# Strong Nash equilibrium in games with common and complementary local utilities

Nikolai S. Kukushkin*

November 23, 2016

**Abstract**

A rather general class of strategic games is described where the coalitional improvements are acyclic and hence strong Nash equilibria exist: The players derive their utilities from the use of certain facilities; all players using a facility extract the same amount of local utility therefrom, which amount depends both on the set of users and on their actions, and is decreasing in the set of users; the ultimate utility of each player is the minimum of the local utilities at all relevant facilities. Two important subclasses are "games with structured utilities," basic properties of which were discovered in 1970s and 1980s, and "bottleneck congestion games," which attracted researchers' attention quite recently. The former games are representative in the sense that every game from the whole class is isomorphic to one of them. The necessity of the minimum aggregation for the existence of strong Nash equilibria, actually, just Pareto optimal Nash equilibria, in all games of this type is established.

*MSC2010 Classification*: 91A10; *JEL Classification*: C 72.

*Key words:* Strong Nash equilibrium; Weakest-link aggregation; Coalitional improvement path; Congestion game; Game with structured utilities

## 1 Introduction

Both motivation for and the structure of this paper closely resemble those of Kukushkin (2007). Moreover, the models considered in either paper, when described in very general terms, sound quite similarly.

The players derive their utilities from the use of certain objects. Rosenthal (1973) called them "factors"; following Monderer and Shapley (1996), we call them "facilities" here. The players are free to choose facilities within certain limits. All the players using a facility extract the same amount of "local utility" therefrom, which amount may depend both on the set of users and on their actions. The "ultimate" utility of each player is an aggregate of the local utilities obtained from all relevant facilities.

Four crucial differences should be listed at the start. First, in Kukushkin (2007), following Rosenthal (1973), each player summed up relevant local utilities (strictly speaking, monotone transformations

*Dorodnicyn Computing Centre, FRC CSC RAS, 40, Vavilova, Moscow 119333 Russia E-mail: ququns@inbox.ru

were allowed); here, each player takes into account only the worst local utility (again, monotone transformations may be allowed).

Second, the main results of Kukushkin (2007) were about the acyclicity of *individual* improvements and, accordingly, the existence of Nash equilibria. Here, it is about the acyclicity of *coalitional* improvements and, accordingly, the existence of strong Nash equilibria.

Thirdly, we have to assume "negative impacts" here, i.e., whenever a new player starts to use a facility, those already there cannot be better off. In Kukushkin (2007), as well as in Rosenthal (1973), there was no need for such an assumption.

Finally, the games considered in Kukushkin (2007) were partitioned into two classes: "generalized congestion games" and "games with structured utilities." In the former class, the players choose *which* facilities to use and do not choose anything else; in the latter, each player chooses *how* to use facilities from a fixed list. Actually, the possibility of certain combinations was overlooked there, see Le Breton and Weber (2011), but the range of permissible combinations is rather limited in any case. Here, both those classes are present too, but "which" and "how" choices could be combined arbitrarily. It should be mentioned that, both here and in Kukushkin (2007), games with structured utilities form a representative subclass.

The idea of games with structured utilities and the minimum aggregation originated in Germeier and Vatel' (1974) although in a much less general form. Their approach was developed further in a series of papers, see Kukushkin et al. (1985) and references therein.

The first, to my knowledge, result on the existence of strong Nash equilibria in congestion games, even though without a reference to Rosenthal (1973), was in Moulin (1982, Chapter 5): pirates were going to a treasure island; each pirate could choose between two ships, and the more pirates on board of either ship, the slower it went. Since each player could only use a single facility (ship), the application of the minimum aggregation may be assumed, and hence that example belongs to the class of games considered here.

A systematic study of conditions under which a congestion game possesses strong Nash equilibria was started by Holzman and Law-Yone (1997), and has been continued (Holzman and Law-Yone, 2003; Rozenfeld and Tennenholtz, 2006; Epstein et al., 2009; Holzman and Monderer, 2015). As is natural in light of the necessity part of our Theorem 6.1, all those results need specific assumptions on available strategies.

The fact that the minimum ("bottleneck") aggregation and negative impacts in congestion games are conducive to coalition stability was gradually noticed quite recently (Fotakis et al., 2008; Harks et al., 2013). The results of those papers are rather similar to our Theorem 4.1, but obtained in a much less general models.

Here, the same fact is expressed in its most general form: As long as each player uses the minimum aggregation and there are negative impacts at each facility, it does not matter which subsets of facilities and what methods of using them are available to each player: all coalitional improvements are acyclic (to be more precise, there exists a "strong $\omega$-potential") and hence strong Nash equilibria exist and, in a sense, attract adaptive dynamics.

Theorem 4.4 shows that every game satisfying the assumptions of Theorem 4.1 is isomorphic to a game with structured utilities and the minimum aggregation. In other words, the main findings of

Kukushkin et al. (1985) remain relevant to every model of this type that has been considered since then. That paper, however, was silent on some important issues, e.g., algorithmic and computational aspects.

Perhaps the most interesting results of this paper are Theorems 6.1 and 6.3, which establish the necessity of the minimum aggregation for the existence, regardless of other characteristics of the game, of Pareto optimal Nash equilibria, to say nothing of strong Nash equilibria, and hence for the acyclicity of coalitional improvements as well. The first result of this kind was in Kukushkin (1992); however, it was designed for a particular class of games, so rather peculiar combinations of the minimum and maximum were allowed, which are not good in a more general case.

The minimum operator is not at all unusual in the theory of production functions. Galbraith (1958, Chapter XVIII) explicitly invoked Leontief's model to justify an attitude to public and private consumption ("social balance") that sounds indistinguishable from the minimum aggregation. Our Theorem 4.1 shows that agents who have internalized this attitude do not need any taxes to provide for an efficient level of public consumption; it is difficult to say whether Galbraith himself expected such a conclusion.

Models of public good provision where the output of the public good is the minimum or maximum of private contributions ("weakest-link" or "best-shot") are considered now and then (Hirshleifer, 1983; Cornes and Hartley, 2007; Boncinelli and Pin, 2012). Such production functions have some nice implications in that context too, but not as good as here; in particular, the existence of a strong Nash equilibrium is not guaranteed.

Section 2 introduces principal improvement relations associated with a strategic game. Section 3 provides a formal description of our basic model as well as its main structural properties. Throughout Section 4, the players use the minimum aggregation. The main results there are Theorems 4.1 and 4.4.

In Section 5, we consider the maximum aggregation rule, which has the same implications in games with positive impacts (Theorem 5.1). The leximin/leximax aggregation of local utilities is also considered there. Its properties are much closer to those of additive aggregation than minimum/maximum ones; it ensures the acyclicity of individual improvements, but not of coalitional ones.

Section 6 contains the characterization results, Theorems 6.1 and 6.3, which establish the necessity of the minimum aggregation for the existence of Pareto optimal Nash equilibria under broad assumptions. In Section 7, several related questions of secondary importance are discussed.

More complicated proofs (of Theorems 2.1, 6.1 and 6.3) are deferred to the Appendix.

## 2 Improvement dynamics in strategic games

A *strategic game* $\Gamma$ is defined by a finite set of players $N$ (we denote $n := \#N$), and strategy sets $X_i$ and utility functions $u_i$ on $X_N := \prod_{i \in N} X_i$ for all $i \in N$. We denote $\mathcal{N} := 2^N \setminus \{\emptyset\}$ (the set of potential coalitions) and $X_I := \prod_{i \in I} X_i$ for each $I \in \mathcal{N}$; instead of $X_{N \setminus \{i\}}$ and $X_{N \setminus I}$, we write $X_{-i}$ and $X_{-I}$, respectively. It is sometimes convenient to consider utility functions $u_i$ as components of a "joint" mapping $u_N \colon X_N \to \mathbb{R}^N$.

With every strategic game, a few improvement relations on $X_N$ are associated ($I \in \mathcal{N}$, $y_N, x_N \in$

$X_N$):

$$y_N \rhd_I x_N \rightleftharpoons \big[y_{-I} = x_{-I} \;\&\; \forall i \in I\, [u_i(y_N) > u_i(x_N)]\big]; \tag{1a}$$

$$y_N \rhd^{\mathrm{Ind}} x_N \rightleftharpoons \exists i \in N\, [y_N \rhd_{\{i\}} x_N] \tag{1b}$$

(*individual improvement* relation);

$$y_N \rhd^{\mathrm{Coa}} x_N \rightleftharpoons \exists I \in \mathcal{N}\, [y_N \rhd_I x_N] \tag{1c}$$

(*strong coalitional improvement* relation).

A *maximizer* of an improvement relation $\rhd$, i.e., a strategy profile $x_N \in X_N$ such that $y_N \rhd x_N$ holds for no $y_N \in X_N$, is an equilibrium: a Nash equilibrium if $\rhd$ is $\rhd^{\mathrm{Ind}}$; a strong Nash equilibrium if $\rhd$ is $\rhd^{\mathrm{Coa}}$.

An *individual improvement path* is a (finite or infinite) sequence $\{x_N^k\}_{k=0,1,\dots}$ such that $x_N^{k+1} \rhd^{\mathrm{Ind}} x_N^k$ whenever $x_N^{k+1}$ is defined; an *individual improvement cycle* is an individual improvement path such that $x_N^m = x_N^0$ for $m > 0$. A strategic game has the *finite individual improvement property* (*FIP*; Monderer and Shapley, 1996) if there exists no infinite individual improvement path; then every individual improvement path, if continued whenever possible, reaches a Nash equilibrium in a finite number of steps.

Replacing $\rhd^{\mathrm{Ind}}$ with $\rhd^{\mathrm{Coa}}$, we obtain the definitions of a *coalitional improvement path*, a *coalitional improvement cycle*, and the *finite coalitional improvement property* (*FCP*). The latter implies that every coalitional improvement path reaches a strong Nash equilibrium in a finite number of steps.

**Remark.** Under our definitions, a single strategy profile is an improvement path (both individual and coalitional) by itself. This peculiarity causes no harm and is helpful in the formulation of Theorem 2.1 below.

For a finite game, the FIP (FCP) is equivalent to the acyclicity of the relation $\rhd^{\mathrm{Ind}}$ ($\rhd^{\mathrm{Coa}}$) and is equivalent to the existence of a "potential" in the following sense. An *order potential* of $\Gamma$ is an irreflexive and transitive relation $\succ$ on $X_N$ satisfying

$$\forall x_N, y_N \in X_N\, [y_N \rhd^{\mathrm{Ind}} x_N \Rightarrow y_N \succ x_N]. \tag{2}$$

A *strong order potential* of $\Gamma$ is an irreflexive and transitive relation $\succ$ on $X_N$ satisfying

$$\forall x_N, y_N \in X_N\, [y_N \rhd^{\mathrm{Coa}} x_N \Rightarrow y_N \succ x_N]. \tag{3}$$

In an infinite game, the absence of finite cycles does not mean very much by itself. One approach is to employ a more demanding notion of a potential. A binary relation $\succ$ on a metric space $X_N$ is *$\omega$-transitive* if it is transitive and the conditions $x_N^\omega = \lim_{k\to\infty} x_N^k$ and $x_N^{k+1} \succ x_N^k$ for all $k = 0, 1, \dots$ always imply $x_N^\omega \succ x_N^0$.

**Remark.** Gillies (1959) and Smith (1974) considered this condition for orderings.

A *strong $\omega$-potential* of $\Gamma$ is an irreflexive and $\omega$-transitive relation $\succ$ on $X_N$ satisfying (3). By Theorem 1 of Kukushkin (2008), $\succ$ admits a maximizer on $X_N$ if the latter is compact; as follows immediately from (3), every maximizer of $\succ$ is a strong Nash equilibrium.

There is still an asymmetry in the implications of the existence of a strong ($\omega$-)potential (3) in a finite game and in an infinite game. In the former case, strong Nash equilibria exist and all myopic adaptive dynamics converge to an equilibrium in a finite number of steps. In the latter case, only the existence of a strong Nash equilibrium was asserted. Actually, something can be said about adaptive dynamics in compact games too.

The simplest picture emerges if we consider improvement paths parameterized with countable ordinals. Then the existence of a strong $\omega$-potential in a compact game implies that every coalitional improvement path, if continued whenever possible, reaches a strong Nash equilibrium at some stage (Kukushkin, 2010, Theorem 3.21). In other words, the only difference between finite and infinite games is that finite paths should be replaced with transfinite ones in the latter case.

For those who believe whatever happens after the first limit to be irrelevant, the situation is much more complicated and some questions remain open. A clear-cut theorem about the possibility to approximate an equilibrium with a finite improvement path in a continuous enough game with a potential is presented in Kukushkin (2011). Although it is about individual improvements and Nash equilibrium, virtually the same argument can be applied to coalitional improvements and strong Nash equilibrium.

To be more precise, the acyclicity of coalitional improvements in a game with compact strategy sets and continuous enough preferences implies the existence of a strong Nash equilibrium and, moreover, the possibility to come arbitrarily close to an equilibrium with a finite number of coalitional improvements starting from an arbitrary strategy profile. What exactly is required from preferences is this condition expressed in terms of the coalitional improvement relation:

$$\forall I \in \mathcal{N}\, \forall y_N, x_N \in X_N \Big[ y_N \rhd_I x_N \Rightarrow \\ \exists O \subseteq X_N \big[ x_N \in O \;\&\; [O \text{ is open}] \;\&\; \forall x'_N \in O\, [(y_I, x'_{-I}) \rhd_I x'_N] \big] \Big]. \quad (4)$$

It is easily seen that the condition is satisfied if each function $u_i$ is continuous, or just upper semicontinuous in $x_N$ and continuous in $x_{-i}$ given $x_i$. The upper semicontinuity in $x_N$ alone is not enough.

**Theorem 2.1.** *Let each $X_i$ in a strategic game $\Gamma$ be compact; let $\rhd^{\mathrm{Coa}}$ be acyclic and satisfy condition (4). Then for every strategy profile $x_N^0 \in X_N$, there is a strong Nash equilibrium $y_N \in X_N$ such that for every open neighborhood $O$ of $y_N$ there exists a finite coalitional improvement path $x_N^0, x_N^1, \ldots, x_N^m$ with $x_N^m \in O$.*

A straightforward modification of the proof from Kukushkin (2011) is given in the Appendix, Section A, just for completeness.

