

On the Choice of Most-Preferred Alternatives*

Nikolai S. Kukushkin[†]

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Abstract

Maximal elements of a binary relation on compact subsets of a metric space define a choice function. Necessary and sufficient conditions are found for: (1) the choice function to have nonempty values and be path independent; (2) the choice function to have nonempty values provided the underlying relation is an interval order. For interval orders and semiorders, the same properties are characterized in terms of representations in a chain.

Keywords: Maximal element; Path independence; Interval order; Semiorder

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1 Introduction

The notion of a choice function plays a central role in the decision theory (Fishburn, 1973; Sen, 1984; Aizerman and Aleskerov, 1995), the most important being the case of a choice function defined by a binary relation. Traditionally, the attention was focused on choice functions on finite sets; connections between properties of such a choice function and properties of the underlying binary relation have been studied in detail. When the number of conceivable alternatives is large enough, a model with an infinite set of alternatives becomes appealing, but then many familiar results become irrelevant and intuitions developed over the finite case may prove misleading.

Let a binary relation be given on a metric space. There is a considerable literature studying conditions for the existence of maximal elements on compact sets (Gillies, 1959; Smith, 1974; Bergstrom, 1975; Mukherji, 1977; Walker, 1977; Kiruta et al., 1980; Danilov and Sotskov, 1985; Campbell and Walker, 1990). However, Kukushkin (2006) showed that no “simple” condition could characterize binary relations possessing undominated points in every compact subset.

Here we bypass the impossibility theorem by assuming (or demanding) a certain degree of rationality behind the binary relation (or the choice function it defines). A similar path was

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[†]Russian Academy of Sciences, Dorodnicyn Computing Center, 40, Vavilova, Moscow 119333 Russia
Tel.: +7-495-420-5707; fax: +7-495-135-6159. E-mail: ququns@pochta.ru

taken before by Smith (1974), who found a condition necessary and sufficient for an *ordering* (weak order) to admit a maximum on every compact subset of its domain. Theorem 1 below gives a condition for the choice function to be nonempty-valued and path independent on all compact subsets. Theorem 2, for the choice function to be nonempty-valued on all compact subsets provided the underlying relation is an interval order. Both conditions are equivalent (and equivalent to Smith's condition) for semiorders (Theorem 3).

The “internal” characterization results are supplemented with “external” ones. As is well known, interval orders and semiorders can be represented with the help of mappings to a chain (Fishburn, 1985). Most popular are numeric representations (which need not exist generally), but, e.g., lexicographic scales may be preferable in certain cases. Theorems 4–6 characterize representations of interval orders ensuring the non-emptiness or path independence of choice; after minimal modifications, the theorems remain valid for semiorders.

The next section contains basic definitions. Section 3 establishes equivalence between “ ω -transitivity” and path independence; Section 4, between “ ω -acyclicity” and the existence of maximal elements for interval orders. Sections 5 and 6 contain characterization results in terms of representations in a chain for interval orders and semiorders, respectively.

2 Basic Notions

A *binary relation* on a set A is a Boolean function on $A \times A$; as usual, we write $y \succ x$ when the relation $\succ$ is true on a pair (y, x) and $y \not\succ x$ when it is false. An *interval order* is an irreflexive relation $\succ$ such that

$$[y \succ x \ \& \ a \succ b] \Rightarrow [y \succ b \ \text{or} \ a \succ x]; \quad (1)$$

every interval order is transitive. An interval order is called a *semiorder* if

$$z \succ y \succ x \Rightarrow \forall a \in A [z \succ a \ \text{or} \ a \succ x]. \quad (2)$$

A relation $\succ$ is called a *strict ordering* if it is irreflexive, transitive and *negatively transitive*, i.e., $z \not\succ y \ \& \ y \not\succ x \Rightarrow z \not\succ x$. It is easy to see that every strict ordering is a semiorder.

Typically, we consider binary relations on a metric space A (a first countable Hausdorff topological space would do as well: what we actually need is that the topology on A be adequately described by convergent sequences). We denote $\mathfrak{B}$ the lattice of all subsets of A and $\mathfrak{C} \subset \mathfrak{B}$ the set of all nonempty compact subsets of A (we never consider different sets A simultaneously).

Let $X \in \mathfrak{B}$; a point $x \in X$ is a *maximizer* for $\succ$ on X if $y \not\succ x$ for any $y \in X$. The set of all maximizers for $\succ$ on X is denoted $M_\succ(X)$; we omit the subscript when the relation is clear from the context. A relation $\succ$ has the *NM-property* on $X \subseteq A$ if for every $x \in X \setminus M_\succ(X)$ there is $y \in M_\succ(X)$ such that $y \succ x$. The property means that $M_\succ(X)$ is a von Neumann–Morgenstern solution on X .

$M_\succ(\cdot)$ defines a *choice function* on $\mathfrak{B}$; we are mostly interested in properties of this function on $\mathfrak{C}$. First, whether all its values $M_\succ(X)$ are not empty (i.e., a choice can be made); second, whether the choice function is *path independent* on $\mathfrak{B}$ (or $\mathfrak{C}$):

$$M(X \cup Y) = M(M(X) \cup Y) \quad (3)$$

for all $X, Y \in \mathfrak{B}$ (or whenever $X \cup Y \in \mathfrak{C}$).

Remark. This equality is usually perceived as a minimal rationality requirement. Plott's (1973) original definition was a bit different from (3), but both are equivalent.

