

Robert Louis Stevenson’s Bottle Imp: A strategic analysis

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Abstract

The background of Stevenson’s story is viewed as a stylized model of participation in a financial pyramid. You are invited to take part in a dubious activity. If you refuse, you neither gain nor lose anything. If you accept, you will gain if you are able to find somebody who will take your place on exactly the same conditions, but suffer a loss otherwise. The catch is that there is a finite number of discrete steps at which the substitution can be done, so the agent who joins in at the last step inevitably loses. Clearly, rational agents will not agree to participate at any step. However, an arbitrarily small probability of a “bailout,” in which case that last agent will get the same gain as every other participant, plus an appropriately asymmetric structure of private information, change everything, so the proposal will be accepted in a (subgame perfect) equilibrium, provided there are sufficiently many steps ahead.

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1 Introduction

Asset booms and crashes are a persistent feature of economic life (Chancellor, 1999). Naturally, economists have spent considerable time and effort trying to reconcile such phenomena with the tenets of general economic equilibrium (e.g., Tirole, 1982; Boldrin and Levine, 2001; Abreu and Brunnermeier 2003; Kocherlakota, 2008; Leoni, 2009; Barlevy, 2014; Miao, 2014).

This paper is only remotely connected with that strand of literature. First, we have in mind *artificial* booms and crashes that result from deliberate actions by market manipulators. In real life, it need not be clear to economic agents whether they face a “natural” boom or an “artificial” one (Schiller, 2000), but here we assume that it *is* clear.

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Second, we only consider the decision making of “rank and file” participants, not the activity of those manipulators themselves, nor conditions favorable or otherwise to them. To be more precise, we analyze the rationality of this attitude: “I know this house of cards will crumble eventually, but I’ll be out of it by then.” To be even more precise, we try to substantiate the intuitively plausible notion that the more distant in time the expected collapse is perceived to be, the more justified is that attitude. As a stylized model of the financial pyramid itself, we employ the background of Stevenson’s (1891) story.

Section 2 contains a general description of our model. The exact technical details are given, and our main theorem formulated, in Section 3. The proofs are in Section 4. A discussion of various related questions in Section 5 concludes the paper.

2 Informal Model

For those readers unfamiliar with Stevenson’s story, there is no spoiler in the following. Only the general background is invoked explicitly.

A magic bottle (or rather an imp inside the bottle) fulfills any desire of its owner; however, that owner is sure to go to hell upon death. The only way to get rid of the bottle is to sell it to somebody else, for “coined money” and at a loss for oneself. Naturally, the imp will not grant eternal life, nor allow any modification of the terms on which the bottle may be transferred.

When the bottle appears in the story for the first time, its owner at that moment suggests the following strategy to a prospective buyer:

All you have to do is to use the power of the imp in moderation, and then sell it to someone else, as I do to you, and finish your life in comfort.

The problem with this strategy becomes obvious when the price of the bottle approaches zero. However, while it is high enough, willing participants are not hard to find in the story. (Stevenson’s story has already been discussed in philosophical literature (Sharvy, 1983; Sorensen, 1986) as a paradox. The difference of the language makes direct comparisons with this paper difficult.)

We replace “every wish” and eternal perdition, which are hard to formalize anyway, with gains and losses, thus obtaining a stylized model of participation in a financial pyramid. Instead of irregular downward movement of the price of the bottle, we assume that costless transfers of the bottle can be executed at discrete time moments and the number of possible future transfers is commonly known beforehand.

The question: Is it possible to justify the view that the more steps are ahead, the easier it is to accept the bottle, in the framework of common knowledge of rationality of all agents involved?

The interaction between the agents is modeled as a finite extensive game. Each move belongs to one of two categories: either a proposal by the current owner to somebody else, or a decision of an agent proposed to whether to take the bottle. The first proposal to take the bottle comes from

outside, from the Devil. The total number of steps, T , is fixed beforehand. An agent who never held the bottle gains nothing and loses nothing. An agent who took the bottle at some stage gains G if he is able to find a replacement, or suffers loss L otherwise. A detailed technical description is given in the next section.

Our basic question may be formulated, now, in this way: May the existence of an equilibrium where the bottle actually changes hands depend on the number of steps available?

Straightforward argument backwards shows that such an equilibrium cannot exist for any T . One does not even have to restrict attention to, say, subgame perfect equilibria. For our purposes, this negative answer is disappointing to say the least.

The situation will change drastically if we assume that a “bailout” (or an “amnesty”) may happen with a (small) probability $p > 0$, in which case the agent left with the bottle at the end obtains the same gain as those who escaped in time.

Although all agents are basically the same, there may be asymmetry in their information. We use the simplest model of knowledge, a *partitioned information structure* (Aumann, 1976; Bacharach, 1985): there is a set of the “states of the world,” a probability distribution on the set, and mappings determining signals about the actual state the agents receive. This model is common knowledge among the agents, only the signals received are private. Thus, every state of the world determines whether the “bailout” will happen, and what each agent knows about that, as well as about the knowledge of other agents.

