Maximizing a binary relation on convex subsets of its domain

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Abstract

The possibility to characterize binary relations which generate, on convex subsets, choice functions possessing various desirable properties (non-emptiness, the outcast axiom, path independence, etc.) in terms of "prohibited configurations" is analyzed. The results are overwhelmingly negative; the impossibility persists even when restricted classes of binary relations are considered: orderings, semiorders, interval orders, etc. *JEL classification*: D 71.

Keywords: Binary relation; Maximal element; Necessary and sufficient condition; Kakutani Theorem

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1 Introduction

There are plenty of well-known reasons to be interested in undominated points (maximal elements) of a binary relation. The relation may describe the preferences of an individual decision maker, or "aggregate" preferences of several individuals who have to make a decision together. It may reflect preferences and strategic possibilities of several agents who can make decisions individually or in coalitions; such concepts as the core or Nash (strong) equilibrium belong here. Abstract fixed point theorems also fit in this framework although it is more usual to argue in the opposite direction, deriving the existence of undominated points from such theorems.

A researcher needing, say, the existence of an undominated point (equilibrium) in her model will be satisfied with any sufficient condition that can be verified in the model. Those who develop such conditions have to keep in mind the needs of various researchers; therefore, the wider applicable, i.e., weaker, is a condition, the better. And a sufficient condition cannot be weakened if it is also necessary. There is a distinct, though related, reason to be interested in necessity results: when the conjunction of several conditions is proven to be sufficient for something, it may happen that some of them are actually irrelevant; proving the necessity rules this unpleasant possibility out.

There is quite a number of sufficient conditions for a binary relation given on a metric space to admit undominated points in every nonempty compact subset (Gillies, 1959; Bergstrom, 1975; Kalai and Schmeidler, 1977; Mukherji, 1977; Walker, 1977; Campbell and Walker, 1990). Smith (1974) found that a very weak version of upper semicontinuity is *necessary* and sufficient for the property provided the relation in question is an ordering (weak order); Kukushkin (2008b) proved a similar characterization result for interval orders.

Every condition in that literature can be loosely described as a combination of some forms of acyclicity and continuity, or, in more general terms, as the "prohibition of some configurations." Formal definitions were given in Kukushkin (2008a). That paper showed that no condition of the form could be necessary and sufficient for a binary relation to admit undominated points in every nonempty compact subset; even an *a priori* restriction to transitive relations does not help.

The situation is not so hopeless if a certain degree of rationality is demanded of the choice function (Fishburn, 1973; Sen, 1984; Aizerman and Aleskerov, 1995; Malishevski, 1998) generated by the binary relation. In the case of choice from finite subsets, the most popular notions of a rational choice function lead to well-known characterization results (non-emptiness is equivalent to acyclicity, path independence to transitivity, etc.). Each of those restrictions on the relation can also be interpreted as the prohibition of certain configurations. Kukushkin (2008b) found a similar characterization of relations ensuring the outcast axiom or, equivalently, path independence on nonempty compact subsets.

In economics literature, attention is often restricted to convex subsets. The existence problem in this framework has also been studied intensely (Mas-Colell, 1974; Shafer, 1974, 1976; Shafer and Sonnenschein, 1975; Kiruta et al., 1980; Yannelis and Prabhakar, 1983; Danilov and Sotskov, 1985). Actually, many of the quoted papers considered economic equilibrium rather than an abstract binary relation, but, technically, the difference is not very important. There is no characterization result in that literature.

This paper strives to extend the approach of Kukushkin (2008a,b) to the choice from convex subsets. We modify the notion of a configuration in accordance with the new context. An overwhelming majority of the results are negative: the impossibility of necessary and sufficient conditions for nonempty choice holds even for linear orders. Moreover, even when attention is restricted to binary relations ensuring a certain level of rationality of the choice function, a characterization of relations ensuring a higher level of rationality is impossible.

Almost all previously found sufficient conditions can be described as the prohibition of configurations; possible exceptions are only Theorems 1.7.4, 1.7.6, and 1.7.7 of Kiruta et al. (1980). It remains unclear at the moment how far one could advance with conditions of more complicated syntactical structures. The only "positive" result obtained in this paper, Theorem 9, can be viewed as a convex analog of Smith's (1974) characterization theorem. A crucial difference is that the necessary and sufficient conditions found here employ an existence quantifier, i.e., checking them requires some "creative" effort. Unfortunately, the theorem is only proven in the finite-dimensional case.

The next section contains basic definitions. In Section 3, previous results are reviewed, providing a justification for our central notion of a configurational condition, which is formally defined in Section 4. Section 5 contains the main impossibility theorems. In Section 6, an *a priori* restriction to quasiconcave relations is imposed. Some topics of secondary importance are discussed in Section 7.

2 Basic notions

A *binary relation* on a set A is a Boolean function on $A \times A$; as usual, we write $y \succ x$ when the relation $\succ$ is true on a pair (y, x) and $y \not\succ x$ when it is false. We denote $\mathfrak{B}$ the lattice of all subsets of A: we never consider different sets A simultaneously. Given $X \in \mathfrak{B}$, a point $x \in X$ is a *maximizer* of $\succ$ on X if $y \not\succ x$ for any $y \in X$. The set of all maximizers of $\succ$ on X is denoted $M_\succ(X)$.

Whenever a binary relation $\succ$ on A is given, $M_\succ(\cdot)$ defines a mapping $\mathfrak{B} \to \mathfrak{B}$ with the property $M_\succ(X) \subseteq X$ for every $X \in \mathfrak{B}$, i.e., a *choice function.* The simplest desirable property of a choice function is

$$M_\succ(X) \neq \emptyset. \tag{1}$$

We also consider three rationality requirements. A choice function $M_\succ$ satisfies the *outcast* (*Nash's*) axiom if

$$M_\succ(X) \subseteq X' \subseteq X \Rightarrow M_\succ(X') = M_\succ(X) \tag{2}$$

for all $X, X' \in \mathfrak{B}$; it satisfies the *path independence* (*Plott's*) axiom if

$$X = X' \cup X'' \Rightarrow M_\succ(X) = M_\succ(M_\succ(X') \cup X'') \tag{3}$$

for all $X, X', X'' \in \mathfrak{B}$; it satisfies the *revealed preference* (*Arrow's*) axiom if

$$[X' \subseteq X \ \&\ M_\succ(X) \cap X' \neq \emptyset] \Rightarrow M_\succ(X') = M_\succ(X) \cap X' \tag{4}$$

for all $X, X' \in \mathfrak{B}$.

Given A and a set $\mathfrak{C} \subseteq \mathfrak{B}$ of admissible subsets, we consider five classes ("levels of rationality") of binary relations:
$\mathcal{R}_\exists(\mathfrak{C})$ consists of all binary relations $\succ$ on A such that (1) holds for all $X \in \mathfrak{C}$;
$\mathcal{R}_{\mathrm{Out}}(\mathfrak{C})$ consists of all binary relations $\succ$ on A such that (1) and (2) hold for all $X, X' \in \mathfrak{C}$;
$\mathcal{R}_{\mathrm{PI}}(\mathfrak{C})$ consists of all binary relations $\succ$ on A such that (1) and (3) hold for all $X, X', X'' \in \mathfrak{C}$;
$\mathcal{R}_{\mathrm{Rat}}(\mathfrak{C})$ consists of all binary relations $\succ$ on A such that (1), (3), and (4) hold for all $X, X', X'' \in \mathfrak{C}$;
$\mathcal{R}_{\exists!}(\mathfrak{C})$ consists of all binary relations $\succ$ on A such that $\#M_\succ(X) = 1$ for all $X \in \mathfrak{C}$ and (3) holds for all $X, X', X'' \in \mathfrak{C}$.

Proposition 2.1. $\mathcal{R}_{\exists!}(\mathfrak{C}) \subseteq \mathcal{R}_{\mathrm{Rat}}(\mathfrak{C}) \subseteq \mathcal{R}_{\mathrm{PI}}(\mathfrak{C}) \subseteq \mathcal{R}_{\mathrm{Out}}(\mathfrak{C}) \subseteq \mathcal{R}_\exists(\mathfrak{C})$ *for every* $\mathfrak{C} \subseteq \mathfrak{B}$.

A routine proof is omitted. In principle, more "levels" could be singled out, but then the linear order may be lost. For instance, (1) and (4) for all $X, X' \in \mathfrak{C}$ define a class of binary relations intermediate between $\mathcal{R}_{\mathrm{Rat}}(\mathfrak{C})$ and $\mathcal{R}_{\mathrm{Out}}(\mathfrak{C})$, which may be incomparable (w.r.t. set inclusion) with $\mathcal{R}_{\mathrm{PI}}(\mathfrak{C})$. The same is true for the conjunction of $\#M_{\succ}(X) = 1$ and (2) for all $X, X' \in \mathfrak{C}$.

The idea of rational preferences can also be expressed by direct restrictions on the binary relation without any reference to choice functions. A *strict order* is an irreflexive and transitive relation. An *interval order* is a strict order $\succ$ such that

$$[y \succ x \ \& \ a \succ b] \Rightarrow [y \succ b \ \text{ or } \ a \succ x]. \tag{5}$$

A *semiorder* is an interval order such that

$$z \succ y \succ x \Rightarrow \forall a \in A \, [z \succ a \ \text{ or } \ a \succ x]. \tag{6}$$

A strict order $\succ$ is called an *ordering* if it is *negatively transitive*, i.e., $z \not\succ y \not\succ x \Rightarrow z \not\succ x$. A strict order $\succ$ is called a *total order* if $y = x$ whenever $y \not\succ x$ and $x \not\succ y$. Every total order is an ordering while every ordering is a semiorder.

Those properties of a binary relation have well-known implications for the choice function, e.g.: if $\succ$ is a total order, then $\#M_{\succ}(X) \leq 1$ for every $X \in \mathfrak{B}$; if $\succ$ is an ordering, then (4) holds for all $X, X' \in \mathfrak{B}$; if $\succ$ is a semiorder and $M_{\succ}(X) \neq \emptyset$, then (3) holds for all $X', X'' \in \mathfrak{B}$. Various implications in the opposite direction are also well known, but they need the assumption $\mathfrak{C} = \mathfrak{B}$.

Throughout the paper, we consider binary relations on a convex subset A of a Banach space. The set of all nonempty compact subsets of A is denoted $\mathfrak{C}_{\mathrm{omp}} \subset \mathfrak{B}$; the set of all nonempty convex subsets, $\mathfrak{C}_{\mathrm{onv}} \subset \mathfrak{B}$; the set of all nonempty compact and convex subsets, $\mathfrak{C}_{\mathrm{mpx}} = \mathfrak{C}_{\mathrm{omp}} \cap \mathfrak{C}_{\mathrm{onv}}$. The convex hull of $X \subseteq A$ is denoted $\mathrm{co}\, X$; the topological closure of X, $\mathrm{cl}\, X$.

3 A review of previous results

Quite a few useful conditions are naturally formulated with the help of "improvement paths." Generally, such paths may be parameterized by arbitrary well ordered sets (Kukushkin, 2003, 2005); here we can restrict ourselves to natural numbers.

Given a binary relation $\succ$, an *improvement path* is a (finite or infinite) sequence $\langle x^k \rangle_{k=0,1,\ldots}$ such that $x^{k+1} \succ x^k$ whenever both sides are defined.

A relation $\succ$ is *acyclic* if it admits no *finite improvement cycle*, i.e., no improvement path such that $x^m = x^0$ for an $m > 0$. A relation is *strongly acyclic* if it admits no infinite improvement path.

It seems impossible to ascribe any authorship to these well-known statements.

Theorem A. *A binary relation $\succ$ on A has the property that $M_{\succ}(X) \neq \emptyset$ for every $X \in \mathfrak{B} \setminus \{\emptyset\}$ if and only if it is strongly acyclic.*

Theorem B. *A binary relation $\succ$ on A has the property that $M_{\succ}(X) \neq \emptyset$ for every finite $X \in \mathfrak{B} \setminus \{\emptyset\}$ if and only if it is acyclic.*

Let us ponder on the usefulness of the theorems. In the case of Theorem A, a straightforward argument comes to mind immediately: In a "normal," continuous model, the cardinality of A is continuum, hence the cardinality of $\mathfrak{B}$ is greater than continuum, while the cardinality of the set of infinite sequences in A is again continuum. Therefore, checking the "left hand side" requires much more operations than checking the "right hand side," hence the equivalence is of considerable practical importance. When it comes to Theorem B, this argument does not work: whether we check the existence of maximizers directly or check the possibility of a cycle, we have to examine all finite subsets of A.

Looking for an alternative justification, we, first of all, notice that the argument is not quite convincing anyway. Although we can imagine a supreme being capable of looking through a continuum of items one by one, such activity has nothing to do with everyday realities of those studying mathematical models: there, the validity of a statement involving an infinite number of items can only be established by a reasoning, a proof. Both Theorems A and B allow us to concentrate attention on the (im)possibility of certain "patterns" rather than taking into account everything that might happen on an arbitrary (finite) subset of A. Theorem A still has an advantage: it refers to just one pattern, while Theorem B to an infinite sequence of them.

Now let us turn to maximization on compact subsets. A binary relation $\succ$ on a metric space is called *ω-transitive* if it is transitive and, whenever $\langle x^k \rangle_{k=0,1,\ldots}$ is an infinite improvement path and $x^k \to x^\omega$, there holds $x^\omega \succ x^0$. The property seems to have been first considered by Gillies (1959), who proved its sufficiency for the existence of maximal elements on compact sets. A binary relation $\succ$ is called *ω-acyclic* if it is acyclic and, whenever $\langle x^k \rangle_{k=0,1,\ldots}$ is an infinite improvement path and $x^k \to x^\omega$, there holds $x^\omega \neq x^0$. The prohibition of such cycles was introduced by Mukherji (1977) as "Condition (A5)."

Theorem C (Smith, 1974). *An ordering $\succ$ on A has the property that $M_\succ(X) \neq \emptyset$ for every compact $X \in \mathfrak{B} \setminus \{\emptyset\}$ [in other words, $\succ$ belongs to $\mathcal{R}_\exists(\mathfrak{C}_{\mathrm{omp}})$] if and only if it is ω-transitive.*

Theorem D (Kukushkin, 2008b). *An interval order $\succ$ on A belongs to $\mathcal{R}_\exists(\mathfrak{C}_{\mathrm{omp}})$ if and only if it is ω-acyclic.*

Remark. ω-transitivity and ω-acyclicity are equivalent for semiorders, but not for interval orders.

Again, the cardinality of $\mathfrak{C}_{\mathrm{omp}}$ is the same as that of the set of all convergent sequences. However, the argument involving "patterns" works: Once we established what may, or may not, happen when an improvement path converges, we have no need to think about arbitrary compact subsets.