Given a binary relation $\succ$, an *improvement path* is a (finite or infinite) sequence $\langle x^k \rangle_{k=0,1,\dots}$ such that $x^{k+1} \succ x^k$ whenever both sides are defined. A relation $\succ$ is *acyclic* if it admits no *finite improvement cycle*, i.e., no improvement path such that $x^m = x^0$ for an $m > 0$. Clearly, a relation is acyclic if and only if its transitive closure is irreflexive. A relation is *strongly acyclic* if it admits no infinite improvement path.

These two statements are well known (and easy to check anyway): a relation $\succ$ is acyclic if and only if $M_\succ(X) \neq \emptyset$ for every finite $X \in \mathfrak{B} \setminus \{\emptyset\}$; a relation $\succ$ is strongly acyclic if and only if $M_\succ(X) \neq \emptyset$ for every $X \in \mathfrak{B} \setminus \{\emptyset\}$.

A binary relation $\succ$ on a metric space is called ω -*transitive* if it is transitive and, whenever $\langle x^k \rangle_{k=0,1,\dots}$ is an infinite improvement path and $x^k \rightarrow x^\omega$, there holds $x^\omega \succ x^0$. It is worth noting that $x^\omega \succ x^k$ is valid for all $k = 0, 1, \dots$ in this situation, once $\succ$ is ω -transitive.

The notion seems to have been first considered by Gillies (1959), who proved its sufficiency for the existence of maximizers on compact sets. Smith (1974) gave it the name of “ σ -transitivity” and proved that it is *necessary* and sufficient for the existence of maximizers on all compact subsets provided $\succ$ is a strict ordering. However, the prefix “ σ ” traditionally refers to the cardinal concept of a countable set whereas what matters here is the order type of the chain of natural numbers, usually referred to as ω .

A binary relation $\succ$ is called ω -*acyclic* if it is acyclic and, whenever $\langle x^k \rangle_{k=0,1,\dots}$ is an infinite improvement path and $x^k \rightarrow x^\omega$, there holds $x^\omega \neq x^0$. It is worth noting that $x^k \succ x^\omega$ is impossible in this situation for any k once $\succ$ is ω -acyclic. The property was considered by Mukherji (1977) as “Condition (A5).”

Let L be a chain and $C \subseteq L$; an *upper bound* for C is $u \in L$ such that $u \geq v$ for every $v \in C$. A *least upper bound* for C ($\sup C$) is an upper bound u such that $u \leq v$ for every upper bound v for C . If $\sup C$ exists at all, it is unique. Note that $\sup \emptyset$ is, by definition, the least point of L . It is well known (Birkhoff, 1967) that a chain contains a least upper bound for every subset if and only if it is compact in its intrinsic topology; henceforth, we call such chains just *compact*.

Let A be a metric space and B be a partially ordered set. A mapping $\varphi : A \rightarrow B$ is called *upper ω -semicontinuous* if $\varphi(x^\omega) > \varphi(x^0)$ whenever $x^k \rightarrow x^\omega$ and $\varphi(x^{k+1}) > \varphi(x^k)$ for all $k = 0, 1, \dots$

3 Transitivity

Theorem 1. *Let $\succ$ be a binary relation on a metric space. Then the following statements are equivalent.*

- 1.1. $\succ$ is irreflexive and ω -transitive.
- 1.2. $\succ$ has the NM property on every $X \in \mathfrak{C}$.
- 1.3. The choice function $M_\succ$ is nonempty-valued and path independent on $\mathfrak{C}$.

Proof. Let us prove the implication [1.1] $\Rightarrow$ [1.2] first. Let $X \in \mathfrak{C}$; for each $x \in X$, we denote $G(x) = \{y \in X \mid y \succ x\}$. Once x^* is not a maximizer, we have $G(x^*) \neq \emptyset$. If we show that $M_{\succ}(G(x^*)) \neq \emptyset$, then we are obviously home.

To apply Zorn's Lemma (see, e.g., Kuratowski, 1966, p. 27), we have to consider an arbitrary chain $C \subseteq G(x^*)$ and show the existence of $y \in X$ such that $y \succ x$ or $y = x$ for each $x \in C$ (hence $y \in G(x^*)$). For every $x \in C$, we denote $G^*(x) = G(x) \cap C$, $F(x)$ the closure of $G^*(x)$ in X , and $F = \bigcap_{x \in C} F(x) \subseteq X$. If there happens to be $y \in C$ such that $G^*(y) = \emptyset$, then, obviously, $y \succ x$ for every $x \in C \setminus \{y\}$, hence y is an upper bound needed. Therefore, we may assume that $G^*(x) \neq \emptyset$ for every $x \in C$.

Since C is a chain, all the sets $G^*(x)$ ($x \in C$), hence $F(x)$ too, contain each other; therefore, every intersection of a finite number of $F(x)$ is not empty. Since X is compact, $F \neq \emptyset$. We pick $y \in F$; let us show that $y \succ x$ for every $x \in C$.

Supposing the contrary, we can pick $x^0 \in C$ for which $y \not\succ x^0$. Then we recursively define a sequence $x^k \in G(x^0)$ so that $x^{k+1} \succ x^k$ ($k = 0, 1, \dots$) and $x^k \rightarrow y$; then the ω -transitivity of $\succ$ implies $y \succ x^0$: a contradiction. Having $x^k \in G^*(x^0)$ already defined, we notice that $y \in F(x^k) \setminus G^*(x^k)$; therefore, we can pick $x^{k+1} \in G^*(x^k) \subseteq G^*(x^0)$ such that $0 < \rho(y, x^{k+1}) < \rho(y, x^k)/2$. On each step, we have $x^{k+1} \succ x^k$; the condition on distances ensures $x^k \rightarrow y$.