We assume that all proposals and decisions to accept or reject are publicly observed. In a sense, we thus have a game of perfect, but incomplete, information.

The gains and losses are measured in a von Neumann–Morgenstern utility rather than money. An important characteristic of our game is $\pi := L/(L + G)$: For an agent to agree to take the bottle at an equilibrium, he must evaluate the probability of safe escape as, at least, π . Generally, that probability depends on the preceding moves in the game and on the agent’s private information. It is not commonly known. Thus, our situation is not related to the no-trade theorem of Milgrom and Stokey (1982) (see also Neeman, 1996).

3 Formal Model

A *BI game* is characterized by the following data:

$$\langle N, T, \Omega, \Sigma, P, \Omega^{\text{Amn}}, \langle \sigma_i \rangle_{i \in N}, G, L \rangle,$$

where N is a finite set (of players); T is an integer (number of steps), Ω is an arbitrary set (of “states of the world”); Σ is a σ -algebra of subsets of Ω (“events”), including \emptyset and Ω itself; P is a probability measure on Σ ; $\Omega^{\text{Amn}} \in \Sigma$ consists of the states of the world at which the “bailout” happens; each σ_i is a mapping $\sigma_i: \Omega \rightarrow S_i$ (private signals); G and L are strictly positive real numbers. If S_i is finite, we assume that each $\sigma_i^{-1}(s_i) \subseteq \Omega$ belongs to Σ ; otherwise a σ -algebra of subsets of S_i should

be defined and the mapping σ_i should be measurable. The complete description of how such a game is played takes some time.

A *history* is a finite string of the kind $\langle i_1, \theta_1, i_2, \theta_2, \dots, i_m \rangle$ or $\langle i_1, \theta_1, i_2, \theta_2, \dots, i_m, \theta_m \rangle$, where $1 \leq m \leq T$, $i_k \in N$ and $\theta_k \in \{\text{Yes}, \text{No}\}$ for all relevant k . Each i_k denotes the player addressed with a proposal to take the bottle at stage k ; θ_k denotes his decision. There is a natural partial order on the set H of all possible histories, $h' \geq h$ meaning that h coincides with an initial segment of h' . Given $h \in H$, we denote $H^\rightarrow(h)$ the set of *immediate successors* of h : $H^\rightarrow(h) := \{h' \in H \mid h' > h \text{ \& \#} h'' \in H [h' > h'' > h]\}$. When the empty string, “zero history,” $h^0 := \langle \rangle$ is added to H , the latter becomes a tree.

H is naturally partitioned into H^1 and $H^?$. The first subset consists of $h \in H$ ending with $\theta_k \in \{\text{Yes}, \text{No}\}$, so a new proposal is in order (unless $m = T$); the second, of $h \in H$ ending with $i_k \in N$, so an answer is expected. Note that $H^\rightarrow(h) \subset H^?$ whenever $h \in H^1$, while $H^\rightarrow(h) \subset H^1$ whenever $h \in H^?$.

Remark. Our definition allows the current owner to propose to himself, in which case the answer, Yes or No, does not matter. Although there is no reason for a player to waste time in such a manner, there is no necessity to prohibit such decisions in the rules of the game either. Actually, the possibility to retain the bottle allows a simpler description of equilibrium strategies in Section 4.2. Similarly, the rules allow a player who has already held the bottle and passed it to another player to take it again later, even though the second taking can only result in a loss.

We denote \bar{H} the set of *complete* histories, i.e., $\bar{h} \in H^1$ such that $m = T$. Given a player $i \in N$, the set \bar{H} is partitioned into three subsets: \bar{H}_0^i , \bar{H}_+^i and \bar{H}_-^i , according as player i never held the bottle, held it and then passed it to somebody else, or is left with the bottle at the end. Technically, $\bar{h} \in \bar{H}_0^i$ if each occurrence of i in \bar{h} is followed by No (in particular, if \bar{h} does not include i at all); $\bar{h} \in \bar{H}_+^i$ if \bar{h} contains a fragment $\langle i, \text{Yes} \rangle$ and the last such fragment is followed by $\langle j, \text{Yes} \rangle$ with $j \neq i$; $\bar{h} \in \bar{H}_-^i$ if \bar{h} contains a fragment $\langle i, \text{Yes} \rangle$ and Yes does not occur in \bar{h} after the last such fragment.