Without the restriction to interval orders, the sufficiency part of Theorem D is just wrong. Moreover, Theorem G below shows that, in a sense, there is no possibility to adjust the theorem for the case of arbitrary binary relations. A characterization result becomes obtainable if we demand a certain degree of rationality, rather than the mere possibility, of choice.

Theorem E (Kukushkin, 2008b). *A binary relation $\succ$ on A belongs to $\mathcal{R}_{\mathrm{Out}}(\mathfrak{C}_{\mathrm{omp}})$ if and only if it is irreflexive and ω-transitive. Moreover, $\mathcal{R}_{\mathrm{PI}}(\mathfrak{C}_{\mathrm{omp}}) = \mathcal{R}_{\mathrm{Out}}(\mathfrak{C}_{\mathrm{omp}})$.*

The following statements, whose routine proofs are omitted, should have been in Kukushkin (2008b).

Proposition 3.1. *A binary relation $\succ$ on A belongs to $\mathcal{R}_{\mathrm{Rat}}(\mathfrak{C}_{\mathrm{omp}})$ if and only if it is an ω-transitive ordering.*

Proposition 3.2. *A binary relation $\succ$ on A belongs to $\mathcal{R}_{\exists!}(\mathfrak{C}_{\mathrm{omp}})$ if and only if it is an ω-transitive total order.*

Naturally, neither strong acyclicity, nor ω-transitivity remain necessary for the existence of maximizers on *convex* (compact) subsets. Moreover, even acyclicity is not necessary.

There are several existence results based on the Kakutani Theorem (Mas-Colell, 1974; Shafer, 1974, 1976; Shafer and Sonnenschein, 1975), which refer to economic equilibrium models or strategic games; we cite one result about abstract binary relations.

A binary relation $\succ$ is *quasiconcave* if $y \succ x$ whenever y is a convex combination of $y', y'' \in A$ such that $y' \succ x$ and $y'' \succ x$. A binary relation $\succ$ is *pseudoconcave* if $x \notin \mathrm{co}\{y \in A \mid y \succ x\}$ for every $x \in A$.

Theorem F (Kiruta et al., 1980, Theorem 1.7.1). *A binary relation $\succ$ on A belongs to $\mathcal{R}_{\exists}(\mathfrak{C}_{\mathrm{mpx}})$ if it is pseudoconcave and has open lower contours.*

4 Configurations

We denote $\mathbb{N} = \{0, 1, \dots\}$ the chain of natural numbers starting from zero. An *abstract configuration* C consists of: $\operatorname{Dom} C \subseteq \mathbb{N}$; $C_{=}, C_{\neq}, C_{\triangleright}, C_{\ntriangleright} \subseteq \operatorname{Dom} C \times \operatorname{Dom} C$; $C_{\pitchfork}, C_{\not\pitchfork} \subseteq \operatorname{Dom} C \times \operatorname{Dom} C \times \operatorname{Dom} C$; $C_{\to}, C_{\not\to} \subseteq (\operatorname{Dom} C)^{\mathbb{N}}$, where $(\operatorname{Dom} C)^{\mathbb{N}}$ means the set of mappings $\mathbb{N} \to \operatorname{Dom} C$, i.e., sequences in $\operatorname{Dom} C$. In the following we use indices $\varkappa \in \{=, \neq, \triangleright, \ntriangleright, \pitchfork, \not\pitchfork, \to, \not\to\}$. We always assume that $\operatorname{Dom} C \neq \emptyset$, while any $C_{\varkappa}$ may be empty.

Let $\succ$ be a binary relation on a convex subset A of a Banach space and C be an abstract configuration. A *realization* of C in A for $\succ$ is a mapping $\mu \colon \operatorname{Dom} C \to A$ such that: $\mu(k') = \mu(k)$ whenever $(k', k) \in C_{=}$; $\mu(k') \neq \mu(k)$ whenever $(k', k) \in C_{\neq}$; $\mu(k') \succ \mu(k)$ whenever $(k', k) \in C_{\triangleright}$; $\mu(k') \not\succ \mu(k)$ whenever $(k', k) \in C_{\ntriangleright}$; $\mu(k)$ is a convex combination of $\mu(k')$ and $\mu(k'')$ whenever $(k, k', k'') \in C_{\pitchfork}$; $\mu(k)$ is not a convex combination of $\mu(k')$ and $\mu(k'')$ whenever $(k, k', k'') \in C_{\not\pitchfork}$; $\mu(\nu(k)) \to \mu(\nu(0))$ whenever $\nu \in C_{\to}$; $\mu(\nu(k)) \not\to \mu(\nu(0))$ whenever $\nu \in C_{\not\to}$.

Many natural properties of binary relations can be expressed as the impossibility to realize a certain configuration. In the following list, all $C_{\varkappa}$ not explicitly mentioned are assumed empty.
Irreflexivity: $\operatorname{Dom} C = \{0\}$; $C_{\triangleright} = \{(0, 0)\}$.
Transitivity: $\operatorname{Dom} C = \{0, 1, 2\}$; $C_{\triangleright} = \{(1, 0), (2, 1)\}$; $C_{\ntriangleright} = \{(2, 0)\}$.
Condition (5): $\operatorname{Dom} C = \{0, 1, 2, 3\}$; $C_{\triangleright} = \{(1, 0), (3, 2)\}$; $C_{\ntriangleright} = \{(1, 2), (3, 0)\}$.
Condition (6): $\operatorname{Dom} C = \{0, 1, 2, 3\}$; $C_{\triangleright} = \{(1, 0), (2, 1)\}$; $C_{\ntriangleright} = \{(3, 0), (2, 3)\}$.
Negative transitivity: $\operatorname{Dom} C = \{0, 1, 2\}$; $C_{\triangleright} = \{(2, 0)\}$; $C_{\ntriangleright} = \{(1, 0), (2, 1)\}$.
Open lower contours (lower continuity): $\operatorname{Dom} C = \mathbb{N}$; $C_{\triangleright} = \{(0, 1)\}$; $C_{\ntriangleright} = \{(0, k)\}_{k \geq 2}$; $C_{\to} = \{\nu^{+}\}$, where $\nu^{+}(k) = k + 1$.
Weak lower continuity (Campbell and Walker, 1990): $\operatorname{Dom} C = \mathbb{N}$; $C_{\triangleright} = \{(0, 1)\} \cup \{(k, 0)\}_{k \geq 2}$; $C_{\to} = \{\nu^{+}\}$ with the same ν^{+}.
Quasiconcavity: $\operatorname{Dom} C = \{0, 1, 2, 3\}$; $C_{\triangleright} = \{(1, 0), (2, 0)\}$; $C_{\ntriangleright} = \{(3, 0)\}$; $C_{\pitchfork} = \{(3, 2, 1)\}$.

A wider range of properties can be described if we include the prohibition of every configuration from a (finite or infinite) list.
ω-Transitivity: transitivity plus $\operatorname{Dom} C^{(\omega)} = \mathbb{N}$; $C^{(\omega)}_{\triangleright} = \{(k + 1, k)\}_{k=1,2,\dots}$; $C^{(\omega)}_{\ntriangleright} = \{(0, 1)\}$; $C^{(\omega)}_{\to} = \{\nu^{0}\}$, where $\nu^{0}(k) = k$.

Acyclicity: ($m \in \mathbb{N}$) $\operatorname{Dom} C^{(m)} = \{0, \ldots, m+1\}$; $C^{(m)}_{\rhd} = \{(1,0), (2,1), \ldots, (m+1, m)\}$; $C^{(m)}_{=} = \{(0, m+1)\}$.
ω-Acyclicity: acyclicity plus $\operatorname{Dom} C^{(\omega)} = \mathbb{N}$; $C^{(\omega)}_{\rhd} = \{(k+1, k)\}_{k \in \mathbb{N}}$; and $C^{(\omega)}_{\to} = \{\nu^0\}$ with the same ν^0 as in the case of ω-transitivity.
Pseudoconcavity: irreflexivity plus ($m \in \mathbb{N}$, $m \geq 2$) $\operatorname{Dom} C^{(m)} = \{0, 1, \ldots, m\}$; $C^{(m)}_{\rhd} = \{(1,0), (2,0), \ \ldots, (m, 0)\}$; $C^{(2)}_{\pitchfork} = \{(0,1,2)\}$; if $m > 2$, then $C^{(m)}_{\pitchfork} = \{(0, 1, 2m-2), (2m-2, 2, 2m-3), \ldots, (m+1, m-1, m)\}$.

Given a mapping $R \colon A \to \mathfrak{B}$, i.e., a *correspondence*, we may define a binary relation on A by $y \succeq x \rightleftharpoons y \in R(x)$. The restrictions on R imposed in the Kakutani Theorem, except for the nonemptiness of the values, can be expressed as the impossibility to realize two configurations in A for $\succeq$. The convexity of values $R(x)$ is equivalent to the quasiconcavity of $\succeq$; the closed graph assumption prohibits this C: $\operatorname{Dom} C = \mathbb{N}$; $C_{\rhd} = \{(2k+1, 2k)\}_{k=1,2,\ldots}$; $C_{\not\rhd} = \{(1,0)\}$; $C_{\to} = \{\nu_0, \nu_1\}$, where $\nu_0(k) = 2k$ and $\nu_1(k) = 2k+1$. Moreover, defining a binary relation on A by $y \succ x \rightleftharpoons x \notin R(x) \ni y$ and assuming $R(x) \neq \emptyset$ for all $x \in A$, we see that a fixed point of R is the same thing as a maximizer of $\succ$ on A. The conditions of the Kakutani Theorem can be expressed in terms of configurations for $\succ$ as well.

A *simple configurational condition* consists of a set A of indices, and a set of abstract configurations $\mathcal{N}(\alpha)$ for every $\alpha \in$ A. We say that such a condition $\mathcal{C}$ *holds* on A for $\succ$ if there is $\alpha \in$ A such that no configuration $C \in \mathcal{N}(\alpha)$ admits a realization in A for $\succ$. The class of all simple configurational conditions is denoted $\mathcal{S}$.

Every condition from $\mathcal{S}$ is "inherited" (Walker, 1977): if such a condition holds on A for $\succ$, then it also holds on every $A' \subseteq A$ for the restriction of $\succ$ to A'. It seems natural, therefore, to use such conditions when trying to characterize properties of binary relations which are inherited by their nature (like the existence of a maximizer on every compact subset).

Theorem G (Kukushkin, 2008a). *There exists no condition $\mathcal{C} \in \mathcal{S}$ such that $\mathcal{C}$ would hold on a subset A of a finite-dimensional vector space for a transitive binary relation $\succ$ on A if and only if $\succ$ belongs to $\mathcal{R}_{\exists}(\mathfrak{C}_{\mathrm{omp}})$.*

Remark. Naturally, a necessary and sufficient condition applicable to arbitrary binary relations is "even more" impossible. Strictly speaking, the definition of an abstract configuration in Kukushkin (2008a) was narrower: $C_{\pitchfork}$ and $C_{\not\pitchfork}$ were absent. However, it is easily seen from the proof that the impossibility holds under our definition as well.

The language of configurations can also be used to formulate conditions on binary relations of a different form. For instance, the class $\mathcal{S}_1$ in Kukushkin (2008a) included requirements that a certain configuration *be* realizable. Kukushkin (2005) developed a theory of "configurational conditions" with arbitrary syntactical structure. Since there is no impossibility result concerning such conditions here, we do not need the whole apparatus. Yet, two basic definitions have to be reproduced because they are essential for Sections 6.2 and 7.2.

Let C and C' be abstract configurations; C' is an *extension* of C (denoted $C' \geq C$) if $\operatorname{Dom} C \subseteq \operatorname{Dom} C'$ and $C_\varkappa \subseteq C'_\varkappa$ for every $\varkappa$. Let $C' \geq C$, and μ and μ' be realizations of C and C', respectively, in the same A for the same $\succ$; then μ' is an *extension* of μ (denoted $\mu' \geq \mu$) if μ coincides with the restriction of μ' to $\operatorname{Dom} C$.

5 Main impossibility theorems

Theorem 1. *There exists no condition $\mathcal{C} \in \mathcal{S}$ such that $\mathcal{C}$ would hold on a convex subset A of a finite-dimensional vector space for a total order $\succ$ on A if and only if $\succ$ belongs to $\mathcal{R}_\exists(\mathfrak{C}_{\mathrm{mpx}})$. The same impossibility holds with respect to $\mathcal{R}_\exists(\mathfrak{C}_{\mathrm{onv}})$.*

Proof. We consider $A = \mathbb{R}$. Viewing $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$ of rational numbers, we fix a linear operator $u\colon \mathbb{R} \to \mathbb{R}$ such that $\operatorname{Ker} u = \mathbb{Q} \subset \mathbb{R}$ and a bijection $\sigma\colon \mathbb{Q} \to \mathbb{N}$.

Then we define total orders $\succ$ and $\succ^*$ on $\mathbb{R}$ by the following lexicographic constructions:

$$y \succ x \rightleftharpoons \big[u(y) > u(x) \ \text{ or } \ [u(y) = u(x) \ \&\ y > x]\big];$$

$$y \succ^* x \rightleftharpoons \big[x \notin \mathbb{Q} \ni y \ \text{ or } \ [y, x \in \mathbb{Q} \ \&\ \sigma(y) < \sigma(x)] \ \text{ or } \ [y, x \notin \mathbb{Q} \ \&\ y \succ x]\big].$$

If $X \in \mathfrak{C}_{\mathrm{onv}}$ and $\#X > 1$, then $M_\succ(X) = \emptyset$ because the set $u^{-1}(v)$ is dense in A for every $v \in u(A)$, which set is unbounded. On the other hand, $M_{\succ^*}(X) \neq \emptyset$ because $X \cap \mathbb{Q} \neq \emptyset$. Thus, $\succ$ does not belong to $\mathcal{R}_\exists(\mathfrak{C}_{\mathrm{mpx}})$, while $\succ^*$ belongs to $\mathcal{R}_\exists(\mathfrak{C}_{\mathrm{onv}})$. If a condition $\mathcal{C}$ characterizing either class existed, it would hold on $\mathbb{R}$ for $\succ^*$, but not for $\succ$.