Throughout the whole paper, we consider games with *ordinal preferences*, i.e., only the order defined by the utility function matters for each player. It is easily checked that the relations $\rhd^{\mathrm{Ind}}$ and $\rhd^{\mathrm{Coa}}$, the sets of individual and coalitional improvement paths, and the set of (strong) Nash equilibria in a game remain the same if a monotone transformation is applied to each utility function. We could assume that each $u_i$ maps $X_N$ to an arbitrary chain rather than $\mathbb{R}$; we even do that in Section 5.2. Accordingly, our players never contemplate mixed strategies, so "an equilibrium" always means "a pure-strategy equilibrium."

# 3 Common local utilities

A *game with common local utilities* may have an arbitrary (finite) set of players $N$, whereas the strategies and utility functions satisfy certain structural requirements. There is a finite set $\mathrm{A}$ of *facilities*; we denote $\mathcal{B}$ the set of all nonempty subsets of $\mathrm{A}$. For each $i \in N$, there is $\mathcal{B}_i \subseteq \mathcal{B}$ describing which combinations of facilities can be chosen by player $i$; for every $\mathrm{B} \in \mathcal{B}_i$, there is a set $\Xi_i^{\mathrm{B}}$ of strategies available to player $i$ when $\mathrm{B}$ has been chosen. The total set of strategies of player $i$, $X_i$, consists of all pairs $x_i = \langle \mathrm{B}, \xi \rangle$ such that $\mathrm{B} \in \mathcal{B}_i$ and $\xi \in \Xi_i^{\mathrm{B}}$. Given $i \in N$ and $x_i \in X_i$, we denote $\mathfrak{b}_i(x_i) \in \mathcal{B}_i$ the first component of the pair.

For every $\alpha \in \mathrm{A}$, we denote $I_\alpha^- := \{i \in N \mid \forall \mathrm{B} \in \mathcal{B}_i\, [\alpha \in \mathrm{B}]\}$, $I_\alpha^+ := \{i \in N \mid \exists \mathrm{B} \in \mathcal{B}_i\, [\alpha \in \mathrm{B}]\}$, and $\mathcal{I}_\alpha := \{I \in \mathcal{N} \mid I_\alpha^- \subseteq I \subseteq I_\alpha^+\}$. We assume $I_\alpha^+ \neq \emptyset$ for every $\alpha \in \mathrm{A}$ – if nobody can use a facility, there is no point in including it in the description of the game – and hence $\mathcal{I}_\alpha \neq \emptyset$ too. Given $i \in N$ and $\alpha \in \mathrm{A}$, we denote $X_i^\alpha := \{\langle \mathrm{B}, \xi \rangle \in X_i \mid \alpha \in \mathrm{B}\}$; clearly, $X_i^\alpha \neq \emptyset \iff i \in I_\alpha^+$. For every $\alpha \in \mathrm{A}$ and $I \in \mathcal{I}_\alpha$, there is a function $\varphi_\alpha(I, \cdot)\colon X_I^\alpha \to \mathbb{R}$, the "local utility function" associated with the facility $\alpha$. For every $i \in N$ and $x_i \in X_i$, there is a mapping $U_i^{x_i}\colon \mathbb{R}^{\mathfrak{b}_i(x_i)} \to \mathbb{R}$, an *aggregation rule*.

**Remark.** We use the notation $\mathbb{R}^{\mathfrak{b}_i(x_i)}$ rather than $\mathbb{R}^{\#\mathfrak{b}_i(x_i)}$, here and later on, to make explicit the connection between arguments of $U_i^{x_i}$ and facilities from $\mathfrak{b}_i(x_i)$. If, e.g., $\alpha, \beta, \gamma \in \mathrm{A}$, $\alpha \neq \beta \neq \gamma \neq \alpha$, $x_i, y_i \in X_i$, $\mathfrak{b}_i(x_i) = \{\alpha, \beta\}$ and $\mathfrak{b}_i(y_i) = \{\beta, \gamma\}$, then it would be unnatural to believe both $U_i^{x_i}$ and $U_i^{y_i}$ to be defined on the same $\mathbb{R}^2$ while their arguments are flagrantly different.

Given a strategy profile $x_N \in X_N$, we denote $N(\alpha, x_N) := \{i \in N \mid \alpha \in \mathfrak{b}_i(x_i)\}$ for each $\alpha \in \mathrm{A}$: the set of players using $\alpha$ at $x_N$. The "ultimate" utility functions of the players are built of the local utilities:

$$u_i(x_N) := U_i^{x_i}\big(\langle \varphi_\alpha(N(\alpha, x_N), x_{N(\alpha,x_N)}) \rangle_{\alpha \in \mathfrak{b}_i(x_i)}\big), \tag{5}$$

for all $i \in N$ and $x_N \in X_N$.

When considering infinite strategy sets, we impose appropriate topological assumptions; the exact conditions may differ from one theorem to another.

We say that player $i$ has a *negative impact* on facility $\alpha$ if, whenever $i \notin I \in \mathcal{I}_\alpha$, $I \cup \{i\} \in \mathcal{I}_\alpha$, $x_i \in X_i^\alpha$, and $x_I^\alpha \in X_I^\alpha$, there holds

$$\varphi_\alpha(I, x_I^\alpha) \geq \varphi_\alpha(I \cup \{i\}, \langle x_I^\alpha, x_i \rangle). \tag{6}$$

We say that player $i$ has a *strictly negative impact* on facility $\alpha$ if the inequality in (6) is strict. We call $\Gamma$ a *game with* (*strictly*) *negative impacts* if the appropriate condition holds for all $i \in N$ and $\alpha \in \mathrm{A}$. A definition of (*strictly*) *positive impacts* is obtained by reversing the inequality sign in (6).

The class of games with common local utilities includes both classes of games considered in Kukushkin (2007): "generalized congestion games" and "games with structured utilities." In a *generalized congestion game*, each set $\Xi_i^{\mathrm{B}}$ is a singleton (i.e., each player chooses just a set of facilities

$B \in \mathcal{B}_i$), and $\varphi_\alpha$ only depends on $\#I$ (so we use the notation $\varphi_\alpha(\#I)$ rather than $\varphi_\alpha(I, x_I^\alpha)$ in this case). Rosenthal's (1973) proper congestion games are distinguished by additive aggregation of local utilities.

If, conversely, each $\mathcal{B}_i$ is a singleton, the game is a *game with structured utilities* as defined in Kukushkin (2007). Dealing with such games, we use the notation $\mathcal{B}_i =: \{\Upsilon_i\}$ and $N(\alpha) := \{i \in N \mid \alpha \in \Upsilon_i\}$, and identify $\Xi_i^{\Upsilon_i}$ with $X_i$. The local utility functions then are just $\varphi_\alpha \colon X_{N(\alpha)} \to \mathbb{R}$. Technically, each facility in a game with structured utilities exhibits both strictly negative and strictly positive impacts since there is no situation where (6) or its reverse would be required.

Henceforth, "a game" always means "a game with common local utilities."

# 4 Games with the minimum aggregation

Throughout this section, we assume that each player always uses the minimum ("weakest-link") aggregation of relevant local utilities:

$$u_i(x_N) = \min_{\alpha \in \mathfrak{b}_i(x_i)} \varphi_\alpha(N(\alpha, x_N), x_{N(\alpha, x_N)}) \tag{7}$$

for all $i \in N$ and $x_N \in X_N$. In economics terms, (7) means that all local utilities are absolute complements.

An important role in the study of such games is played by the leximin ordering on a (finite) Cartesian power of $\mathbb{R}$. Let us recall the standard definition.

Given a finite set $M$, $\#M = m$, and $v_M \in \mathbb{R}^M$, we denote $\pi(v_M) := \langle \pi_1(v_M), \ldots, \pi_m(v_M) \rangle$ the list of the same values $v_h$ for $h \in M$ in the increasing order: $\pi_1(v_M) \leq \cdots \leq \pi_m(v_M)$, and there is a one-to-one mapping $\sigma \colon \{1, \ldots, m\} \to M$ such that $\pi_h(v_M) = v_{\sigma(h)}$ for all $h$. Now we can define the ordering itself:

$$v'_M >_{\text{Lmin}} v_M \rightleftharpoons \exists h \left[\pi_h(v'_M) > \pi_h(v_M) \;\&\; \forall h' < h \left[\pi_{h'}(v'_M) = \pi_{h'}(v_M)\right]\right]. \tag{8}$$

Obviously, $>_{\text{Lmin}}$ is irreflexive and transitive. Two lists $v_M, v'_M \in \mathbb{R}^M$ are incomparable if and only if $\pi(v_M) = \pi(v'_M)$; therefore, incomparability is an equivalence relation.

For future reference, we also define the leximax ordering. The only difference is that we start with the greatest components when comparing two lists.

$$v'_M >_{\text{Lmax}} v_M \rightleftharpoons \exists h \left[\pi_h(v'_M) > \pi_h(v_M) \;\&\; \forall h' > h \left[\pi_{h'}(v'_M) = \pi_{h'}(v_M)\right]\right]. \tag{9}$$

**Theorem 4.1.** *Let $\Gamma$ be a game with negative impacts where each player uses the minimum aggregation, i.e., conditions (6) and (7) hold everywhere. Let each relevant $\Xi_i^{\mathrm{B}}$ be a compact metric space (hence so is each $X_i$) and each relevant function $\varphi_\alpha(I, \cdot)$ be upper semicontinuous. Then $\Gamma$ admits a strong $\omega$-potential, and hence possesses a strong Nash equilibrium.*

*Proof.* Considering utility functions $u_i$ as components of a mapping $u_N \colon X_N \to \mathbb{R}^N$, we define $\succ$ on $X_N$ by

$$y_N \succ x_N \rightleftharpoons u_N(y_N) >_{\text{Lmin}} u_N(x_N),$$

where $>_{\text{Lmin}}$ is the leximin ordering on $\mathbb{R}^N$ defined by (8). Obviously, $\succ$ is irreflexive and transitive.

**Lemma 4.1.1.** $\succ$ *is* $\omega$*-transitive on* $X_N$*.*

*Proof.* For every $x_N \in X_N$, we denote $\vartheta(x_N) := \langle \vartheta_1(x_N), \ldots, \vartheta_n(x_N) \rangle$ the list of values $u_i(x_N)$ for $i \in N$ in the increasing order; in the above notation, $\vartheta_h(x_N) = \pi_h(u_N(x_N))$. Since each function $u_i$ is upper semicontinuous in $x_N$, so is each $\vartheta_h$.

Now let $x_N^{k+1} \succ x_N^k$ for all $k = 0, 1, \ldots$ and $x_N^k \to x_N^\omega$; we have to show $x_N^\omega \succ x_N^0$. For each $k \in \mathbb{N}$, we denote $h(k)$ the $h$ from (8) for $u_N(x_N^{k+1}) >_{\mathrm{Lmin}} u_N(x_N^k)$, i.e., $\vartheta_{h'}(x_N^{k+1}) = \vartheta_{h'}(x_N^k)$ for $h' < h(k)$ and $\vartheta_{h(k)}(x_N^{k+1}) > \vartheta_{h(k)}(x_N^k)$. Since $N$ is finite, we may, replacing $\langle x_N^k \rangle_k$ with a subsequence if needed, assume that $h(k) = h$ does not depend on $k$. Now we have $\vartheta_{h'}(x_N^\omega) \geq \vartheta_{h'}(x_N^0)$ for $h' < h$ and $\vartheta_h(x_N^\omega) > \vartheta_h(x_N^0)$ by the upper semicontinuity; therefore, $u_N(x_N^\omega) >_{\mathrm{Lmin}} u_N(x_N^0)$. $\square$

**Lemma 4.1.2.** *Given* $y_N, x_N \in X_N$*, we denote* $N_+ := \{i \in N \mid u_i(y_N) > u_i(x_N)\}$ *and* $N_- := \{i \in N \mid u_i(y_N) < u_i(x_N)\}$*. Let* $\min_{i \in N_-} u_i(y_N) > \min_{i \in N_+} u_i(x_N)$*, assuming* $\min \emptyset := +\infty$*. Then* $y_N \succ x_N$*.*

A straightforward proof is omitted.

**Lemma 4.1.3.** $\succ$ *satisfies* (3)*.*

*Proof.* Supposing $y_N \rhd_I x_N$, we have to show $y_N \succ x_N$. If $y_N$ Pareto dominates $x_N$, then we are home immediately. Let

$$u_j(y_N) < u_j(x_N); \tag{10}$$

then $j \notin I$, so $y_j = x_j$. By (7), there is $\alpha \in \mathfrak{b}_j(y_j) = \mathfrak{b}_j(x_j)$ such that $u_j(y_N) = \varphi_\alpha(N(\alpha, y_N), y_{N(\alpha, y_N)})$. Suppose $I \cap N(\alpha, y_N) = \emptyset$; then $N(\alpha, y_N) \subseteq N(\alpha, x_N)$ and $x_{N(\alpha,y_N)} = y_{N(\alpha,y_N)}$; hence $\varphi_\alpha(N(\alpha, x_N), x_{N(\alpha,x_N)}) \leq \varphi_\alpha(N(\alpha, y_N), y_{N(\alpha,y_N)})$ by (6); hence $u_j(x_N) \leq u_j(y_N)$, contradicting (10). Therefore, there must be $i \in I \cap N(\alpha, y_N)$ and hence $u_j(y_N) = \varphi_\alpha(N(\alpha, y_N), y_{N(\alpha,y_N)}) \geq u_i(y_N) > u_i(x_N)$. Since $j$ satisfying (10) was arbitrary, Lemma 4.1.2 is applicable, implying $y_N \succ x_N$. $\square$

Now Theorem 4.1 immediately follows from Lemmas 4.1.1 and 4.1.3. $\square$

**Corollary.** *If in Theorem 4.1 all functions* $\varphi_\alpha(I, \cdot)$ *are* continuous*, then for every strategy profile* $x_N^0 \in X_N$*, there is a strong Nash equilibrium* $y_N \in X_N$ *such that for every open neighborhood* $O$ *of* $y_N$*, there is a finite coalitional improvement path starting at* $x_N^0$ *and ending in* $O$*.*

The statement immediately follows from Theorems 4.1 and 2.1.