Given a player $i \in N$, a complete history $\bar{h} \in \bar{H}$, and $\omega \in \Omega$, we define “local” NM utilities $u_i(\bar{h}, \omega)$ in the following way:

$$u_i(\bar{h}, \omega) := \begin{cases} 0, & \bar{h} \in \bar{H}_0^i; \\ G, & \bar{h} \in \bar{H}_+^i; \\ G, & \bar{h} \in \bar{H}_-^i \text{ \& } \omega \in \Omega^{\text{Amn}}; \\ -L, & \bar{h} \in \bar{H}_-^i \text{ \& } \omega \notin \Omega^{\text{Amn}}. \end{cases} \quad (1)$$

To the set of players, we add a “quasi-player,” the Devil, who is denoted 0 (assuming $0 \notin N$). We denote $\bar{N} := N \cup \{0\}$. The Devil’s objective – to get the players to take the bottle – can be expressed with this local utility function (independent of $\omega \in \Omega$):

$$u_0(\bar{h}) := \begin{cases} 0, & \forall i \in N [\bar{h} \in \bar{H}_0^i]; \\ G, & \text{otherwise.} \end{cases}$$

The set $H \setminus \bar{H}$ can also be partitioned into subsets H_i , $i \in \bar{N}$, according as who is to make a decision after that history; the straightforward technical definition is omitted. A *strategy* x_i of player $i \in N$ is a mapping $H_i \times S_i \rightarrow H$ such that $x_i(h, \sigma_i(\cdot)): \Omega \rightarrow H$ is Σ -measurable, and $x_i(h, s_i) \in H^{\rightarrow}(h)$ for each $h \in H_i$ and all $s_i \in S_i$. The Devil's strategy is $\langle i_1, i_2, \dots, i_T \rangle$: at the first step, the bottle is offered to player i_1 ; if refused, at the second step it is offered to i_2 , etc. If the bottle is accepted at some stage, the Devil takes no part in the further play.

Once a total strategy profile $x_{\bar{N}} \in X_{\bar{N}} := \prod_{i \in \bar{N}} X_i$ and a history $h \in H$ are given, a complete history $x_{\bar{N}}(h, \omega) \in \bar{H}$ is defined by an obvious recursion for every $\omega \in \Omega$ – what happens if each player uses strategy x_i starting from h and the state of the world is ω . Given $x_{\bar{N}} \in X_{\bar{N}}$, $i \in N$, $h \in H$, and $\varkappa \in \{0, -, +\}$, we define $p_i^{\varkappa}(x_{\bar{N}} | h) := P(\{\omega \in \Omega \mid x_{\bar{N}}(h, \omega) \in \bar{H}_{\varkappa}^i(x_{\bar{N}})\})$ and $p_i^*(x_{\bar{N}} | h) := P(\{\omega \in \Omega^{\text{Ammn}} \mid x_{\bar{N}}(h, \omega) \in \bar{H}_-^i(x_{\bar{N}})\})$.

Now the NM utility function $U_i: X_{\bar{N}} \rightarrow \mathbb{R}$ of each player $i \in N$ is the expected value of $u_i(\bar{h}, \omega)$:

$$U_i(x_{\bar{N}}) := \mathbb{E}_P [u_i(x_{\bar{N}}(h^0, \omega), \omega)] = [p_i^+(x_{\bar{N}}) + p_i^*(x_{\bar{N}})] \cdot G - [p_i^-(x_{\bar{N}}) - p_i^*(x_{\bar{N}})] \cdot L. \quad (2)$$

The Devil's utility function $U_0: X_{\bar{N}} \rightarrow \mathbb{R}$ is written in this way:

$$U_0(x_{\bar{N}}) := \mathbb{E}_P [u_0(x_{\bar{N}}(h^0, \omega))] = (1 - P(\{\omega \in \Omega \mid x_{\bar{N}}(h^0, \omega) \in \bigcap_{i \in N} \bar{H}_0^i\})) \cdot G. \quad (3)$$

Naturally, a total strategy profile $x_{\bar{N}}^0 \in X_{\bar{N}}$ is a *Nash equilibrium* if

$$U_i(x_{\bar{N}}^0) \geq U_i(x_i, x_{\bar{N} \setminus \{i\}}^0) \quad (4)$$

for all $i \in \bar{N}$ and $x_i \in X_i$.

The notion of a Nash equilibrium in extensive games is often viewed as too lax: if a combination (h, s_i) has probability 0, then condition (4) does not prohibit *any* choice of $x_i(h, s_i)$. Typically, the attention is concentrated on *perfect Bayesian equilibria*, where the decisions in impossible situations are optimal with respect to a consistent system of beliefs. Here, we can impose even stricter restrictions, demanding optimality with respect to a family of explicitly defined beliefs.

The exact formulation requires quite a lot of notations, so we start with a general outline. First of all, our definition of perfectness only applies when each S_i is finite and $P(\sigma_i^{-1}(s_i)) > 0$ for each $s_i \in S_i$ (the last assumption does not restrict generality). Second, no player ever doubts the signal s_i received: if an observed history h is incompatible with $x_{\bar{N}}$ and s_i , then the inference is that somebody has deviated from the prescribed strategy x_j . We consider three variants of belief update in such situations. In the first version, the player assumes that a mistake could have been made by any player at any previous position, so all the choices made before are ignored, and prior beliefs are conditioned only on the signal s_i . In the second version, the player finds a preceding history h^* which is closest to h among those consistent with s_i , and conditions prior beliefs on h^* and s_i . In the third version, the player finds a preceding history h_* which is closest to h^0 among those histories from which h is reached with a strictly positive probability (conditioned on s_i), and then makes a usual Bayesian

update of prior beliefs viewing h_* as the initial position. Clearly, $h^0 \geq h^* > h_* \geq h$ whenever (h, s_i) has zero prior probability, while $h^* = h$ and $h_* = h^0$ whenever (h, s_i) has a strictly positive prior probability. Our notion of perfectness requires that the choice at an impossible combination (h, s_i) should be optimal for all the three variants of updating.