Suppose it exists. Then there must be $\alpha^* \in$ A such that no $C \in \mathcal{N}(\alpha^*)$ admits a realization in $\mathbb{R}$ for $\succ^*$, while at least one $C \in \mathcal{N}(\alpha^*)$ admits a realization μ in $\mathbb{R}$ for $\succ$. We pick $d \in \mathbb{R} \setminus \{r - \mu(k)\}_{r \in \mathbb{Q},\, k \in \operatorname{Dom} \mu}$ and define $\mu^*\colon \operatorname{Dom} \mu \to \mathbb{R}$ by $\mu^*(k) = \mu(k) + d$. Clearly, the following equivalences hold for all $k, k', k'' \in \mathbb{N}$ and $\nu \in \mathbb{N}^{\mathbb{N}}$: $\mu^*(k) = \mu^*(k') \iff \mu(k) = \mu(k')$; $\mu^*(k) \succ \mu^*(k') \iff \mu(k) \succ \mu(k')$; $\mu^*(k)$ is a convex combination of

$\mu^*(k')$ and $\mu^*(k'')$ if and only if the same holds for $\mu(k)$, $\mu(k')$, and $\mu(k'')$; $\mu^*(\nu(k)) \to \mu^*(\nu(0)) \iff \mu(\nu(k)) \to \mu(\nu(0))$. Therefore, μ^* is also a realization of C in $\mathbb{R}$ for $\succ$. By the definition of d, we have $\mu^*(\operatorname{Dom}\mu) \cap \mathbb{Q} = \emptyset$, hence $\succ$ and $\succ^*$ coincide on $\mu^*(\operatorname{Dom}\mu)$. Thus, μ^* is a realization of C in $\mathbb{R}$ for $\succ^*$, contradicting the choice of α^*. □

Remark. Since $\succ^*$ in Theorem 1 is a total order, $\#M_{\succ^*}(X) = 1$ for every $X \in \mathfrak{C}_{\text{onv}}$; therefore, characterization of relations from $\mathcal{R}_\exists(\mathfrak{C}_{\text{onv}})$ or $\mathcal{R}_\exists(\mathfrak{C}_{\text{mpx}})$ with the property that all $M_{\succ^*}(X)$ are convex or closed is also impossible.

Unlike the situation with $\mathfrak{C} = \mathfrak{C}_{\text{omp}}$ (Kukushkin, 2005, 2008b), rationality of choice from convex subsets also allows no characterization by simple configurational conditions. In the following theorems, we show that the impossibility persists even in the presence of an oracle able to verify lower levels of rationality.

Theorem 2. *There exists no condition $\mathcal{C} \in \mathcal{S}$ such that, whenever A is a convex subset of a finite-dimensional vector space while $\succ$ is an interval order from $\mathcal{R}_\exists(\mathfrak{C}_{\text{mpx}})$, $\mathcal{C}$ holds for $\succ$ on A if and only if $\succ$ belongs to $\mathcal{R}_{\text{Out}}(\mathfrak{C}_{\text{mpx}})$.*

Remark. Naturally, the interval order cannot be replaced here with a semiorder, let alone a total order: whatever $\mathfrak{C} \subseteq \mathfrak{B}$, every semiorder belonging to $\mathcal{R}_\exists(\mathfrak{C})$ belongs to $\mathcal{R}_{\text{PI}}(\mathfrak{C})$.

Proof. As in the proof of Theorem 1, we assume $A = \mathbb{R}$ and define two interval orders on A:

$$y \succ x \rightleftharpoons \big[y > x > 0 \text{ or } 0 > y > x \text{ or } y < 0 < x\big];$$

$$y \succ^* x \rightleftharpoons \big[\, y, x \notin \mathbb{Q} \;\&\; y \succ x \,\big].$$

Let $X \in \mathfrak{C}_{\text{mpx}}$. If $0 \in X$, then $0 \in M_\succ(X) \neq \emptyset$; otherwise, $\max X \in M_\succ(X) \neq \emptyset$. On the other hand, for $X = [-1, 1]$ and $X' = [0, 1]$, we have $M_\succ(X) = \{0\} \subset X'$, but $M_\succ(X') = \{0, 1\} \neq \{0\}$, i.e., (2) does not hold for $\succ$. Therefore, $\succ$ belongs to $\mathcal{R}_\exists(\mathfrak{C}_{\text{mpx}}) \setminus \mathcal{R}_{\text{Out}}(\mathfrak{C}_{\text{mpx}})$. For each $X \in \mathfrak{C}_{\text{mpx}}$ such that $\#X > 1$, we have $M_{\succ^*}(X) \supseteq X \cap \mathbb{Q} \neq \emptyset$. Moreover, $M_{\succ^*}(X) \subseteq X' \subseteq X$ and $X' \in \mathfrak{C}_{\text{mpx}}$ immediately imply $X' = X$, hence (2) for $\succ^*$ holds for all $X, X' \in \mathfrak{C}_{\text{mpx}}$.

The rest of the proof is similar to that of Theorem 1. If such a condition $\mathcal{C}$ existed, it would hold for $\succ^*$, but not for $\succ$. On the other hand, if μ is a realization of a configuration C in $\mathbb{R}$ for $\succ$, we pick $d \in \mathbb{R}_{++} \setminus \{r/\mu(k)\}_{r \in \mathbb{Q},\, k \in \operatorname{Dom}\mu,\, \mu(k) \neq 0}$ and define $\mu^*\colon \operatorname{Dom}\mu \to \mathbb{R}$ by $\mu^*(k) = d \cdot \mu(k)$. Clearly, μ^* is a realization of C in $\mathbb{R}$ for $\succ^*$. □

Theorem 3. *There exists no condition $\mathcal{C} \in \mathcal{S}$ such that, whenever A is a convex subset of a finite-dimensional vector space while $\succ$ is an interval order from $\mathcal{R}_{\mathrm{Out}}(\mathfrak{C}_{\mathrm{mpx}})$, $\mathcal{C}$ holds for $\succ$ on A if and only if $\succ$ belongs to $\mathcal{R}_{\mathrm{PI}}(\mathfrak{C}_{\mathrm{mpx}})$. The same impossibility holds with respect to $\mathcal{R}_{\mathrm{Out}}(\mathfrak{C}_{\mathrm{onv}})$ and $\mathcal{R}_{\mathrm{PI}}(\mathfrak{C}_{\mathrm{onv}})$.*

Remark. The interval order cannot be replaced even with a semiorder for the same reasons as in Theorem 2.

Proof. As in the proof of Theorem 1, we perceive $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$ of rational numbers. We fix a subspace $\mathbb{Q}^\perp \subset \mathbb{R}$ such that $\mathbb{R} = \mathbb{Q} \oplus \mathbb{Q}^\perp$; clearly, $\mathbb{Q}^\perp$ is uncountable and dense in $\mathbb{R}$. Then we fix $q \in \mathbb{Q} \setminus \{0\}$ and define $\mathbb{T} = q + \mathbb{Q}^\perp$; note that $\mathbb{T} \cap \mathbb{Q}{=}\emptyset$.

Now we define two interval orders on $A = \mathbb{R}$:

$$y \succ x \rightleftharpoons \bigl[x \notin \mathbb{T} \;\&\; y \notin \mathbb{T} \;\&\; y > x\bigr];$$

$$y \succ^* x \rightleftharpoons \bigl[x \notin \mathbb{Q} \ni y \;\text{ or }\; [x \notin \mathbb{Q} \;\&\; y \notin \mathbb{Q} \;\&\; y \succ x]\bigr].$$

Whenever $X \in \mathfrak{C}_{\mathrm{onv}}$, we have either $\#X = 1$ or $X \cap \mathbb{T} \neq \emptyset \neq X \cap \mathbb{Q}$, hence $M_\succ(X) \neq \emptyset$. Let $X \in \mathfrak{C}_{\mathrm{onv}}$ and $\#X > 1$; then either $M_\succ(X) = X \cap \mathbb{T}$ or $M_\succ(X) = (X \cap \mathbb{T}) \cup \{\max X\}$. If $\sup X \notin X$, then $\sup X' = \sup X \notin X'$ whenever $X' \in \mathfrak{C}_{\mathrm{onv}}$ and $M_\succ(X) \subseteq X' \subseteq X$, hence $M_\succ(X') = M_\succ(X)$, i.e., (2) holds. Otherwise, $\max X \in X'$, hence $M_\succ(X') = (X' \cap \mathbb{T}) \cup \{\max X'\} = M_\succ(X)$ and (2) holds again. On the other hand, picking $a \in \mathbb{T}$, $X = X' = [a-1, a]$, $b \in X \setminus \mathbb{T}$, and $X'' = \{b\}$, we immediately see that $X = X' \cup X''$, but $M_\succ(X) = X \cap \mathbb{T} \neq (X \cap \mathbb{T}) \cup \{b\} = M_\succ(M_\succ(X') \cup X'')$, i.e., (3) does not hold for $\succ$. Therefore, $\succ$ belongs to $\mathcal{R}_{\mathrm{Out}}(\mathfrak{C}_{\mathrm{onv}}) \setminus \mathcal{R}_{\mathrm{PI}}(\mathfrak{C}_{\mathrm{mpx}})$.

Similarly, $M_{\succ^*}(X) = X \cap \mathbb{Q} \neq \emptyset$ whenever $X \in \mathfrak{C}_{\mathrm{onv}}$ and $\#X > 1$, hence $M_{\succ^*}(X) = M_{\succ^*}(X')$ for every $X' \in \mathfrak{B}$ such that $M_{\succ^*}(X) \subseteq X' \subseteq X$. Now path independence (3) follows in a standard way; therefore, $\succ^*$ belongs to $\mathcal{R}_{\mathrm{PI}}(\mathfrak{C}_{\mathrm{onv}})$.

The rest of the proof is again similar to that of Theorem 1. If μ is a realization of a configuration C in $\mathbb{R}$ for $\succ$, we pick $d \in \mathbb{Q}^\perp \setminus \{r - \mu(k)\}_{r \in \mathbb{Q},\, k \in \mathrm{Dom}\,\mu}$ [$\neq \emptyset$ because $\mathbb{Q}^\perp$ is uncountable] and define $\mu^*\colon \mathrm{Dom}\,\mu \to \mathbb{R}$ by $\mu^*(k) = \mu(k) + d$. By the definition of $\mathbb{Q}^\perp$ and $\mathbb{T}$, we have $\mu^*(k) \in \mathbb{T} \iff \mu(k) \in \mathbb{T}$; by the definition of d, $\mu^*(k) \notin \mathbb{Q}$ for any $k \in \mathrm{Dom}\,\mu$. Therefore, μ^* is a realization of C in $\mathbb{R}$ for $\succ^*$. □

Theorem 4. *There exists no condition $\mathcal{C} \in \mathcal{S}$ such that, whenever A is a convex subset of a finite-dimensional vector space while $\succ$ is an interval order from $\mathcal{R}_\exists(\mathfrak{C}_{\mathrm{onv}})$, $\mathcal{C}$ holds for $\succ$ on A if and only if $\succ$ belongs to $\mathcal{R}_{\mathrm{Out}}(\mathfrak{C}_{\mathrm{onv}})$.*

Remark. Logically, this result should have preceded Theorem 3; however, its proof employs more complicated constructions. The replacement of the interval order with a semiorder is again impossible for the same reasons.

Proof. We assume $A = \mathbb{R}^2$, viewing the plane as a vector space over $\mathbb{R}$, and simultaneously introduce polar coordinates by picking $e \in A$ with $\|e\| = 1$ and defining $\varphi(x)$ (for $x \neq 0$) as the angle from e to x measured counterclockwise. Clearly, $\|x\| \geq 0$ and $\varphi(x)$ ($0 \leq \varphi(x) < 2\pi$) uniquely define $x \in A$. Defining $\mathbb{Q}^\perp \subset \mathbb{R}$ and $\mathbb{T} = q + \mathbb{Q}^\perp$ exactly as in the proof of Theorem 3, we denote $T = \{x \in A \setminus \{0\} \mid \log\|x\| \in \mathbb{T}\} \cup \{0\}$ and $Q = \{x \in A \setminus \{0\} \mid \log\|x\| \in \mathbb{Q}\}$.

Then we define two interval orders on A:

$$y \succ x \rightleftharpoons \big[x \notin T \ \&\ y \notin T \ \&\ \big(\|y\| > \|x\| \ \text{ or } \ [\|y\| = \|x\| \ \&\ \varphi(y) > \varphi(x)]\big)\big];$$

$$y \succ^* x \rightleftharpoons \big[x \notin Q \ni y \ \text{ or } \ [x \notin Q \ \&\ y \notin Q \ \&\ y \succ x]\big].$$

Clearly, $\succ$ belongs to $\mathcal{R}_\exists(\mathfrak{C}_{\text{onv}})$ and $\succ^*$ belongs to $\mathcal{R}_{\text{PI}}(\mathfrak{C}_{\text{onv}})$ for the same reasons as in the proof of Theorem 3. To show that $\succ$ does not belong to $\mathcal{R}_{\text{Out}}(\mathfrak{C}_{\text{onv}})$, let us pick $r > 0$ for which $\log r \notin \mathbb{T}$, and define $X = \{x \in A \mid \|x\| \leq r\}$. Clearly, $M_\succ(X) = X \cap T$. Defining $X' = \{x \in A \mid \|x\| < r\} \cup \{r \cdot e\}$, we obtain $X' \in \mathfrak{C}_{\text{onv}}$ and $M_\succ(X) \subset X' \subset X$, while $M_\succ(X') = M_\succ(X) \cup \{r \cdot e\}$, i.e., (2) does not hold.

The rest of the proof is virtually the same as in Theorem 3. If μ is a realization of a configuration C in $\mathbb{R}^2$ for $\succ$, we pick $d \in \mathbb{Q}^\perp \setminus \{r - \log\|\mu(k)\|\}_{r\in\mathbb{Q},\, k\in\text{Dom}\,\mu,\, \mu(k)\neq 0}$ and define $\mu^* \colon \text{Dom}\,\mu \to \mathbb{R}^2$ by $\mu^*(k) = \exp(d)\cdot \mu(k)$. By the definition of $\mathbb{Q}^\perp$ and $\mathbb{T}$, we have $\mu^*(k) \in T \iff \mu(k) \in T$; by the definition of d, $\mu^*(k) \notin Q$ for any $k \in \text{Dom}\,\mu$. Therefore, μ^* is a realization of C in $\mathbb{R}^2$ for $\succ^*$. ☐

Theorem 5. *There exists no condition $\mathcal{C} \in \mathcal{S}$ such that, whenever A is a convex subset of a finite-dimensional vector space while $\succ$ is a semiorder from $\mathcal{R}_{\text{PI}}(\mathfrak{C}_{\text{mpx}})$, $\mathcal{C}$ holds for $\succ$ on A if and only if $\succ$ belongs to $\mathcal{R}_{\text{Rat}}(\mathfrak{C}_{\text{mpx}})$. The same impossibility holds for $\mathcal{R}_{\text{PI}}(\mathfrak{C}_{\text{onv}})$ and $\mathcal{R}_{\text{Rat}}(\mathfrak{C}_{\text{onv}})$.*

Remark. The semiorder cannot be replaced with an ordering, let alone a total order: whatever $\mathfrak{C} \subseteq \mathfrak{B}$, every ordering belonging to $\mathcal{R}_{\text{PI}}(\mathfrak{C})$ belongs to $\mathcal{R}_{\text{Rat}}(\mathfrak{C})$.