Without the negative impacts assumption, the proof of Lemma 4.1.3 collapses; actually, Theorem 4.1 cannot be extended that far. Even a generalized congestion game with the minimum aggregation may fail to possess a strong Nash equilibrium.

**Example 4.2.** Let us consider a two person generalized congestion game with the minimum aggregation (7): $N := \{1, 2\}$; $\mathrm{A} := \{a, b, c\}$; $X_1 := \{\mathrm{A}, \{a\}\}$, $X_2 := \{\mathrm{A}, \{b\}\}$; $\varphi_a(1) := \varphi_b(1) := 1$, $\varphi_a(2) := \varphi_b(2) := 3$, $\varphi_c(1) := 0$ $\varphi_c(2) := 2$ (i.e., every facility exhibits positive impacts). The matrix of the game looks as follows:

$$\begin{array}{ccc} & \text{abc} & \text{b} \\ \text{abc} & (2,2) & (0,3) \\ \text{a} & (3,0) & (1,1). \end{array}$$

We have a prisoner's dilemma.

**Example 4.3.** Let us consider a two person generalized congestion game: $N := \{1, 2\}$, $\mathrm{A} := \{a, b\}$, $X_1 := X_2 := \{\{a\}, \{b\}\}$, $\varphi_a(1) := 0$, $\varphi_a(2) := 2$, $\varphi_b(1) := 3$, $\varphi_b(2) := 1$ (i.e., $a$ exhibits positive impacts; $b$, negative). The matrix of the game looks as follows:

$$\begin{array}{ccc} & \text{a} & \text{b} \\ \text{a} & (2,2) & (0,3) \\ \text{b} & (3,0) & (1,1). \end{array}$$

We have a prisoner's dilemma again.

Among games with the minimum aggregation and negative impacts, games with structured utilities form a representative subclass. We call two strategic games $\Gamma^*$ and $\Gamma$ *isomorphic* if the set $N$ is the same in both games whereas there is a bijection $\sigma_i \colon X_i \to X_i^*$ for each $i \in N$ such that $u_i^*(\sigma_N(x_N)) = u_i(x_N)$ for all $x_N \in X_N$ and $i \in N$.

**Theorem 4.4.** *For every game $\Gamma$ with the minimum aggregation and negative impacts, there exists a game $\Gamma^*$ with structured utilities and also with the minimum aggregation, which is isomorphic to $\Gamma$.*

*Proof.* We define $\mathrm{A}^* := \mathrm{A} \times \mathcal{N}$, and, for each $i \in N$, $\Upsilon_i := \{(\alpha, I) \in \mathrm{A}^* \mid i \in I\}$. Then we define $\Gamma^*$ as follows: the set of players is the same, $N^* := N$; the set of facilities is $\mathrm{A}^*$; each player has a singleton set $\mathcal{B}_i^* := \{\Upsilon_i\}$ (hence $\Gamma^*$ is a game with structured utilities indeed and $N((\alpha, I)) = I$ for every $(\alpha, I) \in \mathrm{A}^*$) and strategies $\Xi_i^{\Upsilon_i} := X_i$. As was agreed at the end of Section 3, we identify $X_i^*$ with $\Xi_i^{\Upsilon_i} = X_i$. For every $(\alpha, I) \in \mathrm{A}^*$ and $x_I \in X_I^* = X_I$, we define

$$\varphi^*_{(\alpha,I)}(x_I) := \begin{cases} \varphi_\alpha(I, x_I), & \text{if } \forall i \in I\,[\alpha \in \mathfrak{b}_i(x_i)], \\ +\infty, & \text{otherwise.} \end{cases}$$

The description of $\Gamma^*$ is accomplished by the assumption that every player always employs the minimum aggregation of local utility functions: $u_i^*(x_N) := \min_{(\alpha,I)\in\Upsilon_i} \varphi^*_{(\alpha,I)}(x_I)$.

**Remark.** The $+\infty$ in the definition of $\varphi^*$ need not be understood literally: anything large enough would do.

Now we define a bijection $\sigma_i \colon X_i \to X_i^*$ for each $i \in N$ by $\sigma_i(x_i) := x_i$. Let us show that $u_i(x_N) = u_i^*(\sigma_N(x_N))$ for every $i \in N$ and $x_N \in X_N$.

Let $u_i(x_N) = \varphi_{\bar\alpha}(N(\bar\alpha, x_N), x_{N(\bar\alpha,x_N)})$ with $i \in N(\bar\alpha, x_N) = M$. We have $(\bar\alpha, M) \in \Upsilon_i$ and $\varphi^*_{(\bar\alpha,M)}(x_M) = \varphi_{\bar\alpha}(M, x_M) = u_i(x_N)$; therefore, $u_i^*(\sigma_N(x_N)) \le u_i(x_N)$.

Now let $(\alpha, I) \in \Upsilon_i$ and $\varphi^*_{(\alpha,I)}(x_I) < +\infty$; then $i \in I \subseteq N(\alpha, x_N)$. If $I \subset N(\alpha, x_N)$, then $\varphi^*_{(\alpha,I)}(x_I) = \varphi_\alpha(I, x_I) \ge \varphi_\alpha(N(\alpha, x_N), x_{N(\alpha,x_N)})$ by (6). If $I = N(\alpha, x_N)$, then $\varphi^*_{(\alpha,I)}(x_I) = \varphi_\alpha(N(\alpha, x_N), x_{N(\alpha,x_N)})$. In either case, $\varphi^*_{(\alpha,I)}(x_I) \ge u_i(x_N)$ by (7). Since $(\alpha, I) \in \Upsilon_i$ was arbitrary, $u_i^*(\sigma_N(x_N)) \ge u_i(x_N)$. $\square$

## 5 Related aggregation rules

### 5.1 Maximum aggregation

The maximum ("best-shot") aggregation is defined "dually" to (7):

$$u_i(x_N) = \max_{\alpha \in \mathfrak{b}_i(x_i)} \varphi_\alpha(N(\alpha, x_N), x_{N(\alpha, x_N)}) \tag{11}$$

for all $i \in N$ and $x_N \in X_N$.

From the economists' viewpoint, an unpleasant feature of the aggregation rule (11) is that $u_i$ need not be concave even if all $\varphi_\alpha$'s are.

**Theorem 5.1.** *Let $\Gamma$ be a game with positive impacts where each player uses the maximum aggregation, i.e., conditions (11) and (6) with the reversed inequality sign hold everywhere. Let each relevant $\Xi_i^{\mathrm{B}}$ be a compact metric space (hence so is each $X_i$) and each relevant function $\varphi_\alpha(I, \cdot)$ be upper semicontinuous. Then $\Gamma$ admits a strong $\omega$-potential, and hence possesses a strong Nash equilibrium.*

*Proof.* Similarly to Theorem 4.1, we define a strong $\omega$-potential by the leximax ordering (9) rather than leximin (8): $y_N \succ x_N \rightleftharpoons u_N(y_N) >_{\mathrm{Lmax}} u_N(x_N)$. Then condition (3) is proven just dually. □

**Corollary.** *If in Theorem 5.1 all functions $\varphi_\alpha(I, \cdot)$ are* continuous, *then for every strategy profile $x_N^0 \in X_N$, there is a strong Nash equilibrium $y_N \in X_N$ such that for every open neighborhood $O$ of $y_N$, there is a finite coalitional improvement path starting at $x_N^0$ and ending in $O$.*

Immediately follows from Theorems 5.1 and 2.1.

**Theorem 5.2.** *For every game $\Gamma$ with the maximum aggregation and positive impacts, there exists a game $\Gamma^*$ with structured utilities and also with the maximum aggregation, which is isomorphic to $\Gamma$.*

The proof, dual to that of Theorem 4.4, is omitted.

The duality between the minimum and maximum aggregation rules obtains the simplest expression when it comes to finite games, with the help of this trivial identity:

$$\max_s v_s = -\min_s(-v_s)$$

for every list of $\langle v_s \rangle_{s \in \Sigma} \in \mathbb{R}^\Sigma$. If every local utility function $\varphi_\alpha$ in a game $\Gamma$ with the maximum aggregation of local utilities is replaced with $-\varphi_\alpha$, and each player's aggregation rule switches to the minimum, then all total utilities will just change their signs. Therefore, every finite (individual or coalitional) improvement cycle in the original game will remain an improvement cycle, one only has to reverse the direction. It should be mentioned also that positive impacts will become negative and vice versa. Naturally, the same trick works in the opposite direction.

Thus, Examples 4.2 and 4.3, transformed in this way, carry the same messages for games with the maximum aggregation.

### 5.2 Lexicographic aggregation

As was mentioned at the end of Section 2, we may consider preferences described by orderings without numeric representations. Then we may consider games where the players use the leximin (or leximax) ordering to aggregate local utilities. Despite the presence of "min" or "max" in their names, these types of aggregation are much closer to the additive one. This similarity is based on the well-known separability of the leximin (leximax) ordering.

We start the introduction of necessary technicalities with the notion of extended real line: $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. Given a finite set $M$, we consider the leximin ordering (8) on $\bar{\mathbb{R}}^M$; identifying incomparable vectors, we obtain a chain $\mathcal{L}_M$.

Now let there be a game with common local utilities $\Gamma$; we say that each player *uses the leximin aggregation* in $\Gamma$ if the total "utility" functions $u_i \colon X_N \to \bar{\mathbb{R}}^{\mathrm{A}}$ are:

$$u_i(x_N)_\alpha := \begin{cases} \varphi_\alpha(N(\alpha, x_N), x_{N(\alpha,x_N)}), & \text{if } i \in N(\alpha, x_N), \\ +\infty, & \text{otherwise.} \end{cases} \tag{12}$$

**Remark.** Strictly speaking, (12) should be supplemented with the identification mapping $\bar{\mathbb{R}}^{\mathrm{A}} \to \mathcal{L}_{\mathrm{A}}$. However, we disregard such technicalities.

**Proposition 5.3.** *Let $\Gamma$ be a game with structured utilities where each player uses the leximin aggregation (12), each relevant $\Xi_i^{\mathrm{B}}$ is a compact metric space, and each relevant function $\varphi_\alpha(I, \cdot)$ is upper semicontinuous. Then $\Gamma$ admits an $\omega$-potential and hence possesses a Nash equilibrium.*

*Proof.* We define a mapping $P \colon X_N \to \bar{\mathbb{R}}^{\mathrm{A}}$ by:

$$P(x_N)_\alpha := \varphi_\alpha(N(\alpha, x_N), x_{N(\alpha,x_N)}). \tag{13}$$

Then we define a binary relation $\succ$ on $X_N$ by:

$$y_N \succ x_N \rightleftharpoons P(y_N) >_{\mathrm{Lmin}} P(x_N),$$

where $>_{\mathrm{Lmin}}$ is the leximin ordering on $\bar{\mathbb{R}}^{\mathrm{A}}$ defined by (8). $\succ$ is irreflexive and $\omega$-transitive for the same reasons as in the proof of Theorem 4.1.

Let $y_N \rhd^{\mathrm{Ind}} x_N$. This implies that $u_i(y_N) >_{\mathrm{Lmin}} u_i(x_N)$ with $u_i$ defined by (12). Since $\Gamma$ is a game with structured utilities, $N(\alpha, y_N) = N(\alpha, x_N) = N(\alpha)$ for all $\alpha \in \mathrm{A}$. Therefore, $P(x_N)_\alpha = u_i(x_N)_\alpha$ and $P(y_N)_\alpha = u_i(y_N)_\alpha$ for $\alpha \in \Upsilon_i$, whereas $P(x_N)_\alpha = P(y_N)_\alpha$ and $u_i(x_N)_\alpha = u_i(y_N)_\alpha$ for $\alpha \in \mathrm{A} \setminus \Upsilon_i$. Now the separability of the the leximin ordering implies that $P(y_N) >_{\mathrm{Lmin}} P(x_N)$ and hence $y_N \succ x_N$. □

**Corollary.** *Let $\Gamma$ satisfy all conditions of Proposition 5.3, and each each relevant function $\varphi_\alpha(I, \cdot)$ be* continuous. *Then for every strategy profile $x_N^0 \in X_N$, there is a Nash equilibrium $y_N \in X_N$ such that for every open neighborhood $O$ of $y_N$, there is a finite individual improvement path starting at $x_N^0$ and ending in $O$.*

Immediately follows from Proposition 5.3 and Theorem 2.1.

**Proposition 5.4.** *Every generalized congestion game where each player uses the leximin aggregation* (12) *admits an order potential, and hence has the FIP and possesses a Nash equilibrium.*

The statement is proven with essentially the same potential as in Rosenthal (1973), see Kukushkin (2004).

**Corollary.** *Every generalized congestion game where each player uses the minimum aggregation* (7) *admits an order potential, and hence has the FIP and possesses a Nash equilibrium.*

**Remark.** The statement of the corollary is much weaker than that of Theorem 4.1, but the negative impacts assumption is not needed here. There are quite a few statements similar to Proposition 5.4 and its corollary in the literature; I cannot say who was the first to say what.

To stress the difference between the leximin and minimum aggregation, let us show that the statement of Theorem 4.1 cannot be derived from the assumptions of Propositions 5.3 or 5.4.

**Example 5.5.** Let us consider a game with structured utilities: $N := \{1,2\}$; $\mathrm{A} := \{a,b,c\}$; $\Upsilon_1 := \{a,c\}$; $\Upsilon_2 := \{b,c\}$; $X_1 := X_2 := \{1,2\}$; $\varphi_a(1) := \varphi_b(1) := 0$; $\varphi_a(2) := \varphi_b(2) := 2$; $\varphi_c(1,1) := 3$; $\varphi_c(2,1) := \varphi_c(1,2) := 1$; $\varphi_c(2,2) := 0$. Assuming that both players use the leximin aggregation, we obtain the $2\times 2$ matrix of the game:

$$\begin{matrix} (\langle 0,3\rangle,\langle 0,3\rangle) & (\langle 0,1\rangle,\langle 1,2\rangle) \\ (\langle 1,2\rangle,\langle 0,1\rangle) & (\langle 0,2\rangle,\langle 0,2\rangle). \end{matrix}$$

We have a prisoner's dilemma: the southeastern corner is a unique Nash equilibrium, which is Pareto dominated by the northwestern corner.

