Acyclicity of Improvements in Games with Common Intermediate Objectives*

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Abstract

Strategic games are considered where the players participate in certain "activities"; each activity generates a "level of satisfaction," shared by all participating players; the utility of each player is an aggregate of the relevant levels. Attention is paid to conditions conducive to the acyclicity of individual or coalition improvements, hence the existence of appropriate (Nash or strong) equilibria. There are two types of such conditions: those concerning how a new participant can affect an activity, and those concerning the aggregation functions. Special rôles of additive aggregation, as well as of the minimum/maximum ones, are demonstrated.

Key words: Nash equilibrium; Individual or coalition improvement path; Congestion game; Participation game; Game with public and private objectives.

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1 Introduction

This paper has originated from a comparison between two approaches started in the 1970s: Rosenthal (1973) and Germeier and Vatel' (1974), respectively; the latter approach was developed further in a series of papers, see Kukushkin et al. (1985) and references therein. When described in very general terms, both approaches sound identical: the players derive their utilities from participation in the functioning of certain "objects"; the state of each object depends on the participating players; the utility of each player is a function of the states of relevant objects; an equilibrium always exists, actually, there is the acyclicity of improvements.

When one looks closer, plenty of differences can be discerned: players in congestion games choose themselves what objects should be relevant to them (actually, they do not choose anything else) whereas the correspondence between players and objects in a game with structured payoffs is fixed forever; in the former games, the players sum up their intermediate utilities whereas in the latter they take the minimum of them; in the former class, the existence of a Nash equilibrium is ensured, while in the latter it is strong equilibrium.

In Moulin (1982, Chapter 5), pirates were going to a treasure island; each pirate could choose between two ships, and the more pirates on board of either ship, the slower it went. The game was a particular case of Rosenthal's model, but the existence of a strong equilibrium, as in Germeier–Vatel's model, was established. It is remarkable that, since each player could only be associated with a single object (ship), we may assume that the minimum aggregation was applied and, therefore, the existence of a strong equilibrium (and even the acyclicity of coalition improvements) was to be expected.

Here we define a rather general class of games including all the above models: the players participate in certain "activities," which they are free to choose within certain limits; each activity generates a "level of satisfaction," shared by all participating players, and which depends on the list of participants as well as on their actions; the utility of each player is an aggregate of the relevant levels.

An arbitrary game from the class need not have any nice properties; however, it is possible to formulate a number of conditions conducive to the acyclicity of individual or coalition improvements. There are two types of such conditions: those concerning how the players aggregate relevant intermediate objectives into their ultimate utility functions, and those concerning how an activity is influenced when a new player joins the list of participants. Each sufficient condition for the acyclicity of an improvement relation consists of two "elementary" conditions, one of each type.

The acyclicity of coalition improvements is ensured by the minimum aggregation and negative impacts (Theorem 1 below), or, dually, by the maximum aggregation and positive impacts (Theorem 2). Theorem 1 strengthens both Theorem 1 of Kukushkin et al. (1985) — acyclicity is added to mere existence — and Moulin's result about the pirates and ships: acyclicity is added to the statement whereas anonymity is dropped from the assumptions.

Holzman and Law-Yone (1997) found restrictions on strategy sets in a congestion game ensuring the existence of a strong equilibrium under negative impacts. They even described a class of coalition improvements that cannot cycle in this situation. However, it remains unclear whether arbitrary coalition improvements can form a cycle; nothing is also known about the possibility to drop or modify the negative impacts assumption. So far, there is no clear relation to Theorem 1 of this paper.

Acyclicity of individual improvements is ensured by additive aggregation of intermediate objectives plus either anonymity or structured utilities (the latter term was borrowed from Kukushkin et al., 1985, but is used in a much more general sense, formally equivalent to strictly negative and strictly positive impacts combined). Both results are already in the literature, in Rosenthal (1973) and Kukushkin (1994), respectively. A new connection between them is established by our Proposition 6.3.

Monderer and Shapley (1996) showed that every finite game admitting an exact potential — the counterpart of the acyclicity of individual improvements in the context of games with cardinal utilities — can be represented as a congestion game. Our Proposition 6.2 shows that every compact-continuous game admitting a continuous exact potential can be represented as a game with structured utilities and additive aggregation; the latter class contains, e.g., the Stag Hunt game of Rousseau considered in Section 5 of Monderer and Shapley (1996).

Perhaps the most important results of this paper are those establishing the necessity of additive or minimum (maximum) aggregation for the "persistent" existence of (Pareto optimal) Nash equilibria, hence for the acyclicity of improvements as well: Theorems 3–7.

The same aggregation rules — sum, min, and max — emerge when separability is of main interest (Segal and Sobel, 2002). The famous Debreu–Gorman Theorem (Debreu, 1960; Gorman, 1968) on additive representation of separable orderings plays an important rôle in the proofs of Theorems 5 and 7 below. Propositions 4.1 and 5.1 show that the separability of aggregation is sufficient for the acyclicity of individual improvements although there is no general necessity result as yet.

The sum, leximin and leximax are often met in the social choice theory, see, e.g., Moulin (1988). Economists naturally dislike the latter rule, but usually find it difficult to get rid of in their axiomatic characterizations (d'Aspremont and Gevers, 1977; Deschamps and Gevers, 1978). However, our Theorem 2 makes sense even for those who would never accept the maximum aggregation: it shows the acyclicity of strong coalition improvements, hence the existence of strong equilibria, in a natural class of group formation games with positive externalities, viz., where all members of a group receive the same utility depending on the list of participants rather than on their number only.

The minimum operator is not at all unusual in the theory of production functions. Galbraith (1958, Chapter XVIII) explicitly invokes Leontief's model to justify an attitude to public and private consumption ("social balance") that sounds indistinguishable from the minimum aggregation. Our Theorem 1 shows that players who have accepted this attitude do not need any taxes to provide for an efficient level of public consumption; it is difficult to say whether Galbraith himself expected such a conclusion.

Our assumption that all participants receive the same intermediate utility from an activity should not be viewed as a simplifying technical condition. Making it, we concentrate on relationships between "fellow travellers," which can be considered as basic as, e.g., those between competitors for a scarce resource. At the moment, there is no evidence to suggest that similar results could hold in a broader context.

There is some literature on group formation games where each utility function only depends on the strategy chosen by the player and on the number of players who have chosen the same strategy, but different players may have different functions. Typically, there is just the existence of equilibria in such models, without acyclicity of improvements (Milchtaich, 1996; Konishi et al., 1997a), so there is no ground to expect a close connection with this paper. When acyclicity happens, the departure of the model from Rosenthal's scheme is either illusory from the start as in Hollard (2000) or gradually assumed away as in Konishi et al. (1997b).

It is funny to notice an analogy between the results of this paper and of Kukushkin (2004): in both cases, each condition for the acyclicity of an improvement relation in strategic games consists of two elementary ones — one on aggregation and one on the "character of impacts" (in most cases, a monotonicity condition). It seems impossible at the moment to put both groups of theorems into the same formal framework. Actually, there is a remarkable difference: all positive results of this paper assume separable (or "quasiseparable") aggregation whereas Dubey et al. (2004) showed that some nonseparable aggregation rules also ensure the acyclicity of best response improvements in games with strategic complements or substitutes.

Section 2 introduces principal improvement relations associated with a strategic game, and provides a formal description of our basic model as well as its main structural properties. In Section 3, sufficient conditions for the acyclicity of strong coalition improvements are presented.

Sections 4 and 5 follow the same plan: first, we prove the sufficiency of separable aggregation for the acyclicity of individual improvements; then, the necessity of the minimum or maximum aggregation for the acyclicity of strong coalition improvements; finally, the necessity of additive aggregation (among those strictly increasing) for the acyclicity of individual improvements. The two sections differ in the context: in Section 4 it is anonymous participation ("generalized congestion") games; in Section 5, games with structured utilities.

Section 6 contains representation results showing that every game where the acyclicity of an improvement relation is ensured by a theorem from this paper can be represented as a game with structured utilities. The last section deals with some subclasses of the latter games, where our general necessity results need modifications.

2 Basic Notions

2.1 Strategic Games

A strategic game Γ is defined by a finite set of players N (we denote $n = \#N$), and strategy sets X_i and utility functions u_i on $X = \prod_{i \in N} X_i$ for all $i \in N$. We introduce a number of binary relations on X ($y, x \in X$, $i \in N$, $\emptyset \neq I \subseteq N$):

$$y \rhd^{\mathrm{Ind}}{}_i\, x \iff [y_{-i} = x_{-i} \;\&\; u_i(y) > u_i(x)]; \tag{2.1a}$$

$$y \rhd^{\mathrm{Ind}} x \iff \exists i \in N\, [y \rhd^{\mathrm{Ind}}{}_i\, x] \tag{2.1b}$$

(individual improvement relation);

$$y \rhd^{\mathrm{sCo}}{}_I\, x \iff \big[y_{-I} = x_{-I} \;\&\; \forall i \in I\, [u_i(y) > u_i(x)]\big]; \tag{2.2a}$$

$$y \rhd^{\mathrm{sCo}} x \iff \exists I \subseteq N\, [y \rhd^{\mathrm{sCo}}{}_I\, x] \tag{2.2b}$$

(strong coalition improvement);

$$y \rhd^{\mathrm{wCo}}{}_I\, x \iff \big[y_{-I} = x_{-I} \;\&\; \forall i \in I\, [u_i(y) \geq u_i(x)] \;\&\; \exists i \in I\, [u_i(y) > u_i(x)]\big]; \tag{2.3a}$$

$$y \rhd^{\mathrm{wCo}} x \iff \exists I \subseteq N\, [y \rhd^{\mathrm{wCo}}{}_I\, x] \tag{2.3b}$$

(weak coalition improvement).

It is often convenient to speak of just "an improvement relation" without specifying which of the relations defined by (2.1), (2.2), or (2.3) is meant. A maximizer for an improvement relation $\rhd$, i.e., a strategy profile $x \in X$ such that $y \rhd x$ is impossible for any $y \in X$, is an equilibrium: a Nash equilibrium if $\rhd$ is $\rhd^{\mathrm{Ind}}$; a ("very") strong equilibrium if $\rhd$ is $\rhd^{\mathrm{sCo}}$ ($\rhd^{\mathrm{wCo}}$). A strategy profile $x \in X$ is a strong (weak) Pareto optimum if and only if it is a maximizer for $\rhd^{\mathrm{wCo}}{}_N$ ($\rhd^{\mathrm{sCo}}{}_N$).

In a finite game, the acyclicity of an improvement relation ensures the existence of an appropriate equilibrium; moreover, all myopic adaptive dynamics converge to an equilibrium in a finite number of steps. When considering infinite games, we have to consider improvement paths parameterized with countable ordinals; the definitions and exact results concerning such paths are to be found in Kukushkin (2000, 2003). The next subsection contains a sketch of the theory.

We always assume that the players have ordinal preferences (the only exception is Proposition 6.2), so each u_i is defined up to a strictly increasing transformation; clearly, our improvement relations are invariant to such transformations. If $\nu : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing (i.e., $\nu(v') > \nu(v)$ whenever $v' > v$), we denote $\nu(+\infty)$ and $\nu(-\infty)$ its supremum and infimum, respectively. Sometimes we even consider auxiliary functions with infinite values; necessary explanations are given at appropriate moments.

2.2 Acyclicity

Let $\rhd$ be a binary relation on a set X. An *improvement path* (for $\rhd$) is a sequence $\{x^k\}_{k=0,1,\ldots}$ such that $x^{k+1} \rhd x^k$ whenever x^{k+1} is defined; an *improvement cycle* is an improvement path such that $x^m = x^0$ for $m > 0$. The relation $\rhd$ has the *finite improvement path (FIP)* property if there exists no infinite improvement path; then every improvement path, if continued whenever possible, reaches a maximizer in a finite number of steps. A relation is *acyclic* if it admits no improvement cycle; on a finite set X, acyclicity is equivalent to the FIP property of the relation.

Generally, the absence of finite cycles does not mean very much and we have to consider transfinite improvement paths. The definition and basic properties of the, well ordered, set of all countable ordinal numbers, denoted K, are to be found, e.g., in Natanson (1974, Chapter XIV).

Let $\rhd$ be a binary relation on a compact metric space X. A *generalized improvement path* for $\rhd$ is a mapping $\pi : \mathrm{Dom}(\pi) \to X$, where $\mathrm{Dom}(\pi)$ is an initial interval of K (possibly the whole K), satisfying these two conditions:

1. $\pi(\alpha + 1) \rhd \pi(\alpha)$ whenever $\alpha + 1 \in \mathrm{Dom}(\pi)$;
2. if $\alpha \in \mathrm{Dom}(\pi)$ and α is a limit ordinal, there exists a sequence $\{\beta^k\}_k$ for which $\beta^{k+1} > \beta^k$ for all $k = 0, 1, \ldots$, $\alpha = \sup_k \beta^k$, and $\pi(\alpha) = \lim_{k\to\infty} \pi(\beta^k)$.

A *generalized improvement cycle* for $\rhd$ is a generalized improvement path π such that $\pi(\alpha) = \pi(0)$ for $\alpha > 0$; $\rhd$ is called *Ω-acyclic* if it admits no generalized improvement cycle. By Theorem 2 of Kukushkin (2003), Ω-acyclicity is equivalent to the *countable improvement path (CIP)* property, i.e., to the impossibility of a generalized improvement path π for $\rhd$ with $\mathrm{Dom}(\pi) = K$ (K itself is uncountable). On a compact space, where the only obstacle to extending an improvement path further is the fact that it has already reached a maximizer, CIP means that every improvement path, if continued whenever possible, ends at a maximizer. The same theorem provides a useful criterion for the property: a relation is Ω-acyclic if and only if it admits a "potential" in the following sense.

A binary relation $\succ$ on X is called *ω-transitive* if it is transitive and the conditions $x^\omega = \lim_{k\to\infty} x^k$ and $x^{k+1} \succ x^k$ for all $k = 0, 1, \ldots$ always imply $x^\omega \succ x^0$. It is worth noting that $x^\omega \succ x^k$ is valid for all $k = 0, 1, \ldots$ in the above situation, once $\succ$ is ω-transitive. A *potential* for $\rhd$ is an irreflexive and ω-transitive relation $\succ$ satisfying $y \rhd x \Rightarrow y \succ x$ for all $y, x \in X$.

For technical reasons, it is useful to develop a theory of "deterioration paths" as well. With every binary relation $\rhd$, a dual relation $\lhd$ can be associated: $x \lhd y \iff y \rhd x$. A (*generalized*) *deterioration path* for $\rhd$ is a (generalized) improvement path for $\lhd$. Clearly, a finite deterioration path is exactly an improvement path read from the end to the beginning, so we obtain no new acyclicity concept; however, there is no straightforward connection between transfinite improvement and deterioration paths.

A binary relation is called Ω^*-*acyclic* if its dual is Ω-acyclic, i.e., if the relation itself admits no generalized deterioration cycle. A binary relation is called ω^*-*transitive* if its dual is ω-transitive (then the relation itself is transitive at least). A *double potential* for $\triangleright$ is an irreflexive, ω-transitive, and ω^*-transitive relation $\succ$ satisfying $y \triangleright x \Rightarrow y \succ x$ for all $y, x \in X$. A relation admitting a double potential is both Ω-acyclic and Ω^*-acyclic.

2.3 Games with Common Intermediate Objectives

A game with common intermediate objectives may have an arbitrary (finite) set of players N and arbitrary sets of strategies X_i whereas the utility functions satisfy certain structural requirements. There is a finite set A of "*activities.*" Rosenthal (1973) called them "factors"; Monderer and Shapley (1996), "facilities." For every $i \in N$, there is a mapping $\mathrm{B}_i : X_i \to 2^{\mathrm{A}} \setminus \{\emptyset\}$; we interpret $\mathrm{B}_i(x_i)$ as the set of activities chosen by player i under the strategy x_i. We denote $X_i(\alpha) = \{x_i \in X_i |\ \alpha \in \mathrm{B}_i(x_i)\}$, $N(\alpha) = \{i \in N |\ X_i(\alpha) \neq \emptyset\}$, and $N_-(\alpha) = \{i \in N |\ X_i(\alpha) = X_i\}$; we may assume $N(\alpha) \neq \emptyset$ for all α (otherwise, the α would be irrelevant). With every $\alpha \in \mathrm{A}$, an *intermediate objective* is associated: a function $\varphi_\alpha(I, x_I^\alpha) \in \mathbb{R}$, defined for all $I \neq \emptyset$ such that $N_-(\alpha) \subseteq I \subseteq N(\alpha)$ and $x_I^\alpha \in X_I(\alpha) = \prod_{i \in I} X_i(\alpha)$.

Let a strategy profile $x \in X = \prod_{i \in N} X_i$ be given; for each $\alpha \in \mathrm{A}$, we denote $N(\alpha, x) = \{i \in N |\ \alpha \in \mathrm{B}_i(x_i)\}$: the set of players having chosen α at x. The "ultimate" utility functions of the players are built of the intermediate objectives:

$$u_i(x) = U_i^{x_i}\big(\langle \varphi_\alpha(N(\alpha, x), x_{N(\alpha,x)}) \rangle_{\alpha \in \mathrm{B}_i(x_i)}\big), \tag{2.4}$$

where $i \in N$, $x \in X$, and $U_i^{x_i}$ is a numeric function defined on the appropriate subset of $\mathbb{R}^{\mathrm{B}_i(x_i)}$.

We assume that each X_i is a compact metric space and all functions B_i, $U_i^{x_i}$, and $\varphi_\alpha(I, \cdot)$ are continuous (i.e., B_i is a constant on each connected component of X_i); therefore, each utility function u_i is continuous too. Each $U_i^{x_i}$ is also increasing in a sense to be specified below.

Throughout the rest of the paper, we only consider games with common intermediate objectives (the only exception is again Proposition 6.2). There is a potential source of ambiguity since the same game (in the sense of Subsection 2.1) may be generated by different constructions (activities, intermediate objectives, etc.). Actually, equivalence results play an important rôle in our theory. We do not try to produce a formal resolution of the ambiguity; the reader should not find it difficult to recognize what is meant by "a game" in each case.

The concept of a *universal aggregator* will be used; it is perceived as an infinite sequence of functions $U^{(m)} : \mathbb{R}^m \to \mathbb{R}$, $m = 1, 2, \dots$, each of which is assumed continuous, symmetric (w.r.t. any permutation of the arguments), and increasing in the sense of

$$\forall s [v'_s > v_s] \Rightarrow U^{(m)}(v') > U^{(m)}(v); \tag{2.5}$$

the continuity of $U^{(m)}$ implies $\forall s [v'_s \geq v_s] \Rightarrow U^{(m)}(v') \geq U^{(m)}(v)$. In some results a stronger monotonicity condition is assumed; an aggregation function *strictly increases* if

$$\big[\forall s[v'_s \geq v_s] \ \&\ \exists s[v'_s > v_s]\big] \Rightarrow U^{(m)}(v') > U^{(m)}(v). \tag{2.6}$$

We say that a player $i \in N$ in a game Γ *uses a universal aggregator* U if the appropriate $U^{(m)}$ is substituted into (2.4):

$$u_i(x) = U^{(\#\mathrm{B}_i(x_i))}\big(\langle \varphi_\alpha(N(\alpha,x), x_{N(\alpha,x)})\rangle_{\alpha\in\mathrm{B}_i(x_i)}\big) \tag{2.7}$$

for every $x \in X$. The assumed symmetry of $U^{(m)}$ ensures an unambiguous meaning of (2.7), which is difficult to achieve without the symmetry. We often omit the superscript $^{(m)}$ when the number of arguments is clear. Our general assumption of ordinal preferences implies that the application of a monotonic transformation to every $U^{(m)}$ cannot turn a "good" universal aggregator into a "bad" one.

Two subclasses play an important rôle in the following.

In a *participation game*, $X_i \subseteq 2^{\mathrm{A}} \setminus \emptyset$, $\mathrm{B}_i(x_i) = x_i$, and φ_α only depends on I. A participation game is *anonymous* if each φ_α only depends on the cardinality of its argument, in which case we use the notation $\varphi_\alpha(k)$ rather then $\varphi_\alpha(I)$. In this terminology, Rosenthal's (1973) congestion games are described as "anonymous participation games with additive aggregation (of intermediate objectives)." A *partition game* is a participation game where $\#x_i = 1$ for each $i \in N$ and each $x_i \in X_i$; dealing with such games, we assume $X_i \subseteq \mathrm{A}$. A partition game may be anonymous or not. Aggregation plays no part in such games, so, technically, each player may be regarded as using any aggregator whatsoever.