Proof. Denoting $Q = \mathbb{Q} \setminus \{-1, 0, 1\}$, we define two semiorders on $A = \mathbb{R}$:

$$y \succ x \rightleftharpoons \big[x \notin \{-1, 0, 1\} \ni y \ \text{ or } \ [x = -1 \ \&\ y = 1]\big];$$

$$y \succ^* x \rightleftharpoons \big[x \notin Q \ni y \text{ or } [x \notin Q \;\&\; y \notin Q \;\&\; y \succ x]\big].$$

Both relations obviously belong to $\mathcal{R}_{\exists}(\mathfrak{C}_{\text{onv}})$; since they are semiorders, they belong to $\mathcal{R}_{\text{PI}}(\mathfrak{C}_{\text{onv}})$ as well. To show that $\succ$ does not belong to $\mathcal{R}_{\text{Rat}}(\mathfrak{C}_{\text{mpx}})$, let us pick $X = [-1,1]$ and $X' = [-1,0]$; clearly, $M_{\succ}(X) = \{0,1\}$ while $M_{\succ}(X') = \{-1,0\}$, i.e., (4) does not hold. On the other hand, whenever $X \in \mathfrak{C}_{\text{onv}}$ and $\#X > 1$, we have $M_{\succ^*}(X) = X \cap Q$, hence (4) for $\succ^*$ holds.

The rest of the proof is standard, but requires plenty of notation. We denote $R_{-2} =]-\infty, -1[$, $R_{-1} =]-1, 0[$, $R_1 =]0,1[$, $R_2 =]1, +\infty[$, and $R = \bigcup_{h \in \{-2,-1,1,2\}} R_h = \mathbb{R} \setminus \{-1,0,1\}$. Then we define a mapping $\mathbb{R}_{++} \times \mathbb{R} \to \mathbb{R}$ by

$$\psi(d,x) = \begin{cases} dx + d - 1, & \text{if } x \le -1; \\ -(-x)^d, & \text{if } -1 \le x < 0; \\ 0, & \text{if } x = 0; \\ x^d, & \text{if } 0 < x \le 1; \\ dx - d + 1, & \text{if } x \ge 1. \end{cases}$$

Clearly, ψ is continuous and strictly increasing in x for every $d > 0$, and $\psi(d,x) \in R_h$ whenever $x \in R_h$ ($h \in \{-2,-1,1,2\}$). Moreover, whenever $v, x \in R_h$, there exists no more than one $d \in \mathbb{R}_{++}$ such that $\psi(d,x) = v$; when it exists, we denote it $\chi(v,x)$.

Now if μ is a realization of a configuration C in $\mathbb{R}$ for $\succ$, we denote $H_\mu = \{\chi(r, \mu(k)) \mid k \in \operatorname{Dom}\mu \;\&\; \exists h \in \{-2,-1,1,2\}\, [r \in Q \cap R_h \;\&\; \mu(k) \in R_h]\}$, pick $d \in \mathbb{R}_{++} \setminus H_\mu$ ($\neq \emptyset$ because H_μ is countable) and define $\mu^* \colon \operatorname{Dom}\mu \to \mathbb{R}$ by $\mu^*(k) = \psi(d, \mu(k))$. By the definition of d, we have $\mu^*(k) \notin Q$ for any $k \in \operatorname{Dom}\mu$; therefore, μ^* is a realization of C in $\mathbb{R}$ for $\succ^*$ since $\psi(d,\cdot)$ is continuous and strictly increasing. □

Theorem 6. *There exists no condition $\mathcal{C} \in \mathcal{S}$ such that, whenever A is a convex subset of a finite-dimensional vector space while $\succ$ is an ordering from $\mathcal{R}_{\text{Rat}}(\mathfrak{C}_{\text{mpx}})$, $\mathcal{C}$ holds for $\succ$ on A if and only if $\succ$ belongs to $\mathcal{R}_{\exists!}(\mathfrak{C}_{\text{mpx}})$. The same impossibility holds for $\mathcal{R}_{\text{Rat}}(\mathfrak{C}_{\text{onv}})$ and $\mathcal{R}_{\exists!}(\mathfrak{C}_{\text{onv}})$.*

Remark. The ordering cannot be replaced with a total order for the same reasons as above.

Proof. We consider $A = \mathbb{R}$ and fix a bijection $\sigma \colon \mathbb{Q} \to \mathbb{N}$. Then we define $\succ$ by $y \not\succ x$ for all $y, x \in \mathbb{R}$ and

$$y \succ^* x \rightleftharpoons \big[x \notin \mathbb{Q} \ni y \text{ or } [y, x \in \mathbb{Q} \;\&\; \sigma(y) < \sigma(x)]\big].$$

Both are obviously orderings.

For every $X \in \mathfrak{C}_{\text{onv}}$, we have $M_{\succ}(X) = X$, hence $\succ$ belongs to $\mathcal{R}_{\text{Rat}}(\mathfrak{C}_{\text{onv}}) \setminus \mathcal{R}_{\exists!}(\mathfrak{C}_{\text{mpx}})$. Whenever $X \in \mathfrak{C}_{\text{onv}}$ and $\#X > 1$, we have $M_{\succ^*}(X) = \text{Argmin}_{x \in X \cap \mathbb{Q}}\, \sigma(x)$, hence $\succ^*$ belongs to $\mathcal{R}_{\exists!}(\mathfrak{C}_{\text{onv}})$.

The rest of the proof is again standard. □

6 Quasiconcave relations

The results of the previous section indicate, in particular, that quasiconcavity does not emerge as an indispensable condition if one wants the existence or rationality of choice from convex subsets. Here we impose the condition exogenously.

6.1 Simple conditions

The characterization by conditions from $\mathcal{S}$ of *arbitrary* quasiconcave binary relations ensuring the existence of maximizers on convex subsets remains impossible.

Theorem 7. *There exists no condition $\mathcal{C} \in \mathcal{S}$ such that $\mathcal{C}$ would hold on a convex subset A of a finite-dimensional vector space for a quasiconcave binary relation $\succ$ on A if and only if $\succ$ belongs to $\mathcal{R}_{\exists}(\mathfrak{C}_{\text{mpx}})$. The same impossibility holds with respect to $\mathcal{R}_{\exists}(\mathfrak{C}_{\text{onv}})$.*

Proof. First, we denote $R_{-1} =]-\infty, -\pi/2[$, $R_0 =]-\pi/2, \pi/2[$, $R_{+1} =]\pi/2, +\infty[$, and $R = R_{-1} \cup R_0 \cup R_{+1} = \mathbb{R} \setminus \{-\pi/2, \pi/2\}$. Similarly to the proof of Theorem 5, we define a mapping $\mathbb{R}_{++} \times \mathbb{R} \to \mathbb{R}$ by

$$\psi(d,x) = \begin{cases} d \cdot (x + \pi/2) - \pi/2, & \text{if } x < -\pi/2; \\ -\pi/2, & \text{if } x = -\pi/2; \\ \arctan(\tan x + d), & \text{if } -\pi/2 < x < \pi/2; \\ \pi/2, & \text{if } x = \pi/2; \\ d \cdot (x - \pi/2) + \pi/2, & \text{if } x > \pi/2. \end{cases}$$

Clearly, ψ is continuous and strictly increasing in x for every $d > 0$, and $\psi(d,x) \in R_h$ whenever $x \in R_h$ ($h = -1, 0, +1$). Moreover, whenever $v, x \in R_h$, there exists no more than one $d \in \mathbb{R}_{++}$ such that $\psi(d,x) = v$; when it exists, we denote it $\chi(v,x)$.

Then we fix $\bar{d} \in]0,1[$ and define a mapping $\varphi \colon \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) = \begin{cases} \psi(\bar{d}, x), & \text{if } x \in R; \\ -x, & \text{otherwise.} \end{cases}$$

Denoting $Q_{-1} = \mathbb{Q} \cap R_{-1}$, $Q_{+1} = \mathbb{Q} \cap R_{+1}$, $Q_0 = \{x \in R_0 \mid \tan x \in \mathbb{Q}\}$, and $Q = Q_{-1} \cup Q_{+1} \cup Q_0$, we define two binary relations on $A = \mathbb{R}$:

$$y \succ x \rightleftharpoons \bigl[y = \varphi(x)\bigr];$$

$$y \succ^* x \rightleftharpoons \bigl[x \notin Q \;\&\; y \notin Q \;\&\; y \succ x\bigr].$$

Both are obviously quasiconcave.

The second relation belongs to $\mathcal{R}_\exists(\mathfrak{C}_{\mathrm{onv}})$ because Q is dense in A. To show that $\succ$ does not belong to $\mathcal{R}_\exists(\mathfrak{C}_{\mathrm{mpx}})$, we pick $X = [-\pi/2, \pi/2]$; clearly, $\varphi(x) \in X$ for every $x \in X$, hence $M_\succ(X) = \emptyset$.

The rest of the proof is again standard. If μ is a realization of a configuration C in $\mathbb{R}$ for $\succ$, we denote $H_\mu = \{\chi(r, \mu(k)) \mid k \in \operatorname{Dom}\mu \;\&\; \exists h \in \{-1, 0, +1\}\,[r \in Q_h \;\&\; \mu(k) \in R_h]\}$, pick $d \in \mathbb{R}_{++} \setminus H_\mu$ ($\neq \emptyset$ because H_μ is countable) and define $\mu^*\colon \operatorname{Dom}\mu \to \mathbb{R}$ by $\mu^*(k) = \psi(d, \mu(k))$. By the definition of d, we have $\mu^*(k) \notin Q$ for any $k \in \operatorname{Dom}\mu$; besides, $y = \varphi(x) \iff \psi(d, y) = \varphi(\psi(d, x))$ for all $y, x \in \mathbb{R}$. Therefore, μ^* is a realization of C in $\mathbb{R}$ for $\succ^*$. $\square$

Unfortunately, there is neither positive nor negative result about any narrower class of preferences. The ω-transitivity (plus irreflexivity) of $\succ$ ensures the nonemptyness of $M_\succ(X)$ for every $X \in \mathfrak{C}_{\mathrm{mpx}}$; however, it is not necessary even when $\succ$ is a total order.

Example 6.1. We introduce polar coordinates on $A = \mathbb{R}^2$ exactly as in the proof of Theorem 4 and define a quasiconcave total order $\succ$ on A by

$$y \succ x \rightleftharpoons \bigl[\, \|y\| < \|x\| \text{ or } [\|y\| = \|x\| \;\&\; \varphi(y) > \varphi(x)] \,\bigr].$$

Whenever $X \in \mathfrak{C}_{\mathrm{mpx}}$, there is a unique $x^* \in X$ where $\|x\|$ is minimized over X; clearly, $M_\succ(X) = \{x^*\} \neq \emptyset$. On the other hand, the improvement path x^k with $\|x^k\| = 1$ and $\varphi(x^k) = 2k\pi/(k+1)$ converges to $x^\omega = x^0$, obviously violating the definition of ω-transitivity.

A transitive binary relation $\succ$ on a convex subset of a Banach space is called *ω-L-transitive* if, whenever $\langle x^k \rangle_{k \in \mathbb{N}}$ is an infinite improvement path such that all vectors $x^k - x^0$ belong to the same one-dimensional subspace and $x^k \to x^\omega$, there holds $x^\omega \succ x^0$. As in the case of ω-transitivity, $x^\omega \succ x^k$ is valid for all $k = 0, 1, \ldots$ in this situation, once $\succ$ is ω-L-transitive. A binary relation $\succ$ on a convex subset of a Banach space is called *strongly L-acyclic* if there is no infinite improvement path $\langle x^k \rangle_{k \in \mathbb{N}}$ such that all vectors $x^k - x^0$ belong to the same one-dimensional subspace.

Proposition 6.2. *Let $\succ$ be a pseudoconcave semiorder on a convex subset of a Banach space. If $\succ$ belongs to $\mathcal{R}_{\exists}(\mathfrak{C}_{\mathrm{mpx}})$, then it is ω-L-transitive. If $\succ$ belongs to $\mathcal{R}_{\exists}(\mathfrak{C}_{\mathrm{onv}})$, then it is strongly L-acyclic.*

Proof. First, we may, without restricting generality, assume $A \subseteq \mathbb{R}$. Since $\succ$ is pseudoconcave, $y' \succ x$ and $y'' \succ x$ are incompatible with $y' \leq x \leq y''$.

Let $\langle x^k \rangle_{k \in \mathbb{N}}$ be an infinite improvement path. We denote $\mathbb{N}^{\downarrow} = \{k \in \mathbb{N} \mid \forall y \in A\, [y \succ x^k \Rightarrow y < x^k]\}$ and $\mathbb{N}^{\uparrow} = \{k \in \mathbb{N} \mid \forall y \in A\, [y \succ x^k \Rightarrow y > x^k]\}$. By the previous argument, $\mathbb{N} = \mathbb{N}^{\downarrow} \cup \mathbb{N}^{\uparrow}$, hence at least one of them must be infinite. Without restricting generality, we may assume that $\mathbb{N} = \mathbb{N}^{\uparrow}$. (We may be forced to delete x^0 itself; however, no problem will be created even in the case of the first statement because $x^k \succ x^0$ for each $k \in \mathbb{N}$.)

Proving the first statement, we assume that $x^k \to x^{\omega}$, denote $X = [x^0, x^{\omega}]$, and pick $y \in M_{\succ}(X)$. For every $k \in \mathbb{N}$, we have $x^{k+2} \succ x^{k+1} \succ x^k$, hence $y \succ x^k$ since $\succ$ is a semiorder, hence $y > x^k$; therefore, $x^{\omega} = y \succ x^0$.

Turning to the second statement, we denote $X = \mathrm{co}\{x^k\}_{k \in \mathbb{N}}$ and again pick $y \in M_{\succ}(X)$. Since $\succ$ is a semiorder, we must have $y \succ x^k$, hence $y > x^k$ for each k; however, this is incompatible with $y \in X$. Therefore, an infinite improvement path is impossible. □

Corollary. *Let $\succ$ be a pseudoconcave semiorder on a convex subset $A \subseteq \mathbb{R}$. Then $\succ$ belongs to $\mathcal{R}_{\exists}(\mathfrak{C}_{\mathrm{mpx}})$ if and only if it is ω-transitive, while $\succ$ belongs to $\mathcal{R}_{\exists}(\mathfrak{C}_{\mathrm{onv}})$ if and only if it is strongly acyclic.*

Example 3 from Kukushkin (2008b) shows that the replacement of the semiorder in the first statement of Proposition 6.2 with an interval order makes it just wrong. What is more unpleasant, the converse statement is wrong even for quasiconcave total orders on the plane.