Given $x_{\bar{N}} \in X_{\bar{N}}$ and $h, h' \in H$, we introduce the notation $\Psi(x_{\bar{N}}, h, h') := \{\omega \in \Omega \mid h' \leq x_{\bar{N}}(\omega, h)\}$ and $\rho(h' \mid (x_{\bar{N}}, h)) := P(\Psi(x_{\bar{N}}, h, h'))$. When $h' \leq h$, $\Psi(x_{\bar{N}}, h, h') = \Omega$; otherwise, $\Psi(x_{\bar{N}}, h, h') = \emptyset$ unless $h' > h$. We only use the notation in the latter case; then, $\rho(h' \mid (x_{\bar{N}}, h))$ is the probability of h' in the event that the players start at h using strategies $x_{\bar{N}}$. We also denote $H(x_{\bar{N}}) := \{h \in H \mid \rho(h \mid (x_{\bar{N}}, h^0)) > 0\}$, the set of histories that have strictly positive prior probability under $x_{\bar{N}}$. If $h \in H(x_{\bar{N}})$, then the conditional distribution $P(\cdot \mid (x_{\bar{N}}, h))$ on Σ is well-defined, viz., for every $A \in \Sigma$,

$$P(A \mid (x_{\bar{N}}, h)) := P(A \cap \Psi(x_{\bar{N}}, h^0, h)) / \rho(h \mid (x_{\bar{N}}, h^0)).$$

Whenever $x_{\bar{N}} \in X_{\bar{N}}$, $h \in H(x_{\bar{N}})$, and $h' \in H \rightarrow (h)$, there hold $h' \in H(x_{\bar{N}}) \iff \rho(h' \mid (x_{\bar{N}}, h)) > 0$ and

$$P(A \mid (x_{\bar{N}}, h)) = \sum_{h' \in H \rightarrow (h) \cap H(x_{\bar{N}})} \rho(h' \mid (x_{\bar{N}}, h)) P(A \mid (x_{\bar{N}}, h')) \quad (5)$$

for every $A \in \Sigma$. It is important to note that $\Psi(x_{\bar{N}}, h^0, h)$ depends exclusively on the restriction of $x_{\bar{N}}$ to $h' < h$; hence the same holds for $P(\cdot \mid (x_{\bar{N}}, h))$.

Let $i \in N$, S_i be finite, and $P(\sigma_i^{-1}(s_i)) > 0$ for each $s_i \in S_i$. Given $x_{\bar{N}} \in X_{\bar{N}}$, we denote $H_i(x_{\bar{N}}) := \{(h, s_i) \in H \times S_i \mid P(\Psi(x_{\bar{N}}, h^0, h) \cap \sigma_i^{-1}(s_i)) > 0\}$; for each $(h, s_i) \in H_i(x_{\bar{N}})$, we define a conditional probability distribution on Σ :

$$P_i(A \mid (x_{\bar{N}}, h, s_i)) := P(A \cap \Psi(x_{\bar{N}}, h^0, h) \cap \sigma_i^{-1}(s_i)) / P(\Psi(x_{\bar{N}}, h^0, h) \cap \sigma_i^{-1}(s_i))$$

for every $A \in \Sigma$.

Given a history $h \in H$ and a probability distribution \tilde{P} on Σ , a *subgame* $\Gamma(h, \tilde{P})$ is a strategic game with the same players and strategies x_i ($i \in N$) restricted to $\{h' \in H_i \mid h' \geq h\} \times S_i$. Similarly to (2), the NM utility function $\tilde{U}_i: \tilde{X}_{\bar{N}} \rightarrow \mathbb{R}$ of each player $i \in N$ in the subgame is the expected value of the local utility $u_i(\bar{h}, \omega)$ defined by (1), $\tilde{U}_i(\tilde{x}_{\bar{N}}) := E_{\tilde{P}}[u_i(\tilde{x}_{\bar{N}}(h, \omega), \omega)]$. If the Devil has already got rid of the bottle at h , he does not participate in the subgame; otherwise, his strategy is a list of $i_k \in N$ of the appropriate length. His utility is also defined as the expected value of the local utility $u_0(\bar{h}, \omega)$, under \tilde{P} rather than P .