Now let us turn to [1.2] $\Rightarrow$ [1.3]. First, we notice that $M(X) \neq \emptyset$ for every $X \in \mathfrak{C}$ by the definition of the NM property. As to path independence, the inclusion $M(X \cup Y) \subseteq M(M(X) \cup Y)$ holds for maximizers of every binary relation and for all sets X and Y . Let $x \in (X \cup Y) \setminus M(X \cup Y)$ and $X \cup Y \in \mathfrak{C}$; by [1.2], there is $z \in M(X \cup Y) \subseteq M(M(X) \cup Y)$ such that $z \succ x$. Therefore, even if $x \in M(X) \cup Y$, $x \notin M(M(X) \cup Y)$.

Finally, let us prove [1.3] $\Rightarrow$ [1.1]. If $x \succ x$, then $M(\{x\}) = \emptyset$. If $z \succ y \succ x$, then $\{z\} = M(\{x, y, z\}) = M(\{y, z\} \cup \{x\}) = M(\{x, z\})$, hence $z \succ x$. Let $x^k \rightarrow x^\omega$ and $x^{k+1} \succ x^k$ for each k . Denoting $X = \{x^\omega\} \cup \{x^k\}_{k=0,1,\dots}$ and $X' = X \setminus \{x^0\}$, we have $x^k \notin M(X)$ for each k , hence $M(X) = \{x^\omega\}$; similarly, $M(X') = \{x^\omega\}$. Now $\{x^\omega\} = M(X' \cup \{x^0\}) = M(M(X') \cup \{x^0\}) = M(\{x^\omega, x^0\})$, hence $x^\omega \succ x^0$. $\square$

Theorem 1 gives us a sufficient condition for the existence of maximizers. A *potential* for $\succ$ is an irreflexive and ω -transitive relation $\succsim$ finer than $\succ$, i.e., satisfying $y \succ x \Rightarrow y \succsim x$ for all $y, x \in A$. The notion was first introduced in Kukushkin (1999).

Corollary. *If $\succ$ admits a potential, then $M_{\succ}(X) \neq \emptyset$ for each $X \in \mathfrak{C}$.*

Proof. Obviously, $M_{\succsim}(X) \subseteq M_{\succ}(X)$ for any potential $\succsim$ for $\succ$. $\square$

The Corollary often helps to establish the existence of maximizers.

The following statement is proven with arguments similar to the proof of Theorem 1 (and similar to the standard proof of the finite analogue).

Proposition 3.1. *Let $\succ$ be a binary relation on a metric space. Then the following statements are equivalent.*

3.1.1. *$\succ$ is transitive and strongly acyclic.*

3.1.2. *$\succ$ has the NM property on every $X \in \mathfrak{B}$.*

3.1.3. *The choice function $M_{\succ}$ is nonempty-valued and path independent on $\mathfrak{B}$.*

4 Acyclicity

Proposition 4.1. *If $M_{\succ}(X) \neq \emptyset$ for each $X \in \mathfrak{C}$, then $\succ$ is ω -acyclic on A .*

Proof. If $\langle x^k \rangle_{k \in \mathbb{N}}$ is an improvement path converging to x^0 , then it is a compact subset without maximizers itself. (This argument was present in the proof of Corollary 3 from Mukherji, 1977, although the formulation was different.) $\square$

The condition is by no means sufficient.

Example 4.1. Let A be a circle parameterized with real numbers from $[-\pi, \pi[$ (with “ $-\pi = \pi$ ”). We define $\succ$ by

$$y \succ x \iff [\pi > y > x \geq 0 \text{ or } 0 > y > x \geq -\pi].$$

It is easily checked that $\succ$ is transitive and ω -acyclic, but not ω -transitive. A itself is compact, but there is no maximizer on A .

Actually, the relation in Example 4.1 admits a cycle a bit different from those prohibited by the definition of ω -acyclicity. Let $x^k = (1 - 1/(k+1))\pi$ and $y^k = -1/(k+1)\pi$ ($k \in \mathbb{N}$); clearly, $x^{k+1} \succ x^k$ and $y^{k+1} \succ y^k$ for all k , while $x^k \rightarrow y^0$ and $y^k \rightarrow x^0$. As in Proposition 4.1, the absence of such cycles is also necessary for the nonemptiness of $M_{\succ}(X)$ for all $X \in \mathfrak{C}$. Similarly, one can define transfinite cycles of a greater length and again find that they must be prohibited too. However, a sufficient condition will never be obtained on this way.

Example 4.2. Let us consider a circle represented as the set of complex numbers with $|z| = 1$; formally, $A = \{e^{it} \mid t \in \mathbb{R}\}$. We define a binary relation $y \succ x \iff y = e^i \cdot x$. The relation is acyclic because 1 is incommensurable with 2π . It is ω -acyclic because no infinite improvement path, i.e., a sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ such that $x^k = e^{ki} \cdot x^0$, can converge. Cycles similar to that from Example 4.1, or of a greater length, are impossible for the same reason. On the other hand, A itself is compact, but $M_{\succ}(A) = \emptyset$.

Here we have cycles in a different sense: every infinite improvement path is dense in A (the Jacobi theorem, see, e.g., Billingsley, 1965), hence its origin is among its limit points. The absence of *such* cycles, however, is not necessary for $M_{\succ}(X) \neq \emptyset$ for all $X \in \mathfrak{C}$.

With any binary relation $\succ$ on A , we associate these two relations:

$$y \succsim x \iff \exists \langle x^k \rangle_{k \in \mathbb{N}} [x^0 = x \text{ \& } \forall k \in \mathbb{N} [x^{k+1} \succ x^k] \text{ \& } x^k \rightarrow y]; \quad (4a)$$

$$y \succcurlyeq x \iff [y \succ x \text{ or } y \succsim x]. \quad (4b)$$

Proposition 4.2. *If $\succ$ is an ω -acyclic interval order, then $\succcurlyeq$ defined by (4) is irreflexive and ω -transitive.*

Proof. Irreflexivity of $\succcurlyeq$ immediately follows from the ω -acyclicity of $\succ$.