**Example 5.6.** Let us consider a generalized congestion game with negative impacts: $N := \{1,2\}$; $\mathrm{A} := \{a,b,c,d,e,f,g\}$; $X_1 := \big\{\{a,b,c\},\{d,e,f\}\big\}$; $X_2 := \big\{\{a,f,g\},\{b,c,d\}\big\}$; $\varphi_a(2) := \varphi_b(2) := \varphi_d(2) := \varphi_e(1) := \varphi_g(1) := 0$; $\varphi_c(2) := 1$; $\varphi_a(1) := \varphi_d(1) := \varphi_f(2) := 2$; $\varphi_b(1) := \varphi_c(1) := \varphi_f(1) := 3$. Assuming that both players use the leximin aggregation, we obtain the $2\times 2$ matrix of the game:

$$\begin{array}{ccc} & \text{afg} & \text{bcd} \\ \text{abc} & (\langle 0,3,3\rangle,\langle 0,0,3\rangle) & (\langle 0,1,2\rangle,\langle 0,1,2\rangle) \\ \text{def} & (\langle 0,2,2\rangle,\langle 0,2,2\rangle) & (\langle 0,0,3\rangle,\langle 0,3,3\rangle). \end{array}$$

We have a prisoner's dilemma again: the northeastern corner is a unique Nash equilibrium, which is Pareto dominated by the southwestern corner.

Dual versions of Propositions 5.3 and 5.4 as well as Examples 5.5 and 5.6, where the leximin aggregation is replaced with the leximax one, are easy to write down.

# 6 Characterization results

A mapping $U\colon \mathbb{R}^{\Sigma(U)} \to \mathbb{R}$, where $\Sigma(U)$ is a finite set, is an *admissible aggregation function* if it is continuous and increasing in the sense of

$$\big[\forall s \in \Sigma(U)[v'_s > v_s]\big] \Rightarrow U(v'_{\Sigma(U)}) > U(v_{\Sigma(U)}). \tag{14}$$

The continuity of $U$ and (14) imply

$$\big[\forall s \in \Sigma(U)[v'_s \geq v_s]\big] \Rightarrow U(v'_{\Sigma(U)}) \geq U(v_{\Sigma(U)}). \tag{15}$$

**Remark.** Exactly as in Kukushkin (2007), all results of this section remain valid if each $U$ is assumed to be defined on a Cartesian power of an open interval in $\mathbb{R}$, e.g., $\mathbb{R}_{++}$, and the attention is restricted to games where all values of local utilities belong to that interval. When the attention is restricted to, say, integer-valued local utilities, nothing is known about the necessity parts of the following theorems; most likely, they are wrong.

To formalize the idea that the players can apply the same aggregation function $U$ to local utilities obtained from various sets of facilities, we need some very technical notations. Let there be a bijection $\mu\colon \Sigma \to \mathrm{B}$ between two finite sets. Even though $\#\Sigma = \#\mathrm{B}$, we distinguish between $\mathbb{R}^{\mathrm{B}}$ and $\mathbb{R}^{\Sigma}$ (cf. the remark at the beginning of Section 3) because the coordinates (may) have different names: $\mathbb{R}^{\mathrm{B}}$ consists of lists $v_{\mathrm{B}} = \langle v_b \rangle_{b\in\mathrm{B}}$, whereas $\mathbb{R}^{\Sigma}$ consists of lists $v_{\Sigma} = \langle v_s \rangle_{s\in\Sigma}$. We define a mapping (actually, a bijection) $\mu^*\colon \mathbb{R}^{\mathrm{B}} \to \mathbb{R}^{\Sigma}$ by $\mu^*(v_{\mathrm{B}})_s = v_{\mu(s)}$ for all $v_{\mathrm{B}} \in \mathbb{R}^{\mathrm{B}}$ and $s \in \Sigma$. Given a function $U\colon \mathbb{R}^{\Sigma} \to \mathbb{R}$, we define a function $\mu * U\colon \mathbb{R}^{\mathrm{B}} \to \mathbb{R}$ by $\mu * U(v_{\mathrm{B}}) = U(\mu^*(v_{\mathrm{B}}))$.

Let $\mathfrak{U}$ be a set of admissible aggregation functions. We say that a game with common local utilities $\Gamma$ *is consistent with the set* $\mathfrak{U}$ if for every $i \in N$ and $x_i \in X_i$, there are $U \in \mathfrak{U}$ and a bijection $\mu_i^{x_i}\colon \Sigma(U) \to \mathfrak{b}_i(x_i)$ such that

$$U_i^{x_i}(v_{\mathfrak{b}_i(x_i)}) = \mu_i^{x_i} * U(v_{\mathfrak{b}_i(x_i)}).$$

**Theorem 6.1.** *Let $\mathfrak{U}$ be a set of admissible aggregation functions such that $\#\Sigma(U) = 1$ for, at most, one $U \in \mathfrak{U}$. Then the following conditions are equivalent.*

1. *Every generalized congestion game with negative impacts which is consistent with $\mathfrak{U}$ has the FCP and hence possesses a strong Nash equilibrium.*
2. *Every generalized congestion game with strictly negative impacts which is consistent with $\mathfrak{U}$ possesses a weakly Pareto optimal Nash equilibrium.*
3. *For every $U \in \mathfrak{U}$, there is a continuous and strictly increasing mapping $\lambda^U : \mathbb{R} \to \mathbb{R}$ such that:*

$$\forall U \in \mathfrak{U}\ \forall v_{\Sigma(U)} \in \mathbb{R}^{\Sigma(U)}\ \big[U(v_{\Sigma(U)}) = \lambda^U(\min\{v_s\}_{s\in\Sigma(U)})\big]; \tag{16}$$

$$\forall U', U \in \mathfrak{U}\ \big[\lambda^{U'} = \lambda^U \ \text{ or } \ \lambda^{U'}(\mathbb{R}) \cap \lambda^U(\mathbb{R}) = \emptyset\big]. \tag{17}$$

The implication [1 ⇒ 2] is trivial. The proofs of [3 ⇒ 1] and [2 ⇒ 3] are deferred to Sections B.1 and B.2, respectively.

As is easily seen from the proof, the uniqueness of $U \in \mathfrak{U}$ with $\#\Sigma(U) = 1$ is not needed for the implication [3 ⇒ 1] in Theorem 6.1 to hold. It is, however, essential for the equivalence between Statements 1 and 2. For instance, if $\#\Sigma(U) = 1$ for all $U \in \mathfrak{U}$, then no restriction on $\lambda^U$ is needed to ensure the existence of even a strong Nash equilibrium (Konishi et al., 1997), but there may be no FIP (Milchtaich, 1996). Without that uniqueness, only a necessity result, without characterization, has been obtained.

**Proposition 6.2.** *Let $\mathfrak{U}$ be a set of admissible aggregation functions such that every generalized congestion game with negative impacts which is consistent with $\mathfrak{U}$ possesses a weakly Pareto optimal Nash equilibrium. Then there is a continuous and strictly increasing mapping $\lambda^U : \mathbb{R} \to \mathbb{R}$, for every $U \in \mathfrak{U}$, such that:*

$$\forall U \in \mathfrak{U}\ \forall v_{\Sigma(U)} \in \mathbb{R}^{\Sigma(U)}\ \big[U(v_{\Sigma(U)}) = \lambda^U(\min\{v_s\}_{s\in\Sigma(U)})\big]; \tag{18}$$

$$\forall U', U \in \mathfrak{U}\ \big[\lambda^{U'} = \lambda^U \ \text{ or }\ \lambda^{U'}(\mathbb{R}) \cap \lambda^U(\mathbb{R}) = \emptyset \ \text{ or }\ \#\Sigma(U) = \#\Sigma(U') = 1\big]. \tag{19}$$

The proof, very similar to that of the implication [2 $\Rightarrow$ 3] in Theorem 6.1, is deferred to Section B.3.

**Theorem 6.3.** *For every set $\mathfrak{U}$ of admissible aggregation functions, the following conditions are equivalent.*

1. *Every game with structured utilities which is consistent with $\mathfrak{U}$ and where the strategy sets are compact and local utility functions are upper semicontinuous admits a strong $\omega$-potential and hence possesses a strong Nash equilibrium.*

2. *Every finite game with structured utilities which is consistent with $\mathfrak{U}$ possesses a weakly Pareto optimal Nash equilibrium.*

3. *For every $U \in \mathfrak{U}$, there is a continuous and strictly increasing mapping $\lambda^U : \mathbb{R} \to \mathbb{R}$ such that either*
$$\forall U \in \mathfrak{U}\ \forall v_{\Sigma(U)} \in \mathbb{R}^{\Sigma(U)}\ \big[U(v_{\Sigma(U)}) = \lambda^U(\min\{v_s\}_{s\in\Sigma(U)})\big], \tag{20}$$
*or*
$$\forall U \in \mathfrak{U}\ \forall v_{\Sigma(U)} \in \mathbb{R}^{\Sigma(U)}\ \big[U(v_{\Sigma(U)}) = \lambda^U(\max\{v_s\}_{s\in\Sigma(U)})\big]; \tag{21}$$
*besides,*
$$\forall U', U \in \mathfrak{U}\ \big[\lambda^{U'} = \lambda^U \ \text{ or }\ \#\Sigma(U) \neq \#\Sigma(U') \ \text{ or }\ \lambda^{U'}(\mathbb{R}) \cap \lambda^U(\mathbb{R}) = \emptyset\big]. \tag{22}$$

The implication [1 $\Rightarrow$ 2] is trivial. The proofs of [3 $\Rightarrow$ 1] and [2 $\Rightarrow$ 3] are deferred to Sections C.1 and C.2, respectively.

# 7 Concluding remarks

**7.1.** It is worth repeating that this paper is about games with ordinal preferences. Theorems 4.1 and 4.4, as well as Theorems 5.1 and 5.2, would remain valid if we assumed that every $\varphi_\alpha(I, \cdot)$ maps $X_I^\alpha$ to an arbitrary chain rather than $\mathbb{R}$, cf. Proposition 5.4. On the other hand, the chain must be the same for all $I$ and $\alpha$; thus, the preferences are actually "co-ordinal" here.

**7.2.** Our assumption that all users obtain the same local utility from a facility should not be viewed as a simplifying technical condition. Making it, we concentrate on relationships between "fellow travelers," which can be considered as basic as, e.g., those between competitors for a scarce resource. At the moment, there is no evidence to suggest that similar results could hold in a broader context.

There are models in the literature where local utilities are *not* common. For instance, congestion games with player-specific payoff functions possess, under certain assumptions, strong Nash equilibria

(Milchtaich, 1996; Konishi et al., 1997). Harks and Klimm (2015) obtained the existence of Nash equilibria in games where the players choose both facilities and local strategies. However, there is, typically, no acyclicity of improvements in such models, and hence no ground to expect a close connection with this paper.

**7.3.** We could define the *weak* coalitional improvement relation similarly to (1):

$$y_N \rhd_I^{\mathrm{wCo}} x_N \rightleftharpoons \big[y_{-I} = x_{-I} \,\&\, \forall i \in I\,[u_i(y_N) \geq u_i(x_N)] \,\&\, \exists i \in I\,[u_i(y_N) > u_i(x_N)]\big]; \tag{23a}$$

$$y_N \rhd^{\mathrm{wCo}} x_N \rightleftharpoons \exists I \in \mathcal{N}\,[y_N \rhd_I^{\mathrm{wCo}} x_N]. \tag{23b}$$

A maximizer of $\rhd^{\mathrm{wCo}}$ can be called a "very" strong Nash equilibrium. The existence of such equilibria could not be asserted in Theorem 4.1 or Theorem 5.1; however, Kukushkin et al. (1985) showed that they, in a sense, exist "more often than not." Feldman and Tennenholtz (2010) obtained a more straightforward existence theorem at the price of restricting the quantifier in (23b) to a subset of $\mathcal{N}$.

**7.4.** Theorems 4.1 and 5.1 imply the acyclicity of strong coalitional improvements, and hence the existence of strong Nash equilibria, in congestion games with singleton strategies under negative, respectively positive, impacts. Both facts were noticed in Holzman and Law-Yone (1997), and Rozenfeld and Tennenholtz (2006), respectively. However, our theorems are equally applicable to group formation games where the wellbeing of each group depends on the set of members rather than on their number only, provided either all impacts are negative or all positive. Without this uniformity of impacts, even the existence of a Nash equilibrium in such games is not guaranteed.

**Example 7.1.** Let us consider a two person game where each player chooses a single facility, but there is no anonymity: $N := \{1, 2\}$, $\mathrm{A} := \{a, b\}$, $X_1 := X_2 := \{\{a\}, \{b\}\}$, $\varphi_a(\{2\}) := 0$, $\varphi_a(\{1\}) := 2$, $\varphi_a(N) := 4$, $\varphi_b(N) := 1$, $\varphi_b(\{2\}) := 3$, $\varphi_b(\{1\}) := 5$ (i.e., $a$ exhibits positive impacts; $b$, negative). The matrix of the game looks as follows:

$$\begin{array}{ccc} & \mathrm{a} & \mathrm{b} \\ \mathrm{a} & (4,4) & (2,3) \\ \mathrm{b} & (5,0) & (1,1). \end{array}$$

There is no Nash equilibrium.

**7.5.** Just as in the case of Kukushkin (2007), some forms of the main results of this paper can be found in Kukushkin (2004). The biggest advances over that paper are in Theorems 6.1 and 6.3 here: a much broader notion of a family of aggregation rules is employed. Under this notion, the special role of "aggregation rules" for the case of a single local utility in generalized congestion games was discerned. It should be stressed that the possibility to reverse the implication in Proposition 6.2 remains unclear.