Proposition 3.1. *Let $x_{\bar{N}}^0$ be a Nash equilibrium. If $h \in H(x_{\bar{N}}^0)$, then, for each $i \in \bar{N}$, x_i^0 satisfies (4) in the subgame $\Gamma(h^0, P(\cdot \mid (x_{\bar{N}}, h)))$; whereas the restriction of x_i^0 to $h' \geq h$ satisfies (4) in the subgame $\Gamma(h, P(\cdot \mid (x_{\bar{N}}, h)))$. If S_i (for an $i \in N$) is finite and $(h, s_i) \in H_i(x_{\bar{N}}^0)$, then x_i^0 satisfies (4) in the subgame $\Gamma(h^0, P_i(\cdot \mid (x_{\bar{N}}, h, s_i)))$; whereas the restriction of x_i^0 to $h' \geq h$ satisfies (4) in the subgame $\Gamma(h, P_i(\cdot \mid (x_{\bar{N}}, h, s_i)))$.*

A straightforward proof is omitted.

The following definitions are only given for BI games where all sets S_i are finite and $P(\sigma_i^{-1}(s_i)) > 0$ for each $s_i \in S_i$. Given $x_{\bar{N}} \in X_{\bar{N}}$, $h \in H$, $i \in N$, and $s_i \in S_i$, we set $h^* := \max\{h' \in H \mid h' \leq h \ \& \ P(\Psi(x_{\bar{N}}, h^0, h') \cap \sigma_i^{-1}(s_i)) > 0\}$ and $h_* := \min\{h' \in H \mid P(\Psi(x_{\bar{N}}, h', h) \cap \sigma_i^{-1}(s_i)) > 0\}$. Then we define two auxiliary probability distributions:

$$P_i^*(A \mid (x_{\bar{N}}, h, s_i)) := P(A \cap \Psi(x_{\bar{N}}, h^0, h^*) \cap \sigma_i^{-1}(s_i)) / P(\Psi(x_{\bar{N}}, h^0, h^*) \cap \sigma_i^{-1}(s_i));$$

$$P_*^i(A \mid (x_{\bar{N}}, h, s_i)) := P(A \cap \Psi(x_{\bar{N}}, h_*, h) \cap \sigma_i^{-1}(s_i)) / P(\Psi(x_{\bar{N}}, h_*, h) \cap \sigma_i^{-1}(s_i))$$

for every $A \in \Sigma$. It is worth noting that $P_i^*(\cdot \mid (x_{\bar{N}}, h, s_i)) = P_*^i(\cdot \mid (x_{\bar{N}}, h, s_i)) = P_i(\cdot \mid (x_{\bar{N}}, h, s_i))$ whenever $(h, s_i) \in H_i(x_{\bar{N}})$. We set $\Lambda := \{(\lambda^0, \lambda^*, \lambda_*) \in [0, 1]^3 \mid \lambda^0 + \lambda^* + \lambda_* = 1\}$. Given $\lambda = (\lambda^0, \lambda^*, \lambda_*) \in \Lambda$, we denote $P_i^\lambda(\cdot \mid (x_{\bar{N}}, h, s_i)) = \lambda^0 P(\cdot) + \lambda^* P_i^*(\cdot \mid (x_{\bar{N}}, h, s_i)) + \lambda_* P_*^i(\cdot \mid (x_{\bar{N}}, h, s_i))$.

Now we call $x_{\bar{N}}^0 \in X_{\bar{N}}$ a *perfect equilibrium* if, for each $h \in H$, $i \in N$ and $s_i \in S_i$, and every $\lambda \in \Lambda$, the restriction of $x_{\bar{N}}^0$ to $h' \geq h$ satisfies (4) in the subgame $\Gamma(h, P_i^\lambda(\cdot \mid (x_{\bar{N}}, h, s_i)))$.

Applying this condition to $h = h^0$, and each $i \in N$ and $s_i \in S_i$, we see that every perfect equilibrium is a Nash equilibrium indeed.

Remark. The optimality with respect to P_i^λ for every $\lambda \in \Lambda$ is obviously equivalent to the optimality with respect to each of P , P_i^* , and P_*^i . The probabilities $P_*^i(\cdot \mid (x_{\bar{N}}, h, s_i))$ describe a consistent system of beliefs; therefore, every perfect equilibrium is a perfect Bayesian equilibrium.

Theorem. *Given T , p , and π , these statements are equivalent:*

$$p \geq \pi^T; \tag{6a}$$

there is a BI game in which the number of steps is T , $L/(L + G) = \pi$, $P(\Omega^{\text{Ammn}}) \leq p$, each S_i is finite, and which possesses a perfect equilibrium such that the Devil's proposal is accepted at the first stage in every state of the world; (6b)

there is a BI game in which the number of steps is T , $L/(L + G) = \pi$, $P(\Omega^{\text{Ammn}}) \leq p$, and which possesses a Nash equilibrium such that the Devil's proposal is accepted at some stage in almost every state of the world. (6c)

The implication (6b) \Rightarrow (6c) is obvious. The implications (6c) \Rightarrow (6a) and (6a) \Rightarrow (6b) are proved in Section 4.1 and Section 4.2, respectively.