Example 6.3. We introduce polar coordinates on $\mathbb{R}^2$ exactly as in Example 6.1 (or in the proof of Theorem 4). There is a unique $e^{\perp} \in \mathbb{R}^2$ such that $\|e^{\perp}\| = 1$ and $\varphi(e^{\perp}) = \pi/2$. We assume $A = \{x \in \mathbb{R}^2 \mid \|x - e^{\perp}\| \leq 1\}$. Clearly, $0 < \varphi(x) < \pi$ for every $x \in A \setminus \{0\}$; for technical convenience, we assume here $\varphi(0) = 0$. Then we define a quasiconcave total order $\succ$ on A by

$$y \succ x \rightleftharpoons \big[\, \varphi(y) > \varphi(x) \ \text{ or } \ [\varphi(y) = \varphi(x) \ \&\ \|y\| > \|x\|] \,\big].$$

Since φ does not attain a maximum on A, we have $M_{\succ}(A) = \emptyset$; since A is compact itself, $\succ$ does not belong to $\mathcal{R}_{\exists}(\mathfrak{C}_{\mathrm{mpx}})$. On the other hand, on every segment of a straight line in A, the order $\succ$ coincides with the order on the real line; therefore, $\succ$ is ω-L-transitive.

The invalidity of the second statement in Proposition 6.2 for an interval order is also easy to show. The converse statement is wrong even for orderings (a quasiconcave total order cannot belong to $\mathcal{R}_\exists(\mathfrak{C}_{\text{onv}})$ unless $\#A = 1$).

Example 6.4. Let $A = \mathbb{R}$. We define a quasiconcave interval order $\succ$ on A by a numeric representation

$$y \succ x \rightleftharpoons f^-(y) > f^+(x),$$

where

$$f^-(x) = \begin{cases} -2, & \text{if } x < 0; \\ \\ -1/(k+1), & \text{if } k \le x < k+1 \, (k \in \mathbb{N}); \end{cases}$$

$$f^+(x) = \begin{cases} -1, & \text{if } x < 0; \\ 0, & \text{if } x \in \mathbb{N}; \\ -1/(k+2), & \text{if } k < x < k+1 \, (k \in \mathbb{N}). \end{cases}$$

Let $X \in \mathfrak{C}_{\text{onv}}$. If $X \cap \mathbb{N} \neq \emptyset$, then $M_\succ(X) = X \cap \mathbb{N}$. Otherwise, both f^- and f^+ are constants on X, hence $M_\succ(X) = X$. Therefore, $\succ$ belongs to $\mathcal{R}_\exists(\mathfrak{C}_{\text{onv}})$ [actually, it even belongs to $\mathcal{R}_{\text{PI}}(\mathfrak{C}_{\text{onv}})$]. On the other hand, $\succ$ is not strongly acyclic: the sequence $x^k = (4k+1)/2$ is an infinite improvement path.

Example 6.5. We consider the same set A as in Example 6.3 and the same polar coordinates on $\mathbb{R}^2$, again assuming $\varphi(0) = 0$. For every $x \in A$, we define $k(x)$ as the greatest $k \in \mathbb{N}$ for which $\pi - \varphi(x) < 1/k$; it is well defined because $\varphi(x) < \pi$. Then we consider the ordering $\succ$ on A defined by the function $k(x)$:

$$y \succ x \rightleftharpoons k(y) > k(x).$$

It is obviously quasiconcave.

Since $k(x)$ does not attain a maximum on A, we have $M_\succ(A) = \emptyset$, hence $\succ$ does not belong to $\mathcal{R}_\exists(\mathfrak{C}_{\text{onv}})$. On the other hand, $k(x)$ is bounded above on every segment of a straight line in A; therefore, $\succ$ is strongly L-acyclic.

6.2 Conditions of the type "$\forall\,\exists$"

Some characterization results can be obtained if we allow the use of conditions with two different quantifiers. A transitive binary relation $\succ$ on a convex subset of a Banach space is called *ω-C-transitive* if, whenever $\langle x^k \rangle_{k\in\mathbb{N}}$ is an infinite improvement path and $x^k \to x^\omega$, there is $y \in \operatorname{cl}\operatorname{co}\{x^k\}_{k\in\mathbb{N}}$ such that $y \succ x^k$ for each $k \in \mathbb{N}$; $\succ$ is called *strongly C-transitive* if, whenever $\langle x^k \rangle_{k\in\mathbb{N}}$ is an infinite improvement path, there is $y \in \operatorname{co}\{x^k\}_{k\in\mathbb{N}}$ such that $y \succ x^k$ for each $k \in \mathbb{N}$. Clearly, strong C-transitivity implies ω-C-transitivity. If either property holds on A, then it holds on every convex (and closed) subset of A.

Although we do not reproduce the whole theory developed in Kukushkin (2005), it seems worthwhile to explain in what sense both conditions belong to the class $\forall\exists$. Let us define abstract configurations C^0 and C' in this way: $\operatorname{Dom} C^0 = \{0\} \cup \{2k+3\}_{k\in\mathbb{N}}$; $C^0_{\triangleright} = \{(2k+5, 2k+3)\}_{k\in\mathbb{N}}$; $C^0_{\rightarrow} = \{\nu^*\}$, where $\nu^*(0) = 0$ while $\nu^*(k+1) = 2k+3$ for all $k \in \mathbb{N}$; $\operatorname{Dom} C' = \mathbb{N}$; $C'_{\triangleright} = C_{\triangleright} \cup \{(2, 2k+3)\}_{k\in\mathbb{N}}$; $C'_{\pitchfork} = \{(2k+6, 2k+4, 2k+5)\}_{k\in\mathbb{N}} \cup \{(4,3,5), (2,1,0)\}$; $C'_{\rightarrow} = C_{\rightarrow} \cup \{\nu^{**}\}$, where $\nu^{**}(0) = 1$ while $\nu^{**}(k+1) = 2k+4$ for all $k \in \mathbb{N}$. Clearly, $C' \geq C$ in the sense of the definition at the end of Section 4.

Proposition 6.6. *A binary relation $\succ$ on a convex subset A of a Banach space is ω-C-transitive if and only if, for every realization μ of C^0 in A for $\succ$, there exists a realization $\mu' \geq \mu$ of C'.*

Proof. Given a realization μ of C^0, the points $x^k = \mu(2k+3)$ form an infinite improvement path converging to $x^\omega = \mu(0)$. If a realization $\mu' \geq \mu$ of C' exists, then $\operatorname{co}\{x^0, \ldots, x^{k+1}\} \ni y^k = \mu'(2k+4)$ and $y^k \to y^\omega = \mu'(1) \in \operatorname{cl}\operatorname{co}\{x^k\}_{k\in\mathbb{N}}$; defining $y = \mu'(2)$, we have $y \succ x^k$ for each $k \in \mathbb{N}$ and $y \in \operatorname{cl}\operatorname{co}\{x^k\}_{k\in\mathbb{N}}$. Conversely, if such a y can be found, we have $y = c_\omega x^\omega + \sum_{k\in\mathbb{N}} c_k x^k$, where all c_k and c_ω are non-negative and $c^\omega + \sum_{k\in\mathbb{N}} c_k = 1$. Denoting $\mu'(2k+4) = \sum_{h=0}^{k} c_h x^h / \sum_{h=0}^{k} c_h$ (if $\sum_{h=0}^{k} c_h = 0$, then, say, $\mu'(2k+4) = x^0$), we have $\mu'(2k+6) \in \operatorname{co}\{\mu'(2k+4), \mu'(2k+5)\}$ for all $k \in \mathbb{N}$; denoting $\mu'(1) = \sum_{k\in\mathbb{N}} c_k x^k / \sum_{k\in\mathbb{N}} c_k$ (with the same agreement about 0) and $\mu'(2) = y$, we have $\mu'(2k+4) \to \mu'(1)$ and $\mu'(2) \in \operatorname{co}\{\mu'(1), \mu'(0)\}$. Thus, μ' is a realization of C'. $\square$

Now we define configurations C^1 and $C^{(m)}$ ($m \in \mathbb{N}$) in this way: $\operatorname{Dom} C^1 = 2 \cdot \mathbb{N}$; $C^1_{\triangleright} = \{(2k+2, 2k)\}_{k\in\mathbb{N}}$; $\operatorname{Dom} C^{(m)} = \operatorname{Dom} C \cup \{1, 3, \ldots, 2m+1\}$; $C^{(m)}_{\triangleright} = C_{\triangleright} \cup \{(2m+1, 2k)\}_{k\in\mathbb{N}}$; $C^{(m)}_{\pitchfork} = \{(2k+3, 2k+1, 2k)\}_{k\in\mathbb{N}} \cup \{(1, 0, 2)\}$.

Proposition 6.7. *A binary relation $\succ$ on a convex subset A of a Banach space is strongly C-transitive if and only if, for every realization μ of C^1 in A for $\succ$, there exist $m \in \mathbb{N}$ and a realization $\mu' \geq \mu$ of $C^{(m)}$.*

The proof is similar to that of Proposition 6.6 and hence omitted.

Theorem 8. *Let $\succ$ be a quasiconcave, irreflexive and transitive binary relation on a convex subset A of a finite-dimensional vector space, and $X \in \mathfrak{B}$. Then either of the following conditions is sufficient for the property that for every $x \in X \setminus M_\succ(X)$ there is $y \in M_\succ(X)$ such that $y \succ x$* ("NM property"):

1. *$\succ$ is ω-C-transitive and $X \in \mathfrak{C}_{\mathrm{mpx}}$;*
2. *$\succ$ is strongly C-transitive and $X \in \mathfrak{C}_{\mathrm{onv}}$.*

Remark. Both conditions are put into the same theorem because there is a lot of similarity in the proofs.

Proof. Let X be an appropriate subset of A, i.e., $X \in \mathfrak{C}_{\mathrm{mpx}}$ under Condition 1, or $X \in \mathfrak{C}_{\mathrm{onv}}$ under Condition 2; let $x^* \in X \setminus M_\succ(X)$. For every $x \in X$, we denote $G(x) = \{y \in X \mid y \succ x\} \in \mathfrak{C}_{\mathrm{onv}}$; note that $G(x^*) \neq \emptyset$. The key step is proving that $M_\succ(G(x^*)) \neq \emptyset$: if $y \in M_\succ(G(x^*))$, then $y \succ x$ by definition while $y \in M_\succ(X)$ by transitivity.

We, naturally, apply Zorn's Lemma to $G(x^*)$ ordered by $\succ$: a maximizer exists if every chain admits an upper bound. Let $L \subseteq G(x^*)$ be a chain w.r.t. $\succ$. If there is a maximum in L, we are home immediately; otherwise, $G(x) \neq \emptyset$ for every $x \in L$. Now the existence of an upper bound is equivalent to $G^L = \bigcap_{x \in L} G(x) \neq \emptyset$.

We denote $F(x) = \operatorname{cl} G(x)$ for every $x \in L$ and $F^L = \bigcap_{x \in L} F(x) \supseteq G^L$. By the Lindelöf theorem, (see, e.g., Kuratowski, 1966, p. 54), there is a countable $L' \subseteq L$ such that $F^L = \bigcap_{x \in L'} F(x)$. Since every countable chain can be embedded into $\mathbb{Q}$, there is an infinite improvement path $\langle x^k \rangle_{k \in \mathbb{N}} \subseteq L$ such that $F^L = \bigcap_{k \in \mathbb{N}} F(x^k)$. We denote $G^\omega = \bigcap_{k \in \mathbb{N}} G(x^k)$; clearly, $G^\omega \subseteq F^L$.

Let us show $G^\omega \neq \emptyset$. Under Condition 2, we immediately apply the definition of strong C-transitivity: since $y \in \operatorname{co}\{x^k\}_{k \in \mathbb{N}}$, we have $y \in X$. Under Condition 1, X is compact, so we may, without restricting generality, assume $x^k \to x^\omega \in X$, and then argue in the same way. Thus, $F^L \neq \emptyset$ too.

If $G^L = G^\omega$, we are home; suppose $G^L \subset G^\omega$. Then $F^L \subseteq \operatorname{cl} G^\omega \subseteq F^L$, hence $\operatorname{cl} G^\omega = F^L = \operatorname{cl} G(y)$ for every $y \in L \cap G^\omega$. Denoting G^∞ the relative interior of F^L, we see that $G^\infty \subseteq G(y)$ for every $y \in L$, hence $G^L \supseteq G^\infty \neq \emptyset$, and we are home again. □

Remark. The restriction to finite-dimensional spaces was needed for G^∞ to be meaningful. It is unclear what could be done without the restriction.

Corollary. *Every quasiconcave, irreflexive and ω-C-transitive binary relation on a convex subset A of a finite-dimensional vector space belongs to*

$\mathcal{R}_{\mathrm{PI}}(\mathfrak{C}_{\mathrm{mpx}})$. *Every quasiconcave, irreflexive and strongly C-transitive binary relation on a convex subset A of a finite-dimensional vector space belongs to* $\mathcal{R}_{\mathrm{PI}}(\mathfrak{C}_{\mathrm{onv}})$.

Proof. Routine derivation of (3) from the NM property is omitted. □

Theorem 9. *Let $\succ$ be a quasiconcave semiorder on a convex subset A of a finite-dimensional vector space. Then $\succ$ belongs to $\mathcal{R}_{\exists}(\mathfrak{C}_{\mathrm{mpx}})$ if and only if it is ω-C-transitive, whereas $\succ$ belongs to $\mathcal{R}_{\exists}(\mathfrak{C}_{\mathrm{onv}})$ if and only if it is strongly C-transitive.*

Proof. Both sufficiency statements immediately follow from Theorem 8. As to necessity, we pick $y \in M_{\succ}(X)$, where $X = \operatorname{cl}\operatorname{co}\{x^k\}_{k\in\mathbb{N}} = \operatorname{co}(\{x^k\}_{k\in\mathbb{N}} \cup \{x^\omega\})$ in the first case, or $X = \operatorname{co}\{x^k\}_{k\in\mathbb{N}}$ in the second. In either case, $y \succ x^k$ for each $k \in \mathbb{N}$ because $\succ$ is a semiorder. □

Neither necessity statement holds for interval orders.