**7.6.** Comparing the formulations of Theorems 6.1 and 6.3 with those of Theorems 1 and 3 from Kukushkin (2007), three essential differences can be noticed (apart from the difference between Nash and strong Nash equilibria): The monotonicity conditions in the latter case are stronger; instead of a set of admissible aggregation functions available to any player who may show up, the set of players was fixed beforehand in Kukushkin (2007), and each player had his own family of available aggregation functions; finally, there was a unique aggregation function for every number of arguments in each such family, whereas here we only assume uniqueness in the case of a single argument. Let us discuss each item separately.

First, under the monotonicity assumption (14), the necessity parts of Theorems 1 and 3 from Kukushkin (2007) would be just wrong, the maximum/minimum aggregation functions being counterexamples. It remains unclear whether a similar characterization result under weaker monotonicity assumptions is at all possible. Consider, e.g., this aggregation rule:

$$U(v_1, v_2, v_3, \ldots, v_m) := \begin{cases} v_1 \cdot v_2 \cdot v_3 \cdots v_m, & \text{if } \forall s\, [v_s > 0], \\ \min\{v_1, v_2, v_3, \ldots, v_m\}, & \text{otherwise.} \end{cases}$$

It is easy to see that the use of such aggregation by all players ensures the acyclicity of individual improvements in any generalized congestion game or game with structured utilities.

Second, it is of crucial importance in the proofs of Lemmas B.2.3, B.2.2, C.2.2, and C.2.3 of this paper that the same aggregation function may be used by different players. I cannot prove the lemmas assuming that each player may have an idiosyncratic family of aggregation functions; but have no counterexample either.

Thirdly, the proofs of Theorems 1 and 3 from Kukushkin (2007) should survive dropping that uniqueness assumption (although I have not checked everything carefully) with a single exception: the uniqueness for single-argument "aggregation" is essential in Theorem 1 for the same reasons as in Theorem 6.1 here.

**7.7.** In Kukushkin (2007), a similarity was noted between the necessity proofs there and the famous Debreu–Gorman Theorem (Debreu, 1960; Gorman, 1968), see also Wakker (1989), on additive representation of separable orderings. There seems to be no general theorem on abstract preference orderings that could display parallel similarities with Theorems 6.1 and 6.3 here. In particular, no connection has been established so far with the axiomatic characterizations of the leximin and leximax orderings in the social choice theory (d'Aspremont and Gevers, 1977; Deschamps and Gevers, 1978).

# Appendix: Proofs

## A Proof of Theorem 2.1

Given $x_N^0 \in X_N$, we denote $Y \subseteq X_N$ the set of strategy profiles that can be reached from $x_N^0$ with finite coalitional improvement paths. Then we define $Z := \operatorname{cl} Y$; clearly, $Z$ is compact. We have to prove that $Z$ contains a strong Nash equilibrium, i.e., a maximizer of $\rhd^{\mathrm{Coa}}$ on $X_N$.

**Lemma A.1.** *If $z_N \in Z$, $I \in \mathcal{N}$, and $y_N \rhd_I z_N$, then $y_N \in Z$ too.*

*Proof.* By (4), we have $(y_I, x_{-I}) \rhd_I x_N$ whenever $x_N$ belongs to an appropriate neighborhood $O$ of $z_N$. Let $V$ be an arbitrary open neighborhood of $y_N$; we pick an open neighborhood $V_{-I}$ of $y_{-I}$ such that $\{y_I\} \times V_{-I} \subseteq V$. By the definition of $Z$, there is a finite coalitional improvement path $\langle x_N^0, x_N^1, \ldots, x_N^m \rangle$ such that $x_N^m \in O \cap (X_I \times V_{-I})$. We define $x_N^{m+1} := (y_I, x_{-I}^m)$. Since $\langle x_N^0, x_N^1, \ldots, x_N^m, x_N^{m+1} \rangle$ remains a finite coalitional improvement path, $x_N^{m+1} \in Y$. Since $x_N^{m+1} \in V$ and $V$ was arbitrary, we have $y_N \in Z$. $\square$

**Lemma A.2.** *There exists a maximizer of $\triangleright^{\text{Coa}}$ on $Z$.*

*Proof.* Supposing the contrary, we have $y_N(x_N) \in Z$ and $I(x_N) \in \mathcal{N}$, for every $x_N \in Z$, such that $y_N(x_N) \triangleright_{I(x_N)} x_N$; therefore, there holds $(y_{I(x_N)}, x'_{-I(x_N)}) \triangleright_{I(x_N)} x'_N$ for every $x'_N$ from an appropriate neighborhood of $x_N$ by (4). Since $Z$ is compact, there are open subsets $O^1, \ldots, O^m \subseteq X_N$, strategy profiles $y_N^1, \ldots, y_N^m \in Z$, and $I(h) \in \mathcal{N}$ for each $h \in \{1, \ldots, m\}$ such that $Z \subseteq \bigcup_{h=1}^m O^h$ and $(y_{I(h)}^h, x_{-I(h)}) \triangleright_{I(h)} x_N$ whenever $x_N \in O^h$ ($h \in \{1, \ldots, m\}$).

Now we recursively construct an infinite sequence $\langle x_N^k \rangle_{k \in \mathbb{N}}$ in $Z$, starting with $x_N^0$ already given. Having $x_N^k \in Z$ defined, we pick $h$ such that $x_N^k \in O^h$ and define $x_N^{k+1} := (y_{I(h)}^h, x_{-I(h)}^k)$. By (4), we have $x_N^{k+1} \triangleright_{I(h)} x_N^k$, hence $x_N^{k+1} \in Z$ by Lemma A.1. Therefore, $\langle x_N^k \rangle_{k \in \mathbb{N}}$ is an infinite coalitional improvement path in $Z$. The way our path is constructed ensures that, for every $i \in N$ and $k \in \mathbb{N}$, $x_i^k$ is either $x_i^0$ or one of $y_i^h$ ($h \in \{1, \ldots, m\}$), i.e., there is a finite number of possible values. Therefore, we must have $x_N^{k'} = x_N^{k''}$ with $k' \neq k''$, which contradicts the supposed acyclicity of $\triangleright^{\text{Coa}}$. $\square$

To finish with the proof of the theorem, we pick a maximizer $z_N$ of $\triangleright^{\text{Coa}}$ on $Z$, existing by Lemma A.2. By Lemma A.1, it is a maximizer of $\triangleright^{\text{Coa}}$ on $X_N$, i.e., a strong Nash equilibrium.

# B Proof of Theorem 6.1

## B.1 Sufficiency

Let $\mathfrak{U}$ be a set of admissible aggregation functions satisfying both conditions (16) and (17) from Theorem 6.1. The condition (17) obviously implies that $\mathfrak{U}$ is partitioned into a (finite or infinite) number of subsets $W^t$ ($t \in T$) such that $\lambda^U = \lambda^{U'}$ whenever $U$ and $U'$ belong to the same $W^t$, and $\lambda^U(\mathbb{R}) \cap \lambda^{U'}(\mathbb{R}) = \emptyset$ whenever they do not. The latter condition, in turn, means that the set $T$ is linearly ordered by the relation $t > t' \rightleftharpoons \big[\lambda^U(u) > \lambda^{U'}(u')$ whenever $U \in W^t$, $U' \in W^{t'}$, and $u, u' \in \mathbb{R}\big]$.

Let $\Gamma$ be a generalized congestion game with negative impacts which is consistent with $\mathfrak{U}$. For each player $i \in N$, the order on $T$ generates an ordering on $X_i$: $x_i \succeq y_i \rightleftharpoons \big[U_i^{x_i} \in W^t \;\&\; U_i^{y_i} \in W^{t'} \;\&\; [t = t' \text{ or } t > t']\big]$. Obviously, $u_i(x_i, z_{-i}) > u_i(y_i, z'_{-i})$ for all $z_{-i}, z'_{-i} \in X_{-i}$ whenever $x_i \succ y_i$. Assuming the possibility of a coalitional improvement cycle in $\Gamma$, we immediately see that all strategies of each player involved in the cycle must be equivalent in that ordering. Denoting $\Gamma^*$ the game with the same players, facilities, and strategies, but with the minimum aggregation (7), we see that the same cycle is a coalitional improvement cycle in $\Gamma^*$ as well; however, this contradicts Theorem 4.1.

## B.2 Necessity

As a first step, we show that every function $U \in \mathfrak{U}$ is symmetric.

**Lemma B.2.1.** *Let $U \in \mathfrak{U}$, $s', s'' \in \Sigma(U)$, $v^-, v^+ \in \mathbb{R}$, and $v'_{\Sigma(U)}, v''_{\Sigma(U)} \in \mathbb{R}^{\Sigma(U)}$ be such that $v^+ > v^-$, $v''_{s''} = v'_{s'} = v^+$, $v''_{s'} = v'_{s''} = v^-$, and $v''_s = v'_s$ for all $s \in \Sigma(U) \setminus \{s', s''\}$. Then $U(v''_{\Sigma(U)}) = U(v'_{\Sigma(U)})$.*

*Proof.* Supposing the contrary, we may, without restricting generality, assume $u^+ := U(v''_{\Sigma(U)}) > U(v'_{\Sigma(U)}) =: u^-$. Now let us consider a generalized congestion game with strictly negative impacts which is consistent with $\mathfrak{U}$: $N := \{1, 2\}$; the facilities are $A := \{a, b, c, d\} \cup E$, where $E := \{e_s\}_{s \in \Sigma(U) \setminus \{s', s''\}}$; $X_1 := \big\{\{a, c\} \cup E, \{b, d\} \cup E\big\}$; $X_2 := \big\{\{d, a\} \cup E, \{c, b\} \cup E\big\}$; $\varphi_z(1) := v^+$ and $\varphi_z(2) := v^-$ for each $z \in \{a, b, c, d\}$, while $\varphi_{e_s}(2) := v'_s$ for all $s \in \Sigma(U) \setminus \{s', s''\}$; $U_i^{x_i}$ is $U$ for both $i \in N$ and all $x_i \in X_i$; $\mu_1^{\{a,c\}\cup E}(s') := c$, $\mu_1^{\{a,c\}\cup E}(s'') := a$, $\mu_1^{\{b,d\}\cup E}(s') := d$, $\mu_1^{\{b,d\}\cup E}(s'') := b$, $\mu_2^{\{d,a\}\cup E}(s') := a$, $\mu_2^{\{d,a\}\cup E}(s'') := d$, $\mu_2^{\{c,b\}\cup E}(s') := b$, $\mu_2^{\{c,b\}\cup E}(s'') := c$, and $\mu_i^{x_i}(s) := e_s$ for both $i \in N$ and all $x_i \in X_i$ and $s \in \Sigma(U) \setminus \{s', s''\}$. The $2 \times 2$ matrix of the game looks as follows:

$$\begin{array}{ccc} & \text{daE} & \text{cbE} \\ \text{acE} & (u^-, u^+) & (u^+, u^-) \\ \text{bdE} & (u^+, u^-) & (u^-, u^+). \end{array}$$

There is no Nash equilibrium in the game. □

**Remark.** Neither continuity, nor monotonicity of $U$ were needed in the proof. On the other hand, in Lemmas B.1 and B.2 of Kukushkin (2007), where the continuity and monotonicity *were* used, there was no need to assume that the same aggregation function is available to all players, cf. Section 7.6.

Lemma B.2.1 shows that the mappings $\mu_i^{x_i}$ do not matter and hence may be dropped in the following. Moreover, we will assume that $\Sigma(U) = \{1, \ldots, m\}$ (with $m$ depending on $U$, naturally; when considering two functions simultaneously, we will assume that $\Sigma(U') = \{1, \ldots, m'\}$). As a next step, we show that the impossibility of a prisoner's dilemma implies that each indifference curve in each two-dimensional section must exhibit a similarity with either minimum or maximum.

**Lemma B.2.2.** *Let $U \in \mathfrak{U}$, $v_1 > v_2$, and*

$$U(v_1, v_2, v_3, \ldots, v_m) > U(v_2, v_2, v_3, \ldots, v_m); \tag{24}$$

*then $U(v_1, \bar{v}_2, v_3, \ldots, v_m) = U(v_1, v_2, v_3, \ldots, v_m)$ for all $\bar{v}_2 \le v_2$.*

*Proof.* A non-strict inequality immediately follows from the monotonicity of $U$. Let us suppose that $U(v_1, \bar{v}_2, \ldots, v_m) =: u' < u := U(v_1, v_2, \ldots, v_m)$ for some $\bar{v}_2 < v_2$. Taking into account (24) and the continuity of $U$, we may, increasing $\bar{v}_2$ if needed, assume $U(v_2, v_2, v_3, \ldots, v_m) < u'$. By the continuity of $U$, there is $\delta > 0$ such that $v_2 + \delta < v_1$ and $U(v_2 + \delta, v_2 + \delta, v_3 + \delta, \ldots, v_m + \delta) =: u'' < u'$; we denote $U(v_1 + \delta, v_2 + \delta, v_3 + \delta, \ldots, v_m + \delta) =: u^+ > u$. Thus,

$$u'' < u' < u < u^+. \tag{25}$$

Now let us consider a generalized congestion game with strictly negative impacts which is consistent with $\mathfrak{U}$: $N := \{1, 2\}$; the facilities are $A := \{a, b, c, d\} \cup E \cup F$, where $E := \{e_s\}_{3 \le s \le m}$ and $F := \{f_s\}_{3 \le s \le m}$; $X_1 := \big\{\{a, c\} \cup E, \{b, d\} \cup F\big\}$; $X_2 := \big\{\{a, d\} \cup E, \{b, c\} \cup F\big\}$; $\varphi_a(1) := v_1 + \delta$, $\varphi_a(2) := \bar{v}_2$, $\varphi_b(1) := v_2 + \delta$, $\varphi_b(2) := v_2$, $\varphi_c(1) := \varphi_d(1) := v_1$, $\varphi_c(2) := \varphi_d(2) := v_2 + \delta$, $\varphi_{e_s}(1) := \varphi_{f_s}(1) := v_s + \delta$

and $\varphi_{e_s}(2) := \varphi_{f_s}(2) := v_s$ $(s = 3, \dots, m)$; $U_i^{x_i} := U$ for both $i \in N$ and all $x_i \in X_i$. The $2 \times 2$ matrix of the game looks as follows:

$$\begin{array}{ccc} & \text{adE} & \text{bcF} \\ \text{acE} & (u', u') & (u^+, u'') \\ \text{bdF} & (u'', u^+) & (u, u). \end{array}$$