In a sense, this theorem answers our question: however small $p > 0$ is, an equilibrium where the bottle actually changes hands may exist if T is large enough. If T is too small, such an equilibrium becomes just impossible.

It must be stressed that only the existence of a suitable model is asserted in the theorem. There is no attempt to describe what may happen in an arbitrary BI game. For instance, if the agents' information is symmetric, then an equilibrium where the bottle changes hands at (almost) every state of the world is only possible if $P(\Omega^{\text{Ammn}}) \geq \pi$.

4 Proof

4.1 Necessity

We fix a BI game and a Nash equilibrium $x_{\bar{N}}$ such that the Devil's proposal is accepted at some stage in almost every state of the world. For every history $h \in H$, we denote $\Omega^*(h)$ the set of $\omega \in \Omega$ such that the bottle changes hands (at least once) after h under $x_{\bar{N}}$. Our assumption about $x_{\bar{N}}$ means that $P(\Omega^*(h^0)) = 1$. Roughly speaking, $P(\Omega^*(h) \cup \Omega^{\text{Amn}} \mid (x_{\bar{N}}, h))$ for $h \in \mathbb{H}(x_{\bar{N}})$ is the prior probability that the owner of the bottle at h will get away with it eventually; note, however, that those $\omega \in \Omega$ where that owner takes the bottle again later are included in $\Omega^*(h)$ regardless of the final outcome.

For every history $h \in H$, we define $l(h)$ as the maximal number of changes of the ownership of the bottle that can happen after h . To be more precise, $l(h) := T - m$ if $h = \langle i_1, \theta_1, \dots, i_m, \theta_m \rangle \in H^!$, whereas $l(h) := T - m + 1$ if $h = \langle i_1, \theta_1, \dots, i_m \rangle \in H^?$.

Lemma 4.1. *For every history $h \in \mathbb{H}(x_{\bar{N}})$, there holds*

$$P(\Omega^{\text{Amn}} \mid (x_{\bar{N}}, h)) \geq P(\Omega^*(h) \cup \Omega^{\text{Amn}} \mid (x_{\bar{N}}, h)) \cdot \pi^{l(h)}. \quad (7)$$

Proof. We go by (backward) induction along the game tree.

Step 4.1.1. *The inequality (7) holds when $h \in \bar{H}$.*

Proof. If $h \in \bar{H}$, then $l(h) = 0$ and $\Omega^*(h) = \emptyset$; hence (7) holds as an equality. \square

Although all penultimate histories belong to $H^?$, it is more convenient to start with $h \in H^!$.

Step 4.1.2. *Let $h \in H^! \cap \mathbb{H}(x_{\bar{N}})$, and let inequality (7) hold for all $h' \in H^{\rightarrow}(h) \cap \mathbb{H}(x_{\bar{N}})$. Then (7) holds for h too.*

Proof. First, we notice that $H^{\rightarrow}(h)$ consists of strings $\langle h, i \rangle$ for $i \in N$, $l(h') = l(h)$ for every $h' \in H^{\rightarrow}(h)$, and $\Omega^*(h)$ is partitioned into $\Omega^*(h')$ for $h' \in H^{\rightarrow}(h)$. Applying (5) first to $A = \Omega^{\text{Amn}}$ and then to $A = \Omega^*(h) \cup \Omega^{\text{Amn}}$, and invoking the induction hypothesis, we obtain

$$\begin{aligned} P(\Omega^{\text{Amn}} \mid (x_{\bar{N}}, h)) &= \sum_{h' \in H^{\rightarrow}(h) \cap \mathbb{H}(x_{\bar{N}})} \rho(h' \mid (x_{\bar{N}}, h)) \cdot P(\Omega^{\text{Amn}} \mid (x_{\bar{N}}, h')) \geq \\ &\sum_{h' \in H^{\rightarrow}(h) \cap \mathbb{H}(x_{\bar{N}})} \rho(h' \mid (x_{\bar{N}}, h)) \cdot P(\Omega^*(h') \cup \Omega^{\text{Amn}} \mid (x_{\bar{N}}, h')) \cdot \pi^{l(h)} = P(\Omega^*(h) \cup \Omega^{\text{Amn}} \mid (x_{\bar{N}}, h)) \cdot \pi^{l(h)}, \end{aligned}$$

which is the inequality (7). \square

Step 4.1.3. *Let $h \in H^? \cap \mathbb{H}(x_{\bar{N}})$, and let inequality (7) hold for all $h' \in H^{\rightarrow}(h) \cap \mathbb{H}(x_{\bar{N}})$. Then (7) holds for h too.*

Proof. First, we notice that $H^\rightarrow(h)$ consists of strings $\langle h, \text{Yes} \rangle$ and $\langle h, \text{No} \rangle$, and that $l(h') = l(h) - 1$ for both $h' \in H^\rightarrow(h)$. If the last proposal in h is addressed to the current owner, i.e., that owner decided to keep the bottle for the time being, then, similarly to the proof of Step 4.1.2, $\Omega^*(h)$ is partitioned into $\Omega^*(h')$ for $h' \in H^\rightarrow(h)$ and hence (7) follows from the induction hypothesis in the same way.