Lemma 4.2.1. *If $y \succsim x$ and $z \succ y$, then $z \succ x$.*

Proof. Let $\langle x^k \rangle_{k \in \mathbb{N}}$ be an appropriate improvement path. From $z \succ y$ and $x^1 \succ x$, we derive that either $z \succ x$ or $x^1 \succ y$. The latter relation would contradict the ω -acyclicity of $\succ$ (consider the sequence $y, x^1, x^2, \dots$). $\square$

Let $z \succcurlyeq y \succcurlyeq x$. If $y \succ x$, then obviously $z \succcurlyeq x$; let $y \asymp x$. If $z \succ y$, then Lemma 4.2.1 applies immediately. Let $z \asymp y$ and $\langle y^k \rangle_{k \in \mathbb{N}}$ be an appropriate sequence. Lemma 4.2.1 implies that $y^1 \succ x$; therefore, $z \asymp x$. Thus, $\succcurlyeq$ is transitive.

Let $x^k \rightarrow x^\omega$ and $x^{k+1} \succcurlyeq x^k$ for all k . If $x^{k+1} \succ x^k$ for all k except for a finite number, a straightforward backward induction based on Lemma 4.2.1 shows that $x^\omega \asymp x^0$. Otherwise, we may assume that $x^{k+1} \asymp x^k$ for all k . Let $\langle x_k^h \rangle_{k,h \in \mathbb{N}}$ be such that $x_k^0 = x^k$, $\langle x_k^h \rangle_{h \in \mathbb{N}}$ is an improvement path, and $x_k^h \xrightarrow{h} x^{k+1}$ for each $k \in \mathbb{N}$. Lemma 4.2.1 implies that $x_{k'}^{h'} \succ x_k^h$ whenever $k' > k$ and $h' > 0$. We denote $y^0 = x^0$, and, for each $k > 0$, pick $h(k) > 0$ such that $\rho(x_k^{h(k)}, x^{k+1}) < 1/k$ and denote $y^k = x_k^{h(k)}$. Now we have $y^k \rightarrow x^\omega$ and $y^{k+1} \succ y^k$ for all k ; therefore, $x^\omega \asymp x^0$ and $\succcurlyeq$ is ω -transitive. $\square$

Theorem 2. *Let $\succ$ be an interval order on a metric space A . Then $M_\succ(X) \neq \emptyset$ for every $X \in \mathfrak{C}$ if and only if $\succ$ is ω -acyclic.*

Proof. The necessity immediately follows from Proposition 4.1; the sufficiency, from Proposition 4.2 and Corollary to Theorem 1. $\square$

Campbell and Walker (1990) called a relation $\succ$ “weak lower continuous” if $y \succ x$ implies the existence of an open neighborhood U of x such that $z \not\succcurlyeq y$ for every $z \in U$. Obviously, the weak lower continuity of $\succ$ implies its ω -acyclicity; therefore, Theorem 1 of Campbell and Walker (when restricted to metric spaces) immediately follows from our Theorem 2. Weak lower continuity is not necessary for an interval order to admit a maximizer on every $X \in \mathfrak{C}$: consider a lexicographic order on a plane with fixed coordinates.

Theorem 3. *Let $\succ$ be a semiorder; then the following statements are equivalent.*

3.1. $M_\succ(X) \neq \emptyset$ for every $X \in \mathfrak{C}$.

3.2. $M_\succ(X) \neq \emptyset$ for every $X \in \mathfrak{C}$ and $M_\succ(\cdot)$ is path independent on $\mathfrak{C}$.

3.3. $\succ$ is ω -transitive.

3.4. $\succ$ is ω -acyclic.

Proof. The implications $[3.2] \Rightarrow [3.1]$ and $[3.3] \Rightarrow [3.4]$ are obvious; $[3.1] \Rightarrow [3.4]$ follows from Proposition 4.1; $[3.2] \iff [3.3]$, from Theorem 1. Thus, it is sufficient to prove $[3.4] \Rightarrow [3.3]$. Let $\succ$ be an ω -acyclic semiorder, $x^k \rightarrow x^\omega$, and $x^{k+1} \succcurlyeq x^k$ for all k . Applying (2) to $x^2 \succcurlyeq x^1 \succcurlyeq x^0$, we obtain that either $x^\omega \succcurlyeq x^0$ or $x^2 \succcurlyeq x^\omega$. The latter would contradict the ω -acyclicity (with the sequence $x^\omega, x^2, x^3, \dots$). $\square$

The restriction of the equivalence $[3.3] \iff [3.1]$ to strict orderings renders the main theorem of Smith (1974, Theorem 4.1).

Example 4.3. Let $A = [0, 1]$ and $y \succ x \iff 1 > y > x$ for all $y, x \in A$. Then $\succ$ is an interval order, ω -acyclic but not ω -transitive. Thus, Theorem 3 does not hold for interval orders. (Actually, there is no NM property on A itself, which is compact: any $x \in [0, 1[$ is neither a maximizer nor dominated by a maximizer.)

Every ω -acyclic interval order has an “ ε -version” of the NM property on every $X \in \mathfrak{C}$.