Taking into account (25), we see that the northwestern corner is a unique Nash equilibrium, which is strongly Pareto dominated by the southeastern corner. □

**Lemma B.2.3.** *Let $U \in \mathfrak{U}$, $v_1 > v_2$, and*

$$U(v_1, v_1, v_3, \dots, v_m) > U(v_1, v_2, v_3, \dots, v_m); \tag{26}$$

*then $U(\bar{v}_1, v_2, v_3, \dots, v_m) = U(v_1, v_2, v_3, \dots, v_m)$ for all $\bar{v}_1 \geq v_1$.*

*Proof.* A non-strict inequality immediately follows from the monotonicity of $U$. Let us suppose

$$U(\bar{v}_1, v_2, \dots, v_m) =: u^+ > u := U(v_1, v_2, \dots, v_m) \tag{27}$$

for some $\bar{v}_1 > v_1$. By the continuity of $U$, (26) and (27) imply the existence of $v_1' \in ]v_2, v_1[$ such that $u < U(v_1', v_1, v_3, \dots, v_m) < u^+$. By the same continuity, we may pick $\delta > 0$ such that $v_1' + \delta < v_1$, $U(v_2 + \delta, v_1 + \delta, v_3 + \delta, \dots, v_m + \delta) =: u' < U(v_1', v_1, v_3, \dots, v_m)$ and $U(v_1' + \delta, v_1 + \delta, v_3 + \delta, \dots, v_m + \delta) =: u'' < u^+$; by monotonicity,

$$u < u' < u'' < u^+. \tag{28}$$

Now let us consider a generalized congestion game with strictly negative impacts which is consistent with $\mathfrak{U}$: $N := \{1, 2\}$; the facilities are $A := \{a, b, c, d\} \cup E \cup F$, where $E := \{e_s\}_{3 \leq s \leq m}$ and $F := \{f_s\}_{3 \leq s \leq m}$; $X_1 := \big\{\{a, c\} \cup E, \{b, d\} \cup F\big\}$; $X_2 := \big\{\{a, d\} \cup F, \{b, c\} \cup E\big\}$; $\varphi_a(1) := \bar{v}_1$, $\varphi_a(2) := v_2 + \delta$, $\varphi_b(1) := v_1$, $\varphi_b(2) := v_1' + \delta$, $\varphi_c(1) := \varphi_d(1) := v_1 + \delta$, $\varphi_c(2) := \varphi_d(2) := v_2$, $\varphi_{e_s}(1) := \varphi_{f_s}(1) := v_s + \delta$ and $\varphi_{e_s}(2) := \varphi_{f_s}(2) := v_s$ $(s = 3, \dots, m)$; $U_i^{x_i} := U$ for both $i \in N$ and all $x_i \in X_i$. The $2 \times 2$ matrix of the game looks as follows:

$$\begin{array}{ccc} & \text{adF} & \text{bcE} \\ \text{acE} & (u', u') & (u^+, u) \\ \text{bdF} & (u, u^+) & (u'', u''). \end{array}$$

Taking into account (28), we see that the northwestern corner is a unique Nash equilibrium, which is strongly Pareto dominated by the southeastern corner. □

As a next step, we establish a restriction on mutual location of combinations of arguments where condition (29), respectively, (30) holds.

**Lemma B.2.4.** *Let $U, U' \in \mathfrak{U}$, $v_1 > v_2$,*

$$U(v_1, v_1, v_3, \dots, v_m) > U(v_1, v_2, v_3, \dots, v_m), \tag{29}$$

*$v_1' > v_2'$, and*

$$U'(v_1', v_2', v_3', \dots, v_{m'}') > U'(v_2', v_2', v_3', \dots, v_{m'}'). \tag{30}$$

*Then $v_1 > v_2'$.*

*Proof.* Supposing the contrary, $v_2' \geq v_1$, we denote $u_1^- := U(v_1', v_2, v_3, \ldots, v_m)$, $u_1^+ := U(v_1, v_1, v_3, \ldots, v_m)$, $u_2^- := U'(v_1, v_1, v_3', \ldots, v_{m'}')$, and $u_2^+ := U'(v_1', v_2, v_3', \ldots, v_{m'}')$. We have $u_1^+ > u_1^-$ by Lemma B.2.3 since $v_1' > v_2' \geq v_1$, and $u_2^+ > u_2^-$ by Lemma B.2.2 since $v_2' \geq v_1 > v_2$.

Now we consider a generalized congestion game with strictly negative impacts which is consistent with $\mathfrak{U}$: $N := \{1, 2\}$; the facilities are $\mathrm{A} := \{a, b, c, d\} \cup E \cup F$, where $E := \{e_s\}_{s\in\{3,\ldots,m\}}$ and $F := \{f_s\}_{s\in\{3,\ldots,m'\}}$; $X_1 := \big\{\{a, b\} \cup E, \{c, d\} \cup E\big\}$; $X_2 := \big\{\{a, c\} \cup F, \{b, d\} \cup F\big\}$; $\varphi_a(2) := \varphi_d(2) := v_2$, $\varphi_a(1) := \varphi_d(1) := \varphi_b(2) := \varphi_c(2) := v_1$, $\varphi_b(1) := \varphi_c(1) := v_1'$, $\varphi_{e_s}(1) := v_s$ for each $s \in \{3, \ldots, m\}$, and $\varphi_{f_s}(1) := v_s'$ for each $s \in \{3, \ldots, m'\}$; $U_1^{x_1}$ is $U$ for each $x_1 \in X_1$ and $U_2^{x_2}$ is $U'$ for each $x_2 \in X_2$. The $2 \times 2$ matrix of the game looks as follows:

$$\begin{array}{ccc} & \text{acF} & \text{bdF} \\ \text{abE} & (u_1^-, u_2^+) & (u_1^+, u_2^-) \\ \text{cdE} & (u_1^+, u_2^-) & (u_1^-, u_2^+). \end{array}$$

There is no Nash equilibrium in the game. $\square$

Given $U \in \mathfrak{U}$, we denote:

$$V_U^{\min} := \{v_1 \in \mathbb{R} \mid \exists v_2, \ldots, v_m \in \mathbb{R}\, [v_1 > v_2 \;\&\; U(v_1, v_1, v_3, \ldots, v_m) > U(v_1, v_2, v_3, \ldots, v_m)]\};$$

$$V_U^{\max} := \{v_2 \in \mathbb{R} \mid \exists v_1, v_3, \ldots, v_m \in \mathbb{R}\, [v_1 > v_2 \;\&\; U(v_1, v_2, v_3, \ldots, v_m) > U(v_2, v_2, v_3, \ldots, v_m)]\};$$

$$v_U^{\min} := \inf V_U^{\min};\; v_U^{\max} := \sup V_U^{\max}.$$

(If $V_U^{\min} = \emptyset$, then we assume $v_U^{\min} = +\infty$; if $V_U^{\max} = \emptyset$, then $v_U^{\max} = -\infty$.) By Lemma B.2.4, $v_U^{\min} \geq v_U^{\max}$. For $v \in \mathbb{R}$, we define

$$\lambda^U(v) := U(v, v, \ldots, v).$$

Clearly, $\lambda^U$ is continuous and strictly increasing.

**Lemma B.2.5.** *For every $U \in \mathfrak{U}$ and $v_1, v_2, v_3, \ldots, v_m \in \mathbb{R}$, there hold*

$$U(v_1, v_2, v_3, \ldots, v_m) = \lambda^U(\min_s v_s) \tag{31}$$

*whenever* $\min_s v_s \geq v_U^{\max}$, *and*

$$U(v_1, v_2, v_3, \ldots, v_m) = \lambda^U(\max_s v_s) \tag{32}$$

*whenever* $\max_s v_s \leq v_U^{\min}$.

*Proof.* Let $\min_s v_s > v_U^{\max}$. Without restricting generality, we may assume $v_1 \geq v_2 \geq \cdots \geq v_m$. By the definition of $v_U^{\max}$ and symmetry of $U$, we have $U(v_1, v_2, \ldots, v_{m-1}, v_m) = U(v_1, v_2, \ldots, v_{m-2}, v_m, v_m) = \cdots = U(v_1, v_m, \ldots, v_m, v_m) = U(v_m, \ldots, v_m) = \lambda^U(\min\{v_1, v_2, \ldots, v_m\})$.

If $\min_s v_s = v_U^{\max}$, we obtain the same equality by continuity. If $\max_s v_s \leq v_U^{\min}$, we argue dually. $\square$

**Lemma B.2.6.** *For every $U \in \mathfrak{U}$, either $v_U^{\min} = v_U^{\max} = +\infty$ or $v_U^{\min} = v_U^{\max} = -\infty$.*

*Proof.* Supposing that $v_U^{\max} < v' < v'' < v_U^{\min}$, we would have $U(v', v'', \dots, v'') = \lambda^U(v')$ by (31) and $U(v', v'', \dots, v'') = \lambda^U(v'')$ by (32), which is impossible since $\lambda^U$ is strictly increasing.

Supposing that $v' < v_U^{\max} = v_U^{\min} < v''$, we would have $U(v_U^{\max}, v'', \dots, v'') = \lambda^U(v_U^{\max})$ by (31) and $U(v', \dots, v', v_U^{\max}) = \lambda^U(v_U^{\max})$ by (32), which contradicts monotonicity (14). □

**Lemma B.2.7.** *Either $U(v_1, v_2, v_3, \dots, v_m) = \lambda^U(\min_s v_s)$ for every $U \in \mathfrak{U}$ and all $v_1, v_2, v_3, \dots, v_m \in \mathbb{R}$, or $U(v_1, v_2, v_3, \dots, v_m) = \lambda^U(\max_s v_s)$ for every $U \in \mathfrak{U}$ and all $v_1, v_2, v_3, \dots, v_m \in \mathbb{R}$.*

Immediately follows from Lemmas B.2.5, B.2.6, and B.2.4.

**Lemma B.2.8.** *For every $U \in \mathfrak{U}$ and all $v_1, v_2, v_3, \dots, v_m \in \mathbb{R}$, there holds $U(v_1, v_2, v_3, \dots, v_m) = \lambda^U(\min_s v_s)$.*

*Proof.* In light of Lemma B.2.7, it is enough to show that the maximum aggregation is not "good" in the case of negative impacts. If $\mathfrak{U}$ contains functions of $m \geq 3$ arguments, the dual to Example 4.2 will do. Otherwise, we need an example more.

Let us consider a generalized congestion game with strictly negative impacts and the maximum aggregation: $N := \{1, 2, 3\}$; the facilities are $\mathrm{A} := \{a, b, c, d, e\}$; $X_1 := \{\{a, e\}, \{b, d\}\}$; $X_2 := \{\{a, c\}, \{d, e\}\}$; $X_3 := \{\{a, b\}, \{c, e\}\}$; $\varphi_a(3) := \varphi_e(3) := 0$, $\varphi_c(2) := \varphi_e(2) := 1$, $\varphi_a(2) := \varphi_d(2) := \varphi_e(1) := 2$, $\varphi_b(2) := 3$, $\varphi_c(1) := 4$, $\varphi_d(1) := 5$, $\varphi_a(1) := 6$, and $\varphi_b(1) := 7$; every $U_i^{x_i}$ is the same $U$ defined by (32). Denoting $u^k := \lambda^U(k)$ for each $k \in \{1, 2, \dots, 7\}$, we obtain the following $2 \times 2 \times 2$ matrix of the game (player 1 chooses rows, player 2 columns, and player 3 matrices):

$$\begin{array}{ccc} & \text{ab} & \text{ce} \\ & \begin{array}{cc} \text{ac} & \text{de} \end{array} & \begin{array}{cc} \text{ac} & \text{de} \end{array} \\ \begin{array}{c} \text{ae} \\ \text{bd} \end{array} & \begin{bmatrix} (u^2, u^4, u^7) & (u^2, u^5, u^7) \\ (u^5, u^4, u^3) & (u^3, u^2, u^6) \end{bmatrix} & \begin{bmatrix} (u^2, u^2, u^1) & (u^6, u^5, u^4) \\ (u^7, u^6, u^2) & (u^7, u^2, u^4) \end{bmatrix} \end{array}.$$

The individual improvement relation is acyclic (as it should be according to the dual version of corollary to Proposition 5.4) and the southwestern corner of the left matrix is a unique Nash equilibrium. However, this equilibrium is strongly Pareto dominated by the northeastern corner of the right matrix. □

**Remark.** I have been unable to prove the lemma with a two-person game where no player ever uses more than two facilities. Probably, there is a "positive" result here, but it has not yet been distilled.

Thus, (16) is proven. Let us turn to (17).

**Lemma B.2.9.** *Let $U, U' \in \mathfrak{U}$ and $\lambda^U \neq \lambda^{U'}$. Then $\lambda^U(\mathbb{R}) \cap \lambda^{U'}(\mathbb{R}) = \emptyset$.*

*Proof.* Let us suppose the contrary, $\lambda^U(\mathbb{R}) \cap \lambda^{U'}(\mathbb{R}) \neq \emptyset$. Since both $\lambda^U$ and $\lambda^{U'}$ are homeomorphisms, $\lambda^U(\mathbb{R}) \cap \lambda^{U'}(\mathbb{R})$ is open and $\{v \in \mathbb{R} \mid \lambda^{U'}(v) = \lambda^U(v)\}$ is closed in $\mathbb{R}$, there must be $v' \neq v$ such that $\lambda^{U'}(v') = \lambda^U(v)$. Let $\#\Sigma(U') =: m' > 1$.

**1.** Supposing first that $v > v'$, we denote $u^1 := \lambda^{U'}(v')$. Then we pick $\underline{v} \in ]v', v[$, denote $u^0 := \lambda^U(\underline{v})$ and $u^3 := \lambda^{U'}(\underline{v})$ (so $u^0 < \lambda^U(v) = u^1 = \lambda^{U'}(v') < u^3$), and pick $\bar{v} > v$ so that $u^2 := \lambda^U(\bar{v}) < u^3$; $u^2 > u^1$ is satisfied automatically.