Otherwise, i.e., when the current owner j asks another player i to take the bottle, $\Omega^*(h) = \Psi(\langle h, \text{Yes} \rangle \mid (x_{\bar{N}}, h)) \cup \Omega^*(\langle h, \text{No} \rangle)$. If $\rho(\langle h, \text{Yes} \rangle \mid (x_{\bar{N}}, h)) = 0$, then we argue in essentially the same way as in the previous paragraph.

Assuming $\rho(\langle h, \text{Yes} \rangle \mid (x_{\bar{N}}, h)) > 0$, we start with this auxiliary optimization problem, where q and π are constants from the closed interval $[0, 1]$ and $s_Y, s_N, p_Y, p_N, q_Y, q_N$ are controlled variables, also from $[0, 1]$:

$$s_Y p_Y + s_N p_N \Rightarrow \min \quad (8)$$

subject to:

$$s_Y + s_N q_N = q; \quad (9a)$$

$$q_Y \geq \pi; \quad (9b)$$

$$p_Y \geq q_Y \pi^{l(h)-1}; \quad (9c)$$

$$p_N \geq q_N \pi^{l(h)-1}; \quad (9d)$$

$$s_Y + s_N = 1. \quad (9e)$$

This is a rather cumbersome, non-linear problem; however, it is not so difficult to unravel. First, we produce a feasible choice: $s_Y = q$, $s_N = 1 - q$, $p_Y = \pi^{l(h)}$, $q_Y = \pi$, and $p_N = q_N = 0$. The value of the left-hand side in (8) is then $s_Y p_Y = q \pi^{l(h)}$. Let us show that this is, actually, the minimum.

Since all variables are non-negative, there would be no point in strict inequalities in (9b), (9c) and (9d). Therefore, we can assume $p_Y = \pi^{l(h)}$ and $p_N = q_N \pi^{l(h)-1}$. Re-arranging (9a), we obtain $s_N q_N = q - s_Y$; hence $s_N p_N = (q - s_Y) \pi^{l(h)-1}$. Substituting this into (8), we re-write the function to be minimized as $s_Y (\pi - 1) \pi^{l(h)-1} + q \pi^{l(h)-1}$. In other words, it decreases in s_Y when all restrictions are taken into account. Since $s_Y \leq q$ by (9a), we must have $s_Y = q$, i.e., the above choice is, indeed, optimal.

Now we return to the assumptions of Step 4.1.3, denoting $q := P(\Omega^*(h) \cup \Omega^{\text{Amm}} \mid (x_{\bar{N}}, h))$, $s_Y := \rho(\langle h, \text{Yes} \rangle \mid (x_{\bar{N}}, h))$, $s_N := \rho(\langle h, \text{No} \rangle \mid (x_{\bar{N}}, h))$, $p_Y := P(\Omega^{\text{Amm}} \mid (x_{\bar{N}}, \langle h, \text{Yes} \rangle))$, $p_N := P(\Omega^{\text{Amm}} \mid (x_{\bar{N}}, \langle h, \text{No} \rangle))$, $q_Y := P(\Omega^*(\langle h, \text{Yes} \rangle) \cup \Omega^{\text{Amm}} \mid (x_{\bar{N}}, \langle h, \text{Yes} \rangle))$, and $q_N := P(\Omega^*(\langle h, \text{No} \rangle) \cup \Omega^{\text{Amm}} \mid (x_{\bar{N}}, \langle h, \text{No} \rangle))$. If $s_N = 0$, then $P(\cdot \mid (x_{\bar{N}}, \langle h, \text{No} \rangle))$ is *not* well-defined; we set $p_N := q_N := 0$ in this case. It is easy to see now that all conditions (9) are satisfied.

Only (9b) may deserve attention. Let us denote $S_i^* := \{s_i \in S_i \mid P(\Omega^*(\langle h, \text{Yes} \rangle) \cup \Omega^{\text{Amm}} \mid (x_{\bar{N}}, \langle h, \text{Yes} \rangle) \& \sigma_i = s_i) < \pi\}$. Supposing, to the contrary, that $q_Y < \pi$, we immediately obtain $P(\sigma_i^{-1}(S_i^*) \mid (x_{\bar{N}}, h)) > 0$. Since $\Omega^*(\langle h, \text{Yes} \rangle) \cup \Omega^{\text{Amm}} \supseteq \{\omega \in \Psi(x_{\bar{N}}, \langle h, \text{Yes} \rangle) \mid u_i(x_{\bar{N}}(\omega), \omega) = G\}$, redefining $x_i(h', s_i) := \text{No}$ for all $s_i \in S_i^*$ and $h' \geq h$, player i would increase his utility, which contradicts the assumption that $x_{\bar{N}}$ is a Nash equilibrium.