If, conversely, each B_i is a constant on the whole X_i, the game is called a *game with structured utilities*; in such games, each $\Upsilon_i = \mathrm{B}_i(x_i)$ is treated as a parameter of the model. For each $\alpha \in \mathrm{A}$, we have $N_-(\alpha) = N(\alpha) = \{i \in N \mid \alpha \in \Upsilon_i\}$; we assume that $\varphi_\alpha : X_{N(\alpha)} \to \mathbb{R}$.

2.4 Negative and Positive Impacts

We say that player i has a *negative impact* on activity α if for each $I \neq \emptyset$ such that $N_-(\alpha) \subseteq I \subset I \cup \{i\} \subseteq N(\alpha)$, each $x_i^\alpha \in X_i(\alpha)$, and each $x_I^\alpha \in X_I(\alpha)$,

$$\varphi_\alpha(I, x_I^\alpha) \geq \varphi_\alpha(I \cup \{i\}, \langle x_I^\alpha, x_i^\alpha\rangle). \tag{2.8}$$

We say that player i has a *strictly negative impact* on activity α if the inequality in (2.8) is strict. We call Γ a *game with* (*strictly*) *negative impacts* if the appropriate condition holds for all $i \in N$ and $\alpha \in \mathrm{A}$. A definition of (*strictly*) *positive impacts* is obtained by reversing the inequality in (2.8) (or in its strict version).

There is kind of duality between negative and positive impacts. With every game Γ, we can associate its *opposite* game $\bar{\Gamma}$: the sets N, A, and X_i ($i \in N$) are the same; $\bar{\varphi}_\alpha(I, x_I) = -\varphi_\alpha(I, x_I)$; $\bar{U}^{x_i}(v_1, \ldots, v_m) = -U^{x_i}(-v_1, \ldots, -v_m)$. It is easy to see that $\bar{u}_i(x) = -u_i(x)$ for all $i \in N$ and $x \in X$. If Γ exhibits negative impacts, then $\bar{\Gamma}$ exhibits positive impacts, and vice versa.

Proposition 2.1. *If an improvement relation in a game Γ is acyclic, then the same relation in the opposite game $\bar{\Gamma}$ is acyclic too.*

Proposition 2.2. *If an improvement relation in a game Γ admits a double potential (hence is Ω-acyclic), then the same relation in $\bar{\Gamma}$ also admits a double potential (hence is Ω-acyclic too).*

Proof of both propositions. Obviously, $y \triangleright x$ in Γ implies $x \triangleright y$ in $\bar{\Gamma}$, hence a (generalized) improvement path in Γ is a (generalized) deterioration path in $\bar{\Gamma}$. Now Proposition 2.1 is obvious whereas Proposition 2.2 follows from the observation that a double potential for $\triangleright$ is simultaneously a double potential for $\triangleleft$. □

It is important to note that both propositions would fail if we replaced the acyclicity of improvements with just the existence of equilibria.

Example 2.1. Let us consider an anonymous participation game with strictly negative impacts where two players use the aggregator $U_i^{x_i}(\langle v_\alpha \rangle_{\alpha \in x_i}) = \frac{1}{\# x_i} \sum_{\alpha \in x_i} v_\alpha$: $N = \{1, 2\}$; $\mathrm{A} = \{a, b, c, d, e\}$; $X_1 = \big\{\{a\},\ \{b, c\}\big\}$; $X_2 = \big\{\{b, d\},\ \{c\},\ \{e\}\big\}$; $\varphi_a(1) = 6$, $\varphi_b(1) = 12$, $\varphi_b(2) = 0$, $\varphi_c(1) = 10$, $\varphi_c(2) = 2$, $\varphi_d(1) = 6$, $\varphi_e(1) = 11$. The 2×3 matrix of the game looks as follows:

$$
\begin{array}{cccc}
 & \text{bd} & \text{c} & \text{e} \\
\text{a} & (6, 9) & (6, 10) & (6, 11) \\
\text{bc} & (5, 3) & (7, 2) & (11, 11).
\end{array}
$$

The southeastern corner is even a strong equilibrium. In the opposite game (where the utilities just change their signs), there is no Nash equilibrium.

Proposition 2.3. Γ *is a game with both strictly negative and strictly positive impacts if and only if* Γ *is a game with structured utilities.*

Proof. If Γ is a game with structured utilities, we have $N_-(\alpha) = N(\alpha)$ for each $\alpha \in \mathrm{A}$. Therefore, the strict negative impacts condition (2.8) holds by default: no i and I can satisfy the preceding conditions. Similarly, Γ is a game with strictly positive impacts.

Conversely, no inequality can be strict in both directions, so a game with both strictly negative and strictly positive impacts must satisfy condition (2.8) by default. Hence, $N_-(\alpha) = N(\alpha)$ for each α, which implies that Γ is a game with structured utilities. □

It is funny to note that "strictly" cannot be dropped from the formulation, see Example 5.2 below.

Remark. Proposition 2.3 gives no answer to a subtler question: If Γ can be generated by a construction with strictly negative impacts and by another construction with strictly positive impacts, must it be possible to generate Γ by a construction with structured utilities?

3 Games with the Minimum (Maximum) Aggregation

Let us consider games where each player uses the minimum aggregation:

$$u_i(x) = \min_{\alpha \in \mathrm{B}_i(x_i)} \varphi_\alpha(N(\alpha, x), x_{N(\alpha,x)}) \tag{3.1}$$

for all $i \in N$ and $x \in X$.

In economic terms, (3.1) means that all intermediate objectives are perfect complements.

Theorem 1. *Let Γ be a game with negative impacts where each player uses the minimum aggregation, i.e., conditions* (3.1) *and* (2.8) *hold for all $i \in N$ and $x \in X$. Then the strong coalition improvement relation $\triangleright^{\mathrm{sCo}}$ in Γ, defined by* (2.2)*, admits a double potential, hence is Ω-acyclic.*

Proof. Having $n = \#N$ utility functions u_i on the space X, we denote $\succ$ the leximin ordering defined in the standard way. For every $x \in X$, we denote $\vartheta(x) = \langle \vartheta_1(x), \dots, \vartheta_n(x) \rangle$ the vector of values $u_i(x)$ for $i \in N$ in the increasing order: $\vartheta_1(x) \le \dots \le \vartheta_n(x)$, and there is a one-to-one mapping $\sigma : \{1, \dots, n\} \to N$ such that $\vartheta_k(x) = u_{\sigma(k)}(x)$ for all k. Now $y \succ x$ if there is k such that $\vartheta_k(y) > \vartheta_k(x)$ whereas $\vartheta_h(y) \ge \vartheta_h(x)$ for all $h < k$. Since each function u_i is continuous in x, so is each ϑ_k; it is easy to check that both $\succ$ and its dual $\prec$ are ω-transitive — a reference to Proposition 3.9 from Kukushkin (2003) is sufficient anyway. An alternative definition is available: it is easily verified that

$$y \succ x \iff \exists w \in \mathbb{R}\big[\#\{i \in N | \ u_i(x) \le w\} > \#\{i \in N | \ u_i(y) \le w\} \ \& \\ \forall v < w[\#\{i \in N | \ u_i(x) \le v\} \ge \#\{i \in N | \ u_i(y) \le v\}]\big] \tag{3.2}$$

for all $y, x \in X$.

Now let $y \triangleright^{\mathrm{sCo}} x$; we have to show $y \succ x$. Supposing $y \triangleright^{\mathrm{sCo}}{}_I \ x$, we denote $w = \min_{i \in I} u_i(x)$. Let

$$u_j(y) < u_j(x); \tag{3.3}$$

then $j \notin I$, so $y_j = x_j$. By (3.1), there is $\alpha \in \mathrm{B}_j(y_j)$ such that $u_j(y) = \varphi_\alpha(N(\alpha, y), y_{N(\alpha,y)})$. Suppose $I \cap N(\alpha, y) = \emptyset$; then $N(\alpha, y) \subseteq N(\alpha, x)$ and $x_{N(\alpha,y)} = y_{N(\alpha,y)}$, hence $\varphi_\alpha(N(\alpha, x), x_{N(\alpha,x)}) \le \varphi_\alpha(N(\alpha, y), y_{N(\alpha,y)})$ by (2.8), hence $u_j(x) \le u_j(y)$, contradicting (3.3). Therefore, there must be $i \in I \cap N(\alpha, y)$; by (3.1), $\varphi_\alpha(N(\alpha, y), y_{N(\alpha,y)}) \ge u_i(y) > u_i(x) \ge w$, hence $u_j(y) > w$. It follows immediately that the right-hand side of (3.2) holds (with the w already defined), hence $y \succ x$. □

The maximum aggregation is defined "dually":

$$u_i(x) = \max_{\alpha \in \mathrm{B}_i(x_i)} \varphi_\alpha(N(\alpha, x), x_{N(\alpha,x)}) \tag{3.4}$$

for all $i \in N$ and $x \in X$.

From the economic viewpoint, there is a big difference between (3.1) and (3.4): the former satisfies the "decreasing marginal utility" condition, while the latter does not.

Proposition 3.1. *Let* Γ *be a game with positive impacts where each player uses the maximum aggregation* (3.4)*. Then the strong coalition improvement relation* $\triangleright^{\mathrm{sCo}}$ *in* Γ*, defined by* (2.2)*, admits a double potential, hence is* Ω*-acyclic.*

Proof. $\bar{\Gamma}$ satisfies the conditions of Theorem 1, hence admits a double potential. Now Proposition 2.2 applies (to $\bar{\bar{\Gamma}} = \Gamma$). □

Theorem 1 and Proposition 3.1 obviously cover all partition games with negative or positive impacts. The acyclicity of strong coalition improvements in *anonymous* partition games with negative impacts was noticed in Holzman and Law-Yone (1997); they do not mention positive impacts.

If both negative and positive impacts are possible in a game, coalition (or even individual) improvement cycles may emerge.

Example 3.1. Consider a two person anonymous partition game: $N = \{1, 2\}$, $\mathrm{A} = \{a, b\} = X_1 = X_2$, $\varphi_a(1) = 0$, $\varphi_a(2) = 2$, $\varphi_b(1) = 3$, $\varphi_b(2) = 1$ (i.e., a exhibits positive impacts; b, negative). The matrix of the game looks as follows:

$$\begin{array}{ccc} & \text{a} & \text{b} \\ \text{a} & (2,2) & (0,3) \\ \text{b} & (3,0) & (1,1). \end{array}$$

We have a prisoner's dilemma.

Remark. The existence of a Nash equilibrium in the example was inevitable because it is a congestion game. Without anonymity, there may be no equilibrium at all.

Example 3.2. Consider a two person (non-anonymous) partition game: $N = \{1, 2\}$, $\mathrm{A} = \{a, b\} = X_1 = X_2$, $\varphi_a(\{2\}) = 0$, $\varphi_a(\{1\}) = 2$, $\varphi_a(N) = 4$, $\varphi_b(N) = 1$, $\varphi_b(\{2\}) = 3$, $\varphi_b(\{1\}) = 5$. The matrix of the game looks as follows:

$$\begin{array}{ccc} & \text{a} & \text{b} \\ \text{a} & (4,4) & (2,3) \\ \text{b} & (5,0) & (1,1). \end{array}$$

There is no Nash equilibrium.

Let us consider games with infinite sets of activities. We assume that A is a separable metric space, each $\mathrm{B}_i(x_i)$ is compact, each $\mathrm{B}_i : X_i \to 2^A$ is continuous in the Hausdorff metric in its image, and each $\varphi_{(\cdot)}(I, \cdot)$ (for $I \subseteq N$) is continuous on the (closed) subset of $\mathrm{A} \times X_I$ where it is defined.

Theorem 2. *Let* Γ *be a game with a space of activities* A *where each player uses the maximum aggregation* (3.4) *and all impacts are positive. Then the strong coalition improvement relation* $\triangleright^{\mathrm{sCo}}$ *in* Γ*, defined by* (2.2)*, is* Ω*-acyclic.*

Proof.

Lemma 3.1. *Let $i \in N$, $x^k \to x$ and $\alpha^k \in \mathrm{B}_i(x_i^k)$ for all $k = 0, 1, \ldots$ Then there is $\alpha \in \mathrm{B}_i(x_i)$ such that*

$$\varphi_\alpha(N(\alpha, x), x_{N(\alpha,x)}) \geq \overline{\lim}_{k\to\infty} \varphi_{\alpha^k}(N(\alpha^k, x^k), x_{N(\alpha^k,x^k)}). \tag{3.5}$$

Proof. First, replacing $\{\alpha^k\}_k$ with a subsequence if needed, we may assume that the upper limit in the right-hand side of (3.5) is just the limit. Since N is finite, we may (again replacing $\{\alpha^k\}_k$ with a subsequence if needed) assume that $N(\alpha^k, x^k) = I$ is the same for all k. The condition $x^k \to x$ implies $\mathrm{B}_i(x_i^k) \to \mathrm{B}_i(x_i)$ in the Hausdorff metric. Let $r^k \to 0$ (e.g., $r^k = 1/k$); for each $k = 0, 1, \ldots$, there is $\beta^k \in \mathrm{B}_i(x_i)$ and $h(k)$ for which $\rho(\beta^k, \alpha^{h(k)}) < r^k$. Since $\mathrm{B}_i(x_i)$ is compact, we may assume $\beta^k \to \alpha \in \mathrm{B}_i(x_i)$, hence $\alpha^{h(k)} \to \alpha$; therefore, we may assume $\alpha^k \to \alpha$ too.

Let $j \in I$; if $\alpha \notin \mathrm{B}_j(x_j)$, then $\rho(\{\alpha\}, \mathrm{B}_j(x_j)) > 0$, hence $\rho(\{\alpha^k\}, \mathrm{B}_j(x_j^k)) > 0$ for all k large enough, hence $\alpha^k \notin \mathrm{B}_j(x_j^k)$, which contradicts $j \in N(\alpha^k, x^k)$. Thus, $I \subseteq N(\alpha, x)$, hence $\varphi_\alpha(N(\alpha, x), x_{N(\alpha,x)}) \geq \varphi_\alpha(I, x_I) = \lim_{k\to\infty} \varphi_{\alpha^k}(I, x_I)$, the inequality following from the positive impacts assumption, the equality from the continuity assumption. Taking into account the first step of the proof and the definition of I, we have (3.5). □

To prove that the maximum in (3.4) is attained for any $i \in N$ and $x \in X$, we define $x^k = x$ for all k and pick a maximizing sequence for $\psi(\alpha) = \varphi_\alpha(N(\alpha, x), x_{N(\alpha,x)})$ as $\{\alpha^k\}_k$; then the α from Lemma 3.1 obviously maximizes $\psi(\alpha)$.

Given $i \in N$ and $x^k \to x$, we pick $\alpha^k \in \mathrm{B}_i(x_i^k)$ such that $u_i(x^k) = \varphi_{\alpha^k}(N(\alpha^k, x^k), x_{N(\alpha^k,x^k)})$; then Lemma 3.1 means that u_i is upper semicontinuous at x. It is easy to check that each function ϑ_h defined in the proof of Theorem 1 is upper semicontinuous too. Therefore, the leximax ordering $\succ$ is ω-transitive by Proposition 3.7 of Kukushkin (2003). The condition $y \rhd^{\mathrm{sCo}} x \Rightarrow y \succ x$ is proven exactly as in (or rather dually to) Theorem 1. Therefore, $\rhd^{\mathrm{sCo}}$ is Ω-acyclic. □

For games with negative impacts and the minimum aggregation, the dual of Lemma 3.1 is valid, but it implies the lower semicontinuity of u_i, which is not a useful property of the utility function.

Example 3.3. Let us consider an anonymous partition game with negative impacts: $N = \{1, 2\}$, $\mathrm{A} = [0, 1] = X_1 = X_2$, $\varphi_\alpha(1) = \alpha + 1$, $\varphi_\alpha(2) = \alpha$ for all $\alpha \in [0, 1]$. Suppose (x_1, x_2) to be a Nash equilibrium. If $1 \notin \{x_1, x_2\}$, then player 1 can switch to $y_1 = 1$ increasing his utility level. Let, say, $x_1 = 1$; then player 2 does not have a best response (the supremum is $\lim_{\alpha\to 1} \varphi_\alpha(1) = 2$, but it is not attained).

4 Anonymous Participation Games

4.1 Sufficiency of Quasiseparable Aggregation

A *universal separable order* is a sequence of strict orders $\succ^m$ on $\mathbb{R}^m$ ($m = 1, 2, \dots$) such that

1. $\succ^1$ is the standard order $>$ on $\mathbb{R}$;
2. for every two one-to-one mappings σ, σ' of $\{1, \dots, m\}$ to itself,
$$\langle v_1, \dots, v_m \rangle \succ^m \langle v'_1, \dots, v'_m \rangle \iff \langle v_{\sigma(1)}, \dots, v_{\sigma(m)} \rangle \succ^m \langle v'_{\sigma'(1)}, \dots, v'_{\sigma'(m)} \rangle$$
(invariance to permutations);
3. for every $m' > m \geq 1$, every $\langle v_1, \dots, v_{m'} \rangle \in \mathbb{R}^{m'}$, and every $\langle v'_1, \dots, v'_m \rangle \in \mathbb{R}^m$,
$$\langle v_1, \dots, v_m, v_{m+1}, \dots, v_{m'} \rangle \succ^{m'} \langle v'_1, \dots, v'_m, v_{m+1}, \dots, v_{m'} \rangle \iff \langle v_1, \dots, v_m \rangle \succ^m \langle v'_1, \dots, v'_m \rangle$$
(separability).

A universal aggregator is *consistent* with a universal separable order if there is a sequence $\{\bar{v}_m \in \mathbb{R}\}_{m=2,3,\dots}$ such that for every $m' \geq m$, every $\langle v_1, \dots, v_m \rangle \in \mathbb{R}^m$, and every $\langle v'_1, \dots, v'_{m'} \rangle \in \mathbb{R}^{m'}$,

$$U^{(m)}(v_1, \dots, v_m) > U^{(m')}(v'_1, \dots, v'_{m'}) \Rightarrow \langle v_1, \dots, v_m, \bar{v}_{m+1}, \dots, \bar{v}_{m'} \rangle \succ^{m'} \langle v'_1, \dots, v'_{m'} \rangle \tag{4.1a}$$

and

$$U^{(m)}(v_1, \dots, v_m) < U^{(m')}(v'_1, \dots, v'_{m'}) \Rightarrow \langle v'_1, \dots, v'_{m'} \rangle \succ^{m'} \langle v_1, \dots, v_m, \bar{v}_{m+1}, \dots, \bar{v}_{m'} \rangle. \tag{4.1b}$$

A universal aggregator is *quasiseparable* if it is consistent with a universal separable order. A family of quasiseparable universal aggregators is called *consistent* if they are all consistent with the same universal separable order.

Proposition 4.1. *Let $\mathfrak{U}$ be a consistent family of quasiseparable universal aggregators and let Γ be an anonymous participation game where each player i uses an aggregator $U_i \in \mathfrak{U}$. Then the individual improvement relation $\triangleright^{\mathrm{Ind}}$ in Γ, defined by (2.1), is acyclic.*

Proof. Let $\bar{v}^i_s$ be constants associated with the aggregator used by player i; we denote $K_i = \max_{x_i \in X_i} \# x_i$, $K = \sum_{i \in N} K_i$. Each relation $\succ^m$ can be perceived as defined on the set of unordered corteges of the length m (a reader interested in an exhaustive formalism can easily provide all the details him(her)self). With every $x \in X$, we associate an unordered cortege:

$$\varkappa(x) = \Big\langle \langle \varphi_\alpha(k) \rangle_{\alpha \in \mathrm{A}, k = \# N_-(\alpha), \dots, \# N(\alpha, x)}, \langle \bar{v}^i_s \rangle_{i \in N, s = \# x_i + 1, \dots, K_i} \Big\rangle$$

(assuming the convention that activities $\alpha \in \mathrm{A}$ with $\#N(\alpha, x) = 0$ are not represented in $\varkappa(x)$ at all). It is easy to check that $\sum_i \#x_i = \sum_\alpha \#N(\alpha, x)$; denoting $D = \sum_{\alpha\in\mathrm{A}}\bigl(\#N_-(\alpha)-1\bigr)_+$, we see that the length of $\varkappa(x)$ is $K - D$ for every $x \in X$. If we show that $y \rhd^{\mathrm{Ind}} x$ implies $\varkappa(y) \succ^{K-D} \varkappa(x)$, the acyclicity of $\rhd^{\mathrm{Ind}}$ will follow immediately.

