated in an intermediate sense.

Example 9.10. Let us consider the following bimatrix game:

$$\begin{array}{ccc} (3,3) & (2,2) & (0,0) \\ (0,0) & (1,1) & (1,1) \end{array}.$$

The middle column weakly dominates the right one; when the latter is deleted, the upper row becomes strictly dominant. Therefore, the game is weakly dominance solvable. On the other hand, none of the strategies is strictly BR-dominated in an intermediate sense: each row is the unique best response to a column; the left column is the unique best response to the upper row; both other columns are only best responses to the bottom row.

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