Proposition 4.3. *Let $\succ$ be an ω -acyclic interval order on a compact metric space X and $x^* \in X \setminus M_\succ(X)$. Then there is $z \in M_\succ(X)$ for which either $z \succ x^*$ or there is an infinite improvement path $\langle z^k \rangle_{k \in \mathbb{N}}$ such that $z^0 = x^*$ and $z^k \rightarrow z$.*

Proof. Applying Theorem 1 to $\succ$ defined by (4), we obtain $z \in M_\succ(X)$ such that $z \succ x^*$. A reference to (4b) completes the proof. $\square$

Remark. If A is finite, then every interval order generates a path independent choice function with nonempty values, so there is no analogue of the distinction between ω -transitivity and ω -acyclicity.

5 Representation of Interval Orders

Neither Proposition 5.1, nor its proof can claim any originality. However, both are needed in Theorems 4–6 and their corollaries.

Proposition 5.1. *Let $\succ$ be a binary relation on a set A . Then $\succ$ is an interval order if and only if there are a chain L and two mappings $\varphi^+, \varphi^- : A \rightarrow L$ such that, for all $x, y \in A$,*

$$\varphi^+(x) \geq \varphi^-(x); \quad (5a)$$

$$y \succ x \iff \varphi^-(y) > \varphi^+(x). \quad (5b)$$

Proof. Let $\succ$ be defined by (5b), $y \succ x$, and $b \succ a$. Since L is a chain, we may, without restricting generality, assume $\varphi^-(y) \geq \varphi^-(b)$; then $\varphi^-(y) > \varphi^+(a)$, hence $y \succ a$. Thus, (5) imply (1).

Let $\succ$ be an interval order. We consider $\mathfrak{B}$ with set inclusion as a partially ordered set and define two mappings $\varphi^+, \varphi^- : A \rightarrow \mathfrak{B}$ by $\varphi^-(y) = \{x \in A \mid y \succ x\}$ and $\varphi^+(y) = \bigcup_{z \not\succ y} \varphi^-(z)$. By definition, $\varphi^-(y) \subseteq \varphi^+(y)$ for every $y \in A$; we denote $L = \varphi^-(A) \cup \varphi^+(A) \subseteq \mathfrak{B}$.

Let us show that L is a chain. If $\varphi^-(y_1) \setminus \varphi^-(y_2) \neq \emptyset \neq \varphi^-(y_2) \setminus \varphi^-(y_1)$ for some $y_1, y_2 \in A$, then there are $x_1, x_2 \in A$ such that $y_i \succ x_i$, but $y_i \not\succ x_j$ ($i, j \in \{1, 2\}$, $i \neq j$); clearly, this contradicts (1). If $\varphi^+(y_1) \setminus \varphi^+(y_2) \neq \emptyset \neq \varphi^+(y_2) \setminus \varphi^+(y_1)$ for some $y_1, y_2 \in A$, there are $x_1, x_2, z_1, z_2 \in A$ such that $z_i \succ x_i$, $z_i \not\succ y_i$, and $z_i \succ y_j$ ($i, j \in \{1, 2\}$, $i \neq j$); the configuration formed by z_1, z_2, y_1, y_2 violates (1). Comparing $\varphi^-(y_1)$ with $\varphi^+(y_2)$, we have two alternatives: if $\varphi^-(y_1) \subset \varphi^-(z)$ for some $z \not\succ y_2$, then $\varphi^-(y_1) \subset \varphi^+(y_2)$; if $\varphi^-(y_1) \supseteq \varphi^-(z)$ for every $z \not\succ y_2$, then $\varphi^-(y_1) \supseteq \varphi^+(y_2)$.

Let us check (5b). If $y \succ x$, then $\varphi^-(y) \ni x \notin \varphi^+(x)$; since L is a chain, there must be $\varphi^+(x) \subset \varphi^-(y)$. If $\varphi^+(x) \subset \varphi^-(y)$, we pick $z \in \varphi^-(y) \setminus \varphi^+(x)$. If $y \not\succ x$, then $z \in \varphi^+(x)$ by definition; therefore, $y \succ x$. $\square$

Let $\succ$ be an interval order represented in the sense of (5) with a compact chain L . For every $v \in L$, we define $\Xi(v) = \{x \in A \mid \varphi^+(x) \leq v\}$ and $\lambda(v) = \sup \varphi^-(\Xi(v))$; obviously, $v \geq \lambda(v)$ for every $v \in L$. A representation in the sense of (5) is called *regular* if L is compact and $\varphi^+(x) > \lambda \circ \varphi^+(x)$ for every $x \in A$.

Proposition 5.2. *Every interval order admits a regular representation.*

Proof. Let $\succ$ be an interval order; by Proposition 5.1, there is a representation in the sense of (5). We define $\mathcal{L} = \{V \subseteq L \mid [v \in V \ \& \ v > w] \Rightarrow w \in V\}$ and $\iota : L \rightarrow \mathcal{L}$ by $\iota(v) = \{w \in L \mid v > w\}$. $\mathcal{L}$ with set inclusion is a compact chain – $\sup C$ for $C \subseteq \mathcal{L}$ is the set union of C – while ι is an order-preserving embedding. Then we define $\mathcal{L}^* = \mathcal{L} \times \{0, 1\}$ with a lexicographic order: $(V', \vartheta') > (V, \vartheta) \iff [V' \supset V \text{ or } [V' = V \ \& \ \vartheta' > \vartheta]]$ for all $V', V \in \mathcal{L}$ and $\vartheta', \vartheta \in \{0, 1\}$; clearly, $\mathcal{L}^*$ is a compact chain as well. Finally, we define $\psi^+, \psi^- : A \rightarrow \mathcal{L}^*$ by $\psi^+(x) = (\iota \circ \varphi^+(x), 1)$ and $\psi^-(x) = (\iota \circ \varphi^-(x), 0)$. Clearly, $\psi^-(y) > \psi^+(x) \iff \varphi^-(y) > \varphi^+(x)$ for all $y, x \in A$, hence ψ^- and ψ^+ provide a representation of $\succ$ in the sense of (5). Besides, we have $\lambda \circ \psi^+(x) \leq (\iota \circ \varphi^+(x), 0) < \psi^+(x)$ for every $x \in A$. $\square$