Let $y \rhd^{\mathrm{Ind}}{}_i\, x$, i.e., $u_i(y) > u_i(x)$ and $y_{-i} = x_{-i}$. A is partitioned into four disjoint subsets: $\mathrm{A}^0 = x_i \cap y_i$, $\mathrm{A}^+ = y_i \setminus x_i$, $\mathrm{A}^- = x_i \setminus y_i$, $\mathrm{A}' = \mathrm{A} \setminus (x_i \cup y_i)$. We denote

$$\varkappa_{-i} = \Bigl\langle \langle\varphi_\alpha(k)\rangle_{\alpha\in\mathrm{A}^0,k=\#N_-(\alpha),\dots,\#N(\alpha,x)-1=\#N(\alpha,y)-1}, \langle\varphi_\alpha(k)\rangle_{\alpha\in\mathrm{A}^+,k=\#N_-(\alpha),\dots,\#N(\alpha,x)=\#N(\alpha,y)-1}, \\ \langle\varphi_\alpha(k)\rangle_{\alpha\in\mathrm{A}^-,k=\#N_-(\alpha),\dots,\#N(\alpha,y)=\#N(\alpha,x)-1}, \langle\varphi_\alpha(k)\rangle_{\alpha\in\mathrm{A}',k=\#N_-(\alpha),\dots,\#N(\alpha,x)=\#N(\alpha,y)}, \\ \langle\bar{v}^j_s\rangle_{j\in N,j\neq i,s=\#x_j+1,\dots,K_j}, \langle\bar{v}^i_s\rangle_{s=\max\{\#x_i,\#y_i\}+1,\dots,K_i}\Bigr\rangle$$

(under a similar convention). If $\#y_i \geq \#x_i$ (then $\#\mathrm{A}^+ \geq \#\mathrm{A}^-$), we denote

$$\varkappa_i(x) = \Bigl\langle \langle\varphi_\alpha(N(\alpha,x))\rangle_{\alpha\in\mathrm{A}^0\cup\mathrm{A}^-}, \langle\bar{v}^i_s\rangle_{s=\#x_i+1,\dots,\#y_i}\Bigr\rangle$$

and

$$\varkappa_i(y) = \Bigl\langle \varphi_\alpha(N(\alpha,y))\Bigr\rangle_{\alpha\in\mathrm{A}^0\cup\mathrm{A}^+}.$$

If $\#y_i \leq \#x_i$ (then $\#\mathrm{A}^+ \leq \#\mathrm{A}^-$), we denote

$$\varkappa_i(x) = \Bigl\langle \varphi_\alpha(N(\alpha,x))\Bigr\rangle_{\alpha\in\mathrm{A}^0\cup\mathrm{A}^-}$$

and

$$\varkappa_i(y) = \Bigl\langle \langle\varphi_\alpha(N(\alpha,y))\rangle_{\alpha\in\mathrm{A}^0\cup\mathrm{A}^+}, \langle\bar{v}^i_s\rangle_{s=\#y_i+1,\dots,\#x_i}\Bigr\rangle.$$

In either case, $\varkappa(x) = \langle\varkappa_{-i}, \varkappa_i(x)\rangle$ and $\varkappa(y) = \langle\varkappa_{-i}, \varkappa_i(y)\rangle$; by separability, $\varkappa(y) \succ^{K-D} \varkappa(x)$ if and only if $\varkappa_i(y) \succ \varkappa_i(x)$. On the other hand, $u_i(x) = U^{(\#x_i)}(\langle\varphi_\alpha(N(\alpha,x))\rangle_{\alpha\in\mathrm{A}^0\cup\mathrm{A}^-})$ and $u_i(y) = U^{(\#y_i)}(\langle\varphi_\alpha(N(\alpha,y))\rangle_{\alpha\in\mathrm{A}^0\cup\mathrm{A}^+})$. By condition (4.1a) or (4.1b), from $u_i(y) > u_i(x)$ we obtain $\varkappa_i(y) \succ \varkappa_i(x)$, hence $\varkappa(y) \succ \varkappa(x)$. □

The simplest and most important example of a universal separable order is given by the additive aggregation rule:

$$v' \succ^m v \iff \sum_{s=1}^m \nu(v'_s) > \sum_{s=1}^m \nu(v_s), \tag{4.2}$$

where $\nu : \mathbb{R} \to \mathbb{R}$ is strictly increasing. Thus, Rosenthal's (1973) congestion games are covered by Proposition 4.1 with this order, $\nu(v) = v$, and $\bar{v}_m = 0$ for all m.

At a first glance, different $\nu(\cdot)$ and $\bar{v}_m \neq 0$ provide a more general result, but this is just an illusion; it may be worthwhile to consider the situation in more detail. Let each player i in an anonymous participation game Γ use a universal aggregator U_i consistent with the additive order (4.2); the conditions (4.1) imply that player i's utility function is (up to a monotonic transformation)

$$u_i(x) = \sum_{\alpha\in x_i} \nu(\varphi_\alpha(\#N(\alpha,x))) \; + \sum_{s=\#x_i+1}^{K_i} \bar{v}^i_s.$$

Obviously, we can represent Γ as a congestion game, redefining $\varphi^*_\alpha(k) = \nu(\varphi_\alpha(k))$, adding to A new activities (i, m), $i \in N$, $1 \le m \le K_i$, defining $\varphi^*_{(i,m)}(1) = \bar{v}^i_m$, and replacing each $x_i \in X_i$ with $x_i \cup \{(i, \#x_i + 1), \ldots, (i, K_i)\}$.

Similarly, the group formation games considered by Hollard (2000) are congestion games. Each player there chooses an "action" $a_i \in A$ and the utility is

$$u_i(a_1, \ldots, a_n) = v^i(a_i) + I_{a_i}(n(a_i)) \; + \sum_{z \in A \setminus \{a_i\}} E_z(n(z)), \tag{4.3}$$

where $n(a)$ is the number of players having chosen a at the given strategy profile, and $v^i(\cdot)$, $I_a(\cdot)$, and $E_a(\cdot)$ are given functions. Let us define an extended set of activities as the union of $A \times N$ and two copies of A: $\mathrm{A} = (A \times N) \cup \{a^{\mathrm{Int}}\}_{a \in A} \cup \{a^{\mathrm{Ext}}\}_{a \in A}$ with $\varphi_{(a,i)}(k) = v^i(a)$, $\varphi_{a^{\mathrm{Int}}}(k) = I_a(k)$, and $\varphi_{a^{\mathrm{Ext}}}(k) = E_a(n-k)$; the revised strategy sets will be $X_i = \big\{\{(a,i)\} \cup \{a^{\mathrm{Int}}\} \cup \{b^{\mathrm{Ext}}\}_{b \in A \setminus \{a\}}\big\}_{a \in A}$. Obviously, we have (4.3) for the utilities in the congestion game; in other words, Theorem 1 of Hollard (2000) is a particular case of Rosenthal's (1973) theorem, so there was no need to prove it again.

The additive utilities in Section 5 of Konishi et al. (1997b) are a particular case of (4.3), so a reference to Rosenthal (1973) would have been also sufficient to prove their implication Lemma 4.2 $\Rightarrow$ Proposition 4.1 (Lemma 4.2 itself belongs to a quite different set of ideas).

Another natural example of a universal separable order is provided by the leximin (or, dually, leximax) ordering; then Proposition 4.1 implies the acyclicity of the individual improvement relation $\triangleright^{\mathrm{Ind}}$, defined by (2.1), in every anonymous participation game with the minimum (maximum) aggregation. It should be noted, however, that Proposition 4.1 as such is not necessary to reach the conclusion: Rosenthal's theorem on congestion games is sufficient again. Having an anonymous participation game with the minimum aggregation, we denote Φ the finite set of all feasible $\varphi_\alpha(k)$. Now let $\nu : \mathbb{R} \to \mathbb{R}$ satisfy the condition:

$$v'', v' \in \Phi \;\&\; v'' > v' \Rightarrow \nu(v'') - \nu(v') > \max_{i \in N} K_i \cdot [\nu(\max \Phi) - \nu(v'')]; \tag{4.4}$$

clearly, the minimum aggregation is consistent (for feasible values) with the order defined by (4.2) with (4.4). To satisfy (4.4), we can define a piecemeal linear, concave function $\nu : \mathbb{R} \to \mathbb{R}$, going inductively from $\max \Phi$ downwards.

Generalizing our basic notions, we may consider games where the players use the leximin (leximax) ordering to aggregate intermediate objectives. Proposition 4.1, applied to the ordering itself, ensures the acyclicity of individual improvements. Again, the fact follows from Rosenthal's theorem if we consider aggregation (4.2) satisfying (4.4). Leximin aggregation and minimum aggregation may seem very similar, but there is no analogue of Theorem 1 for the former case.

Example 4.1. Let us consider an anonymous participation game with negative impacts: $N = \{1, 2\}$, $\mathrm{A} = \{a, b, c, d, e, f, g\}$; $X_1 = \big\{\{a, b, c\},\ \{d, e, f\}\big\}$; $X_2 = \big\{\{a, f, g\},\ \{b, c, d\}\big\}$; $\varphi_a(2) = \varphi_b(2) = \varphi_d(2) = \varphi_e(1) = \varphi_g(1) = 0$; $\varphi_c(2) = 1$; $\varphi_a(1) = \varphi_d(1) = \varphi_f(2) = 2$;

$\varphi_b(1) = \varphi_c(1) = \varphi_f(1) = 3$. Assuming that both players use the leximin aggregation, we obtain the 2×2 matrix of the game:

$$\begin{array}{ccc} & \text{afg} & \text{bcd} \\ \text{abc} & (\langle 0,3,3\rangle, \langle 0,0,3\rangle) & (\langle 0,1,2\rangle, \langle 0,1,2\rangle) \\ \text{def} & (\langle 0,2,2\rangle, \langle 0,2,2\rangle) & (\langle 0,0,3\rangle, \langle 0,3,3\rangle). \end{array}$$

We have a prisoner's dilemma: the northeastern corner is a unique Nash equilibrium, which is Pareto dominated by the southwestern corner.

To summarize, Proposition 4.1 shows that Rosenthal's (1973) theorem hinges on the separability of additive aggregation; however, it does not add much to the latter's content. It is unclear whether anything at all is added: although separable orderings admitting no additive representation, even on finite sets, are well known, usually they are not symmetric. Finally, let us show that Proposition 4.1 cannot be extended beyond anonymous games.

Example 4.2. Consider a two person (non-anonymous) participation game with additive aggregation: $N = \{1,2\}$, $\mathrm{A} = \{a,b,c,d\}$, $X_1 = \big\{\{a,b\},\{c,d\}\big\}$, $X_2 = \big\{\{a,c\},\{b,d\}\big\}$, $\varphi_a(\{2\}) = \varphi_d(\{2\}) = \varphi_b(\{1\}) = \varphi_c(\{1\}) = 1$, $\varphi_a(\{1\}) = \varphi_d(\{1\}) = \varphi_b(\{2\}) = \varphi_c(\{2\}) = 2$, $\varphi_\alpha(N) = 0$ for all $\alpha \in \mathrm{A}$. The matrix of the game looks as follows:

$$\begin{array}{ccc} & \text{ac} & \text{bd} \\ \text{ab} & (1,2) & (2,1) \\ \text{cd} & (2,1) & (1,2). \end{array}$$

There is no Nash equilibrium.

Remark. Unlike Example 3.2, this game exhibits strictly negative impacts, which do not help.

4.2 Necessity of the Minimum (Maximum) Aggregation

Theorem 3. *Let $\mathfrak{U}$ be a set of universal aggregators such that every anonymous participation game with strictly negative impacts where each player uses an aggregator from $\mathfrak{U}$ possesses a weakly Pareto optimal Nash equilibrium. Then for every $U \in \mathfrak{U}$,*

1. *for every $m \geq 1$, there is a continuous and strictly increasing mapping $\lambda_m^U : \mathbb{R} \to \mathbb{R}$ such that*
$$U^{(m)}(v_1, \ldots, v_m) = \lambda_m^U(\min\{v_1, \ldots, v_m\})$$
for all $v_1, \ldots, v_m \in \mathbb{R}$;

2. *for every $m, m' \geq 1$, either $\lambda_{m'}^U = \lambda_m^U$ or $\lambda_m^U(\mathbb{R}) \cap \lambda_{m'}^U(\mathbb{R}) = \emptyset$.*

Remark. The proof below remains valid, virtually without any modification, if each $U^{(m)}$ is assumed defined on R^m, where R is an open interval (bounded or not) in $\mathbb{R}$; e.g., $R = \mathbb{R}_{++}$. If R is not open (e.g., if only integer-valued φ_α are considered), the proof collapses; I have no idea whether the theorem itself remains valid in this case.

Proof. Since the statement of the theorem does not include any connection between different aggregators from $\mathfrak{U}$, we fix $U \in \mathfrak{U}$ and later drop the superscript ^{U} at λ. As a first step we show that the impossibility of a prisoner's dilemma implies that each indifference curve in each two-dimensional section must exhibit a similarity with either minimum or maximum.

Lemma 4.2.1. *Let $m \geq 2$, $v_1 > v_2$, and*

$$U^{(m)}(v_2, v_2, v_3, \dots, v_m) < U^{(m)}(v_1, v_2, v_3, \dots, v_m); \tag{4.5}$$

then $U^{(m)}(v_1, \bar{v}_2, v_3, \dots, v_m) = U^{(m)}(v_1, v_2, v_3, \dots, v_m)$ for all $\bar{v}_2 \leq v_2$.

Proof. A nonstrict inequality immediately follows from the monotonicity of $U^{(m)}$. Let us suppose $U(v_1, \bar{v}_2, \dots, v_m) = u' < u = U(v_1, v_2, \dots, v_m)$ for some $\bar{v}_2 < v_2$. Taking into account (4.5) and the continuity of U, we may, increasing $\bar{v}_2$ if needed, assume $U(v_2, v_2, v_3, \dots, v_m) < u'$. By the continuity of U, there are $\delta_s > 0$ $(s = 1, \dots, m)$ such that $v_2 + \delta_2 < v_1$ and $U(v_2 + \delta_1, v_2 + \delta_2, v_3 + \delta_3, \dots, v_m + \delta_m) = u'' < u'$; we denote $U(v_1 + \delta_1, v_2 + \delta_2, v_3 + \delta_3, \dots, v_m + \delta_m) = u^+ > u$. Thus,

$$u'' < u' < u < u^+. \tag{4.6}$$

Now let us consider an anonymous participation game with strictly negative impacts where both players use the aggregator U: $N = \{1, 2\}$; there are $2m$ activities, $\mathrm{A} = \{a, b, c, d, e_3, \dots, e_m, f_3, \dots, f_m\}$; $X_1 = \big\{\{a, c, e_3, \dots, e_m\}, \{b, d, f_3, \dots, f_m\}\big\}$; $X_2 = \big\{\{a, d, e_3, \dots, e_m\}, \{b, c, f_3, \dots, f_m\}\big\}$; $\varphi_a(1) = v_1 + \delta_1$, $\varphi_a(2) = \bar{v}_2$, $\varphi_b(1) = v_2 + \delta_1$, $\varphi_b(2) = v_2$, $\varphi_c(1) = \varphi_d(1) = v_1$, $\varphi_c(2) = \varphi_d(2) = v_2 + \delta_2$, $\varphi_{e_s}(1) = \varphi_{f_s}(1) = v_s + \delta_s$ $(s = 3, \dots, m)$, $\varphi_{e_s}(2) = \varphi_{f_s}(2) = v_s$ $(s = 3, \dots, m)$. The 2×2 matrix of the game looks as follows:

	ade	bcf
ace	(u', u')	(u^+, u'')
bdf	(u'', u^+)	(u, u).

Taking into account (4.6), we see that the northwestern corner is a unique Nash equilibrium, which is strongly Pareto dominated by the southeastern corner. □

Lemma 4.2.2. *Let $m \geq 2$, $v_1 > v_2$, and*

$$U^{(m)}(v_1, v_2, v_3, \dots, v_m) < U^{(m)}(v_1, v_1, v_3, \dots, v_m); \tag{4.7}$$

then $U^{(m)}(\bar{v}_1, v_2, v_3, \dots, v_m) = U^{(m)}(v_1, v_2, v_3, \dots, v_m)$ for all $\bar{v}_1 \geq v_1$.

Proof. A nonstrict inequality immediately follows from the monotonicity of $U^{(m)}$. Let us suppose

$$U(\bar{v}_1, v_2, \dots, v_m) = u^+ > u = U(v_1, v_2, \dots, v_m) \tag{4.8}$$

for some $\bar{v}_1 > v_1$. Since $U(v_2, v_1, v_3, \dots, v_m) = u$ by symmetry, (4.7) and the continuity of U imply the existence of $v_1' \in]v_2, v_1[$ such that $u < U(v_1', v_1, v_3, \dots, v_m) < u^+$. By the continuity of U, we may pick $\delta_s > 0$ $(s = 1, \dots, m)$ such that $v_1' + \delta_1 < v_1$, $U(v_2 + \delta_1, v_1 +$

$\delta_2, v_3+\delta_3, \ldots, v_m+\delta_m) = u' < U(v_1', v_1, v_3, \ldots, v_m)$ and $U(v_1'+\delta_1, v_1+\delta_2, v_3+\delta_3, \ldots, v_m+\delta_m) = u'' < u^+$; by monotonicity,

$$u < u' < u'' < u^+. \tag{4.9}$$

Now let us consider an anonymous participation game with strictly negative impacts where both players use the aggregator U: $N = \{1,2\}$; there are $2m$ activities, $\mathrm{A} = \{a, b, c_2, \ldots, c_m,\ d_2, \ldots, d_m\}$; $X_1 = \big\{\{a, c_2, \ldots, c_m\},\ \{b, d_2, \ldots, d_m\}\big\}$; $X_2 = \big\{\{a, d_2, \ldots, d_m\}, \{b, c_2, \ldots, c_m\}\big\}$; $\varphi_a(1) = \bar{v}_1$, $\varphi_a(2) = v_2+\delta_1$, $\varphi_b(1) = v_1$, $\varphi_b(2) = v_1'+\delta_1$, $\varphi_{c_2}(1) = \varphi_{d_2}(1) = v_1 + \delta_2$, $\varphi_{c_s}(1) = \varphi_{d_s}(1) = v_s + \delta_s$ $(s = 3, \ldots, m)$, $\varphi_{c_s}(2) = \varphi_{d_s}(2) = v_s$ $(s = 2, \ldots, m)$. The 2×2 matrix of the game looks as follows:

$$\begin{array}{ccc} & \text{ad} & \text{bc} \\ \text{ac} & (u', u') & (u^+, u) \\ \text{bd} & (u, u^+) & (u'', u''). \end{array}$$

Taking into account (4.9), we see that the northwestern corner is a unique Nash equilibrium, which is strongly Pareto dominated by the southeastern corner. $\square$

As a second step, we fix a two-dimensional section and show that the whole indifference map is "minimum-like." Let us fix $m \geq 2$ and $v_3, \ldots, v_m \in \mathbb{R}$. We will study possible indifference maps of the function $U(\cdot, \cdot, v_3, \ldots, v_m)$. The symmetry and continuity allow us to restrict attention to $\mathbb{R}^2_> = \{(v_1, v_2) \in \mathbb{R}^2 |\ v_1 > v_2\}$.

Lemma 4.2.3. *If $(v_1, v_2) \in \mathbb{R}^2_>$, $u = U(v_1, v_2, v_3, \ldots, v_m) > U(v_2, v_2, v_3, \ldots, v_m)$, $(v_1', v_2') \in \mathbb{R}^2_>$, and $u' = U(v_1', v_2', v_3, \ldots, v_m) < u$, then $U(v_1', v_1', v_3, \ldots, v_m) = u'$.*

Proof. By Lemma 4.2.1, $U(v_1, \bar{v}_2, v_3, \ldots, v_m) = u$ for all $\bar{v}_2 \leq v_2$, hence

$$U(\bar{v}_1, \bar{v}_2, v_3, \ldots, v_m) \geq u \quad \text{for all } \bar{v}_1 \geq v_1 \text{ and } \bar{v}_2 \leq \bar{v}_1. \tag{4.10}$$

Since $u' < u$, $v_1' < v_1$. Supposing $U(v_1', v_1', v_3, \ldots, v_m) > u'$, we, by Lemma 4.2.2, obtain $U(\bar{v}_1, v_2', v_3, \ldots, v_m) = u'$ for all $\bar{v}_1 \geq v_1'$, in particular, $U(v_1, v_2', v_3, \ldots, v_m) = u' < u$, contradicting (4.10). $\square$

Lemma 4.2.4. *For every $(v_1, v_2) \in \mathbb{R}^2_>$, $U(v_1, v_2, v_3, \ldots, v_m) = U(v_2, v_2, v_3, \ldots, v_m)$.*

Proof. Suppose the contrary: $U(v_1, v_2, v_3, \ldots, v_m) > U(v_2, v_2, v_3, \ldots, v_m)$. By the continuity of U, we can choose $v^0 < v^1 < \cdots < v^7$ such that $v_2 < v^0$, $v_1 > v^7$, and $U(v^s, v_2, v_3, \ldots, v_m) < U(v^{s+1}, v_2, v_3, \ldots, v_m)$ for $s = 0, \ldots, 6$, and $U(v^7, v_2, v_3, \ldots, v_m) < U(v_1, v_2, v_3, \ldots, v_m)$. Denoting $u^s = U(v^s, v_2, v_3, \ldots, v_m)$ $(s = 0, \ldots, 7)$, we obtain $u^0 < u^1 < \cdots < u^7$. By Lemma 4.2.3, $U(v^s, v^s, v_3, \ldots, v_m) = u^s$ for $s = 0, \ldots, 7$, hence $U(v^s, v^{s'}, v_3, \ldots, v_m) = u^s$ whenever $s > s'$.