Example 6.8. We assume $A = \{x \in \mathbb{R}^2 \mid \|x\| \le 1\}$ and introduce polar coordinates on $\mathbb{R}^2$ exactly as in Example 6.1 (or in the proof of Theorem 4). Then we define a mapping $k\colon A \setminus \{0\} \to \mathbb{N}$ by the condition $2(1 - 1/2^{k(x)})\pi \le \varphi(x) < 2(1 - 1/2^{k(x)+1})\pi$ for every $x \in A \setminus \{0\}$. Finally, we define a quasiconcave interval order $\succ$ on A by a numeric representation

$$y \succ x \rightleftharpoons f^-(y) > f^+(x),$$

where

$$f^-(x) = \begin{cases} -1/2^{k(x)}, & \text{if } \|x\| = 1; \\ -1/2^{k(x)+1}, & \text{if } 0 < \|x\| < 1; \\ -1/2, & \text{if } x = 0; \end{cases} \quad f^+(x) = \begin{cases} -1/2^{k(x)}, & \text{if } \|x\| = 1; \\ 0, & \text{if } \|x\| < 1. \end{cases}$$

Let us show that $\succ$ possesses the NM property on every $X \in \mathfrak{C}_{\mathrm{onv}}$ [hence belongs to $\mathcal{R}_{\mathrm{PI}}(\mathfrak{C}_{\mathrm{onv}})$]. Whenever $x \in A$ and $\|x\| < 1$, we have $y \not\succ x$ for all $y \in A$. Let $X \in \mathfrak{C}_{\mathrm{onv}}$ and $x \in X \setminus M_{\succ}(X)$, say, $X \ni y \succ x$. If $y \in M_{\succ}(X)$, we are home; otherwise, $\|y\| = 1$ and $k(y) > k(x)$. Picking a convex combination z of x and y, we immediately see that $k(z) \ge k(x)$ while $\|z\| < 1$; therefore, $M_{\succ}(X) \ni z \succ x$. On the other hand, $\succ$ is not even ω-C-transitive. Let us consider the same sequence x^k with $\|x^k\| = 1$ and $\varphi(x^k) = 2k\pi/(k+1)$ as in Example 6.1. It is still an improvement path converging to $x^\omega = x^0$; however, for every $y \in A$, there is $k \in \mathbb{N}$ such that $y \not\succ x^k$.

It is worthwhile to note that both statements of Theorem 8, as well as both sufficiency statements of Theorem 9, become wrong if the quasi-concavity assumption is dropped. The theorems from Section 5 are silent on the possibility to characterize those properties of binary relations with conditions of the type $\forall\,\exists$; nothing is known about that as of now.

Example 6.9. We assume $A = \mathbb{R}$. Exactly as in the proof of Theorem 3, we fix $\mathbb{Q}^{\perp} \subset \mathbb{R}$ such that $\mathbb{R} = \mathbb{Q} \oplus \mathbb{Q}^{\perp}$ viewed as vector spaces over $\mathbb{Q}$. For every $x \in \mathbb{R}$, its projections $p(x) \in \mathbb{Q}$ and $p_{\perp}(x) \in \mathbb{Q}^{\perp}$ such that $x = p(x) + p_{\perp}(x)$ are uniquely defined.

Clearly, the cardinality of $\mathbb{Q}^{\perp}$ is continuum. Let Λ be a well ordered set such that: (i) the cardinality of Λ is continuum; (ii) for every $\alpha \in \Lambda$, the cardinality of $\{\beta \in \Lambda \mid \alpha > \beta\}$ is less than continuum. Let $\pi\colon \mathbb{Q}^{\perp} \to \Lambda$ be a bijection. We define a total order $\succ$ on $\mathbb{R}$ by

$$y \succ x \rightleftharpoons \big[\pi \circ p_{\perp}(y) > \pi \circ p_{\perp}(x) \ \text{ or } \ [\pi \circ p_{\perp}(y) = \pi \circ p_{\perp}(x) \ \& \ p(y) > p(x)]\big].$$

Since $(\pi \circ p_{\perp})^{-1}(\alpha)$ is dense in $\mathbb{R}$ for every $\alpha \in \Lambda$, and there is no maximum in Λ, $M_{\succ}(X) = \emptyset$ for every $X \in \mathfrak{C}_{\mathrm{onv}}$ such that $\#X > 1$. On the other hand, $\succ$ is strongly C-transitive. Let $\langle x^k \rangle_{k \in \mathbb{N}}$ be an infinite improvement path. Then $\Lambda^* = \{\pi \circ p_{\perp}(x^k)\}_{k \in \mathbb{N}} \subset \Lambda$ is countable, hence there is $\beta \in \Lambda$ such that $\beta > \alpha$ for every $\alpha \in \Lambda^*$. Since $(\pi \circ p_{\perp})^{-1}(\beta)$ is dense in $\mathbb{R}$, there is $y \in \mathrm{co}\{x^k\}_{k \in \mathbb{N}}$ such that $\pi \circ p_{\perp}(y) = \beta$, hence $y \succ x^k$ for each $k \in \mathbb{N}$.

7 Miscellany

7.1 Countability

Theorems A and D have straightforward corollaries.

Corollary. *Let $\succ$ be a binary relation on a set A. Then $M_{\succ}(X) \neq \emptyset$ for every $X \in \mathfrak{B} \setminus \{\emptyset\}$ if and only if $M_{\succ}(X) \neq \emptyset$ for every countable $X \in \mathfrak{B} \setminus \{\emptyset\}$.*

Corollary. *Let $\succ$ be an interval order on a metric space A. Then $M_{\succ}(X) \neq \emptyset$ for every $X \in \mathfrak{C}_{\mathrm{omp}}$ if and only if $M_{\succ}(X) \neq \emptyset$ for every countable $X \in \mathfrak{C}_{\mathrm{omp}}$.*

The proof of Theorem 2 from Kukushkin (2008a), quoted above as Theorem G, shows that the interval order in the second corollary cannot be replaced with an arbitrary transitive binary relation. It is funny to note that Theorem B admits no convincing interpretation of this type.

Naturally, one cannot find much use for "countable convex subsets"; however, we may consider convex, or closed and convex, hulls of countable subsets. Since *every* $X \in \mathfrak{C}_{\text{mpx}}$ belongs to the latter class, one of two potential corollaries from Theorem 9 is tautological. The other holds under weaker assumptions.

Proposition 7.1. *A semiorder* $\succ$ *on a Banach space* A *belongs to* $\mathcal{R}_{\exists}(\mathfrak{C}_{\text{onv}})$ *if and only if* $M_{\succ}(\text{co}\{x^k\}_{k\in\mathbb{N}}) \neq \emptyset$ *whenever* $\{x^k\}_{k\in\mathbb{N}} \subseteq A$.

Proof. The necessity is straightforward. To prove sufficiency, we suppose the contrary: $\succ$ admits a maximizer on the convex hull of every countable subset, $Y \in \mathfrak{C}_{\text{onv}}$, but $M_{\succ}(Y) = \emptyset$. We pick $x^0 \in Y$ and recursively construct two infinite sequences $\langle x^k\rangle_{k\in\mathbb{N}}$, $\langle y^k\rangle_{k\in\mathbb{N}}$ in Y such that, for each $k \in \mathbb{N}$,

$$y^k \in M_{\succ}(\text{co}\{x^0, \dots x^k\}); \tag{7a}$$

$$y^{k+1} \succ y^k. \tag{7b}$$

First, we define $y^0 = x^0$; then (7a) for $k = 0$ holds trivially. Supposing x^h and y^h defined for all $h \leq k$, we pick $x^{k+1} \in Y$ such that $x^{k+1} \succ y^k$; this is possible because $M_{\succ}(Y) = \emptyset$. Now we have $y^k \in \text{co}\{x^0, \dots, x^{k+1}\} \setminus M_{\succ}(\text{co}\{x^0, \dots, x^{k+1}\})$. On the other hand, $M_{\succ}(\text{co}\{x^0, \dots, x^{k+1}\}) \neq \emptyset$ by our assumption. Since $\succ$ is a semiorder, there is $y^{k+1} \in M_{\succ}(\text{co}\{x^0, \dots, x^{k+1}\})$ such that $y^{k+1} \succ y^k$. Thus, both (7a) for $k+1$ and (7b) hold.

Having x^k and y^k defined for all k, we pick $y \in M_{\succ}(\text{co}\{x^k\}_{k\in\mathbb{N}})$, which is possible by our assumption. By the definition of the convex hull, there is $k \in \mathbb{N}$ such that $y \in \text{co}\{x^0, \dots x^k\}$. Applying (6) to $y^{k+2} \succ y^{k+1} \succ y^k$ from (7b), we see that either $y \succ y^k$ or $y^{k+2} \succ y$ must hold. However, the first relation contradicts (7a) and the second, the choice of y. □

Example 7.2. We introduce polar coordinates on $\mathbb{R}^2$ exactly as in the proof of Theorem 4, and define $A = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$, $A^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$, $A^0 = \{x \in \mathbb{R}^2 \mid \|x\| = 1/2\}$, and $B = A \setminus (A^1 \cup A^0)$. Then we fix a $\varphi_0 \in]0, \pi[$ which is incommensurable with π. For every $x, y \in A$, we set $y \succ x$ if and only if (at least) one of the three conditions holds: $y \in A^0$, $x \in B$, and $\varphi(y) = \varphi(x)$; $y \in A^1$, $x \in A^0$, and $\varphi(y) = \varphi(x)$; $y \in A^1$ and there are $k, m \in \mathbb{N}$ such that $k > 0$ and $\varphi(y) = \varphi(x) + k \cdot \varphi_0 + 2m \cdot \pi$. The relation is transitive and irreflexive. Clearly, $M_{\succ}(A) = \emptyset$, i.e., $\succ$ does not belong to $\mathcal{R}_{\exists}(\mathfrak{C}_{\text{onv}})$, nor to $\mathcal{R}_{\exists}(\mathfrak{C}_{\text{mpx}})$.

Given $X = \text{co}\{x^k\}_{k\in\mathbb{N}} \subset A$, let us show that $M_{\succ}(X) \neq \emptyset$. If $X \subset B$, then $M_{\succ}(X) = X$; if $X \cap A^0 \neq \emptyset = X \cap A^1$, then $M_{\succ}(X) = X \cap A^0$. Let $X^1 = X \cap A^1 \neq \emptyset$. Since $M_{\succ}(X) \supseteq M_{\succ}(X^1)$, we are home if $M_{\succ}(X^1) \neq \emptyset$.

Otherwise, X^1 is dense in A^1, hence $A^0 \subset X$. On the other hand, X^1 is countable because A^1 consists of extreme points, while A^1 is uncountable. Defining an equivalence relation on A^1 by $y \sim x \rightleftharpoons \exists k, m \in \mathbb{N}\, [\varphi(y) = \varphi(x) + k \cdot \varphi_0 + 2m \cdot \pi]$, we see that A^1 is partitioned into a continuum of countable equivalence classes. Therefore, we may pick $x \in A^1 \setminus X^1$ such that $y \not\succ x$ for any $y \in X^1$; then $y \not\succ x/2$ for any $y \in X$, i.e., $x/2 \in M_{\succ}(X) \neq \emptyset$.

Abandoning the transitivity of $\succ$, we can slightly modify the example and have $y \succ x$ for a unique $y \in A$, given $x \in A$; in other words, $\succ$ can be made quasiconcave. The gap between Proposition 7.1 and Example 7.2 is considerable: nothing is known in this respect about interval orders or transitive quasiconcave relations.

7.2 "Tautological" characterizations

Theorem 7 of Kukushkin (2005) showed that the class $\mathcal{R}_{\exists}(\mathfrak{C}_{\text{omp}})$ *can* be characterized, in a tautological way, by a configurational condition of the type $\forall\, \exists\, \nexists$. Here we demonstrate the possibility of similar characterizations of the classes $\mathcal{R}(\mathfrak{C}_{\text{mpx}})$.

To the end of the subsection, we employ notation: $\Lambda = \{\lambda \colon \mathbb{N} \to \mathbb{N} \mid k' > k \Rightarrow \lambda(k') > \lambda(k)\}$; $\mathbb{N}_2 = \{2^{k+1}\}_{k \in \mathbb{N}}$; $\mathrm{N}_2 = \{\nu \colon \mathbb{N} \to \mathbb{N}_2\}$; $\mathbb{N}_3 = \{3^{k+1}\}_{k \in \mathbb{N}}$; $\mathrm{N}_3 = \{\nu \colon \mathbb{N} \to \mathbb{N}_3\}$; $\mathrm{N} = \mathrm{N}_2 \cup \mathrm{N}_3$; $\mathbb{N}_5 = \{5^{k+1}\}_{k \in \mathbb{N}}$. Defining various configurations, we always assume $C_{\varkappa} = \emptyset$ unless explicitly defined otherwise. These mappings $\nu \colon \mathbb{N} \to \mathbb{N}$ are invoked throughout: $\nu^m(0) = m$ ($m = 0, 1, 5$); $\nu^7_m(0) = 7$ ($m = 2, 3$); $\nu^0(k+1) = 2^{k+1} \cdot 3^{k+1}$, $\nu^1(k+1) = 2^{k+1} \cdot 5^{k+1}$, $\nu^5(k+1) = 3^{k+1} \cdot 5^{k+1}$, $\nu^7_2(k+1) = 2^{k+1} \cdot 5^{k+1} \cdot 3$, and $\nu^7_3(k+1) = 3^{k+1} \cdot 5^{k+1} \cdot 2$ for all $k \in \mathbb{N}$. Given $\nu \in \mathrm{N}$ and $\lambda \in \Lambda$, we define $\nu^{\lambda} \colon \mathbb{N} \to \mathbb{N}$ by $\nu^{\lambda}(0) = 5$ and $\nu^{\lambda}(k+1) = \nu \circ \lambda(k)$ for all $k \in \mathbb{N}$.