Given a regular representation of an interval order $\succ$, we define subsets $S, T^-, T^+ \subseteq L$: $S = \{v \in L \mid v > \lambda(v)\} \cup \{\min L\}$ (the regularity implies $\varphi^+(A) \subseteq S$); $v \in T^-$ if and only if, whenever $x^k \rightarrow x^\omega$, $\varphi^+(x^{k+1}) > \varphi^+(x^k)$ and $\varphi^-(x^{k+1}) > \varphi^-(x^k)$ for all $k = 0, 1, \dots$, and $v = \sup_k \varphi^+(x^k) = \sup_k \varphi^-(x^k)$, there holds $\varphi^-(x^\omega) \geq v$; $v \in T^+$ if and only if, whenever $x^k \rightarrow x^\omega$, $\varphi^+(x^{k+1}) > \varphi^+(x^k)$ and $\varphi^-(x^{k+1}) > \varphi^-(x^k)$ for all $k = 0, 1, \dots$, and $v = \sup_k \varphi^+(x^k) = \sup_k \varphi^-(x^k)$, there holds $\varphi^+(x^\omega) \geq v$. Obviously, $T^- \subseteq T^+$; if $v \in S$, then $v \in T^-$ by default.

Theorem 4. *Let $\succ$ be an interval order; then the following statements are equivalent.*

4.1. $S = L$ for every regular representation of $\succ$.

4.2. There is a regular representation of $\succ$ with $S = L$.

4.3. $\succ$ is strongly acyclic.

Proof. The implication $[4.1] \Rightarrow [4.2]$ immediately follows from Proposition 5.2.

$[4.2] \Rightarrow [4.3]$: Suppose there is an infinite improvement path $x^0, x^1, \dots$; then $\varphi^+(x^{k+1}) > \varphi^+(x^k)$ and $\varphi^-(x^{k+1}) > \varphi^-(x^k)$ for all k . We denote $v^+ = \sup_k \varphi^+(x^k)$ and $v^- = \sup_k \varphi^-(x^k)$; since $v^+ \in S$, we have $v^+ > \lambda(v^+) \geq v^-$. Therefore, there is k such that $\varphi^+(x^k) > v^- \geq \varphi^-(x^{k+1})$, but this contradicts the supposed $x^{k+1} \succ x^k$.

$[4.3] \Rightarrow [4.1]$: Let there be a regular representation of $\succ$ and $v \in L \setminus S \subseteq L \setminus \varphi^+(A)$. We pick $x^0 \in \Xi(v)$ arbitrarily, and then define an infinite sequence of $x^k \in \Xi(v)$ inductively, in the following way. Since $\varphi^+(x^k) < v$ and $\lambda(v) = v$, there must be $x^{k+1} \in \Xi(v)$ such that $\varphi^-(x^{k+1}) > \varphi^+(x^k)$. Now we have $x^{k+1} \succ x^k$, i.e., a contradiction with the strong acyclicity of $\succ$. $\square$

The conditions of the theorem are satisfied if, e.g., $\succ$ is represented by two numeric functions $\varphi^+, \varphi^- : A \rightarrow \mathbb{R}$ such that φ^+ is bounded above and $\varphi^+(x) \geq \varphi^-(x) + \varepsilon$ for all $x \in A$ with $\varepsilon > 0$.

Theorem 5. *Let $\succ$ be an interval order on a metric space A ; then the following statements are equivalent.*

5.1. $T^- = L$ for every regular representation of $\succ$.

5.2. There is a regular representation of $\succ$ with $T^- = L$.

5.3. $\succ$ is ω -transitive.

Proof. [5.2] $\Rightarrow$ [5.3]: Let $x^0, x^1, \dots$ be an improvement path such that $x^k \rightarrow x^\omega$. Denoting $v = \sup_k \varphi^+(x^k)$, we immediately see that $v = \sup_k \varphi^-(x^k)$ as well; by [5.2], $v \in T^-$. Therefore, $\varphi^-(x^\omega) \geq v > \varphi^+(x^0)$, hence $x^\omega \succ x^0$.

[5.3] $\Rightarrow$ [5.1]: Let there be a regular representation of $\succ$ and $v \in L \setminus T^- \subseteq L \setminus \varphi^+(A)$. By definition, there is a sequence $x^k \rightarrow x^\omega$ such that $\varphi^+(x^{k+1}) > \varphi^+(x^k)$ and $\varphi^-(x^{k+1}) > \varphi^-(x^k)$ for all k , and $v = \sup_k \varphi^+(x^k) = \sup_k \varphi^-(x^k)$, but $\varphi^-(x^\omega) < v$; therefore, $\varphi^-(x^\omega) < \varphi^+(x^h)$ for all $h \in \mathbb{N}$ large enough. Since $\sup_k \varphi^+(x^k) = \sup_k \varphi^-(x^k)$, we may, deleting superfluous members from the sequence if needed, assume that $x^{k+1} \succ x^k$ for all k . Now the improvement path $x^h, x^{h+1}, \dots$ violates the supposed ω -transitivity. $\square$

Proposition 5.3. *Let $\succ$ be an interval order on a metric space A , represented in the sense of (5) by two mappings $\varphi^+, \varphi^- : A \rightarrow L$. Then $\succ$ is ω -transitive if φ^- is upper ω -semicontinuous.*