Now let us consider an anonymous participation game with strictly negative impacts where all players use the aggregator U: $N = \{1,2,3\}$; there are $m+3$ activities, $\mathrm{A} = \{a, b, c, d, e, f_3, \ldots, f_m\}$; $X_1 = \big\{\{a, e, f_3, \ldots, f_m\},\ \{b, d, f_3, \ldots, f_m\}\big\}$; $X_2 = \big\{\{a, c, f_3, \ldots, f_m\},\ \{d, e, f_3, \ldots, f_m\}\big\}$; $X_3 = \big\{\{a, b, f_3, \ldots, f_m\},\ \{c, e, f_3, \ldots, f_m\}\big\}$; $\varphi_a(3) =$

$\varphi_e(3) = v^0$, $\varphi_c(2) = \varphi_e(2) = v^1$, $\varphi_a(2) = \varphi_d(2) = \varphi_e(1) = v^2$, $\varphi_b(2) = v^3$, $\varphi_c(1) = v^4$, $\varphi_d(1) = v^5$, $\varphi_a(1) = v^6$, $\varphi_b(1) = v^7$, and $\varphi_{f_s}(3) = v_s$ $(s = 3, \dots, m)$. The $2 \times 2 \times 2$ matrix of the game looks as follows:

$$\begin{array}{c} \\ \\ \text{ae} \\ \text{bd} \end{array} \begin{array}{c} \text{ab} \\ \begin{array}{cc} \text{ac} & \text{de} \end{array} \\ \begin{bmatrix} (u^2, u^4, u^7) & (u^2, u^5, u^7) \\ (u^5, u^4, u^3) & (u^3, u^2, u^6) \end{bmatrix} \end{array} \begin{array}{c} \text{ce} \\ \begin{array}{cc} \text{ac} & \text{de} \end{array} \\ \begin{bmatrix} (u^2, u^2, u^1) & (u^6, u^5, u^4) \\ (u^7, u^6, u^2) & (u^7, u^2, u^4) \end{bmatrix} \end{array}.$$

The individual improvement relation is acyclic (as it should be according to Proposition 4.1) and the southwestern corner of the left matrix is a unique Nash equilibrium. However, this equilibrium is strongly Pareto dominated by the northeastern corner of the right matrix. $\square$

Let us address the first statement of the theorem. For each $m \geq 1$ and $u \in \mathbb{R}$, we define $\lambda_m(u) = U^{(m)}(u, \dots, u)$; for $m = 1$, there is nothing to prove. Without restricting generality, we may assume $v_1 \geq v_2 \geq \cdots \geq v_m$. Applying Lemma 4.2.4 consequently to pairs (v_{m-1}, v_m), $(v_{m-2}, v_m), \dots$, (v_1, v_m), we obtain $U(v_1, v_2, \dots, v_m) = U(v_1, v_2, \dots, v_{m-2}, v_m, v_m) = \cdots = U(v_1, v_m, \dots, v_m, v_m) = U(v_m, \dots, v_m) = \lambda_m(\min\{v_1, v_2, \dots, v_m\})$.

Turning to the second statement, let $m' > m$ and $\lambda_m(\mathbb{R}) \cap \lambda_{m'}(\mathbb{R}) \neq \emptyset$. Since $\lambda_m(\mathbb{R}) \cap \lambda_{m'}(\mathbb{R})$ is open and $\{v \in \mathbb{R} | \ \lambda_{m'}(v) = \lambda_m(v)\}$ is closed in $\mathbb{R}$, either $\lambda_{m'} = \lambda_m$ or there is $v' \neq v$ such that $\lambda_{m'}(v') = \lambda_m(v)$; let us show the impossibility of the second alternative.

Supposing $v > v'$, we denote $u^1 = \lambda_{m'}(v')$. Then we pick $\underline{v} \in]v', v[$, denote $u^0 = \lambda_m(\underline{v})$ and $u^3 = \lambda_{m'}(\underline{v})$ (so $u^0 < \lambda_m(v) = u^1 = \lambda_{m'}(v') < u^3$), and pick $\bar{v} > v$ so that $u^2 = \lambda_m(\bar{v}) < u^3$; $u^2 > u^1$ is satisfied automatically.

Let us consider an anonymous participation game with strictly negative impacts where both players use the aggregator U: $N = \{1, 2\}$; there are $m' + 2$ activities, $\mathrm{A} = \{a, b, c, d_2, \dots,\ d_{m'}\}$; $X_1 = \big\{\{a, d_2, \dots, d_m\},\ \{b, c, d_3, \dots, d_{m'}\}\big\}$; $X_2 = \big\{\{a, b, d_3, \dots, d_{m'}\}, \{c, d_2, \dots, d_m\}\big\}$; $\varphi_a(1) = \bar{v} = \varphi_c(1)$, $\varphi_a(2) = \underline{v} = \varphi_c(2)$, $\varphi_b(2) = v' < \varphi_b(1) = \varphi_{d_s}(2) < \varphi_{d_s}(1)$ $(s = 2, \dots, m')$. The 2×2 matrix of the game looks as follows:

$$\begin{array}{ccc} & \text{abd} & \text{cd} \\ \text{ad} & (u^0, u^3) & (u^2, u^2) \\ \text{bcd} & (u^1, u^1) & (u^3, u^0). \end{array}$$

We have a prisoner's dilemma: "longer" strategies are dominant, but the northeastern corner strongly Pareto dominates the southwestern one.

Supposing $v' > v$, we denote $u^0 = \lambda_{m'}(v)$ and $u^4 = \lambda_m(v) > u^0$; then we pick $\underline{v} \in]v, v'[$ and $v^+ > \bar{v} > v'$, and denote $u^3 = \lambda_{m'}(\underline{v}) < u^4 < \lambda_{m'}(\bar{v}) = u^6 < \lambda_{m'}(v^+) = u^7$. Then we pick $v'' \in]v, \underline{v}[$ so that $u^5 = \lambda_m(v'') < u^6$; $u^5 > u^4$ is satisfied automatically. Finally, we pick $v''' \in]v, v''[$, and denote $u^1 = \lambda_{m'}(v''')$ and $u^2 = \lambda_{m'}(v'')$; we have $u^0 < u^1 < \cdots < u^7$.

Now we consider an anonymous participation game with strictly negative impacts where all players use the aggregator U: $N = \{1, 2, 3\}$; there are $m' + 3$ activities, $\mathrm{A} =$

$\{a,b,c,d,e_2,\ldots,\ e_{m'}\}$; $X_1 = \{\{a,e_2,\ldots,e_m\},\ \{d,e_2,\ldots,e_{m'}\}\}$; $X_2 = \{\{a,b,e_3,\ldots,e_{m'}\},\ \{c,e_2,\ldots,e_{m'}\}\}$; $X_3 = \{\{d,e_2,\ldots,e_{m'}\},\ \{b,e_2,\ldots,e_{m'}\}\}$; $\varphi_a(2) = v$, $\varphi_a(1) = \varphi_b(2) = v''$, $\varphi_b(1) = v^+$, $\varphi_d(2) = \underline{v}$, $\varphi_d(1) = \bar{v}$, $\varphi_c(1) = v'''$, $v^+ < \varphi_{e_s}(3) < \varphi_{e_s}(2)$ $(s = 2,\ldots,m')$. The $2 \times 2 \times 2$ matrix of the game looks as follows:

$$
\begin{array}{ccc}
 & \mathrm{de}_{(m')} & \mathrm{be}_{(m')} \\
 & \begin{array}{cc} \mathrm{abe}_{(m')} & \mathrm{ce}_{(m')} \end{array} & \begin{array}{cc} \mathrm{abe}_{(m')} & \mathrm{ce}_{(m')} \end{array} \\
\begin{array}{c} \mathrm{ae}_{(m)} \\ \mathrm{de}_{(m')} \end{array} &
\begin{bmatrix} (u^4,u^0,u^6) & (u^5,u^1,u^6) \\ (u^3,u^2,u^3) & (u^3,u^1,u^3) \end{bmatrix} &
\begin{bmatrix} (u^4,u^0,u^2) & (u^5,u^1,u^7) \\ (u^6,u^2,u^2) & (u^6,u^1,u^7) \end{bmatrix}.
\end{array}
$$

There is no Nash equilibrium in the game. □

Proposition 4.2. *Let $\mathfrak{U}$ be a set of universal aggregators satisfying the conditions 1 and 2 from Theorem 3. Then in every anonymous participation game Γ with negative impacts where each player uses an aggregator from $\mathfrak{U}$, the strong coalition improvement relation, defined by* (2.2)*, is acyclic.*

Proof. Let $x^0,\ldots,x^{\bar{m}} = x^0$ be a coalition improvement cycle in Γ. Let i be a player involved in the cycle and using an aggregator $U \in \mathfrak{U}$. The condition 2 obviously implies that $\mathbb{N} = \{1,2,\ldots\}$ is partitioned into a (finite or infinite) number of subsets M_k such that $\lambda^U_{m'} = \lambda^U_m$ whenever m and m' belong to the same M_k, and $\lambda^U_m(\mathbb{R}) \cap \lambda^U_{m'}(\mathbb{R}) = \emptyset$ whenever they do not. The latter condition, in turn, means that the subsets M_k are ordered in the sense that $M_k > M_{k'} \iff \big[\lambda^U_m(u) > \lambda^U_{m'}(u')$ whenever $m \in M_k$, $m' \in M_{k'}$, and $u,u' \in \mathbb{R}\big]$; it follows immediately that, whenever $\#x_i \in M_k > M_{k'} \ni \#y_i$, $u_i(x_i,z_{-i}) > u_i(y_i,z'_{-i})$ for all $z_{-i}, z'_{-i} \in X_{-i}$. Therefore, for each $i \in N$, only strategies from the same element of the partition can be involved in the cycle. Denoting Γ^* the game with the same players, activities, and strategies, but with the minimum aggregation, we see that $x^0,\ldots,x^{\bar{m}} = x^0$ is a coalition improvement cycle in Γ^* as well; however, this contradicts Theorem 1. □

Theorem 4. *Let $\mathfrak{U}$ be a set of universal aggregators such that every anonymous participation game with strictly positive impacts where each player uses an aggregator from $\mathfrak{U}$ possesses a weakly Pareto optimal Nash equilibrium. Then for every $U \in \mathfrak{U}$,*

1. *for every $m \geq 1$, there is a continuous and strictly increasing mapping $\lambda^U_m : \mathbb{R} \to \mathbb{R}$ such that*
$$U^{(m)}(v_1,\ldots,v_m) = \lambda^U_m(\max\{v_1,\ldots,v_m\})$$
for all $v_1,\ldots,v_m \in \mathbb{R}$;

2. *for every $m,m' \geq 1$, either $\lambda^U_{m'} = \lambda^U_m$ or $\lambda^U_m(\mathbb{R}) \cap \lambda^U_{m'}(\mathbb{R}) = \emptyset$.*

Remark. The comment to the formulation of Theorem 3 is appropriate here as well.

Proof. The scheme of the proof is the same as in Theorem 3; all auxiliary games must be replaced with their opposites. (We cannot simply refer to Proposition 2.1 because the condition is just the existence of equilibria.) □

Proposition 4.3. *Let $\mathfrak{U}$ be a set of universal aggregators satisfying the conditions 1 and 2 from Theorem 4. Then in every anonymous participation game with positive impacts where each player uses an aggregator from $\mathfrak{U}$, the strong coalition improvement relation, defined by* (2.2)*, is acyclic.*

The fact follows from Proposition 3.1 in exactly the same way as Proposition 4.2 from Theorem 1.

4.3 Necessity of Additive Aggregation

Theorem 5. *Let N be a finite set with $n = \#N \geq 2$; let $\langle U_i \rangle_{i \in N}$ be a list of universal aggregators such that every function $U_i^{(m)}$ is symmetric, continuous, and strictly increasing in the sense of* (2.6)*. If every anonymous participation game with strictly negative impacts where N is the set of players and each player i uses the aggregator U_i possesses a Nash equilibrium, then*

1. *there is a continuous and strictly increasing mapping $\nu : \mathbb{R} \to \mathbb{R}$ and a continuous and strictly increasing mapping $\lambda_i^m : m \cdot \nu(\mathbb{R}) \to \mathbb{R}$ for every $i \in N$ and $m \geq 1$ such that*
$$U_i^{(m)}(v_1, \ldots, v_m) = \lambda_i^m \Big(\sum_{s=1}^{m} \nu(v_s) \Big) \tag{4.11a}$$
for all $v_1, \ldots, v_m \in \mathbb{R}$;
2. *for every $i \in N$ and $m, m' \geq 1$, there is a constant $\bar{u}_i^{mm'} \in \mathbb{R} \cup \{-\infty, +\infty\}$ such that*
$$\operatorname{sign}\big(\lambda_i^{m'}(u') - \lambda_i^m(u)\big) = \operatorname{sign}(u' - u - \bar{u}_i^{mm'}) \tag{4.11b}$$
for all $u' \in m' \cdot \nu(\mathbb{R})$ and $u \in m \cdot \nu(\mathbb{R})$.

Remark. The comment to the formulation of Theorem 3 is appropriate here as well.

Proof.

Lemma 4.3.1. *Let $i, j \in N$, $m, m' \geq 2$, $v_s \in \mathbb{R}$ for $s = 1, \ldots, m$, and $v'_s \in \mathbb{R}$ for $s = 1, \ldots, m'$; let*
$$U_i^{(m)}(v_1, v'_2, v_3, \ldots, v_m) = U_i^{(m)}(v'_1, v_2, v_3, \ldots, v_m). \tag{4.12a}$$
Then
$$U_j^{(m')}(v_1, v'_2, v'_3, \ldots, v'_{m'}) = U_j^{(m')}(v'_1, v_2, v'_3, \ldots, v'_{m'}). \tag{4.12b}$$

Proof. Suppose first that $i \neq j$; without restricting generality, $v'_s > v_s$ for $s = 1, 2$. The negation of (4.12b) can be written as $U_j^{(m')}(v_1, v'_2, v'_3, \ldots, v'_{m'}) > U_j^{(m')}(v'_1, v_2, v'_3, \ldots, v'_{m'})$. Pick $\delta > 0$ such that $u_j^2 = U_j^{(m')}(v_1, v'_2, v'_3, \ldots, v'_{m'}) > U_j^{(m')}(v'_1 + \delta, v_2, v'_3, \ldots, v'_{m'}) = u_j^1$; by monotonicity from (4.12a), $u_i^1 = U_i^{(m)}(v_1, v'_2, v_3, \ldots, v_m) < U_i^{(m)}(v'_1 + \delta, v_2, v_3, \ldots, v_m) = u_i^2$.

Let us consider an anonymous participation game with strictly negative impacts where each player $k \in N$ uses the aggregator U_k: $\mathrm{A} = \{a, b, c, d, e_3, \ldots, e_m, f_3, \ldots, f_{m'}, g\}$; $X_i = \{\{a, b, e_3, \ldots, e_m\}, \{c, d, e_3, \ldots, e_m\}\}$; $X_j = \{\{a, c, f_3, \ldots, f_{m'}\}, \{b, d, f_3, \ldots, f_{m'}\}\}$; $X_k = \{\{g\}\}$ for $k \in N \setminus \{i, j\}$; $\varphi_a(1) = \varphi_d(1) = v'_1 + \delta$, $\varphi_a(2) = \varphi_d(2) = v_1$; $\varphi_b(1) = \varphi_c(1) = v'_2$, $\varphi_b(2) = \varphi_c(2) = v_2$; $\varphi_{e_s}(1) = v_s$ $(s = 3, \ldots, m)$, $\varphi_{f_s}(1) = v'_s$ $(s = 3, \ldots, m')$. The 2×2 matrix of the essential part of the game looks as follows:

$$\begin{array}{ccc} & \text{ac} & \text{bd} \\ \text{ab} & (u_i^1, u_j^2) & (u_i^2, u_j^1) \\ \text{cd} & (u_i^2, u_j^1) & (u_i^1, u_j^2). \end{array}$$

Since $u_k^2 > u_k^1$ $(k = i, j)$, the game possesses no Nash equilibrium.

If $i = j$, we pick $k \neq i$ (we have assumed $n \geq 2$!) and obtain

$$U_k^{(m)}(v_1, v'_2, v_3, \ldots, v_m) = U_k^{(m)}(v'_1, v_2, v_3, \ldots, v_m)$$

first, and then (4.12b). □

Clearly, the exact analogues of Lemma 4.3.1 with inequalities (of the same sign) instead of equalities are valid as well, i.e., all two-dimensional sections of all functions $U_i^{(m)}$ $(i \in N, m = 1, 2, \ldots)$ have the same indifference map. In other words, the ordering defined by each function $U_i^{(m)}$ $(i \in N,\ m > 2)$ on $\mathbb{R}^m$ admits a separable projection to every two-dimensional subspace; by the main result of Gorman (1968), it admits a separable projection to *every* subspace and also an additive representation on the whole $\mathbb{R}^m$. For $m = 2$, some extra work is needed.

Lemma 4.3.2. *Let $i \in N$ and $v'_s, v''_s, v'''_s \in \mathbb{R}$ for $s = 1, 2$; let*

$$U_i^{(2)}(v'_1, v''_2) = U_i^{(2)}(v''_1, v'_2) \tag{4.13a}$$

and

$$U_i^{(2)}(v'_1, v'''_2) = U_i^{(2)}(v''_1, v''_2) = U_i^{(2)}(v'''_1, v'_2). \tag{4.13b}$$

Then

$$U_i^{(2)}(v''_1, v'''_2) = U_i^{(2)}(v'''_1, v''_2). \tag{4.14}$$

Proof. Supposing the contrary, we may, without restricting generality, assume $U_i(v'''_1, v''_2) > U_i(v''_1, v'''_2)$. By continuity, there exists $\delta_1 > 0$ such that

$$U_i(v'''_1 - \delta_1, v''_2) > U_i(v''_1, v'''_2). \tag{4.15a}$$

Pick $j \neq i$; by Lemma 4.3.1, the equalities (4.13) are valid for $U_j^{(2)}$ as well. By monotonicity from (4.13b) for j, $U_j(v''_1, v''_2) > U_j(v'''_1 - \delta_1, v'_2)$; therefore, there is $\delta_2 > 0$ such that $U_j(v''_1, v''_2) > U_j(v'''_1 - \delta_1, v'_2 + \delta_2)$; by continuity, there is $\delta'_1 > 0$ such that

$$U_j(v''_1 - \delta'_1, v''_2) > U_j(v'''_1 - \delta_1, v'_2 + \delta_2). \tag{4.15b}$$

By monotonicity from (4.15a), $U_i(v_1''' - \delta_1, v_2'') > U_i(v_1'' - \delta_1', v_2''')$, hence, by continuity, there is $\delta_2' > 0$ such that

$$U_i(v_1''' - \delta_1, v_2'' - \delta_2') > U_i(v_1'' - \delta_1', v_2'''). \tag{4.15c}$$

By monotonicity from (4.13a),

$$U_i(v_1'', v_2' + \delta_2) > U_i(v_1', v_2''); \tag{4.15d}$$

by monotonicity from (4.13b) for j,

$$U_j(v_1', v_2''') > U_j(v_1'', v_2'' - \delta_2'). \tag{4.15e}$$

Now we denote $u_i^1 = U_i(v_1', v_2'')$, $u_i^2 = U_i(v_1'', v_2' + \delta_2)$, $u_j^1 = U_j(v_1''' - \delta_1, v_2' + \delta_2)$, $u_j^2 = U_j(v_1'' - \delta_1', v_2'')$, $u_j^3 = U_j(v_1'', v_2'' - \delta_2')$, $u_j^4 = U_j(v_1', v_2''')$, $u_i^3 = U_i(v_1'' - \delta_1', v_2''')$, and $u_i^4 = U_i(v_1''' - \delta_1, v_2'' - \delta_2')$. We have $u_i^2 > u_i^1$ by (4.15d), $u_j^2 > u_j^1$ by (4.15b), $u_i^4 > u_i^3$ by (4.15c), and $u_j^4 > u_j^3$ by (4.15e).