Let us consider a generalized congestion game with strictly negative impacts, which is consistent with $\mathfrak{U}$: $N := \{1,2\}$; the facilities are $\mathrm{A} := \{a,b,c\} \cup D \cup E$, where $D := \{d_s\}_{2 \le s \le m}$ and $E := \{e_s\}_{3 \le s \le m'}$; $X_1 := \big\{\{a\} \cup D, \{b,c\} \cup E\big\}$; $X_2 := \big\{\{a,b\} \cup E, \{c\} \cup D\big\}$; $\varphi_a(2) := \varphi_b(1) := \varphi_c(2) := \underline{v}$, $\varphi_a(1) := \varphi_c(1) := \bar{v}$, $\varphi_b(2) := v'$, $\varphi_{d_s}(2) := \varphi_{e_{s'}}(2) := \bar{v}$ and $\varphi_{d_s}(1) := \varphi_{e_{s'}}(1) > \bar{v}$ for all appropriate $s$ and $s'$; $U_i^{x_i}$ is $U'$ if $x_i$ includes two facilities from $\{a,b,c\}$ and $U$ otherwise. The $2 \times 2$ matrix of the game looks as follows:

$$\begin{array}{ccc} & \mathrm{abE} & \mathrm{cD} \\ \mathrm{aD} & (u^0, u^3) & (u^2, u^2) \\ \mathrm{bcE} & (u^1, u^1) & (u^3, u^0). \end{array}$$

We have a prisoner's dilemma: strategies with the "$U'$ aggregation" are dominant, but the northeastern corner strongly Pareto dominates the southwestern one.

**2.** Supposing $v' > v$, we denote $u^0 := \lambda^{U'}(v)$ and $u^4 := \lambda^U(v) > u^0$; then we pick $\underline{v} \in ]v, v'[$ and $v^+ > \bar{v} > v'$, and denote $u^3 := \lambda^{U'}(\underline{v}) < u^4 < \lambda^{U'}(\bar{v}) =: u^6 < \lambda^{U'}(v^+) =: u^7$. Then we pick $v'' \in ]v, \underline{v}[$ so that $u^5 := \lambda^U(v'') < u^6$; $u^5 > u^4$ is satisfied automatically. Finally, we pick $v''' \in ]v, v''[$, and denote $u^1 := \lambda^{U'}(v''')$ and $u^2 := \lambda^{U'}(v'')$; we have $u^0 < u^1 < \cdots < u^7$.

Let us consider a generalized congestion game with strictly negative impacts, which is consistent with $\mathfrak{U}$: $N := \{1,2,3\}$; the facilities are $\mathrm{A} := \{a,b,c,d\} \cup E \cup F$, where $E := \{e_s\}_{2 \le s \le m}$ and $F := \{f_s\}_{2 \le s \le m'}$; $X_1 := \big\{\{a\} \cup E, \{d\} \cup F\big\}$; $X_2 := \big\{\{a,b\} \cup F \setminus \{f_2\}, \{c\} \cup F\big\}$; $X_3 := \big\{\{d\} \cup F, \{b\} \cup F\big\}$; $\varphi_a(2) := v$, $\varphi_a(1) := \varphi_b(2) := v''$, $\varphi_b(1) := v^+$, $\varphi_c(1) := v'''$, $\varphi_d(2) := \underline{v}$, $\varphi_d(1) := \varphi_{e_{s'}}(1) := \varphi_{f_s}(3) := \bar{v}$, $\varphi_{f_s}(2) := v^+$ and $\varphi_{f_s}(1) > v^+$ for all appropriate $s$ and $s'$; $U_i^{x_i}$ is $U$ if $i = 1$ and $x_i$ includes $a$, and $U'$ otherwise. The $2 \times 2 \times 2$ matrix of the game looks as follows (again, player 1 chooses rows, player 2 columns, and player 3 matrices):

$$\begin{array}{ccc} & \mathrm{dF} & \mathrm{bF} \\ & \begin{array}{cc} \mathrm{abF} & \mathrm{cF} \end{array} & \begin{array}{cc} \mathrm{abF} & \mathrm{cF} \end{array} \\ \begin{array}{c} \mathrm{aE} \\ \mathrm{dF} \end{array} & \begin{bmatrix} (u^4, u^0, u^6) & (u^5, u^1, u^6) \\ (u^3, u^2, u^3) & (u^3, u^1, u^3) \end{bmatrix} & \begin{bmatrix} (u^4, u^0, u^2) & (u^5, u^1, u^7) \\ (u^6, u^2, u^2) & (u^6, u^1, u^6) \end{bmatrix}. \end{array}$$

There is no Nash equilibrium in the game. □

### B.3 Proof of Proposition 6.2

The condition here is the same as Statement 2 of Theorem 6.1. Therefore, we can argue exactly as in Section B.2 until we reach Lemma B.2.9, where the uniqueness of $U \in \mathfrak{U}$ for which $\#\Sigma(U) = 1$ was relied upon indeed. However, if $\#\Sigma(U') = \#\Sigma(U) = 1$, then (19), unlike (17), does not require anything of such $U'$ and $U$, so the lemma is not needed.

## C Proof of Theorem 6.3

### C.1 Sufficiency

Let $\mathfrak{U}$ be a set of admissible aggregation functions satisfying Condition 3 from Theorem 6.3. Denoting $\mathfrak{U}^m := \{U \in \mathfrak{U} \mid \#\Sigma(U) = m\}$ for every $m \in \mathbb{N}$, we may argue in the same way as in Section B.1

and obtain the partitioning of each (nonempty) $\mathfrak{U}^m$ into subsets $W^t$ ($t \in T(m)$) such that $\lambda^U = \lambda^{U'}$ whenever $U$ and $U'$ belong to the same $W^t$ and the set $T(m)$ is linearly ordered by the relation $t > t' \rightleftharpoons \left[\lambda^U(u) > \lambda^{U'}(u') \text{ whenever } U \in W^t,\ U' \in W^{t'}, \text{ and } u, u' \in \mathbb{R}\right]$.

Let $\Gamma$ be a game with structured utilities which is consistent with $\mathfrak{U}$ and where the strategy sets are compact and utility functions upper semicontinuous. We have to prove that $\Gamma$ admits an $\omega$-potential.

For each $i \in N$, we have $\#\Sigma(U_i^{x_i}) = \#\Upsilon_i$ for all $x_i \in X_i$. Therefore, the order on $T(\#\Upsilon_i)$ generates an ordering on $X_i$ (exactly as in Section B.1): $y_i \succeq_i x_i \rightleftharpoons \left[U_i^{y_i} \in W^t \;\&\; U_i^{x_i} \in W^{t'} \;\&\; t \geq t'\right]$. Obviously, $u_i(y_i, z_{-i}) > u_i(x_i, z'_{-i})$ for all $z_{-i}, z'_{-i} \in X_{-i}$ whenever $y_i \succ_i x_i$. It follows immediately that $y_i \succeq_i x_i$ whenever $y_N \rhd_I x_N$ and $i \in I$.

Now we define a preorder on $X_N$ by

$$y_N \succeq_N x_N \rightleftharpoons \forall i \in N\, [y_i \succeq_i x_i],$$

and denote $\succ_N$ and $\sim_N$ its asymmetric and symmetric components. The upper semicontinuity of $u_i$ implies that each $\succeq_i$ is $\omega$-transitive and hence $\succeq_N$ is $\omega$-transitive as well.

Apart from "genuine" utilities $u_i$, we introduce, for each $i \in N$, "neutral" utility functions $u_i^0$ by (7), i.e., "without $\lambda$'s."

Let (20) hold. We denote $>_{\mathrm{Lmin}}$ the leximin ordering on $X_N$ defined by utility functions $u_i^0$ as in the proof of Theorem 4.1. Now we define our potential as a lexicography:

$$y_N \ggcurly x_N \rightleftharpoons \left[y_N \succ_N x_N \;\text{ or }\; [y_N \sim_N x_N \;\&\; y_N >_{\mathrm{Lmin}} x_N]\right]. \tag{33}$$

Obviously, $\ggcurly$ is irreflexive and transitive. To show its $\omega$-transitivity, we assume that $x_N^k \to x_N^\omega$ and $x_N^{k+1} \ggcurly x_N^k$ for all $k \in \mathbb{N}$. Then, by definition, $x_N^{k+1} \succeq_N x_N^k$ for all $k$, and hence $x_N^\omega \succeq_N x_N^0$ since that relation is $\omega$-transitive. If $x_N^\omega \succ_N x_N^0$, we are home by the first component in (33). Otherwise, we have $x_N^{k+1} \sim_N x_N^k$ for all $k$, and hence are home by the second component in (33) since $>_{\mathrm{Lmin}}$ is $\omega$-transitive.

Finally, let $y_N \rhd^{\mathrm{Coa}} x_N$; we have to show that $y_N \ggcurly x_N$. First, $y_N \succeq_N x_N$. If $y_N \succ_N x_N$, then we are home immediately. Otherwise, the same $\lambda$'s are applied to each $u_i^0$ in both cases; hence $y_N >_{\mathrm{Lmin}} x_N$ exactly as in the proof of Theorem 4.1.

If (21) holds, we argue dually, replacing $>_{\mathrm{Lmin}}$ with $>_{\mathrm{Lmax}}$.

## C.2 Necessity

More than one half of the proof goes along exactly the same lines as in Section B.2. We show that every function $U \in \mathfrak{U}$ is symmetric (Lemma C.2.1); hence we ignore the mappings $\mu_i^{x_i}$ and assume that $\Sigma(U) = \{1, \ldots, m\}$; every indifference curve in every two-dimensional section is either a "minimum-like" angle or a "maximum-like" one (Lemmas C.2.2 and C.2.3); every "minimum-like" angle must be on the right of every "maximum-like" one (Lemma C.2.4). The only difference with the relevant part of the proof of Theorem 6.1 is that each time we produce a game with structured utilities rather than a generalized congestion game. Unlike Theorem 6.1, both minimum and maximum are equally good here.

**Lemma C.2.1.** *Let $U \in \mathfrak{U}$, $s', s'' \in \Sigma(U)$, $v^-, v^+ \in \mathbb{R}$, and $v'_{\Sigma(U)}, v''_{\Sigma(U)} \in \mathbb{R}^{\Sigma(U)}$ be such that $v^+ > v^-$, $v''_{s''} = v'_{s'} = v^+$, $v''_{s'} = v'_{s''} = v^-$, and $v''_s = v'_s$ for all $s \in \Sigma(U) \setminus \{s', s''\}$. Then $U(v''_{\Sigma(U)}) = U(v'_{\Sigma(U)})$.*

*Proof.* Supposing the contrary, we may, without restricting generality, assume $u^+ := U(v''_{\Sigma(U)}) > U(v'_{\Sigma(U)}) =: u^-$. Now let us consider a finite game with structured utilities which is consistent with $\mathfrak{U}$: $N := \{1,2\}$; the facilities are $\mathrm{A} := \{a,b\} \cup C$, where $C := \{c_s\}_{s \in \Sigma(U) \setminus \{s',s''\}}$; $\Upsilon_1 := \Upsilon_2 := \mathrm{A}$; $X_1 := X_2 := \{1,2\}$; $\varphi_a(x_1,x_2) := v^-$ if $x_1 = x_2$ and $\varphi_a(x_1,x_2) := v^+$ otherwise; $\varphi_b(x_1,x_2) := v^-$ if $x_1 \neq x_2$ and $\varphi_a(x_1,x_2) := v^+$ otherwise; $\varphi_{c_s}(x_1,x_2) := v_s$ for all $s \in \Sigma(U) \setminus \{s',s''\}$ and $(x_1,x_2) \in X_N$; $U_i^{x_i}$ is $U$ for both $i \in N$ and all $x_i \in X_i$; $\mu_1^{x_1}(s') := b$, $\mu_1^{x_1}(s'') := a$, $\mu_2^{x_2}(s') := a$, $\mu_2^{x_2}(s'') := b$, and $\mu_i^{x_i}(s) := c_s$ for both $i \in N$ and all $x_i \in X_i$ and $s \in \Sigma(U) \setminus \{s',s''\}$. The $2 \times 2$ matrix of the game (as usual, player 1 chooses rows, numbered from top to bottom, while player 2 chooses columns, numbered from left to right) looks as follows:

$$\begin{matrix} (u^-,u^+) & (u^+,u^-) \\ (u^+,u^-) & (u^-,u^+). \end{matrix}$$

There is no Nash equilibrium in the game. ∎

**Lemma C.2.2.** *Let $U \in \mathfrak{U}$, $v_1 > v_2$, and*

$$U(v_1,v_2,v_3,\dots,v_m) > U(v_2,v_2,v_3,\dots,v_m);$$

*then $U(v_1,\bar{v}_2,v_3,\dots,v_m) = U(v_1,v_2,v_3,\dots,v_m)$ for all $\bar{v}_2 \leq v_2$.*

*Proof.* A non-strict inequality immediately follows from the monotonicity of $U$. Let us suppose that $U(v_1,\bar{v}_2,\dots,v_m) =: u' < u := U(v_1,v_2,\dots,v_m)$ for some $\bar{v}_2 < v_2$. As in Lemma B.2.2, we may assume that $u^- := U(v_2,v_2,v_3,\dots,v_m) < u'$. By the continuity of $U$, there is $\bar{v}_1 \in ]v_2,v_1[$ such that $u' < U(\bar{v}_1,v_2,v_3,\dots,v_m) =: u'' < u$. Thus,

$$u^- < u' < u'' < u. \tag{34}$$

Now let us consider a finite game with structured utilities which is consistent with $\mathfrak{U}$: $N := \{1,2\}$; the facilities are $\mathrm{A} := \{a_1,a_2,b\} \cup C$, where $C := \{c_s\}_{s \in \{3,\dots,m\}}$; $\Upsilon_i := \{a_i,b\} \cup C$ for both $i$; $X_1 := X_2 := \{1,2\}$; $U_i^{x_i}$ is $U$ for both $i \in N$ and all $x_i \in X_i$; $\varphi_{a_i}(1) := v_2$, $\varphi_{a_i}(2) := v_1$; $\varphi_b(1,1) := \bar{v}_1$, $\varphi_b(1,2) := \varphi_b(2,1) := v_2$, $\varphi_b(2,2) := \bar{v}_2$; $\varphi_{c_s}(x_1,x_2) := v_s$ $(s = 3,\dots,m)$. The $2 \times 2$ matrix of the game looks as follows:

$$\begin{matrix} (u'',u'') & (u^-,u) \\ (u,u^-) & (u',u'). \end{matrix}$$

Taking into account (34), we see that the southeastern corner ($x_1 = x_2 = 2$) is a unique Nash equilibrium, which is strongly Pareto dominated by the northwestern corner ($x_1 = x_2 = 1$). ∎

**Lemma C.2.3.** *Let $U \in \mathfrak{U}$, $v_1 > v_2$, and*

$$U(v_1,v_1,v_3,\dots,v_m) > U(v_1,v_2,v_3,\dots,v_m); \tag{35}$$

*then $U(\bar{v}_1,v_2,v_3,\dots,v_m) = U(v_1,v_2,v_3,\dots,v_m)$ for all $\bar{v}_1 \geq v_1$.*

*Proof.* A non-strict inequality immediately follows from the monotonicity of $U$. Let us suppose that $U(\bar{v}_1, v_2, \ldots, v_m) =: u'' > u := U(v_1, v_2, \ldots, v_m)$ for some $\bar{v}_1 > v_1$; we may assume, without restricting generality, that $u'' < u^+ := U(v_1, v_1, v_3, \ldots, v_m)$.