We define an abstract configuration $C^{\exists_0}$ by $\operatorname{Dom} C^{\exists_0} = \mathbb{N}_2$ and $C^{\exists_0}_{\varkappa} = \emptyset$ for all $\varkappa$. For every $\nu \in \mathrm{N}_2$ and $\lambda \in \Lambda$, we define a configuration $C^{\exists_1}[\nu, \lambda]$ by $\operatorname{Dom} C^{\exists_1}[\nu, \lambda] = \operatorname{Dom} C^{\exists_0} \cup \{5\}$ and $C^{\exists_1}[\nu, \lambda]_{\to} = \{\nu^{\lambda}\}$. Finally, we define two more configurations, $C^{\exists_2}$ and $C^{\exists'_2}$. $\operatorname{Dom} C^{\exists_2} = \operatorname{Dom} C^{\exists_0} \cup (\mathbb{N}_2 \cdot \mathbb{N}_3) \cup \{0\}$; $C^{\exists_2}_{\pitchfork} = \{(2^{k+1} \cdot 3, 2, 4)\}_{k \in \mathbb{N}} \cup \{(2^{k+1} \cdot 3^{h+2}, 2^{k+1} \cdot 3^{h+1}, 2^{h+3})\}_{k,h \in \mathbb{N},\, h+1 \leq k}$; $C^{\exists_2}_{\to} = \{\nu^0\}$. $\operatorname{Dom} C^{\exists'_2} = \operatorname{Dom} C^{\exists_2} \cup (\mathbb{N}_2 \cdot \mathbb{N}_5) \cup \{1\}$; $C^{\exists'_2}_{\triangleright} = \{(1, 0)\}$; $C^{\exists'_2}_{\pitchfork} = C^{\exists_2}_{\pitchfork} \cup \{(2^{k+1} \cdot 5, 2, 4)\}_{k \in \mathbb{N}} \cup \{(2^{k+1} \cdot 5^{h+2}, 2^{k+1} \cdot 5^{h+1}, 2^{h+3})\}_{k,h \in \mathbb{N},\, h+1 \leq k}$; $C^{\exists'_2}_{\to} = C^{\exists_2}_{\to} \cup \{\nu^1\}$.

Proposition 7.3. *Let $\succ$ be a binary relation on a convex subset A of a Banach space. Then $\succ$ belongs to $\mathcal{R}_{\exists}(\mathfrak{C}_{\text{mpx}})$ if and only if, for every realization μ of $C^{\exists_0}$ in A for $\succ$, either there exists $\nu \in \mathrm{N}_2$ such that no configuration*

$C^{\exists_1}[\nu,\lambda]$ $(\lambda \in \Lambda)$ *admits a realization* $\mu' \geq \mu$, *or there exists a realization* $\mu' \geq \mu$ *of* $C^{\exists_2}$ *such that there is no realization* $\mu'' \geq \mu'$ *of* $C^{\exists'_2}$.

Proof. The proof is somewhat similar to that of Proposition 6.6. A realization μ of $C^{\exists_0}$ defines a countable subset $\{\mu(k)\}_{k\in\mathbb{N}_2} \subset A$; let us denote X its convex closure. The first alternative in the formulation means that X is not compact – there is a sequence ν no subsequence ν^λ of which converges – hence nothing is required of it. If $\mu' \geq \mu$ is a realization of $C^{\exists_2}$, then $\mu'(2^{k+1}\cdot 3^{k+1}) \in \mathrm{co}\{\mu(2^{h+1})\}_{h\leq k+1}$ for each $k \in \mathbb{N}$, hence $\mu'(0) \in X$. Similarly, the existence of a realization $\mu'' \geq \mu'$ of $C^{\exists'_2}$ would mean that $X \ni \mu''(1) \succ \mu'(0)$. Thus, the impossibility of such μ'' is equivalent to $\mu'(0) \in M_\succ(X)$, hence the second alternative is equivalent to $M_\succ(X) \neq \emptyset$. □

Let us define abstract configurations C^0 and $C^1[\nu,\lambda]$ $(\nu \in \mathrm{N},\ \lambda \in \Lambda)$ to be used in the characterization of various rationality requirements. $\mathrm{Dom}\, C^0 = \mathbb{N}_2 \cup \mathbb{N}_3 \cup (\mathbb{N}_2 \cdot \mathbb{N}_3) \cup \{0\}$; $C^0_\pitchfork = \{(3^{k+1}\cdot 2, 3, 9)\}_{k\in\mathbb{N}} \cup \{(3^{k+1}\cdot 2^{h+2}, 3^{k+1}\cdot 2^{h+1}, 3^{h+3})\}_{k,h\in\mathbb{N},\, h+1\leq k}$; $C^0_\to = \{\nu^0\}$. $\mathrm{Dom}\, C^1[\nu,\lambda] = \mathrm{Dom}\, C^0 \cup \{5\}$; $C^1[\nu,\lambda]_\pitchfork = C^0_\pitchfork$; $C^1[\nu,\lambda]_\to = C^0_\to \cup \{\nu^\lambda\}$.

To characterize the class $\mathcal{R}_{\mathrm{Out}}(\mathfrak{C}_{\mathrm{mpx}})$, we first define two configurations, C^{Out_2} and $C^{\mathrm{Out}'_2}$. $\mathrm{Dom}\, C^{\mathrm{Out}_2} = \mathrm{Dom}\, C^0 \cup (\mathbb{N}_3 \cdot \mathbb{N}_5) \cup \{5\}$; $C^{\mathrm{Out}_2}_\pitchfork = C^0_\pitchfork \cup \{(3^{k+1}\cdot 5, 3, 9)\}_{k\in\mathbb{N}} \cup \{(3^{k+1}\cdot 5^{h+2}, 3^{k+1}\cdot 5^{h+1}, 3^{h+3})\}_{k,h\in\mathbb{N},\, h+1\leq k}$; $C^{\mathrm{Out}_2}_\to = C^0_\to \cup \{\nu^5\}$. $\mathrm{Dom}\, C^{\mathrm{Out}'_2} = \mathrm{Dom}\, C^{\mathrm{Out}_2} \cup (\mathbb{N}_2 \cdot \mathbb{N}_5) \cup \{1\}$; $C^{\mathrm{Out}'_2}_= = \{(1,5)\}$; $C^{\mathrm{Out}'_2}_\pitchfork = C^{\mathrm{Out}_2}_\pitchfork \cup \{(2^{k+1}\cdot 5, 2, 4)\}_{k\in\mathbb{N}} \cup \{(2^{k+1}\cdot 5^{h+2}, 2^{k+1}\cdot 5^{h+1}, 2^{h+3})\}_{k,h\in\mathbb{N},\, h+1\leq k}$; $C^{\mathrm{Out}'_2}_\to = C^{\mathrm{Out}_2}_\to \cup \{\nu^1\}$.

Then we define three configurations, C^{Out_3}, $C^{\mathrm{Out}'_3}$, and $C^{\mathrm{Out}''_3}$. $\mathrm{Dom}\, C^{\mathrm{Out}_3} = \mathrm{Dom}\, C^0 \cup (\mathbb{N}_2 \cdot \mathbb{N}_5) \cup \{1\}$; $C^{\mathrm{Out}_3}_\pitchfork = C^0_\pitchfork \cup \{(2^{k+1}\cdot 5, 2, 4)\}_{k\in\mathbb{N}} \cup \{(2^{k+1}\cdot 5^{h+2}, 2^{k+1}\cdot 5^{h+1}, 2^{h+3})\}_{k,h\in\mathbb{N},\, h+1\leq k}$; $C^{\mathrm{Out}_3}_\to = C^0_\to \cup \{\nu^1\}$. $\mathrm{Dom}\, C^{\mathrm{Out}'_3} = \mathrm{Dom}\, C^{\mathrm{Out}_3} \cup (\mathbb{N}_3 \cdot \mathbb{N}_5) \cup \{5\}$; $C^{\mathrm{Out}'_3}_= = \{(1,5)\}$; $C^{\mathrm{Out}'_3}_\pitchfork = C^{\mathrm{Out}_3}_\pitchfork \cup \{(3^{k+1}\cdot 5, 3, 9)\}_{k\in\mathbb{N}} \cup \{(3^{k+1}\cdot 5^{h+2}, 3^{k+1}\cdot 5^{h+1}, 3^{h+3})\}_{k,h\in\mathbb{N},\, h+1\leq k}$; $C^{\mathrm{Out}'_3}_\to = C^{\mathrm{Out}_3}_\to \cup \{\nu^5\}$. $\mathrm{Dom}\, C^{\mathrm{Out}''_3} = \mathrm{Dom}\, C^{\mathrm{Out}_3} \cup (\mathbb{N}_2 \cdot \mathbb{N}_5 \cdot 3) \cup \{7\}$; $C^{\mathrm{Out}''_3}_\triangleright = \{(7,1)\}$; $C^{\mathrm{Out}''_3}_\pitchfork = C^{\mathrm{Out}_3}_\pitchfork \cup \{(2^{k+1}\cdot 15, 2, 4)\}_{k\in\mathbb{N}} \cup \{(2^{k+1}\cdot 5^{h+2}\cdot 3, 2^{k+1}\cdot 5^{h+1}\cdot 3, 2^{h+3})\}_{k,h\in\mathbb{N},\, h+1\leq k}$; $C^{\mathrm{Out}''_3}_\to = C^{\mathrm{Out}_3}_\to \cup \{\nu^7_2\}$.

Finally, we define two more configurations, C^{Out_4} and C^{Out_5}. $\mathrm{Dom}\, C^{\mathrm{Out}_4} = \mathrm{Dom}\, C^{\mathrm{Out}_2}$; $C^{\mathrm{Out}_4}_\triangleright = \{(5,0)\}$; $C^{\mathrm{Out}_4}_\pitchfork = C^{\mathrm{Out}_2}_\pitchfork$; $C^{\mathrm{Out}_4}_\to = C^{\mathrm{Out}_2}_\to$. $\mathrm{Dom}\, C^{\mathrm{Out}_5} = \mathrm{Dom}\, C^{\mathrm{Out}_3}$; $C^{\mathrm{Out}_5}_\triangleright = \{(1,0)\}$; $C^{\mathrm{Out}_5}_\pitchfork = C^{\mathrm{Out}_3}_\pitchfork$; $C^{\mathrm{Out}_5}_\to = C^{\mathrm{Out}_3}_\to$.

Proposition 7.4. *Let* $\succ$ *be a binary relation on a convex subset* A *of a Banach space. Then* $\succ$ *belongs to* $\mathcal{R}_{\mathrm{Out}}(\mathfrak{C}_{\mathrm{mpx}})$ *if and only if it belongs to*

$\mathcal{R}_\exists(\mathfrak{C}_{\text{mpx}})$ *and at least one of the following alternatives holds for every realization* μ *of* C^0 *in* A *for* $\succ$.

1. *There exists* $\nu \in \mathrm{N}$ *such that no configuration* $C^1[\nu, \lambda]$ $(\lambda \in \Lambda)$ *admits a realization* $\mu' \geq \mu$.

2. *There exists a realization* $\mu' \geq \mu$ *of the configuration* C^{Out_2} *in* A *for* $\succ$ *such that there is no realization* $\mu'' \geq \mu'$ *of* $C^{\text{Out}_2{}'}$.

3. *There exists a realization* $\mu' \geq \mu$ *of the configuration* C^{Out_3} *in* A *for* $\succ$ *such that there is no realization* $\mu'' \geq \mu'$ *of* $C^{\text{Out}_3'}$ *and no realization* $\mu'' \geq \mu'$ *of* $C^{\text{Out}_3''}$.

4. *There exists a realization* $\mu' \geq \mu$ *of the configuration* C^{Out_4} *in* A *for* $\succ$.

5. *There exists no realization* $\mu' \geq \mu$ *of the configuration* C^{Out_5} *in* A *for* $\succ$.

A sketch of the proof. The equality in (2) is equivalent to a set inclusion: $M_\succ(X') \subseteq M_\succ(X)$ whenever $M_\succ(X) \subseteq X' \subseteq X$ and $X, X' \in \mathfrak{C}_{\text{mpx}}$. An interpretation of the configuration C^0 goes along the same lines as in the proof of Proposition 7.3: X is the convex closure of $\{\mu(2^{k+1})\}_{k\in\mathbb{N}}$ in A; X' is the convex closure of $\{\mu(3^{k+1})\}_{k\in\mathbb{N}}$; $\mu(0)$ belongs to X' [and putatively belongs to $M_\succ(X')$]. The first alternative in the list means that either X or X' is not compact; the second, that X' is not a subset of X. If alternative 3 holds, then $\mu'(1) \in M_\succ(X) \setminus X'$. Thus any of the first three alternatives means a violation of the conditions in (2), hence nothing is required at all. Alternative 4 means that $\mu(0) \notin M_\succ(X')$, hence nothing is required of it; alternative 5, that $\mu(0) \in M_\succ(X)$ as it should if none of the preceding alternatives holds. If (2) is violated, we can pick $\{\mu(2^{k+1})\}_{k\in\mathbb{N}}$ generating X, $\{\mu(3^{k+1})\}_{k\in\mathbb{N}}$ generating X', $\mu(0) \in M_\succ(X') \setminus M_\succ(X)$, and see that none of the five alternatives holds. □

Similar constructions allow us to characterize the class $\mathcal{R}_{\text{PI}}(\mathfrak{C}_{\text{mpx}})$. We define four new configurations, C^{PI_2}, $C^{\text{PI}_2'}$, $C^{\text{PI}_2''}$, and $C^{\text{PI}_3'}$. $\operatorname{Dom} C^{\text{PI}_2} = \operatorname{Dom} C^0 \cup (\mathbb{N}_2 \cdot \mathbb{N}_5) \cup (\mathbb{N}_3 \cdot \mathbb{N}_5) \cup \{1, 5, 7\}$; $C^{\text{PI}_2}_{\pitchfork} = C^0_{\pitchfork} \cup \{(7, 1, 5)\} \cup \{(2^{k+1} \cdot 5, 2, 4), (3^{k+1} \cdot 5, 3, 9)\}_{k\in\mathbb{N}} \cup \{(2^{k+1} \cdot 5^{h+2}, 2^{k+1} \cdot 5^{h+1}, 2^{h+3}), (3^{k+1} \cdot 5^{h+2}, 3^{k+1} \cdot 5^{h+1}, 3^{h+3})\}_{k,h\in\mathbb{N},\, h+1\leq k}$; $C^{\text{PI}_2}_{\to} = C^0_{\to} \cup \{\nu^1, \nu^5\}$. $\operatorname{Dom} C^{\text{PI}_2'} = \operatorname{Dom} C^{\text{PI}_2} \cup (\mathbb{N}_2 \cdot \mathbb{N}_5 \cdot 3)$; $C^{\text{PI}_2'}_{\pitchfork} = C^{\text{PI}_2}_{\pitchfork} \cup \{(2^{k+1} \cdot 15, 2, 4)\}_{k\in\mathbb{N}} \cup \{(2^{k+1} \cdot 5^{h+2} \cdot 3, 2^{k+1} \cdot 5^{h+1} \cdot 3, 2^{h+3})\}_{k,h\in\mathbb{N},\, h+1\leq k}$; $C^{\text{PI}_2'}_{\to} = C^{\text{PI}_2}_{\to} \cup \{\nu_2^7\}$. $\operatorname{Dom} C^{\text{PI}_2''} = \operatorname{Dom} C^{\text{PI}_2} \cup (\mathbb{N}_3 \cdot \mathbb{N}_5 \cdot 2)$; $C^{\text{PI}_2''}_{\pitchfork} = C^{\text{PI}_2}_{\pitchfork} \cup \{(3^{k+1} \cdot 10, 3, 9)\}_{k\in\mathbb{N}} \cup \{(3^{k+1} \cdot 5^{h+2} \cdot 2, 3^{k+1} \cdot 5^{h+1} \cdot$

$2, 3^{h+3})\}_{k,h\in\mathbb{N},\, h+1\le k}$; $C_{\to}^{\mathrm{PI}_2''} = C_{\to}^{\mathrm{PI}_2} \cup \{\nu_3^7\}$. $\mathrm{Dom}\, C^{\mathrm{PI}_3'} = \mathrm{Dom}\, C^{\mathrm{PI}_3} \cup (\mathbb{N}_2 \cdot \mathbb{N}_5 \cdot 3) \cup \{7\}$; $C_{\triangleright}^{\mathrm{PI}_3'} = C_{\triangleright}^{\mathrm{PI}_3} \cup \{(7,1)\}$; $C_{\pitchfork}^{\mathrm{PI}_3'} = C_{\pitchfork}^{\mathrm{PI}_3} \cup \{(2^{k+1} \cdot 15, 2, 4)\}_{k\in\mathbb{N}} \cup \{(2^{k+1} \cdot 5^{h+2} \cdot 3, 2^{k+1} \cdot 5^{h+1} \cdot 3, 2^{h+3})\}_{k,h\in\mathbb{N},\, h+1\le k}$; $C_{\to}^{\mathrm{PI}_3'} = C_{\to}^{\mathrm{PI}_3} \cup \{\nu_2^7\}$.