Let us consider an anonymous participation game with strictly negative impacts where N is the set of players and each player k uses the aggregator U_k: $\mathrm{A} = \{a, b, c, d, e\}$; $X_i = \big\{\{a, c\},\ \{b, d\}\big\}$; $X_j = \big\{\{b, c\},\ \{a, d\}\big\}$; $X_k = \big\{\{e\}\big\}$ for $k \in N \setminus \{i, j\}$; $\varphi_a(2) = v_1'$, $\varphi_a(1) = v_1''' - \delta_1$; $\varphi_b(2) = v_1'' - \delta_1'$, $\varphi_b(1) = v_1''$; $\varphi_c(2) = v_2'' - \delta_2'$, $\varphi_c(1) = v_2''$; $\varphi_d(2) = v_2' + \delta_2$, $\varphi_d(1) = v_2'''$. The 2×2 matrix of the essential part of the game looks as follows:

$$\begin{array}{ccc} & \text{bc} & \text{ad} \\ \text{ac} & (u_i^4, u_j^3) & (u_i^1, u_j^4) \\ \text{bd} & (u_i^3, u_j^2) & (u_i^2, u_j^1). \end{array}$$

There is no Nash equilibrium in the game. □

The lemma implies that the condition depicted in Figure 1(a) of Debreu (1960) holds; therefore, by the Blaschke–Thomsen results cited by Debreu, we have the additive representation for $m = 2$ as well. Fishburn (1970; Chapter 5) provides closed proofs for both Theorem 3 of Debreu (1960), which assumes separable projections to all subspaces, and this two-dimensional result. It is easily seen from the proofs that the same function $\nu(\cdot)$ can be used for all $i \in N$ and all $m \geq 1$ because of Lemma 4.3.1; $\nu(\cdot)$ is the same for all coordinates because of the assumed symmetry. Thus, we have (4.11a).

Now let us turn to the second statement of the theorem. If $m' = m$, then $\bar{u}^{mm} = 0$ obviously satisfies (4.11b). If $\lambda_i^m(m \cdot \nu(\mathbb{R})) \cap \lambda_i^{m'}(m' \cdot \nu(\mathbb{R})) = \emptyset$, then either $\lambda_i^{m'}(u') > \lambda_i^m(u)$ for all $u' \in m' \cdot \nu(\mathbb{R})$ and $u \in m \cdot \nu(\mathbb{R})$, or vice versa. In the first case, we define $\bar{u}^{mm'} = -\infty$; in the second, $\bar{u}^{mm'} = +\infty$. The condition (4.11b) obviously holds.

Let us fix $i \in N$ and $m' > m \geq 1$ such that $V = \lambda_i^m(m \cdot \nu(\mathbb{R})) \cap \lambda_i^{m'}(m' \cdot \nu(\mathbb{R})) \neq \emptyset$. Obviously, V is an open interval (bounded or not), hence $W' = (\lambda_i^{m'})^{-1}(V)$ and $W = (\lambda_i^m)^{-1}(V)$ are open intervals too.

Let $u^1 > u^2$ and $u^t \in V$ for $t = 1, 2$, i.e.,

$$u^t = \lambda_i^m(\sigma_t) = \lambda_i^{m'}(\sigma_t') \ \& \ \sigma_t \in m \cdot \nu(\mathbb{R}) \ \& \ \sigma_t' \in m' \cdot \nu(\mathbb{R}) \text{ for } t = 1, 2. \tag{4.16a}$$

There exist $v^t \in \mathbb{R}$ such that $\sigma_t = m \cdot \nu(v^t)$ $(t = 1, 2)$. If u^1 and u^2 are close enough to each other, then

$$\sigma'_1 - \sigma'_2 < (m' - m) \cdot [\nu(+\infty) - \nu(-\infty)] \tag{4.16b}$$

(if $\nu(\mathbb{R}) = \mathbb{R}$, the inequality holds for all u^1, u^2). It can be rewritten as $\sigma'_2 - (m' - m) \cdot \nu(-\infty) > \sigma'_1 - (m' - m) \cdot \nu(+\infty)$. Since the left-hand side is greater than $m \cdot \nu(-\infty)$ whereas the right-hand side is less than $m \cdot \nu(+\infty)$, there is $\sigma_0 \in m \cdot \nu(\mathbb{R})$ such that $\sigma'_t \in \sigma_0 + (m' - m) \cdot \nu(\mathbb{R})$ for both t. Therefore, there are $v^0, \bar{v}^1, \bar{v}^2 \in \mathbb{R}$ such that $\sigma'_t = m \cdot \nu(v^0) + (m' - m) \cdot \nu(\bar{v}^t)$ for both t.

Lemma 4.3.3. *If both conditions* (4.16) *hold, then* $\sigma_1 - \sigma_2 = \sigma'_1 - \sigma'_2$.

Proof. Let us suppose first that $\sigma_1 - \sigma_2 > \sigma'_1 - \sigma'_2$. We pick $\delta > 0$ such that

$$\sigma'_1 - \sigma'_2 < (m' - m) \cdot [\nu(\bar{v}^1 + \delta) - \nu(\bar{v}^2 - \delta)] < \sigma_1 - \sigma_2$$

(the first inequality holds automatically). Denoting $\sigma_j^2 = \sigma_2 + (m' - m) \cdot \nu(\bar{v}^1 + \delta)$ and $\sigma_j^1 = \sigma_1 + (m' - m) \cdot \nu(\bar{v}^2 - \delta)$, we see that $\sigma_j^1 > \sigma_j^2$; since both belong to $m' \cdot \nu(\mathbb{R})$, there is $\sigma_j^0 \in m' \cdot \nu(\mathbb{R})$ such that

$$\sigma_j^1 > \sigma_j^0 > \sigma_j^2; \tag{4.17a}$$

clearly, $\sigma_j^0 = m' \cdot \nu(\hat{v})$ for $\hat{v} \in \mathbb{R}$. We denote $u'' = \lambda_i^{m'}(m \cdot \nu(v^0) + (m' - m) \cdot \nu(\bar{v}^2 - \delta))$ and $u' = \lambda_i^{m'}(m \cdot \nu(v^0) + (m' - m) \cdot \nu(\bar{v}^1 + \delta))$; clearly,

$$u'' < u^2 < u^1 < u'. \tag{4.17b}$$

Let us pick $j \in N$, $j \neq i$, and consider an anonymous participation game with strictly negative impacts where N is the set of players and each player k uses the aggregator U_k: A $= \{a_1, \ldots, a_m,\ b_{m+1}, \ldots, b_{m'},\ c_1, \ldots, c_m,\ d_1, \ldots, d_{m'}, e\}$; $X_i = \big\{\{a_1, \ldots, a_m\}, \{c_1, \ldots, c_m,\ b_{m+1}, \ldots, b_{m'}\}\big\}$; $X_j = \big\{\{a_1, \ldots, a_m,\ b_{m+1}, \ldots, b_{m'}\},\ \{d_1, \ldots, d_{m'}\}\big\}$; $X_k = \big\{\{e\}\big\}$ for $k \in N \setminus \{i, j\}$; $\varphi_{a_s}(t) = v^t$ $(t = 1, 2;\ s = 1, \ldots, m)$; $\varphi_{b_s}(2) = \bar{v}^2 - \delta$, $\varphi_{b_s}(1) = \bar{v}^1 + \delta$ $(s = m + 1, \ldots, m')$; $\varphi_{c_s}(1) = v^0$ $(s = 1, \ldots, m)$; $\varphi_{d_s}(1) = \hat{v}$ $(s = 1, \ldots, m')$. The 2×2 matrix of the essential part of the game looks as follows:

$$\begin{array}{ccc} & \text{ab} & \text{d} \\ \text{a} & \big(u^2, \lambda_j^{m'}(\sigma_j^2)\big) & \big(u^1, \lambda_j^{m'}(\sigma_j^0)\big) \\ \text{bc} & \big(u'', \lambda_j^{m'}(\sigma_j^1)\big) & \big(u', \lambda_j^{m'}(\sigma_j^0)\big). \end{array}$$

The inequalities (4.17) imply that there is no Nash equilibrium in the game.

Now let $\sigma_1 - \sigma_2 < \sigma'_1 - \sigma'_2$. We pick $\delta > 0$ such that

$$\sigma_1 - \sigma_2 < (m' - m) \cdot [\nu(\bar{v}^1 - \delta) - \nu(\bar{v}^2 + \delta)] < \sigma'_1 - \sigma'_2$$

(the second inequality holds automatically). Denoting $\sigma_j^1 = \sigma_2 + (m' - m) \cdot \nu(\bar{v}^1 - \delta)$ and $\sigma_j^2 = \sigma_1 + (m' - m) \cdot \nu(\bar{v}^2 + \delta)$, we see that $\sigma_j^1 > \sigma_j^2$; since both belong to $m' \cdot \nu(\mathbb{R})$, there is $\sigma_j^0 \in m' \cdot \nu(\mathbb{R})$ such that

$$\sigma_j^1 > \sigma_j^0 > \sigma_j^2; \tag{4.18a}$$

clearly, $\sigma_j^0 = m' \cdot \nu(\hat{v})$ for $\hat{v} \in \mathbb{R}$. We denote $u'' = \lambda_i^{m'}(m \cdot \nu(v^0) + (m' - m) \cdot \nu(\bar{v}^2 + \delta))$ and $u' = \lambda_i^{m'}(m \cdot \nu(v^0) + (m' - m) \cdot \nu(\bar{v}^1 - \delta))$; clearly,

$$u^2 < u'' < u' < u^1. \tag{4.18b}$$

Now we pick $j \in N$, $j \neq i$, and consider an anonymous participation game with strictly negative impacts where N is the set of players and each player k uses the aggregator U_k: $\mathrm{A} = \{a_1, \ldots, a_m,\ b_{m+1}, \ldots, b_{m'},\ c_1, \ldots, c_m,\ d_1, \ldots, d_{m'}, e\}$; $X_i = \bigl\{\{a_1, \ldots, a_m\}, \{c_1, \ldots, c_m,\ b_{m+1}, \ldots, b_{m'}\}\bigr\}$; $X_j = \bigl\{\{a_1, \ldots, a_m,\ b_{m+1}, \ldots, b_{m'}\},\ \{d_1, \ldots, d_{m'}\}\bigr\}$; $X_k = \bigl\{\{e\}\bigr\}$ for $k \in N \setminus \{i, j\}$; $\varphi_{a_s}(t) = v^t$ ($t = 1, 2;\ s = 1, \ldots, m$); $\varphi_{b_s}(2) = \bar{v}^2 + \delta$, $\varphi_{b_s}(1) = \bar{v}^1 - \delta$ ($s = m+1, \ldots, m'$); $\varphi_{c_s}(1) = v^0$ ($s = 1, \ldots, m$); $\varphi_{d_s}(1) = \hat{v}$ ($s = 1, \ldots, m'$). The 2×2 matrix of the essential part of the game looks as follows:

$$\begin{array}{ccc} & \text{ab} & \text{d} \\ \text{a} & \bigl(u^2, \lambda_j^{m'}(\sigma_j^1)\bigr) & \bigl(u^1, \lambda_j^{m'}(\sigma_j^0)\bigr) \\ \text{bc} & \bigl(u'', \lambda_j^{m'}(\sigma_j^2)\bigr) & \bigl(u', \lambda_j^{m'}(\sigma_j^0)\bigr). \end{array}$$

The inequalities (4.18) imply that there is no Nash equilibrium in the game. □

Lemma 4.3.3 implies that the function $\Lambda(u) = (\lambda_i^{m'})^{-1}(u) - (\lambda_i^m)^{-1}(u)$ is locally constant on V; therefore, it is a constant on V. Let us denote it $\bar{u}^{mm'}$ and show that (4.11b) holds for all $u' \in m' \cdot \nu(\mathbb{R})$ and $u \in m \cdot \nu(\mathbb{R})$. Note that $W' = W + \bar{u}^{mm'}$ by the same Lemma 4.3.3.

Let $u \in W$, i.e., $\lambda_i^m(u) = \lambda_i^{m'}(u + \bar{u}^{mm'})$; then, for every $u' \in m' \cdot \nu(\mathbb{R})$, we have $\operatorname{sign}\bigl(\lambda_i^{m'}(u') - \lambda_i^m(u)\bigr) = \operatorname{sign}\bigl(\lambda_i^{m'}(u') - \lambda_i^{m'}(u + \bar{u}^{mm'})\bigr) = \operatorname{sign}(u' - u - \bar{u}^{mm'})$ since $\lambda_i^{m'}$ is strictly increasing.

Let $u \notin W$, say, $u \geq \sup W$, hence $\lambda_i^m(u) > \lambda_i^{m'}(u')$ for all $u' \in m' \cdot \nu(\mathbb{R})$, hence the left hand side of (4.11b) equals -1. Suppose there is $u' \in m' \cdot \nu(\mathbb{R})$ such that $u' \geq u + \bar{u}^{mm'}$, hence $u' \geq \sup W'$. We see that $(\sup W) \in m \cdot \nu(\mathbb{R})$ and $(\sup W') \in m' \cdot \nu(\mathbb{R})$; therefore, $\lambda_i^m(\sup W) = \lambda_i^{m'}(\sup W')$ by continuity, hence $(\sup W) \in W$, which is impossible for an open interval.

The case of $u \leq \inf W$ is treated dually. □

Proposition 4.4. *Let $\langle U_i \rangle_{i \in N}$ be a list of universal aggregators satisfying both conditions* (4.11) *from Theorem 5. Let Γ be an anonymous participation game where each player $i \in N$ uses the aggregator U_i. Then the individual improvement relation $\triangleright^{\mathrm{Ind}}$ in Γ, defined by* (2.1)*, is acyclic.*

Proof. Let $x^0, \ldots, x^{\bar{m}} = x^0$ be an improvement cycle in Γ. We define $N^* = \{i \in N \mid \exists k \in \{0, \ldots, \bar{m} - 1\} [x^{k+1} \triangleright^{\mathrm{Ind}}{}_i\ x^k]\}$ and $M_i = \{\# x_i^k\}_{k \in \{0, \ldots, \bar{m}\}}$ for each $i \in N^*$. Let us fix an $i \in N^*$.

We say that m and m' *overlap* if $\lambda_i^m(m \cdot \nu(\mathbb{R})) \cap \lambda_i^{m'}(m' \cdot \nu(\mathbb{R})) \neq \emptyset$. In this case $\bar{u}_i^{mm'}$ satisfying (4.11b) is unique; in particular, $\bar{u}_i^{mm} = 0$. An *overlap path* is a sequence $m_0, m_1, \ldots, m_k$ such that $m_h \in M_i$ and m_h and m_{h+1} overlap for each $h \in \{0, \ldots, k-1\}$. We call m and m' *contiguous* if there is an overlap path $m = m_0, m_1, \ldots, m_k = m'$.

Let m and m' not overlap, say, $\lambda_i^{m'}(u') > \lambda_i^m(u)$ for all $u' \in m' \cdot \nu(\mathbb{R})$ and $u \in m \cdot \nu(\mathbb{R})$; let them even not be contiguous. It is easy to see that $\lambda_i^{m'}(u') > \lambda_i^{m''}(u'')$ for every m'' contiguous with m, and all $u' \in m' \cdot \nu(\mathbb{R})$ and $u'' \in m'' \cdot \nu(\mathbb{R})$. Therefore, all $m \in M_i$ are contiguous.

Lemma 4.3.4. *Let $m_0, m_1, \ldots, m_k$ be an overlap path and $\bar{u}_i^{m_0 m_k} = \sum_{h=0}^{k-1} \bar{u}_i^{m_h m_{h+1}}$. Then $\bar{u}_i^{m_0 m_k}$ satisfies* (4.11b) *(with $m = m_0$ and $m' = m_k$).*

Proof. We argue by induction. For $k = 1$, the statement is tautological. Let it hold for overlap paths of the "length" $k \geq 1$ or less; we have to prove it for any path of the length $k+1$. For each $s = 0, 1, \ldots, k+1$, we denote $W^s = \lambda_i^{m_s}(m_s \cdot \nu(\mathbb{R})) \subseteq \mathbb{R}$.

Supposing first that $W^{k+1} \cap W^0 = \emptyset$, we may assume that $w'' > w$ for all $w'' \in W^{k+1}$ and $w \in W^0$ (the case of opposite inequalities is treated dually). Since $\bigcup_{s=1}^k W^s$ is an open interval which intersects with both W^{k+1} and W^0, there are s and $w' \in W^s$ such that $1 \leq s \leq k$ and $w'' > w' > w$ for all $w'' \in W^{k+1}$ and $w \in W^0$. Let $w' = \lambda_i^{m_s}(u')$. By the induction hypothesis, we have $u' > u + \sum_{h=0}^{s-1} \bar{u}^{m_h m_{h+1}}$ for all $u \in m_0 \cdot \nu(\mathbb{R})$, and $u'' > u' + \sum_{h=s}^{k} \bar{u}^{m_h m_{h+1}}$ for all $u'' \in m_{k+1} \cdot \nu(\mathbb{R})$; therefore, $u'' > u + \sum_{h=0}^{k} \bar{u}^{m_h m_{h+1}}$, i.e., (4.11b) holds.

Now let $W = W^{k+1} \cap W^0 \neq \emptyset$; then $\bar{u}^{m_0 m_{k+1}}$ satisfying (4.11b) is unique. Since $\bigcup_{s=1}^k W^s$ is an open interval which intersects with both W^{k+1} and W^0, there is s ($1 \leq s \leq k$) such that $W^s \cap W \neq \emptyset$; therefore, there are $u \in m_0 \cdot \nu(\mathbb{R})$, $u' \in m_s \cdot \nu(\mathbb{R})$, and $u'' \in m_{k+1} \cdot \nu(\mathbb{R})$ such that $\lambda_i^{m_0}(u) = \lambda_i^{m_s}(u') = \lambda_i^{m_{k+1}}(u'')$. By the induction hypothesis, we have $u' = u + \sum_{h=0}^{s-1} \bar{u}^{m_h m_{h+1}}$ and $u'' = u' + \sum_{h=s}^{k} \bar{u}^{m_h m_{h+1}}$, hence $u'' = u + \sum_{h=0}^{k} \bar{u}^{m_h m_{h+1}}$, hence $\bar{u}^{m_0 m_{k+1}} = \sum_{h=0}^{k} \bar{u}^{m_h m_{h+1}}$.

Thus, the induction step is completed, hence the lemma is proven. □

Lemma 4.3.4 immediately implies that, whenever $m_0, m_1, \ldots, m_k = m_0$ is an overlap cycle, we have $\sum_{h=0}^{k-1} \bar{u}_i^{m_h m_{h+1}} = 0$. Now, for each $i \in N^*$ and each $m, m' \in M_i$, we define $\bar{u}_i^{mm'} = \sum_{h=0}^{k-1} \bar{u}_i^{m_h m_{h+1}}$ for an overlap path $m = m_0, m_1, \ldots, m_k = m'$; the value does not depend on the choice of a particular path. Moreover, $\bar{u}_i^{mm''} = \bar{u}_i^{mm'} + \bar{u}_i^{m'm''}$ for all $m, m', m'' \in M_i$.

We define $\bar{v}_m^i = 0$ for $i \notin N^*$ and all m. When $i \in N^*$, we define $\bar{v}_m^i = 0$ for $m \notin M_i$ and for the minimal $m \in M_i$; for other $m \in M_i$, we define $\bar{v}_m^i = \bar{u}_i^{m'm}$, where m' is the preceding element of M_i. Let $\bar{m}_i = \max\{\#x_i |\ x_i \in X_i\}$. Denoting Γ^* the game with the same players, activities, and strategies, but with the aggregation functions $U_i^{(m)}(v_1, \ldots, v_m) = \sum_{s=1}^m \nu(v_s) + \sum_{s=m+1}^{\bar{m}_i} \bar{v}_s^i$, we see that $x^0, \ldots, x^{\bar{m}} = x^0$ is an improvement cycle in Γ^* as well; but this contradicts Proposition 4.1, actually, even Rosenthal's (1973) theorem. □

5 Games with Structured Utilities

5.1 Sufficiency of Quasiseparable Aggregation

We use a modification of the concepts introduced at the start of Subsection 4.1. For the reader's convenience, complete definitions are given.

A *universal separable order* is a sequence of strict orders $\succ^m$ on $\mathbb{R}^m$ ($m = 1, 2, \dots$) such that

1. $\succ^1$ is the standard order $>$ on $\mathbb{R}$;
2. every $\succ^m$ is ω-transitive on $\mathbb{R}^m$;
3. for every two one-to-one mappings σ, σ' of $\{1, \dots, m\}$ to itself,
$$\langle v_1, \dots, v_m \rangle \succ^m \langle v'_1, \dots, v'_m \rangle \iff \langle v_{\sigma(1)}, \dots, v_{\sigma(m)} \rangle \succ^m \langle v'_{\sigma'(1)}, \dots, v'_{\sigma'(m)} \rangle$$
(invariance to permutations);
4. for every $m' > m \geq 1$, every $\langle v_1, \dots, v_{m'} \rangle \in \mathbb{R}^{m'}$, and every $\langle v'_1, \dots, v'_m \rangle \in \mathbb{R}^m$,
$$\langle v_1, \dots, v_m, v_{m+1}, \dots, v_{m'} \rangle \succ^{m'} \langle v'_1, \dots, v'_m, v_{m+1}, \dots, v_{m'} \rangle \iff \langle v_1, \dots, v_m \rangle \succ^m \langle v'_1, \dots, v'_m \rangle$$
(separability).