By the continuity of $U$, (35) implies the existence of $v'_1 \in ]v_2, v_1[$ such that $u < u' := U(v'_1, v_1, v_3, \ldots, v_m) < u''$. Thus,

$$u < u' < u'' < u^+. \tag{36}$$

Now let us consider a finite game with structured utilities which is consistent with $\mathfrak{U}$: $N := \{1, 2\}$; the facilities are $\mathrm{A} := \{a_1, a_2, b\} \cup C$, where $C := \{c_s\}_{s \in \{3,\ldots,m\}}$; $\Upsilon_i := \{a_i, b\} \cup C$ for both $i$; $X_1 := X_2 := \{1, 2\}$; $U_i^{x_i}$ is $U$ for both $i \in N$ and all $x_i \in X_i$; $\varphi_{a_i}(1) := v_2$, $\varphi_{a_i}(2) := v_1$; $\varphi_b(1, 1) := \bar{v}_1$, $\varphi_b(1, 2) := \varphi_b(2, 1) := v_1$, $\varphi_b(2, 2) := v'_1$; $\varphi_{c_s}(x_1, x_2) := v_s$ $(s = 3, \ldots, m)$. The $2 \times 2$ matrix of the game looks as follows:

$$\begin{array}{ll} (u'', u'') & (u, u^+) \\ (u^+, u) & (u', u'). \end{array}$$

Taking into account (36), we see that the southeastern corner ($x_1 = x_2 = 2$) is a unique Nash equilibrium, which is strongly Pareto dominated by the northwestern corner ($x_1 = x_2 = 1$). □

**Lemma C.2.4.** *Let $U, U' \in \mathfrak{U}$, $v_1 > v_2$,*

$$U(v_1, v_1, v_3, \ldots, v_m) > U(v_1, v_2, v_3, \ldots, v_m), \tag{37}$$

*$v'_1 > v'_2$, and*

$$U'(v'_1, v'_2, v'_3, \ldots, v'_{m'}) > U'(v'_2, v'_2, v'_3, \ldots, v'_{m'}). \tag{38}$$

*Then $v_1 > v'_2$.*

*Proof.* Supposing the contrary, $v'_2 \geq v_1$, we denote $u_1^- := U(v'_1, v_2, v_3, \ldots, v_m)$, $u_1^+ := U(v_1, v_1, v_3, \ldots, v_m)$, $u_2^- := U'(v_1, v_1, v'_3, \ldots, v'_{m'})$, and $u_2^+ := U'(v'_1, v_2, v'_3, \ldots, v'_{m'})$. We have $u_1^+ > u_1^-$ by Lemma C.2.3 since $v'_1 > v'_2 \geq v_1$, and $u_2^+ > u_2^-$ by Lemma C.2.2 since $v'_2 \geq v_1 > v_2$.

Now we consider a finite game with structured utilities which is consistent with $\mathfrak{U}$: $N := \{1, 2\}$; the facilities are $\mathrm{A} := \{a, b\} \cup C \cup D$, where $C := \{c_s\}_{s \in \{3,\ldots,m\}}$ and $D := \{d_s\}_{s \in \{3,\ldots,m'\}}$; $\Upsilon_1 := \{a, b\} \cup C$, $\Upsilon_2 := \{a, b\} \cup D$; $X_1 := X_2 := \{1, 2\}$; $U_1^{x_1}$ is $U$ for both $x_1 \in X_1$ and $U_2^{x_2}$ is $U'$ for both $x_2 \in X_2$; $\varphi_a(x_1, x_2) := v_1$ if $x_1 = x_2$, $\varphi_a(x_1, x_2) := v'_1$ otherwise; $\varphi_b(x_1, x_2) := v_1$ if $x_1 = x_2$, $\varphi_b(x_1, x_2) := v_2$ otherwise; $\varphi_{c_s}(x_1) := v_s$ for both $x_1 \in X_1$ and all $s = 3, \ldots, m$; $\varphi_{d_s}(x_2) := v'_s$ for both $x_2 \in X_2$ and all $s = 3, \ldots, m'$. The $2 \times 2$ matrix of the game looks as follows:

$$\begin{array}{ll} (u_1^+, u_2^-) & (u_1^-, u_2^+) \\ (u_1^-, u_2^+) & (u_1^+, u_2^-). \end{array}$$

There is no Nash equilibrium in the game. □

**Lemma C.2.5.** *Either $U(v_1, v_2, v_3, \ldots, v_m) = \lambda^U(\min_s v_s)$ for every $U \in \mathfrak{U}$ and all $v_1, v_2, v_3, \ldots, v_m \in \mathbb{R}$, or $U(v_1, v_2, v_3, \ldots, v_m) = \lambda^U(\max_s v_s)$ for every $U \in \mathfrak{U}$ and all $v_1, v_2, v_3, \ldots, v_m \in \mathbb{R}$.*

The statement follows from Lemma C.2.4 in the same way as Lemma B.2.7 followed from Lemma B.2.4.

Finally, let us turn to (22).

**Lemma C.2.6.** *Let $U, U' \in \mathfrak{U}$, $\#\Sigma(U) = \#\Sigma(U')$, and $\lambda^U \neq \lambda^{U'}$. Then $\lambda^U(\mathbb{R}) \cap \lambda^{U'}(\mathbb{R}) = \emptyset$.*

*Proof.* Supposing the contrary, we, exactly as in the proof of Lemma B.2.9, obtain the existence of $v' > v$ such that $\lambda^{U'}(v') = \lambda^U(v)$. We denote $u^+ := \lambda^U(v)$ and $u^- := \lambda^{U'}(v)$; obviously, $u^- < u^+$. Then we pick $v'' < v$ such that $\lambda^U(v'') =: u \in ]u^-, u^+[$, and pick $v^0 < v'$ such that $\lambda^{U'}(v^0) =: u^0 \in ]u, u^+[$. Thus, $u^- < u < u^0 < u^+$.

Now we consider a finite game with structured utilities which is consistent with $\mathfrak{U}$: $N := \{1, 2\}$; the facilities are $\mathrm{A} := \{a_s\}_{s\in\{1,\ldots,m\}}$, where $m := \#\Sigma(U) := \#\Sigma(U')$; $\Upsilon_i := \mathrm{A}$ for both $i$; $X_1 := X_2 := \{1, 2\}$; $U_i^1$ is $U$ and $U_i^2$ is $U'$ for both $i$; for each $s \in \{1, \ldots, m\}$, $\varphi_{a_s}(1, 1) := v''$, $\varphi_{a_s}(2, 1) := \varphi_{a_s}(1, 2) := v$, and $\varphi_{a_s}(2, 2) := v^0$. Since $\varphi_{a_s}(x_N)$ does not depend on $s$, the $2 \times 2$ matrix of the game is the same whether (20) or (21) holds:

$$\begin{array}{cc} (u, u) & (u^+, u^-) \\ (u^-, u^+) & (u^0, u^0). \end{array}$$

We have a prisoner's dilemma: strategies with the "$U$ aggregation" ($x_i = 1$) are dominant, but the southeastern corner strongly Pareto dominates the northwestern one. □

**Remark.** Unlike Lemma B.2.9, there is nothing special about the case of $\#\Sigma(U) = 1$ here.

## Acknowledgments

Financial support from the Russian Science Foundation (project 16-11-10246) is acknowledged. The foundation was not involved in the design of this research, nor in any decision concerning submission for publication. I thank three anonymous referees for their effort in reading earlier versions and their helpful suggestions.

## References

d'Aspremont, C., and L. Gevers, 1977. Equity and the informational basis of collective choice. Review of Economic Studies 44, 199–209.

Boncinelli, L., and P. Pin, 2012. Stochastic stability in best shot network games. Games and Economic Behavior 75, 538–554.

Cornes, R., and R. Hartley, 2007. Weak links, good shots and other public good games: Building on BBV. Journal of Public Economics 91, 1684–1707.

Debreu, G., 1960. Topological methods in cardinal utility. In: Arrow, K.J., S. Karlin, and P. Suppes, (Eds.) Mathematical Methods in Social Sciences. Stanford University Press, Stanford, 16–26.

Deschamps, R., and L. Gevers, 1978. Leximin and utilitarian rules: A joint characterization. Journal of Economic Theory 17, 143–163.

Epstein, A., M. Feldman, and Y. Mansour, 2009. Strong equilibrium in cost sharing connection games. Games and Economic Behavior 67, 51–68.

Feldman, M., and M. Tennenholtz, 2010. Structured coalitions in resource selection games. ACM Transactions on Intelligent Systems and Technology 1 (1), Art. 4.

Fotakis, D, S. Kontogiannis, and P. Spirakis, 2008. Atomic congestion games among coalitions. ACM Transactions on Algorithms 4 (4), Art. 52.

Galbraith, G.K., 1958. The Affluent Society. The New American Library, New York and Toronto.

Germeier, Yu.B., and I.A. Vatel', 1974. On games with a hierarchical vector of interests. Izvestiya Akademii Nauk SSSR, Tekhnicheskaya Kibernetika, 3, 54–69 [in Russian; English translation in Engineering Cybernetics 12(3), 25–40].

Gillies, D.B., 1959. Solutions to general non-zero-sum games. In: Tucker, A.W., and R.D. Luce (Eds.) Contributions to the Theory of Games 4 (Annals of Mathematical Studies 40). Princeton University Press, 47–85.

Gorman, W.M., 1968. The structure of utility functions. Review of Economic Studies 35, 367–390.

Harks, T., and M. Klimm, 2015. Equilibria in a class of aggregative location games. Journal of Mathematical Economics 61, 211–220.

Harks, T., M. Klimm, and R.H. Möhring, 2013. Strong equilibria in games with the lexicographical improvement property. International Journal of Game Theory 42, 461–482.

Hirshleifer, J., 1983. From weakest-link to best-shot: The voluntary provision of public goods. Public Choice 41, 371–386.

Holzman, R., and N. Law-Yone, 1997. Strong equilibrium in congestion games. Games and Economic Behavior 21, 85–101.

Holzman, R., and N. Law-Yone, 2003. Network structure and strong equilibrium in route selection games. Mathematical Social Sciences 46, 193–205.

Holzman, R., and D. Monderer, 2015. Strong equilibrium in network congestion games. International Journal of Game Theory 44, 647–666.

Konishi, H., M. Le Breton, and S. Weber, 1997. Equilibria in a model with partial rivalry. Journal of Economic Theory 72, 225–237.

Kukushkin, N.S., 1992. On existence of stable and efficient outcomes in games with public and private objectives. International Journal of Game Theory 20, 295–303.

Kukushkin, N.S., 2004. Acyclicity of Improvements in Games with Common Intermediate Objectives. Russian Academy of Sciences, Dorodnicyn Computing Center, Moscow. (A revised pdf version available at http://www.ccas.ru/mmes/mmeda/ququ/Common3.pdf)

Kukushkin, N.S., 2007. Congestion games revisited. International Journal of Game Theory 36, 57–83.

Kukushkin, N.S., 2008. Maximizing an interval order on compact subsets of its domain. Mathematical Social Sciences 56, 195–206.

Kukushkin, N.S., 2010. Improvement Paths in Strategic Games: A Topological Approach. Russian Academy of Sciences, Dorodnicyn Computing Center, Moscow. (A pdf version available at http://www.ccas.ru/mmes/mmeda/ququ/ImPath.pdf)

Kukushkin, N.S., 2011. Nash equilibrium in compact-continuous games with a potential. International Journal of Game Theory 40, 387–392.

Kukushkin, N.S., I.S. Men'shikov, O.R. Men'shikova, and N.N. Moiseev, 1985. Stable compromises in games with structured payoffs. Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki 25, 1761–1776 [in Russian; English translation in USSR Computational Mathematics and Mathematical Physics 25, 108–116].

Le Breton, M. and S. Weber, 2011. Games of social interactions with local and global externalities. Economics Letters 111, 88–90.

Milchtaich, I., 1996. Congestion games with player-specific payoff functions. Games and Economic Behavior 13, 111–124.

Monderer, D., and L.S. Shapley, 1996. Potential games. Games and Economic Behavior 14, 124–143.

Moulin, H., 1982. Game Theory for the Social Sciences. New York University Press, New York.

Rosenthal, R.W., 1973. A class of games possessing pure-strategy Nash equilibria. International Journal of Game Theory 2, 65–67.

Rozenfeld, O., and M. Tennenholtz, 2006. Strong and correlated strong equilibria in monotone congestion games. In: Spirakis, P., M. Mavronicolas, and S. Kontogiannis (Eds.) Lecture Notes in Computer Science 4286 Proceedings of the Second International Workshop on Internet and Network Economics. Springer, 74–86.

Smith, T.E., 1974. On the existence of most-preferred alternatives. International Economic Review 15, 184–194.

Wakker, P.P., 1989. Additive Representations of Preferences. Kluwer Academic Publishers, Dordrecht.