Proposition 7.5. *Let $\succ$ be a binary relation on a convex subset A of a Banach space. Then $\succ$ belongs to $\mathcal{R}_{\mathrm{PI}}(\mathfrak{C}_{\mathrm{mpx}})$ if and only if it belongs to $\mathcal{R}_{\exists}(\mathfrak{C}_{\mathrm{mpx}})$ and at least one of the following alternatives holds for every realization μ of C^0 in A for $\succ$.*

1. *There exists $\nu \in \mathrm{N}$ such that no configuration $C^1[\nu, \lambda]$ $(\lambda \in \Lambda)$ admits a realization $\mu' \ge \mu$.*
2. *There exists a realization $\mu' \ge \mu$ of the configuration C^{PI_2} in A for $\succ$ such that there is no realization $\mu'' \ge \mu'$ of $C^{\mathrm{PI}_2'}$ and no realization $\mu'' \ge \mu'$ of $C^{\mathrm{PI}_2''}$.*
3. *There exists a realization $\mu' \ge \mu$ of the configuration C^{Out_5} in A for $\succ$ such that there is no realization $\mu'' \ge \mu'$ of $C^{\mathrm{PI}_3'}$.*
4. *There exists a realization $\mu' \ge \mu$ of the configuration C^{Out_4} in A for $\succ$.*
5. *There exists no realization $\mu' \ge \mu$ of the configuration C^{Out_5} in A for $\succ$.*

A sketch of the proof. The equality in (3) is equivalent to a set inclusion: $M_{\succ}(M_{\succ}(X') \cup X'') \subseteq M_{\succ}(X)$ whenever $X = X' \cup X''$ and $X, X', X'' \in \mathfrak{C}_{\mathrm{mpx}}$. An interpretation of the configuration C^0 goes along the same lines again: X' is the convex closure of $\{\mu(2^{k+1})\}_{k\in\mathbb{N}}$ in A; X'' is the convex closure of $\{\mu(3^{k+1})\}_{k\in\mathbb{N}}$; $\mu(0)$ belongs to X''. The first alternative in the list means that either X or X' is not compact; the second, that $X = X' \cup X''$ is not convex: $\mu'(1) \in X'$, $\mu'(5) \in X''$, $\mu'(7)$ is a convex combination of $\mu'(1)$ and $\mu'(5)$, but $\mu'(7) \notin X$. Thus, either alternative means a violation of the conditions in (3), hence nothing is required at all. Alternative 3 means that $M_{\succ}(X') \ni \mu'(1) \succ \mu(0)$; alternative 4, that $X'' \ni \mu'(5) \succ \mu(0)$. In either case, $\mu(0) \notin M_{\succ}(M_{\succ}(X') \cup X'')$. Finally, alternative 5 means that $\mu(0) \in M_{\succ}(X)$ as it should if none of the preceding alternatives holds. $\square$

For the characterization of $\mathcal{R}_{\mathrm{Rat}}(\mathfrak{C}_{\mathrm{mpx}})$, we need one more configuration. $\mathrm{Dom}\, C^{\mathrm{Rat}_3'} = \mathrm{Dom}\, C^{\mathrm{Out}_2} \cup (\mathbb{N}_2 \cdot \mathbb{N}_5) \cup \{1\}$; $C_{\triangleright}^{\mathrm{Rat}_3'} = \{(1,5)\}$; $C_{\pitchfork}^{\mathrm{Rat}_3'} = C_{\pitchfork}^{\mathrm{Out}_2} \cup \{(2^{k+1} \cdot 5, 2, 4)\}_{k\in\mathbb{N}} \cup \{(2^{k+1} \cdot 5^{h+2}, 2^{k+1} \cdot 5^{h+1}, 2^{h+3})\}_{k,h\in\mathbb{N},\, h+1\le k}$; $C_{\to}^{\mathrm{Rat}_3'} = C_{\to}^{\mathrm{Out}_2} \cup \{\nu^1\}$.

Proposition 7.6. *Let $\succ$ be a binary relation on a convex subset A of a Banach space. Then $\succ$ belongs to $\mathcal{R}_{\mathrm{Rat}}(\mathfrak{C}_{\mathrm{mpx}})$ if and only if it belongs to $\mathcal{R}_{\mathrm{PI}}(\mathfrak{C}_{\mathrm{mpx}})$ and at least one of the following alternatives holds for every realization μ of C^0 in A for $\succ$.*

1. *There exists $\nu \in \mathrm{N}$ such that no configuration $C^1[\nu, \lambda]$ ($\lambda \in \Lambda$) admits a realization $\mu' \geq \mu$.*
2. *There exists a realization $\mu' \geq \mu$ of the configuration C^{Out_2} in A for $\succ$ such that there is no realization $\mu'' \geq \mu'$ of $C^{\mathrm{Out}'_2}$.*
3. *For every realization $\mu' \geq \mu$ of the configuration C^{Out_2} in A for $\succ$, there is a realization $\mu'' \geq \mu'$ of $C^{\mathrm{Rat}'_3}$.*
4. *There exists a realization $\mu' \geq \mu$ of the configuration C^{Out_4} in A for $\succ$.*
5. *There exists no realization $\mu' \geq \mu$ of the configuration C^{Out_5} in A for $\succ$.*

Remark. Taking into account alternative 3, we see that this condition belongs to the class $\forall\forall\exists\nexists$.

A sketch of the proof. The configuration C^0 is interpreted in the same way as in the proof of Proposition 7.4: X is the convex closure of $\{\mu(2^{k+1})\}_{k\in\mathbb{N}}$ in A; X' is the convex closure of $\{\mu(3^{k+1})\}_{k\in\mathbb{N}}$; $\mu(0)$ belongs to X'. Thus any of the first three alternatives means a violation of the conditions in (4) (either X or X' is not compact; X' is not a subset of X; $M_{\succ}(X) \cap X' = \emptyset$), hence nothing is required at all. Alternative 4 means that $X' \ni \mu'(5) \succ \mu(0)$, hence $\mu(0) \notin M_{\succ}(X')$ and nothing is required of it. Finally, alternative 5 means that $\mu(0) \in M_{\succ}(X)$ as it should if none of the preceding alternatives holds. □

The characterization of $\mathcal{R}_{\exists!}(\mathfrak{C}_{\mathrm{mpx}})$ is similar, but simpler. Abstract configurations $C^{\exists!_0}$ and $C^{\exists!_1}[\nu, \lambda]$ for all $\nu \in \mathrm{N}_2$ and $\lambda \in \Lambda$ are defined in this way. $\mathrm{Dom}\, C^{\exists!_0} = \mathbb{N}_2 \cup (\mathbb{N}_2 \cdot \mathbb{N}_3) \cup (\mathbb{N}_2 \cdot \mathbb{N}_5) \cup \{0, 1\}$; $C^{\exists!_0}_{\neq} = \{0, 1\}$; $C^{\exists!_0}_{\pitchfork} = \{(2^{k+1} \cdot 3, 2, 4), (2^{k+1} \cdot 5, 2, 4)\}_{k\in\mathbb{N}} \cup \{(2^{k+1} \cdot 3^{h+2}, 2^{k+1} \cdot 3^{h+1}, 2^{h+3}), (2^{k+1} \cdot 5^{h+2}, 2^{k+1} \cdot 5^{h+1}, 2^{h+3})\}_{k,h\in\mathbb{N},\, h+1\leq k}$; $C^{\exists!_0}_{\to} = \{\nu^0, \nu^1\}$. $\mathrm{Dom}\, C^{\exists!_1}[\nu, \lambda] = \mathrm{Dom}\, C^{\exists!_0} \cup \{5\}$; $C^{\exists!_1}[\nu, \lambda]_{\neq} = C^{\exists!_0}_{\neq}$; $C^{\exists!_1}[\nu, \lambda]_{\pitchfork} = C^{\exists!_0}_{\pitchfork}$; $C^{\exists!_1}[\nu, \lambda]_{\to} = C^{\exists!_0}_{\to} \cup \{\nu^{\lambda}\}$.

Then we define two more configurations, $C^{\exists!_2}$ and $C^{\exists!_3}$. $\mathrm{Dom}\, C^{\exists!_2} = \mathrm{Dom}\, C^{\exists!_3} = \mathrm{Dom}\, C^{\exists!_0} \cup (\mathbb{N}_2 \cdot \mathbb{N}_5 \cdot 3) \cup \{7\}$; $C^{\exists!_2}_{\triangleright} = \{(7, 0)\}$; $C^{\exists!'_2}_{\triangleright} = \{(7, 1)\}$;

$C_{\neq}^{\exists!_2} = C_{\neq}^{\exists!_3} = C_{\neq}^{\exists!_0}$; $C_{\pitchfork}^{\exists!_2} = C_{\pitchfork}^{\exists!_3} = C_{\pitchfork}^{\exists!_0} \cup \{(2^{k+1} \cdot 15, 2, 4)\}_{k \in \mathbb{N}} \cup \{(2^{k+1} \cdot 5^{h+2} \cdot 3, 2^{k+1} \cdot 5^{h+1} \cdot 3, 2^{h+3})\}_{k,h \in \mathbb{N},\, h+1 \leq k}$; $C_{\rightarrow}^{\exists!_2} = C_{\rightarrow}^{\exists!_3} = C_{\rightarrow}^{\exists!_0} \cup \{\nu_2^7\}$.

Proposition 7.7. *Let $\succ$ be a binary relation on a convex subset A of a Banach space. Then $\succ$ belongs to $\mathcal{R}_{\exists!}(\mathfrak{C}_{\text{mpx}})$ if and only if it belongs to $\mathcal{R}_{\text{PI}}(\mathfrak{C}_{\text{mpx}})$ and at least one of the following alternatives holds for every realization μ of $C^{\exists!_0}$ in A for $\succ$.*

1. *There exists $\nu \in \text{N}_2$ such that no configuration $C^{\exists!_1}[\nu, \lambda]$ ($\lambda \in \Lambda$) admits a realization $\mu' \geq \mu$.*
2. *There exists a realization $\mu' \geq \mu$ of the configuration $C^{\exists!_2}$ in A for $\succ$.*
3. *There exists a realization $\mu' \geq \mu$ of the configuration $C^{\exists!_3}$ in A for $\succ$.*

A sketch of the proof. Again, X is the convex closure of $\{\mu(2^{k+1})\}_{k \in \mathbb{N}}$ in A; besides, $\mu(0), \mu(1) \in X$ and $\mu(0) \neq \mu(1)$. Alternative 2 means that X is not compact, hence nothing is required. Alternatives 2 and 3 mean that either $\mu(0) \notin M_{\succ}(X)$ or $\mu(1) \notin M_{\succ}(X)$, as it should be. □

Proposition 7.1 allows us to give a similar tautological characterization of semiorders belonging to $\mathcal{R}_{\exists}(\mathfrak{C}_{\text{onv}})$. What could be done about the whole $\mathcal{R}_{\exists}(\mathfrak{C}_{\text{onv}})$ or higher levels of rationality remains unclear.

7.3 Existence via extensions

A binary relation $\succ^{\prime}$ on A is called an *extension* of a relation $\succ$ if $y \succ x \Rightarrow y \succ^{\prime} x$ for all $x, y \in A$; then $M_{\succ^{\prime}}(X) \subseteq M_{\succ}(X)$ for every $X \in \mathfrak{B}$. Therefore, every sufficient condition for the existence of maximizers on all $X \in \mathfrak{C}$ immediately generates a weaker one: a relation belongs to $\mathcal{R}_{\exists}(\mathfrak{C})$ if it admits an extension satisfying the condition. The derivation of Theorem 1.7.4 from Theorem 1.7.1 in Kiruta et al. (1980) follows this scheme; it is also worth noting that a relation is pseudoconcave if and only if it admits a quasiconcave and irreflexive extension. The most popular sufficient condition for a binary relation to belong to $\mathcal{R}_{\exists}(\mathfrak{C}_{\text{omp}})$ (Bergstrom, 1975; Walker, 1977) can be derived from Theorem E in the same way: whenever $\succ$ is acyclic and has open lower contours, it admits an irreflexive and ω-transitive extension.

As shown in Kukushkin (2003, 2005), a relation admits an extension satisfying the conditions of Theorem E if and only if it is "Ω-acyclic," the latter property being the prohibition of the realization of a certain (uncountable) set of configurations. It remains unclear whether the existence of an extension satisfying, e.g., the conditions of Theorem F could be described with configurations.

Concerning the rationality requirements, there is no such straightforward connection between the properties of a relation and its extensions.

7.4 Final remark

A comparison between Theorems C, D, and E, on one hand, and Theorems 1–7 on the other, may justify this (deliberately provocative) conclusion: the Weierstrass Theorem (on the existence of a maximum) addresses, more or less, the root of the issue, while the Kakutani Theorem is somewhat superficial. The existence of such a viewpoint does not preclude the possibility of other, including the opposite, views. Moreover, I have no intention to suggest that the Kakutani Theorem should be thrown overboard: whatever aesthetical deficiencies may be found in the tool, we have no replacement at hand. Still, the very possibility of this interpretation is interesting and deserves attention.

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