A universal aggregator is *consistent* with a universal separable order if for every $m = 1, 2, \dots$, every $\langle v_1, \dots, v_m \rangle \in \mathbb{R}^m$, and every $\langle v'_1, \dots, v'_m \rangle \in \mathbb{R}^m$,
$$U^{(m)}(v_1, \dots, v_m) > U^{(m)}(v'_1, \dots, v'_m) \Rightarrow \langle v_1, \dots, v_m \rangle \succ^m \langle v'_1, \dots, v'_m \rangle. \tag{5.1}$$

A universal aggregator is *quasiseparable* if it is consistent with a universal separable order. A family of quasiseparable universal aggregators is called *consistent* if they are all consistent with the same universal separable order.

Proposition 5.1. *Let $\mathfrak{U}$ be a consistent family of quasiseparable universal aggregators and let Γ be a game with structured utilities where each player i uses an aggregator $U_i \in \mathfrak{U}$. Then the individual improvement relation $\triangleright^{\mathrm{Ind}}$ in Γ, defined by* (2.1)*, is Ω-acyclic.*

Proof. Each relation $\succ^m$ can be perceived as defined on the set of unordered corteges of the length m. With every $x \in X$, we associate an unordered cortege $\varkappa(x) = \langle \varphi_\alpha(x_{N(\alpha)}) \rangle_{\alpha \in \mathrm{A}}$. If we show that $y \triangleright^{\mathrm{Ind}} x$ implies $\varkappa(y) \succ^{\#\mathrm{A}} \varkappa(x)$, the Ω-acyclicity of $\triangleright^{\mathrm{Ind}}$ will follow immediately.

Let $y \triangleright^{\mathrm{Ind}}{}_i\, x$, i.e., $u_i(y) > u_i(x)$ and $y_{-i} = x_{-i}$. We denote $\varkappa_{-i} = \langle \varphi_\alpha(x_{N(\alpha)}) \rangle_{\alpha \notin \Upsilon_i}$, $\varkappa_i(x) = \langle \varphi_\alpha(x_{N(\alpha)}) \rangle_{\alpha \in \Upsilon_i}$, and $\varkappa_i(y) = \langle \varphi_\alpha(y_{N(\alpha)}) \rangle_{\alpha \in \Upsilon_i}$. Clearly, $\varkappa(x) = \langle \varkappa_{-i}, \varkappa_i(x) \rangle$ and $\varkappa(y) = \langle \varkappa_{-i}, \varkappa_i(y) \rangle$ whereas $u_i(x) = U^{(\#\Upsilon_i)}(\varkappa_i(x))$ and $u_i(y) = U^{(\#\Upsilon_i)}(\varkappa_i(y))$. Therefore, (5.1) implies $\varkappa_i(y) \succ^{\#\Upsilon_i} \varkappa_i(x)$, hence $\varkappa(y) \succ^{\#\mathrm{A}} \varkappa(x)$ by separability. $\square$

As in Subsection 4.1, the simplest and most important example of a universal separable order is given by additive aggregation (4.2). Now infinite games are allowed, so the leximin (leximax) ordering admits no additive representation; still, every game with structured utilities and the minimum (maximum) aggregation satisfies the conditions of Theorem 1 (Proposition 3.1).

Nonetheless, the scope of Proposition 5.1 is not exhausted by its additive version: let us consider, e.g., the aggregation rule $U(v_1, \dots, v_m) = v_1 \times \cdots \times v_m$ if $v_1, \dots, v_m \geq 0$ and $U(v_1, \dots, v_m) = \min\{v_1, \dots, v_m\}$ otherwise [cf. Proposition 3 of Kukushkin, 1994; the same aggregation rule was considered in Segal and Sobel, 2002, Eq. (19)]. The rule is consistent with the universal separable order which is multiplicative (i.e., additive in logarithms) for all positive values and the leximin otherwise.

As in Subsection 4.1, we may consider games where the players use the leximin (leximax) ordering to aggregate intermediate objectives. Again, Proposition 5.1, applied to the ordering itself, ensures the Ω-acyclicity of individual improvements. And again, there is no analogue of Theorem 1 for this aggregation.

Example 5.1. Let us consider a game with structured utilities and the leximin aggregation: $N = \{1, 2\}$, $\mathrm{A} = \{a_1, a_2, b\}$; $X_1 = X_2 = \{0, 1\}$; $\Upsilon_i = \{a_i, b\}$, $\varphi_{a_i}(0) = 0$, $\varphi_{a_i}(1) = 2$ ($i \in N$); $\varphi_b(0, 0) = 3$, $\varphi_b(0, 1) = \varphi_b(1, 0) = 1$, $\varphi_b(1, 1) = 0$. The 2×2 matrix of the game looks as follows:

$$\begin{array}{cc} (\langle 0,3\rangle, \langle 0,3\rangle) & (\langle 0,1\rangle, \langle 1,2\rangle) \\ (\langle 1,2\rangle, \langle 0,1\rangle) & (\langle 0,2\rangle, \langle 0,2\rangle). \end{array}$$

We have a prisoner's dilemma: the southeastern corner is a unique Nash equilibrium, which is Pareto dominated by the northwestern corner.

In accordance with Proposition 2.3, we could replace the assumption in Proposition 5.1 that Γ is a game with structured utilities with that of both strictly negative and strictly positive impacts. "Strictly" cannot be dropped.

Example 5.2. Let $N = \{1, 2\}$, $\mathrm{A} = \{a, b\}$, $X_1 = X_2 = \{1, 2\}$, $\mathrm{B}_1(1) = \{a\}$, $\mathrm{B}_1(2) = \mathrm{A}$, $\mathrm{B}_2(1) = \mathrm{B}_2(2) = \mathrm{A}$, $\varphi_a(N, 1, 1) = 3$, $\varphi_a(N, 1, 2) = 0$, $\varphi_a(N, 2, 1) = 4$, $\varphi_a(N, 2, 2) = -1$, $\varphi_b(N, 2, 1) = \varphi_b(\{2\}, 1) = -2$, $\varphi_b(N, 2, 2) = \varphi_b(\{2\}, 2) = 2$, and both players use additive aggregation.

The inequality (2.8) need only be checked for $\alpha = b$, $I = \{2\}$, $i = 1$, $x_i^\alpha = 2$, and $x_I^\alpha \in X_2$. In both cases, it holds as an equality, so the game exhibits both negative and positive impacts. The 2×2 matrix of the game looks as follows:

$$\begin{array}{cc} (3,1) & (0,2) \\ (2,2) & (1,1). \end{array}$$

There is no Nash equilibrium in the game.

5.2 Necessity of the Minimum (Maximum) Aggregation

Theorem 6. *Let $\mathfrak{U}$ be a set of universal aggregators such that every game with structured utilities where each player uses an aggregator from $\mathfrak{U}$ possesses a weakly Pareto optimal*

Nash equilibrium. Then either for every $U \in \mathfrak{U}$ and $m \geq 1$, there is a continuous and strictly increasing mapping $\lambda_m^U : \mathbb{R} \to \mathbb{R}$ such that

$$U^{(m)}(v_1, \ldots, v_m) = \lambda_m^U(\min\{v_1, \ldots, v_m\})$$

for all $v_1, \ldots, v_m \in \mathbb{R}$, or for every $U \in \mathfrak{U}$ and $m \geq 1$, there is a continuous and strictly increasing mapping $\lambda_m^U : \mathbb{R} \to \mathbb{R}$ such that

$$U^{(m)}(v_1, \ldots, v_m) = \lambda_m^U(\max\{v_1, \ldots, v_m\})$$

for all $v_1, \ldots, v_m \in \mathbb{R}$.

Remark. The comment to the formulation of Theorem 3 is appropriate here as well.

Proof. There is a considerable similarity with the proof of Theorem 3. First, we consider two-dimensional sections and show that each indifference curve must exhibit a similarity with either minimum or maximum. Then we show that every two-dimensional aggregation function must be either minimum or maximum (Lemma 5.2.5). Finally, if one such function is minimum, then all two-dimensional sections of all functions must be minimum too; and similarly for the maximum.

Lemma 5.2.1. *Let $m \geq 2$, $v_1 > v_2$, and*

$$U^{(m)}(v_2, v_2, v_3, \ldots, v_m) < U^{(m)}(v_1, v_2, v_3, \ldots, v_m); \tag{5.2}$$

then $U^{(m)}(v_1, \bar{v}_2, v_3, \ldots, v_m) = U^{(m)}(v_1, v_2, v_3, \ldots, v_m)$ for all $\bar{v}_2 \leq v_2$.

Proof. A nonstrict inequality immediately follows from the monotonicity of $U^{(m)}$. Let us suppose $U(v_1, \bar{v}_2, \ldots, v_m) = u' < u = U(v_1, v_2, \ldots, v_m)$ for some $\bar{v}_2 < v_2$; since (5.2) holds, we may assume, as in Lemma 4.2.1, that $u^- = U(v_2, v_2, v_3, \ldots, v_m) < u'$. By the continuity of U, there is $\bar{v}_1 \in]v_2, v_1[$ such that $u' < U(\bar{v}_1, v_2, v_3, \ldots, v_m) = u'' < u$. Thus,

$$u^- < u' < u'' < u. \tag{5.3}$$

Now let us consider a game with structured utilities where both players use the aggregator U: $N = \{1, 2\}$; there are $m + 1$ activities, $\mathrm{A} = \{a_1, a_2, b, \ c_3, \ldots, c_m\}$; $\Upsilon_i = \{a_i, b, \ c_3, \ldots, c_m\}$ $(i = 1, 2)$; $X_1 = X_2 = \{1, 2\}$; $\varphi_{a_i}(1) = v_2$, $\varphi_{a_i}(2) = v_1$; $\varphi_b(1, 1) = \bar{v}_1$, $\varphi_b(1, 2) = \varphi_b(2, 1) = v_2$, $\varphi_b(2, 2) = \bar{v}_2$; $\varphi_{c_s}(x_1, x_2) = v_s$ $(s = 3, \ldots, m)$. The 2×2 matrix of the game looks as follows:

$$\begin{matrix} (u'', u'') & (u^-, u) \\ (u, u^-) & (u', u'). \end{matrix}$$

Taking into account (5.3), we see that the southeastern corner ($x_1 = x_2 = 2$) is a unique Nash equilibrium, which is strongly Pareto dominated by the northwestern corner ($x_1 = x_2 = 1$). □

Lemma 5.2.2. *Let $m \geq 2$, $v_1 > v_2$, and*

$$U^{(m)}(v_1, v_2, v_3, \ldots, v_m) < U^{(m)}(v_1, v_1, v_3, \ldots, v_m); \tag{5.4}$$

then $U^{(m)}(\bar{v}_1, v_2, v_3, \ldots, v_m) = U^{(m)}(v_1, v_2, v_3, \ldots, v_m)$ for all $\bar{v}_1 \geq v_1$.

Proof. A nonstrict inequality immediately follows from the monotonicity of $U^{(m)}$. Let us suppose $U(\bar{v}_1, v_2, \ldots, v_m) = u'' > u = U(v_1, v_2, \ldots, v_m)$ for some $\bar{v}_1 > v_1$; we may assume, without restricting generality, that $u'' < u^+ = U(v_1, v_1, v_3, \ldots, v_m)$. Since $U(v_2, v_1, v_3, \ldots, v_m) = u$ by symmetry, (5.4) and the continuity of U imply the existence of $v_1' \in]v_2, v_1[$ such that $u < u' = U(v_1', v_1, v_3, \ldots, v_m) < u''$. Thus,

$$u < u' < u'' < u^+. \tag{5.5}$$

Now let us consider a game with structured utilities where both players use the aggregator U: $N = \{1, 2\}$; there are $m + 1$ activities, $\mathrm{A} = \{a_1, a_2, b, c_3, \ldots, c_m\}$; $\Upsilon_i = \{a_i, b, c_3, \ldots, c_m\}$ $(i = 1, 2)$; $X_1 = X_2 = \{1, 2\}$; $\varphi_{a_i}(1) = v_2$, $\varphi_{a_i}(2) = v_1$; $\varphi_b(1, 1) = \bar{v}_1$, $\varphi_b(1, 2) = \varphi_b(2, 1) = v_1$, $\varphi_b(2, 2) = v_1'$; $\varphi_{c_s}(x_1, x_2) = v_s$ $(s = 3, \ldots, m)$. The 2×2 matrix of the game looks as follows:

$$\begin{array}{cc} (u'', u'') & (u, u^+) \\ (u^+, u) & (u', u'). \end{array}$$

Taking into account (5.5), we see that the southeastern corner ($x_1 = x_2 = 2$) is a unique Nash equilibrium, which is strongly Pareto dominated by the northwestern corner ($x_1 = x_2 = 1$). $\square$

Lemma 5.2.3. *If $(v_1, v_2) \in \mathbb{R}^2_>$, $u = U(v_1, v_2, v_3, \ldots, v_m) > U(v_2, v_2, v_3, \ldots, v_m)$, $(v_1', v_2') \in \mathbb{R}^2_>$, and $u' = U(v_1', v_2', v_3, \ldots, v_m) < u$, then $U(v_1', v_1', v_3, \ldots, v_m) = u'$.*

Proof. By Lemma 5.2.1, $U(v_1, \bar{v}_2, v_3, \ldots, v_m) = u$ for all $\bar{v}_2 \leq v_2$, hence

$$U(\bar{v}_1, \bar{v}_2, v_3, \ldots, v_m) \geq u \quad \text{for all } \bar{v}_1 \geq v_1 \text{ and } \bar{v}_2 \in \mathbb{R}. \tag{5.6}$$

Since $u' < u$, $v_1' < v_1$. Supposing $U(v_1', v_1', v_3, \ldots, v_m) > u'$, we, by Lemma 5.2.2, obtain $U(\bar{v}_1, v_2', v_3, \ldots, v_m) = u'$ for all $\bar{v}_1 \geq v_1'$, in particular, $U(v_1, v_2', v_3, \ldots, v_m) = u' < u$, contradicting (5.6). $\square$

Lemma 5.2.4. *If $(v_1, v_2) \in \mathbb{R}^2_>$, $u = U(v_1, v_2, v_3, \ldots, v_m) < U(v_1, v_1, v_3, \ldots, v_m)$, $(v_1', v_2') \in \mathbb{R}^2_>$, and $u' = U(v_1', v_2', v_3, \ldots, v_m) > u$, then $U(v_2', v_2', v_3, \ldots, v_m) = u'$.*

Proof. By Lemma 5.2.2, $U(\bar{v}_1, v_2, v_3, \ldots, v_m) = u$ for all $\bar{v}_1 \geq v_1$, hence

$$U(\bar{v}_1, \bar{v}_2, v_3, \ldots, v_m) \leq u \quad \text{for all } \bar{v}_2 \leq v_2 \text{ and } \bar{v}_1 \in \mathbb{R}. \tag{5.7}$$

Since $u' > u$, $v_2' > v_2$. Supposing $U(v_2', v_2', v_3, \ldots, v_m) < u'$, we, by Lemma 5.2.1, obtain $U(v_1', \bar{v}_2, v_3, \ldots, v_m) = u'$ for all $\bar{v}_2 \leq v_2'$, in particular, $U(v_1', v_2, v_3, \ldots, v_m) = u' > u$, contradicting (5.7). $\square$

Lemma 5.2.5. *For every $U \in \mathfrak{U}$, there is $\lambda_2^U : \mathbb{R} \to \mathbb{R}$ such that either $U(v_1, v_2) = \lambda_2^U(\min\{v_1, v_2\})$ for all $v_1, v_2 \in \mathbb{R}$ or $U(v_1, v_2) = \lambda_2^U(\max\{v_1, v_2\})$ for all $v_1, v_2 \in \mathbb{R}$.*

Proof. We define $\lambda_2^U(u) = U^{(2)}(u, u)$. It follows immediately from (2.5) that $\lambda_2^U(\cdot)$ is strictly increasing. Let $(v_1, v_2) \in \mathbb{R}^2_>$; if $U(v_2, v_2) < U(v_1, v_2) < U(v_1, v_1)$, then, by Lemmas 5.2.1 and 5.2.2, $U(v_1, \bar{v}_2) = U(v_1, v_2) = U(\bar{v}_1, v_2)$ for all $\bar{v}_1 > v_1$ and $\bar{v}_2 < v_2$, but this contradicts (2.5). Therefore, either $U(v_1, v_2) = U(v_1, v_1) = \lambda_2^U(\max\{v_1, v_2\})$ or $U(v_1, v_2) = U(v_2, v_2) = \lambda_2^U(\min\{v_1, v_2\})$. Since either condition obviously defines a closed subset of $\mathbb{R}^2_>$, one of them must be empty. □

Lemma 5.2.6. *If there is $\bar{U} \in \mathfrak{U}$ such that $\bar{U}(v_1, v_2) = \lambda_2^{\bar{U}}(\min\{v_1, v_2\})$ for all $v_1, v_2 \in \mathbb{R}$, then $U(v_1, v_2, v_3, \dots, v_m) = U(v_2, v_2, v_3, \dots, v_m)$ for all $U \in \mathfrak{U}$, $m \geq 2$, $v_1 \geq v_2$, and $v_3, \dots, v_m \in \mathbb{R}$.*

Proof. Suppose the contrary: $v_1 > v_2$ and $U(v_1, v_2, v_3, \dots, v_m) > U(v_2, v_2, v_3, \dots, v_m)$. By the continuity of U, we can choose $v^0 < v^1 < \dots < v^4$ such that $v_2 < v^0$, $v_1 > v^4$, and $U(v^s, v_2, v_3, \dots, v_m) < U(v^{s+1}, v_2, v_3, \dots, v_m)$ for $s = 0, \dots, 3$, and $U(v^4, v_2, v_3, \dots, v_m) < U(v_1, v_2, v_3, \dots, v_m)$. Denoting $u^s = U(v^s, v_2, v_3, \dots, v_m)$ ($s = 0, \dots, 4$), we obtain $u^0 < u^1 < \dots < u^4$. By Lemma 5.2.3, $U(v^s, v^{s'}, v_3, \dots, v_m) = u^s$ whenever $s > s'$ ($s' = 0, \dots, 3$).

Now let us consider a two person game with structured utilities where player 1 uses the aggregator $\bar{U}$ and player 2 uses U: $N = \{1, 2\}$; there are $m + 1$ activities, $\mathrm{A} = \{a_1, a_2, b, c_3, \dots, c_m\}$; $\Upsilon_1 = \{a_1, b\}$; $\Upsilon_2 = \{a_2, b, c_3, \dots, c_m\}$; $X_1 = X_2 = \{1, 2\}$; $\varphi_{a_1}(1) = v^1$, $\varphi_{a_1}(2) = v^2$; $\varphi_{a_2}(1) = v^2$, $\varphi_{a_2}(2) = v^3$; $\varphi_b(1, 1) = v^4$, $\varphi_b(1, 2) = v^3$, $\varphi_b(2, 1) = v^2$, $\varphi_b(2, 2) = v^0$; $\varphi_{c_s}(x_2) = v_s$ ($s = 3, \dots, m$). The 2×2 matrix of the game looks as follows:

$$\begin{matrix} \big(\lambda_2^{\bar{U}}(v^1), u^4\big) & \big(\lambda_2^{\bar{U}}(v^1), u^3\big) \\ \big(\lambda_2^{\bar{U}}(v^2), u^2\big) & \big(\lambda_2^{\bar{U}}(v^0), u^3\big). \end{matrix}$$

Clearly, there is no Nash equilibrium in the game. □

Lemma 5.2.7. *If there is $\bar{U} \in \mathfrak{U}$ such that $\bar{U}(v_1, v_2) = \lambda_2^{\bar{U}}(\max\{v_1, v_2\})$ for all $v_1, v_2 \in \mathbb{R}$, then $U(v_1, v_2, v_3, \dots, v_m) = U(v_1, v_1, v_3, \dots, v_m)$ for all $U \in \mathfrak{U}$, $m \geq 2$, $v_1 \geq v_2$, and $v_3, \dots, v_m \in \mathbb{R}$.*

Proof. The proof is "dual" to that of Lemma 5.2.6. Suppose the contrary: $v_1 > v_2$ and $U(v_1, v_2, v_3, \dots, v_m) < U(v_1, v_1, v_3, \dots, v_m)$. By the continuity of U, we can choose $v^0 < v^1 < \dots < v^4$ such that $v_2 < v^0$, $v_1 > v^4$, and $U(v_1, v^s, v_3, \dots, v_m) < U(v_1, v^{s+1}, v_3, \dots, v_m)$ for $s = 0, \dots, 3$, and $U(v_1, v^0, v_3, \dots, v_m) > U(v_1, v_2, v_3, \dots, v_m)$. Denoting $u^s = U(v_1, v^s, v_3, \dots, v_m)$ ($s = 0, \dots, 4$), we obtain $u^0 < u^1 < \dots < u^4$. By Lemma 5.2.4, $U(v^s, v^{s'}, v_3, \dots, v_m) = u^s$ whenever $s < s'$ ($s = 0, \dots, 3$).