Proof. If $x^0, x^1, \dots$ is an improvement path such that $x^k \rightarrow x^\omega$, then $\varphi^-(x^{k+1}) > \varphi^-(x^k)$ for all k , and $x^1, x^2, \dots$ also converges to x^ω . Therefore, $\varphi^-(x^\omega) > \varphi^-(x^1) > \varphi^+(x^0)$, hence $x^\omega \succ x^0$. $\square$

Example 5.1. Let $A =]0, 3[$ and $y \succ x \iff [2 \neq y > x + 1 \text{ or } y \neq x = 2]$. Clearly, $\succ$ is transitive and strongly acyclic, hence ω -transitive by default. The relation can be represented in the sense of (5) by functions $\varphi^+(x) = x + 1$ for $x \neq 2$, $\varphi^+(2) = 2$, and $\varphi^-(x) = \varphi^+(x) - 1$ for all $x \in A$, hence is an interval order. The representation is by no means unique, but it is easy to see that both φ^+ and φ^- must strictly increase on $]1, 2[$ and then jump down. In other words, the relation cannot be represented with an upper ω -semicontinuous function φ^- (or φ^+ for that matter).

Theorem 6. *Let $\succ$ be an interval order on a metric space A ; then the following statements are equivalent.*

6.1. $T^+ = L$ for every regular representation of $\succ$.

6.2. There is a regular representation of $\succ$ with $T^+ = L$.

6.3. $\succ$ is ω -acyclic.

Proof. [6.2] $\Rightarrow$ [6.3]: Let $x^0, x^1, \dots$ be an improvement path such that $x^k \rightarrow x^\omega$. Arguing as in the proof of [5.2] $\Rightarrow$ [5.3] above, we obtain $\varphi^+(x^\omega) > \varphi^+(x^0)$, hence $x^\omega \neq x^0$.

[6.3] $\Rightarrow$ [6.1]: Let there be a regular representation of $\succ$ and $v \in L \setminus T^+$. Arguing as in the proof of [5.3] $\Rightarrow$ [5.1] above, we obtain an improvement path $x^k \rightarrow x^\omega$ such that $\varphi^+(x^\omega) < \varphi^+(x^h) < \varphi^-(x^{h+1})$, hence $x^{h+1} \succ x^\omega$. Now $x^\omega, x^{h+1}, \dots$ is an improvement path violating the supposed ω -acyclicity. $\square$

Proposition 5.4. *Let $\succ$ be an interval order on a metric space A , represented in the sense of (5) by two mappings $\varphi^+, \varphi^- : A \rightarrow L$. Then $\succ$ is ω -acyclic if φ^+ is upper ω -semicontinuous.*

The proof is similar to that of Proposition 5.3. Example 5.1 shows that Proposition 5.4 also cannot be reversed.

An upper semicontinuous numeric function $\varphi : A \rightarrow \mathbb{R}$ gives an obvious example of an upper ω -semicontinuous mapping to a chain. A bit more sophisticated examples are provided

by lexicographic orders. Let $L_1, \dots, L_n$ be chains and $L = \prod_{i=1}^n L_i$. The *lexicographic order* on L is defined in a usual way:

$$(v'_1, \dots, v'_n) > (v_1, \dots, v_n) \iff \exists m [\forall i < m [v'_i = v_i] \& v'_m > v_m].$$

Clearly, L with the order is a chain. A list of mappings $\varphi_i : A \rightarrow L_i$ ($i = 1, \dots, n$) defines a mapping $\varphi : A \rightarrow L$, their Cartesian product.

Proposition 5.5. *Let A be a metric space and each $\varphi_i : A \rightarrow \mathbb{R}$ ($i = 1, \dots, n$) be an upper semicontinuous function. Then the Cartesian product $\varphi : A \rightarrow \mathbb{R}^n$ (with the lexicographic order on $\mathbb{R}^n$) is upper ω -semicontinuous.*

Proof. Let $x^k \rightarrow x^\omega$ and $\varphi(x^{k+1}) > \varphi(x^k)$ for all $k = 0, 1, \dots$. For each k we denote μ_k ($1 \leq \mu_k \leq n$) the corresponding m from the definition of the lexicographic order. Without restricting generality, we may assume that $\mu_k = m$ for all k – we only have to delete a finite number of initial steps and then all steps with $\mu_k > m$. Therefore $\varphi_i(x^k)$ ($i < m$) does not depend on k , while $\varphi_m(x^{k+1}) > \varphi_m(x^k)$. From the upper semicontinuity of φ_m , we obtain $\varphi_m(x^\omega) > \varphi_m(x^0)$; from the upper semicontinuity of φ_i for $i < m$, $\varphi_i(x^\omega) \geq \varphi_i(x^0)$. Clearly, $\varphi(x^\omega) > \varphi(x^0)$. $\square$

6 Representation of Semiorders

The comment preceding Proposition 5.1 is relevant here as well.