Now let us consider a two person game with structured utilities where player 1 uses the aggregator $\bar{U}$ and player 2 uses U: $N = \{1, 2\}$; there are $m + 1$ activities, $\mathrm{A} = \{a_1, a_2, b, c_3, \dots, c_m\}$; $\Upsilon_1 = \{a_1, b\}$; $\Upsilon_2 = \{a_2, b, c_3, \dots, c_m\}$; $X_1 = X_2 = \{1, 2\}$; $\varphi_{a_1}(1) = v^2$,

$\varphi_{a_1}(2) = v^3$; $\varphi_{a_2}(1) = v^1$, $\varphi_{a_2}(2) = v^2$; $\varphi_b(1,1) = v^4$, $\varphi_b(2,1) = v^3$, $\varphi_b(1,2) = v^2$, $\varphi_b(2,2) = v^0$; $\varphi_{c_s}(x_2) = v_s$ $(s = 3, \ldots, m)$. The 2×2 matrix of the game looks as follows:

$$\begin{matrix} \big(\lambda_2^{\bar{U}}(v^4), u^1\big) & \big(\lambda_2^{\bar{U}}(v^2), u^2\big) \\ \big(\lambda_2^{\bar{U}}(v^3), u^1\big) & \big(\lambda_2^{\bar{U}}(v^3), u^0\big). \end{matrix}$$

Clearly, there is no Nash equilibrium in the game. □

Let us address the statement of the theorem. For each $U \in \mathfrak{U}$, $m \geq 1$, and $u \in \mathbb{R}$, we define $\lambda_m(u) = U^{(m)}(u, \ldots, u)$; for $m = 1$, there is nothing to prove. By Lemma 5.2.5, either $U(v_1, v_2) = \lambda_2^U(\min\{v_1, v_2\})$ or $U(v_1, v_2) = \lambda_2^U(\max\{v_1, v_2\})$. In the first case, Lemma 5.2.6 implies that all two-dimensional sections of all aggregation functions are minimum-like; then the equality $U(v_1, v_2, \ldots, v_m) = \lambda_2^U(\min\{v_1, v_2, \ldots, v_m\})$ is derived exactly as in the proof of Theorem 3. In the second case, a dual reasoning establishes $U(v_1, v_2, \ldots, v_m) = \lambda_2^U(\max\{v_1, v_2, \ldots, v_m\})$ for all $U \in \mathfrak{U}$, $m \geq 1$, and $v_1, v_2, \ldots, v_m \in \mathbb{R}$. □

If a family of aggregators $\mathfrak{U}$ satisfies the statement of Theorem 6, then in every game with structured utilities where each player uses an aggregator from $\mathfrak{U}$, the strong coalition improvement relation, defined by (2.2), is Ω-acyclic. The fact immediately follows from Theorem 1 and Proposition 3.1.

5.3 Necessity of Additive Aggregation

Theorem 7. *Let N be a finite set with $\#N \geq 2$; let $\langle U_i \rangle_{i \in N}$ be a list of universal aggregators such that every function $U_i^{(m)}$ is symmetric, continuous, and strictly increasing in the sense of* (2.6)*. If every game with structured utilities where N is the set of players and each player i uses the aggregator U_i possesses a Nash equilibrium, then there is a continuous and strictly increasing mapping $\nu : \mathbb{R} \to \mathbb{R}$ and a continuous and strictly increasing mapping $\lambda_i^m : m \cdot \nu(\mathbb{R}) \to \mathbb{R}$ for every $i \in N$ and $m \geq 1$ such that*

$$U_i^{(m)}(v_1, \ldots, v_m) = \lambda_i^m\Big(\sum_{s=1}^{m} \nu(v_s)\Big) \tag{5.8}$$

for all $v_1, \ldots, v_m \in \mathbb{R}$.

Remark. The comment to the formulation of Theorem 3 is appropriate here as well.

Proof. The general scheme of the proof is the same as in Theorem 5 (and even simpler because we do not have to prove the second statement).

Lemma 5.3.1. *Let $i, j \in N$, $m, m' \geq 2$, $v_s \in \mathbb{R}$ for $s = 1, \ldots, m$, and $v'_s \in \mathbb{R}$ for $s = 1, \ldots, m'$; let*

$$U_i^{(m)}(v_1, v'_2, v_3, \ldots, v_m) = U_i^{(m)}(v'_1, v_2, v_3, \ldots, v_m). \tag{5.9a}$$

Then

$$U_j^{(m')}(v_1, v'_2, v'_3, \ldots, v'_{m'}) = U_j^{(m')}(v'_1, v_2, v'_3, \ldots, v'_{m'}). \tag{5.9b}$$

Proof. As in the proof of Lemma 4.3.1, we suppose first that $i \neq j$. The negation of (5.9b) can be written as $U_j^{(m')}(v_1, v_2', v_3', \dots, v_{m'}') > U_j^{(m')}(v_1', v_2, v_3', \dots, v_{m'}')$. Pick $\delta > 0$ such that $u_j^2 = U_j^{(m')}(v_1, v_2', v_3', \dots, v_{m'}') > U_j^{(m')}(v_1' + \delta, v_2, v_3', \dots, v_{m'}') = u_j^1$; by monotonicity from (5.9a), $u_i^1 = U_i^{(m)}(v_1, v_2', v_3, \dots, v_m) < U_i^{(m)}(v_1' + \delta, v_2, v_3, \dots, v_m) = u_i^2$.

Let us consider a game with structured utilities where N is the set of players and each player k uses the aggregator U_k: $\mathrm{A} = \{a, b, \; c_3, \dots, c_m, \; d_3, \dots, d_{m'}, e\}$; $\Upsilon_i = \{a, b, c_3, \dots, c_m\}$, $\Upsilon_j = \{a, b, d_3, \dots, d_{m'}\}$, $\Upsilon_k = \{e\}$ for $k \in N \setminus \{i, j\}$; $X_i = X_j = \{1, 2\}$, $X_k = \{1\}$ for $k \neq i, j$; $\varphi_a(x_i, x_j) = v_1' + \delta$ if $x_i = x_j$, $\varphi_a(x_i, x_j) = v_1$ if $x_i \neq x_j$; $\varphi_b(x_i, x_j) = v_2$ if $x_i = x_j$, $\varphi_b(x_i, x_j) = v_2'$ if $x_i \neq x_j$; $\varphi_{c_s}(x_i) = v_s$ $(s = 3, \dots, m)$; $\varphi_{d_s}(x_j) = v_s'$ $(s = 3, \dots, m')$. The 2×2 matrix of the essential part of the game looks as follows:

$$\begin{matrix} (u_i^2, u_j^1) & (u_i^1, u_j^2) \\ (u_i^1, u_j^2) & (u_i^2, u_j^1). \end{matrix}$$

Since $u_k^2 > u_k^1$ $(k = i, j)$, the game possesses no Nash equilibrium.

If $i = j$, we pick $k \neq i$ and obtain

$$U_k^{(m)}(v_1, v_2', v_3, \dots, v_m) = U_k^{(m)}(v_1', v_2, v_3, \dots, v_m)$$

first, and then (5.9b). □

Lemma 5.3.2. *Let $i \in N$ and let $v_s' < v_s'' < v_s''' \in \mathbb{R}$ for $s = 1, 2$; let*

$$U_i^{(2)}(v_1', v_2'') = U_i^{(2)}(v_1'', v_2') \tag{5.10a}$$

and

$$U_i^{(2)}(v_1', v_2''') = U_i^{(2)}(v_1'', v_2'') = U_i^{(2)}(v_1''', v_2'). \tag{5.10b}$$

Then

$$U_i^{(2)}(v_1'', v_2''') = U_i^{(2)}(v_1''', v_2''). \tag{5.11}$$

The statement immediately follows from Lemma 2 of Kukushkin (1994).

As in the proof of Theorem 5, the equality (5.8) now follows from the main theorem of Gorman (1968) for $m > 2$ or from the Blaschke–Thomsen results cited by Debreu (1960) for $m = 2$. □

A family of aggregators satisfying (5.8) is obviously consistent with the universal separable order defined by the sums $\sum_{s=1}^{m} \nu(v_s)$; therefore, the Ω-acyclicity of the individual improvement relation is ensured by Proposition 5.1.

6 Representation Theorems

Among games with the minimum aggregation and negative impacts, games with structured utilities form a representative subclass.

Proposition 6.1. *Let Γ be a game with the minimum aggregation and negative impacts; then Γ can be represented as a game with structured utilities (and also with the minimum aggregation).*

Proof. We define $\mathrm{A}^* = \{(\alpha, I) \in \mathrm{A} \times (2^N \setminus \{\emptyset\}) |\ N_-(\alpha) \subseteq I \subseteq N(\alpha)\}$, $\Upsilon_i^* = \{(\alpha, I) \in \mathrm{A}^* |\ i \in I\}$,

$$\varphi^*_{(\alpha,I)}(x_I) = \begin{cases} \varphi_\alpha(I, x_I), & I \subseteq N(\alpha, x), \\ +\infty, & \text{else}, \end{cases}$$

and $u_i^*(x) = \min_{(\alpha,I)\in\Upsilon_i} \varphi^*_{(\alpha,I)}(x_I)$.

Remark. The $+\infty$ in the definition of φ^* need not be understood literally: anything large enough would do.

We only have to show $u_i(x) = u_i^*(x)$ for every $i \in N$ and $x \in X$. Let $u_i(x) = \varphi_\alpha(N(\alpha, x),\ x_{N(\alpha,x)})$ with $i \in N(\alpha, x) = M$. We have $(\alpha, M) \in \Upsilon_i^*$ and $\varphi^*_{(\alpha,M)}(x_M) = \varphi_\alpha(M, x_M) = u_i(x)$; therefore, $u_i^*(x) \le u_i(x)$.

Now let $(\alpha, I) \in \Upsilon_i^*$ and $\varphi^*_{(\alpha,I)}(x) < +\infty$; then $i \in I \subseteq N(\alpha, x)$. If $I \subset N(\alpha, x)$, then $\varphi^*_{(\alpha,I)}(x) = \varphi_\alpha(I, x_I) \ge \varphi_\alpha(N(\alpha, x), x_{N(\alpha,x)})$ by (2.8). If $I = N(\alpha, x)$, then $\varphi^*_{(\alpha,I)}(x) = \varphi_\alpha(N(\alpha, x), x_{N(\alpha,x)})$. In either case, $\varphi^*_{(\alpha,I)}(x) \ge u_i(x)$ by (3.1); therefore, $u_i^*(x) \ge u_i(x)$. $\square$

An exact analogue of Proposition 6.1 for games with the maximum aggregation and positive impacts is obviously valid as well.

Propositions 4.1 and 5.1 show that there are two distinct classes of games with additive aggregation where the acyclicity of individual improvements is ensured: anonymous participation (i.e., congestion) games and games with structured utilities. However, if we understand "a game" as defined in Subsection 2.1, we shall see that the former class is a subclass of the latter (actually, the subclass of all finite games).

First of all, the proof of Proposition 5.1 and Theorem 3.2 of Monderer and Shapley (1996) show that every *finite* game from the latter class can be interpreted as a congestion game (up to monotonic transformations of utilities). Representation in the opposite direction is even simpler and can be done in two independent ways.

Proposition 6.2. *Let Γ be a strategic game with compact metric spaces as X_i, continuous (cardinal) utilities, and a continuous exact potential (as defined by Monderer and Shapley, 1996). Then Γ can be represented as a game with structured utilities and additive aggregation.*

Proof. By definition, there are continuous functions $P : X \to \mathbb{R}$ and $Q_{-i} : X_{-i} \to \mathbb{R}$ $(i \in N)$ such that $u_i(x) = P(x) + Q_{-i}(x_{-i})$ for all $i \in N$ and $x \in X$. We define $\mathrm{A} = N \cup \{N\}$, $\Upsilon_i = \mathrm{A} \setminus \{i\}$ (i.e., there are $n+1$ activities; each player is engaged in n of them; one activity is associated with all players; each of the other activities is associated with $n-1$ players), $\varphi_N(x) = P(x) + \sum_{j\in N} Q_{-j}(x_{-j})$, and $\varphi_i(x_{-i}) = -Q_{-i}(x_{-i})$. Denoting $u_i^*(x)$ the structured utilities, we have

$$u_i^*(x) = \sum_{\alpha\in\Upsilon_i} \varphi_\alpha(x_{N(\alpha)}) = \varphi_N(x) + \sum_{j\neq i} \varphi_j(x_{-j}) =$$
$$P(x) + \sum_{j\in N} Q_{-j}(x_{-j}) - \sum_{j\neq i} Q_{-j}(x_{-j}) = P(x) + Q_{-i}(x_{-i}) = u_i(x)$$

for all $i \in N$ and $x \in X$. □

Remark. Our topological assumptions were not used in the proof. Without them, however, the constructed game with structured utilities need not satisfy the basic assumptions of Subsection 2.3.

Proposition 6.3. *Let Γ be a congestion game; then it can also be represented as a game with structured utilities and additive aggregation.*

Proof. We define $\mathrm{A}^* = \{(\alpha, I) \in \mathrm{A} \times (2^N \setminus \{\emptyset\}) |\ N_-(\alpha) \subseteq I \subseteq N(\alpha)\}$, $\Upsilon_i^* = \{(\alpha, I) \in \mathrm{A}^* |\ i \in I\}$,

$$\varphi^*_{(\alpha,I)}(x_I) = \begin{cases} \Phi_\alpha(\#I), & I \subseteq N(\alpha, x), \\ 0, & \text{else}, \end{cases}$$

where $\Phi_\alpha(m) = \sum_{k=1}^m (-1)^{m-k}\binom{m-1}{k-1}\varphi_\alpha(k)$ (if $\varphi_\alpha(k)$ is not defined, we set $\varphi_\alpha(k) = 0$), and $u_i^*(x) = \sum_{(\alpha,I)\in\Upsilon_i} \varphi^*_{(\alpha,I)}(x_I)$.

Let us show that $u_i(x) = u_i^*(x)$ for every $i \in N$ and $x \in X$. By definition,

$$u_i^*(x) = \sum_{\alpha\in x_i} \sum_{I\subseteq N(\alpha,x),\, I\ni i} \Phi_\alpha(\#I).$$

Let $\alpha \in x_i$ and $m = \#N(\alpha, x) \geq 1$. We have

$$\sum_{I\subseteq N(\alpha,x),\, I\ni i} \Phi_\alpha(\#I) = \sum_{h=1}^m \binom{m-1}{h-1} \Phi_\alpha(h) =$$
$$\sum_{h=1}^m \sum_{k=1}^h (-1)^{h-k} \binom{m-1}{h-1}\binom{h-1}{k-1} \varphi_\alpha(k) = \sum_{k=1}^m \frac{(m-1)!}{(k-1)!}\varphi_\alpha(k) \cdot \sum_{h=k}^m \frac{(-1)^{h-k}}{(m-h)!(h-k)!} =$$
$$\varphi_\alpha(m) + \sum_{k=1}^{m-1} \frac{(m-1)!}{(m-k)!(k-1)!}\varphi_\alpha(k) \cdot \sum_{s=0}^{m-k} \frac{(-1)^s (m-k)!}{(m-k-s)!s!} =$$
$$\varphi_\alpha(m) + \sum_{k=1}^{m-1} \frac{(m-1)!}{(m-k)!(k-1)!}\varphi_\alpha(k) \cdot (1-1)^{m-k} = \varphi_\alpha(m).$$

Therefore, $u_i^*(x) = \sum_{\alpha\in x_i} \sum \varphi_\alpha(\#N(\alpha, x)) = u_i(x)$. □

Remark. A straightforward inductive reasoning shows that the potential for Γ^* defined in the proof of Proposition 5.1, $\sum_{(\alpha,I)\in\mathrm{A}^*} \varphi^*_{(\alpha,I)}(x_I)$, coincides with Rosenthal's (1973) potential for Γ, $\sum_{\alpha\in\mathrm{A}} \sum_{k=1}^{\#N(\alpha,x)} \varphi_\alpha(k)$, used in the proof of Proposition 4.1.

The application of Proposition 6.2 to a congestion game produces a game with structured utilities different from that constructed in Proposition 6.3. (A "game" here is interpreted as in Subsection 2.3).

7 Games with a Fixed Structure of Objectives

Let a set of players N and a set of activities A be fixed; a *structure of objectives* consists of sets $\Upsilon_i \subseteq \mathrm{A}$ for $i \in N$. If we assume that our players may, in principle, find themselves participating in any game with these sets Υ_i of relevant objectives, we might be interested in aggregation functions guaranteeing certain nice properties. The concept of a universal aggregator then becomes inadequate because we know beforehand that each player i will face $\#\Upsilon_i$ objectives; moreover, every objective has an identity of its own, so the symmetry assumption would look arbitrary.

7.1 Games with Public and Private Objectives

In a *game with public and private objectives* there are $n + 1$ activities: one (private) associated with each player and one (public) associated with all players. In other words, the strategy sets X_i may be arbitrary, while the utilities are of the form

$$u_i(x) = U_i(\varphi_N(x), \varphi_i(x_i)) \tag{7.1}$$

for all $i \in N$ and $x \in X$, where $\varphi_N : X \to \mathbb{R}$, $\varphi_i : X_i \to \mathbb{R}$, and $U_i : \mathbb{R}^2 \to \mathbb{R}$ are given functions.

We shall be interested in conditions on the aggregation functions U_i ensuring the Ω-acyclicity of weak coalition improvements regardless of all other characteristics. We assume that the first argument v_1 always corresponds to the public objective $\varphi_N(x)$, while v_2 to the private objective $\varphi_i(x_i)$.

A family $\mathfrak{U}$ of functions $U : \mathbb{R}^2 \to \mathbb{R}$ is called *strongly stable* if for any finite set N, compact metric spaces X_i and functions $U_i \in \mathfrak{U}$ for each $i \in N$, and continuous functions $\varphi_N : X \to \mathbb{R}$ and $\varphi_i : X_i \to \mathbb{R}$ ($i \in N$), the weak coalition improvement relation $\triangleright^{\mathrm{wCo}}$, defined by (2.3), in the game Γ defined by (7.1) is Ω-acyclic.

A family $\mathfrak{U}$ of functions $U : \mathbb{R}^2 \to \mathbb{R}$ is called *weakly stable* if for any finite set N, compact metric spaces X_i and functions $U_i \in \mathfrak{U}$ for each $i \in N$, and continuous functions $\varphi_N : X \to \mathbb{R}$ and $\varphi_i : X_i \to \mathbb{R}$ ($i \in N$), the game Γ defined by (7.1) possesses a Pareto efficient Nash equilibrium.

We use the notion of a continuous and strictly increasing one-variable function having infinite values. More precisely, any such function $\nu(\cdot)$ is defined on an open interval $]\underline{v}, \bar{v}[\subseteq \mathbb{R}$, is continuous and strictly increasing on it in the usual sense, $\nu(v) \to -\infty$ as $v \to \underline{v}$ if $\underline{v} > -\infty$, and $\nu(v) \to +\infty$ as $v \to \bar{v}$ if $\bar{v} < +\infty$. The expression $\nu(v)$ for $v < \underline{v}$ is understood as $-\infty$, for $v > \bar{v}$ as $+\infty$. Any such function with $\underline{v} < \bar{v}$ has the inverse, which is also continuous and increasing, takes the value $+\infty$ for the arguments greater than $\nu(+\infty)$ and $-\infty$ for those less than $\nu(-\infty)$.

Functions $U \in \mathfrak{U}$ are *all properly kinked* if $\mathfrak{U} = \mathfrak{U}^1 \cup \mathfrak{U}^2$, for each $U \in \mathfrak{U}^2$ there exists a continuous and strictly increasing function $\lambda^U : \mathbb{R} \to \mathbb{R}$ such that

$$U(v_1, v_2) = \lambda^U(v_2) \tag{7.2a}$$

for all $(v_1, v_2) \in \mathbb{R}^2$, and for each $U \in \mathfrak{U}^1$ there exist continuous and strictly increasing functions ν_1^U, ν_2^U, and $\nu_3^U : \mathbb{R} \to \mathbb{R}$ such that ν_1^U has a finite value, one function of each pair ν_1^U, ν_2^U may have $-\infty$ as a value, one function of each pair ν_1^U, ν_3^U may have $+\infty$ as a value,

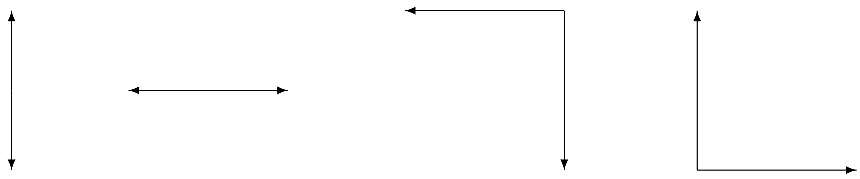
$$U(v_1, v_2) = \min\{\max\{\nu_1^U(v_1), \nu_2^U(v_2)\}, \nu_3^U(v_2)\} \tag{7.2b}$$

for all $(v_1, v_2) \in \mathbb{R}^2$, and

$$\sup_{U \in \mathfrak{U}^1} (\nu_1^U)^{-1} \circ \nu_2^U(+\infty) = v^- \leq v^+ = \inf_{U \in \mathfrak{U}^1} (\nu_1^U)^{-1} \circ \nu_3^U(-\infty). \tag{7.2c}$$

Remark. A function satisfying (7.2a) can be represented in the form (7.2b) with ν_1^U taking no finite value, in which case $(\nu_1^U)^{-1}$ in (7.2c) makes no sense.

There is a kind of geometric interpretation for the property. First, every function $U \in \mathfrak{U}$ may only have indifference curves of the following four types:



(the open ends should be extended to infinity). Second, the projection on the first (public) axis of any maximum-like corner is situated to the left of the projection on the same axis of any (for any $U \in \mathfrak{U}$) minimum-like corner.

Theorem 8. *For any set $\mathfrak{U}$ of continuous functions $U : \mathbb{R}^2 \to \mathbb{R}$ increasing in the sense of* (2.5)*, the following statements are equivalent:*

8.1. *$\mathfrak{U}$ is strongly stable;*

8.2. *$\mathfrak{U}$ is weakly stable;*

8.3. *all functions from $\mathfrak{U}$ are properly kinked.*

The implication [8.1] $\Rightarrow$ [8.2] is trivial.

Sufficiency proof. Given a set of functions $\mathfrak{U}$, all properly kinked, a finite set N, compact metric spaces X_i and functions $U_i \in \mathfrak{U}$ for each $i \in N$, and continuous functions $\varphi_N : X \to \mathbb{R}$ and $\varphi_i : X_i \to \mathbb{R}$ $(i \in N)$, we have to show the Ω-acyclicity of the weak coalition improvement relation $\triangleright^{\mathrm{wCo}}$ in the game defined by (7.1). As in the proof of Theorem 1, we show that $\triangleright^{\mathrm{wCo}}$ admits a potential.

We have $N = N^1 \cup N^2$, where $N^s = \{i \in N |\ U_i \in \mathfrak{U}^s\}$ $(s = 1, 2)$. First, we denote $\succ^{\min}$ the leximin ordering built out of modified utility functions $(\nu_1^{U_i})^{-1} \circ u_i$, $i \in N^1$, as in the proof of Theorem 1, and $\succ^{\max}$ the leximax ordering (again, for $(\nu_1^{U_i})^{-1} \circ u_i$ and $i \in N^1$) defined dually. Then we define

$$y \sim' x \iff \forall i \in N^2\, [u_i(y) = u_i(x)],$$
$$y \succ' x \iff \big[\forall i \in N^2\, [u_i(y) \ge u_i(x)] \;\&\; \exists i \in N^2\, [u_i(y) > u_i(x)]\big],$$

$$\begin{aligned} y \succ x \iff \Big(y \succ' x \;\text{ or }\; \big[y \sim' x \;\&\; \big(&[\varphi_N(y) > v^+ \;\&\; \varphi_N(x) < v^-] \;\text{ or}\\ &[y \succ^{\max} x \;\&\; \varphi_N(y) \le v^+ \;\&\; \varphi_N(x) \le v^+] \;\text{ or}\\ &[y \succ^{\min} x \;\&\; \varphi_N(y) \ge v^- \;\&\; \varphi_N(x) \ge v^-]\big)\big]\Big). \end{aligned} \tag{7.3}$$

Lemma 7.1.1. *The relation $\succ$ is irreflexive and ω-transitive.*

Proof. For every $U \in \mathfrak{U}^1$, we have, by (7.2b), that $(\nu_1^U)^{-1} \circ U(v_1, v_2) = v_1$ whenever $(\nu_1^U)^{-1} \circ \nu_2^U(+\infty) \le v_1 \le (\nu_1^U)^{-1} \circ \nu_3^U(-\infty)$; therefore,

$$v^- \le v_1 \le v^+ \Rightarrow U_i(v_1, v_2) = \nu_1^{U_i}(v_1) \tag{7.4}$$

for every $i \in N^1$.

Now each component in (7.3) is obviously transitive; (7.4) implies that there cannot emerge a contradiction when two components are applicable simultaneously. Irreflexivity is obvious; for ω-transitivity, one can refer to Proposition 3.7 from Kukushkin (2003). □

Lemma 7.1.2. *$y \triangleright^{\mathrm{wCo}} x \Rightarrow y \succ x$.*

Proof. Let $y \triangleright^{\mathrm{wCo}}{}_I\, x$; we have to show $y \succ x$. We denote $I^+ = \{i \in I |\ u_i(y) > u_i(x)\} \neq \emptyset$. If $I^+ \cap N^2 \neq \emptyset$, then $y \succ' x$, hence $y \succ x$. If $\varphi_N(y) > \varphi_N(x)$, then $u_i(y) \ge u_i(x)$ for all $i \in I$ with a strict inequality for some of them by our assumption, while $u_i(y) \ge u_i(x)$ for all $i \in N \setminus I$ because $\varphi_i(x_i) = \varphi_i(y_i)$; therefore, y Pareto dominates x, hence $y \succ x$.

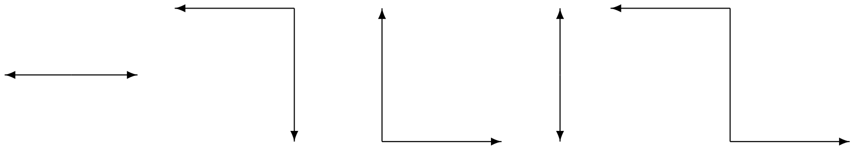
Let $\varphi_N(y) \le \varphi_N(x)$ and $I^+ \subseteq N^1$. (7.4) implies that either $\varphi_N(x) < v^-$ or $\varphi_N(y) > v^+$. In the latter case, we denote $w = \min_{i \in I^+} (\nu_1^{U_i})^{-1} \circ u_i(x)$. Whenever $u_i(y) > u_i(x)$, we have $i \in N^1$ and $(\nu_1^{U_i})^{-1} \circ u_i(x) = (\nu_1^{U_i})^{-1} \circ \nu_3^{U_i} \circ \varphi_i(x_i) < \min\{\varphi_N(y), (\nu_1^{U_i})^{-1} \circ \nu_3^{U_i} \circ \varphi_i(y_i)\} \le \varphi_N(y)$; therefore, $w < \varphi_N(y)$. Whenever $u_j(y) < u_j(x)$, we have $j \in N^1 \setminus I$ and $(\nu_1^{U_j})^{-1} \circ u_j(y) = \varphi_N(y) > w$, so we can argue quite similarly to the proof of Theorem 1. In the former case, we argue exactly dually. □

A reference to Theorem 2 of Kukushkin (2003) ([2.2] $\Rightarrow$ [2.1]) completes the proof. □

Necessity proof. Suppose that a set $U \in \mathfrak{U}$ of continuous increasing functions $U : \mathbb{R}^2 \to \mathbb{R}$ is weakly stable. We have to show that all the functions are properly kinked.

Lemma 7.1.3. *For every $U \in \mathfrak{U}$ and $u \in U(\mathbb{R}^2)$, the indifference curve $U^{-1}(u)$ follows one of the five patterns:*

[1] [2] [3] [4] [5]



where the open ends should be extended to infinity.

Proof. The lemma is proved in Kukushkin (1992, Lemma 2.2). That paper dealt with a particular case of our current problem: the set $\mathfrak{U}$ was a singleton. Since all players may use the same aggregator, the necessary conditions established there remain valid in our more general case. $\square$

A point (v_1^0, v_2^0) of the plane is called a northeastern corner for $U \in \mathfrak{U}$ if the inequalities $v_1^0 - \varepsilon < v_1 < v_1^0$, $v_2^0 - \varepsilon < v_2 < v_2^0$ (for some $\varepsilon > 0$) imply $U(v_1, v_2^0) = U(v_1^0, v_2^0) = U(v_1^0, v_2)$. It follows from Lemma 7.1.3 that in this case $U(v_1, v_2^0) = U(v_1^0, v_2^0)$ for any $v_1 < v_1^0$. A point (v_1^0, v_2^0) is called a southwestern corner for $U \in \mathfrak{U}$ if the same equalities follow from $v_1^0 < v_1 < v_1^0 + \varepsilon$, $v_2^0 < v_2 < v_2^0 + \varepsilon$. (Again, then the first equality holds for any $v_1 > v_1^0$).

Lemma 7.1.4. *If (v_1', v_2') is a northeastern corner for $U \in \mathfrak{U}$, then for any $v_1'' \leq v_1'$, $v_2'' \in \mathbb{R}$, and $U' \in \mathfrak{U}$, the point (v_1'', v_2'') is not a southwestern corner for U'.*

Proof. Suppose the contrary. It follows easily from Lemma 7.1.3 that in a rectangular neighbourhood of the point (v_1', v_2') every indifference curve of U (locally) follows the pattern [2], while in a rectangular neighbourhood of the point (v_1'', v_2'') every indifference curve of U' (locally) follows the pattern [3]. The expression "is close enough" below means "belongs to the appropriate neighbourhood." We may suppose $v_1'' < v_1'$ without any loss of generality, because otherwise we could slightly decrease v_1'' and find a new southwestern corner for the same U'.

We denote $v_1^{21} = v_1'$, $v_2^{21} = v_2''$, and $v_2^{12} = v_2'$. Then we pick $v_1^{11} > v_1'$ close enough to it, $v_1^{12} \in]v_1'', v_1'[$, $v_1^{22} < v_1''$ close enough to it, $v_2^{11} < v_2'$ close enough to it, and $v_2^{22} > v_2''$ close enough to it. Now we define $N = \{1, 2\}$, $X_1 = X_2 = \{1, 2\}$, $\varphi_1(1) = v_2^{11}$, $\varphi_1(2) = v_2^{12}$, $\varphi_2(1) = v_2^{21}$, $\varphi_2(2) = v_2^{22}$, $\varphi_N(1,1) = v_1^{11}$, $\varphi_N(1,2) = v_1^{12}$, $\varphi_N(2,1) = v_1^{21}$, $\varphi_N(2,2) = v_1^{22}$. Let player 1 use the aggregation function U and player 2, U'. It is easy to verify the following inequalities:

$$u_2(1,1) = U'(v_1^{11}, v_2^{21}) = U'(v_1'', v_2'') < U'(v_1^{12}, v_2^{22}) = u_2(1,2);$$

$$u_1(1,2) = U(v_1^{12}, v_2^{11}) < U(v_1', v_2') = U(v_1^{22}, v_2^{12}) = u_1(2,2);$$

$$u_2(2,2) = U'(v_1^{22}, v_2^{22}) < U'(v_1'', v_2'') = U'(v_1^{21}, v_2^{21}) = u_2(2,1);$$

$$u_1(2,1) = U(v_1^{21}, v_2^{12}) = U(v_1', v_2') < U(v_1^{11}, v_2^{11}) = u_1(1,1).$$

Thus, the game has no Nash equilibrium. $\square$

It follows immediately that no indifference curve of any function U follows the pattern [5]. In fact, Lemma 7.1.4 incorporates both Lemmas 2.3 and 2.4 of Kukushkin (1992).

The image $U(\mathbb{R}^2)$ must be an open interval. For $k = 1, \dots, 4$, denoting $R_k^U = \{u \in U(\mathbb{R}^2) | \ U^{-1}(u)$ follows the pattern $[k]\}$, we have

$$U(\mathbb{R}^2) = \bigcup_{k=1}^{4} R_k^U.$$

As in Kukushkin (1992), the following statements are evidently true.

$$\text{For each } U \in \mathfrak{U} \text{ at least one of the sets } R_1^U \text{ or } R_4^U \text{ is empty.} \tag{7.5a}$$

$$\text{The sets } R_1^U \text{ and } R_4^U \text{ are closed in } U(\mathbb{R}^2). \tag{7.5b}$$

$$\text{The sets } R_2^U \text{ and } R_3^U \text{ are open.} \tag{7.5c}$$

$$\text{If } u \in R_2^U,\ u' \in U(\mathbb{R}^2), \text{ and } u' < u, \text{ then } u' \in R_2^U. \tag{7.5d}$$

$$\text{If } u \in R_3^U,\ u' \in U(\mathbb{R}^2), \text{ and } u' > u, \text{ then } u' \in R_3^U. \tag{7.5e}$$

We define $\mathfrak{U}^2 = \{U \in \mathfrak{U} | \ R_2^U = R_3^U = R_4^U = \emptyset\}$ and $\mathfrak{U}^1 = \mathfrak{U} \setminus \mathfrak{U}^2$. Representation (7.2a) is obvious for every $U \in \mathfrak{U}^2$.

Let $U \in \mathfrak{U}^1$; for $u \in R_1^U$, we define $\mu^U(u)$ by the condition $U^{-1}(u) = \{(v_1, v_2) \in \mathbb{R}^2 | \ v_2 = \mu^U(u)\}$; for $u \in R_4^U$, $\mu_1^U(u)$ by $U^{-1}(u) = \{(v_1, v_2) \in \mathbb{R}^2 | \ v_1 = \mu_1^U(u)\}$. If $u \in R_2^U$, we denote $\big(\mu_1^U(u), \mu_2^U(u)\big)$ the coordinates of the unique northeastern corner for U on $U^{-1}(u)$; for $u \in R_3^U$, $\big(\mu_1^U(u), \mu_3^U(u)\big)$ are the coordinates of the unique southwestern corner for U on $U^{-1}(u)$. If $R_2^U = \emptyset$, we define $\nu_2^U \equiv -\infty$; if $R_3^U = \emptyset$, $\nu_3^U \equiv +\infty$. If $R_1^U = R_2^U = R_3^U = \emptyset$, we define $\nu_1^U = (\mu_1^U)^{-1}$; (7.2b) and (7.2c) are obvious.

If $R_2^U \neq \emptyset$, there are three alternatives: $R_2^U = U(\mathbb{R}^2)$ (i.e., $R_1^U = R_3^U = R_4^U = \emptyset$), or $\sup R_2^U \in R_1^U$, or $\sup R_2^U \in R_4^U$. In the first case, we define $\nu_1^U = (\mu_1^U)^{-1}$ and $\nu_2^U = (\mu_2^U)^{-1}$, having $\nu_3^U \equiv +\infty$; (7.2b) holds as $U(v_1, v_2) = \max\{\nu_1^U(v_1), \nu_2^U(v_2)\}$. In the second case, we extend μ_2^U to R_1^U by $\mu_2^U(u) = \mu^U(u)$, notice that the extended μ_2^U is still continuous and strictly increasing, and define ν_1^U and ν_2^U exactly as in the previous case; (7.2b) holds in the same form. In either case, there are northeastern corners for U with arbitrarily large v_1; by Lemma 7.1.4, $R_3^{U'} = \emptyset$ for all $U' \in \mathfrak{U}$, hence $\nu_3^{U'} \equiv +\infty$ and (7.2c) holds as $v^- = +\infty = v^+$. In the third case, we notice that $\mu_1^U(u) \to \mu_1^U(\sup R_2^U)$ as $u \to \sup R_2^U - 0$, hence μ_1^U is continuous and strictly increasing on $R_2^U \cup R_4^U$. If $R_3^U = \emptyset$, we define ν_1^U and ν_2^U exactly as above, and obtain (7.2b) and (7.2c) in the same form.

When $R_3^U \neq \emptyset$, we reproduce the previous paragraph with obvious modifications. (7.2b) in full generality emerges when all the three R_2^U, R_3^U, and R_4^U are not empty.

Finally, we notice that $(\nu_1^U)^{-1} \circ \nu_2(+\infty) = \mu_1^U \circ \nu_2(+\infty) = \sup\{v_1 \in \mathbb{R} | \ \exists v_2 \in \mathbb{R}\, [(v_1, v_2)$ is a northeastern corner for $U]\}$ whereas $(\nu_1^U)^{-1} \circ \nu_3(-\infty) = \mu_1^U \circ \nu_3(-\infty) = \inf\{v_1 \in \mathbb{R} | \ \exists v_2 \in \mathbb{R}\, [(v_1, v_2)$ is a southwestern corner for $U]\}$. Now (7.2c) follows directly from Lemma 7.1.4. □

7.2 On Necessity

In Theorem 8, as in Theorems 3, 4, and 6, our players picked their aggregators from a common *set*, in particular, two players could pick the same aggregator, and it was essential for the proofs. In Theorems 5 and 7, we considered a *list* of aggregators leaving no freedom of choice to the players. In the context of games with public and private objectives, introducing a list into the formulation of Theorem 8 would make it just wrong; in more general cases, no such counterexample is known.

Proposition 7.1. *Let $N = \{1, 2\}$, $U_1(v_1, v_2) = v_1$, and $U_2(v_1, v_2)$ be an arbitrary (continuous and increasing) function. Then the weak coalition improvement relation $\triangleright^{\mathrm{wCo}}$ in every game with public and private objectives where player 1 uses U_1 and player 2, U_2 is Ω-acyclic.*

Proof. We define $y \succ x \iff \big[u_2(y) > u_2(x)$ or $[u_2(y) = u_2(x)\ \&\ u_1(y) > u_1(x)]\big]$. The relation $\succ$ is obviously a potential for $\triangleright^{\mathrm{wCo}}$: increasing his utility, player 1 cannot decrease that of player 2. □

Remark. An analysis of the proofs of Theorems 5, 7, and 8 shows that they remain valid if we consider a list of aggregators assuming that each of them enters the list, at least, twice.

We define the Nash–Pareto improvement relation in a strategic game by

$$y \triangleright^{\mathrm{NP}} x \iff [y \triangleright^{\mathrm{wCo}}{}_N\, x \ \text{ or } \ y \triangleright^{\mathrm{Ind}} x], \tag{7.6}$$

where $\triangleright^{\mathrm{Ind}}$ is defined by (2.1) and $\triangleright^{\mathrm{wCo}}{}_N$, by (2.3a). A maximizer for $\triangleright^{\mathrm{NP}}$ is a Pareto optimal Nash equilibrium.

A structure of objectives $\langle \Upsilon_i \rangle_{i \in N}$ is *balanced* if $\#N(\alpha) = m$ is the same for all $\alpha \in \mathrm{A}$.

Proposition 7.2. *If Γ is a game with a balanced structure of objectives and with additive aggregation, then the Nash–Pareto improvement relation in Γ is Ω-acyclic.*

Proof. As in the proof of Proposition 5.1, $P(x) = \sum_{\alpha \in \mathrm{A}} \varphi_\alpha(x_{N(\alpha)})$ defines a potential for individual improvements. Since the structure of objectives is balanced, we have $P(x) = \frac{1}{m} \sum_{i \in N} u_i(x)$, hence P is also a potential for Pareto improvements. □

The proposition shows that the presence of a *universal* aggregator in the formulation of Theorem 6 was essential: when restricted to games with a balanced structure of objectives, the theorem becomes just wrong.

As an example of a balanced structure, we may consider a completely ordered N, $\mathrm{A} = \{(i, j) \in N \times N \mid i < j\}$, and $\Upsilon_i = \{(i, j)\}_{i<j} \cup \{(j, i)\}_{j<i}$. In other words, every pair of players is engaged in an activity of their own, and each player sums up intermediate utilities derived from all relevant activities.

Another example: the players are arranged on an oriented circle, and each player is engaged in two relationships, one with each neighbour. Again, additive aggregation of intermediate utilities ensures the existence of a Pareto optimal Nash equilibrium in discord with Theorem 6. Alternatively, every three consecutive players may be engaged in an activity (so each player has three relevant objectives).

As to the necessity of additive aggregation for the "persistent" existence of a Nash equilibrium (provided strong monotonicity is assumed), it was established for games with public and private objectives in Kukushkin (1994). It seems plausible that a modification of that proof [which, by the way, cannot be based on Debreu (1960) or even Gorman (1968)] could be developed if each player is only engaged in one or two activities. With a greater number of relevant objectives, nothing seems clear at the moment.

8 References

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