Proposition 6.1. *Let $\succ$ be a binary relation on a set A . Then $\succ$ is a semiorder if and only if there are a chain L and two mappings $\varphi : A \rightarrow L$ and $\lambda : \varphi(A) \rightarrow L$ such that, for all $x, y \in A$:*

$$\varphi(y) > \varphi(x) \Rightarrow \lambda \circ \varphi(y) \geq \lambda \circ \varphi(x); \quad (6a)$$

$$\varphi(x) \geq \lambda \circ \varphi(x); \quad (6b)$$

$$y \succ x \iff \lambda \circ \varphi(y) > \varphi(x). \quad (6c)$$

Proof. Let (6) hold; we define two mappings $\varphi^+, \varphi^- : A \rightarrow L$ by $\varphi^+(x) = \varphi(x)$ and $\varphi^-(x) = \lambda \circ \varphi(x)$. Conditions (5) immediately follow from (6), hence $\succ$ is an interval order by Proposition 5.1. Let $z \succ y \succ x$ and $a \in A$. If $\varphi(a) < \lambda \circ \varphi(z)$, then $z \succ a$; otherwise, we have $\varphi(a) \geq \lambda \circ \varphi(z) > \varphi(y)$, hence $\lambda \circ \varphi(a) \geq \lambda \circ \varphi(y) > \varphi(x)$, hence $a \succ x$.

Let $\succ$ be a semiorder on A . Since $\succ$ is an interval order, the necessity part of Proposition 5.1 applies. Let L, φ^+ and φ^- be as in the proof of the latter. We define $\mathcal{L} = L \times \{0, 1\} \times L$ with the lexicographic order, $\varphi : A \rightarrow \mathcal{L}$ by $\varphi(x) = (\varphi^+(x), 1, \varphi^-(x))$, and $\lambda : \varphi(A) \rightarrow \mathcal{L}$ by $\lambda(\varphi^+(x), 1, \varphi^-(x)) = (\varphi^-(x), 0, \varphi^-(x))$. By definition, (6b) holds as a strict inequality. To check (6c), it is enough to note that $\lambda \circ \varphi(y) > \varphi(x) \iff \varphi^-(y) > \varphi^+(x)$. To check (6a), let us show that the inequalities $\varphi^+(y) > \varphi^+(x)$ and $\varphi^-(x) > \varphi^-(y)$ are incompatible for any $x, y \in A$. Supposing the contrary, we obtain the existence of $z \in A$ such that $x \succ z$ and $y \not\succ z$ ($z \in \varphi^-(x) \setminus \varphi^-(y)$), and $a, b \in A$ such that $b \succ a$, $b \succ x$, and $b \not\succ y$ ($a \in \varphi^+(y) \setminus \varphi^+(x)$). Now we have $b \succ x \succ z$ and $b \not\succ y \not\succ z$, i.e., a configuration violating (2). $\square$

A representation in the sense of (6) is called *regular* if L is compact and (6b) holds as a strict inequality.

Proposition 6.2. *Every semiorder admits a regular representation.*

Proof. The representation defined in the necessity proof of Proposition 6.1 satisfies (6b) as a strict inequality. Embedding into a compact chain can be performed exactly as in the proof of Proposition 5.2. $\square$

Given a regular representation of a semiorder $\succ$, we define $\Xi(v) = \{x \in A \mid \varphi(x) \leq v\}$ for every $v \in L$, and extend λ to $L \setminus \varphi(A)$ by $\lambda(v) = \sup \lambda \circ \varphi(\Xi(v))$; note that λ satisfies the same equality on $\varphi(A)$ as well. Now λ is increasing in the sense of $v' > v \Rightarrow \lambda(v') \geq \lambda(v)$ and satisfies $v \geq \lambda(v)$ on the whole L . We define subsets $S, T \subseteq L$: $S = \{v \in L \mid v > \lambda(v)\} \cup \{\min L\}$ (the regularity implies $\varphi^+(A) \subseteq S$); $v \in T$ if and only if, whenever $x^k \rightarrow x^\omega$, $\varphi(x^{k+1}) > \varphi(x^k)$ for all k , and $v = \sup_k \varphi(x^k) = \sup_k \lambda \circ \varphi(x^k)$, there holds $\varphi(x^\omega) \geq v$. If $v \in S$, then $v \in T$ by default.

Corollary to Theorem 4. *Let $\succ$ be a semiorder; then the following statements are equivalent.*

1. $S = L$ for every regular representation of $\succ$.
2. There is a regular representation of $\succ$ with $S = L$.
3. $\succ$ is strongly acyclic.

Let $\varphi : A \rightarrow \mathbb{R}$ be bounded above and $\varepsilon > 0$; consider a relation defined by $y \succ x \iff \varphi(y) > \varphi(x) + \varepsilon$. A regular representation of $\succ$ with $S = L$ is obtained if we define $L = [-\infty, u]$, where u is an upper bound for φ , and $\lambda(v) = v - \varepsilon$. Maximizers for $\succ$ are exactly ε -maxima of φ .

Example 6.1. Let $x^k = k/(k+1)$ and $A = \{x^k\}_{k=0,1,\dots} \subset \mathbb{R}$. Clearly, the standard order on A is an ω -transitive semiorder, which is not strongly acyclic. A representation in the sense of (6) is given, e.g., by $L = A$, the identity mapping as φ , and $\lambda(x^{k+1}) = x^k$ for all k while $\lambda(x^0) = x^0$. We have $S = L$, but L is not compact. If we redefine $L = A \cup \{1\}$, we will have the compactness, but it will be impossible to extend λ to the new L keeping a strict inequality in (6b).

Corollary to Theorem 5. *Let $\succ$ be a semiorder; then the following statements are equivalent.*

1. $T = L$ for every regular representation of $\succ$.
2. There is a regular representation of $\succ$ with $T = L$.
3. $\succ$ is ω -transitive.

Proposition 6.3. *Let $\succ$ be a semiorder on a metric space A , represented in the sense of (6) by two mappings $\varphi : A \rightarrow L$ and $\lambda : \varphi(A) \rightarrow L$. Then $\succ$ is ω -transitive if φ is upper ω -semicontinuous.*

The relation in Example 5.1 is an ω -transitive semiorder which admits no representation with an upper ω -semicontinuous function φ .

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