Since the left-hand side of (8) is $P(\Omega^{\text{Amm}} \mid (x_{\bar{N}}, h))$, we obtain (7). \square

These three steps imply (7) for all $h \in H$. Lemma 4.1 is proven. \square

Finally, (7) for $h = h^0$ combined with the assumption $P(\Omega^*(h^0)) = 1$ gives us (6a). Thus, the necessity part of the theorem is proven.

4.2 Sufficiency

Let $G, L \in \mathbb{R}_{++}$, $p \in [0, 1]$ and $T \in \mathbb{N}$ be such that $p \geq \pi^T$. We have to produce a BI game with these parameters and a perfect equilibrium in the game such that the Devil's proposal is accepted at once in every state of the world.

We set $N := \{1, \dots, T\}$, $\Omega := N \cup \{0\}$, $P(0) := \pi^T$, and $P(i) := (1 - \pi) \cdot \pi^{T-i}$ for every $i \in N$; $\Omega^{\text{Amm}} := \{0\}$, so $P(\Omega^{\text{Amm}}) \leq p$; for every $i \in N$, $S_i := \{\text{Yes}, \text{No}\}$, $\sigma_i(\omega) := \text{Yes}$ if $\omega \leq i$ and $\sigma_i(\omega) := \text{No}$ otherwise.

To define the equilibrium strategies, we need auxiliary notation. Given $h \in H \setminus \bar{H}$, we define $N(h)$ as the set of $i \in N$ who held the bottle before, or holds at, h , and $N^\rightarrow(h) := \{i \in N \mid \{1, \dots, i\} \cap N(h) = \emptyset\}$. Note that $N \setminus N(h) \neq \emptyset$ since the total number of steps is $T = \#N$, whereas $N^\rightarrow(h)$ may well be empty.

Now, the Devil's strategy is $x_0^0 := \langle T, T-1, \dots, 1 \rangle$. Given $i \in N$, $h \in H_i$, and $s_i \in S_i$, if $h \in H^?$, then

$$x_i^0(h, s_i) := \begin{cases} \langle h, \text{No} \rangle, & s_i = \text{No}; \\ \langle h, \text{No} \rangle, & i \notin N^\rightarrow(h); \\ \langle h, \text{No} \rangle, & l(h) < i; \\ \langle h, \text{Yes} \rangle, & \text{otherwise}; \end{cases} \quad (10)$$

if $h \in H^!$, then

$$x_i^0(h, s_i) := \begin{cases} \langle h, \min\{\max N^\rightarrow(h), l(h)\} \rangle, & N^\rightarrow(h) \neq \emptyset; \\ \langle h, i \rangle, & \text{otherwise}. \end{cases} \quad (11)$$

Suppose all players follow the strategies defined in (10) and (11). Since player T receives no private signal (technically, $\sigma_T(\omega) = \text{Yes}$ for all $\omega \in \Omega$), he accepts the Devil's proposal at the first step. If $\omega = 0$, i.e., there is the bailout, the bottle goes through all players and ends with player 1; everybody gets the gain G . If $\omega = k > 0$, then there is no bailout, while the bottle reaches player k and gets rejected by each player $i < k$, who knows that no bailout is forthcoming. The player k gets $-L$; each player $i > k$ gets G ; each player $i < k$ gets 0. Thus, each player's expected utility is 0.

It is convenient to start checking the equilibrium conditions with (11). Let player i hold the bottle at h . It is clear from (10) that player j would take the bottle from player i only if $j \leq l(h)$, $j \in N^\rightarrow(h)$ (hence $j < i$), and $s_j = \text{Yes}$. Thus, if $N^\rightarrow(h) = \emptyset$ or $s_i = \text{No}$ (hence $s_j = \text{No}$ for all $j < i$), then player i is stuck with the bottle and his actions do not matter. Otherwise, the probability that $s_j = \text{Yes}$

strictly increases in j , so the maximum in the first line of (11) defines the best choice for player i indeed. (The last statement is equally valid for P , P_i^* , or P_*^i .)

Let us turn to (10). If $s_i = \text{No}$, then there will be no bailout, and each player $j < i$ knows there will be no bailout and would not take the bottle from player i . If $i \in N(h)$, then player i cannot gain anything by taking the bottle again. If $i - 1 \notin N^{\rightarrow}(h)$ or $i > l(h)$, then player $i - 1$ would never take the bottle, while the probability that any other player would take it is strictly less than π (again, regardless of whether the probability is assessed under P , P_i^* , or P_*^i). In each of those cases, the only decision for player i compatible with (4) is to reject. Finally, if the last line of (10) is applicable, then the probability of $s_{i-1} = \text{Yes}$ for $i > 1$, or of the bailout for $i = 1$, is just π , so the decision to take the bottle is compatible with the equilibrium condition (4) for player i .

The sufficiency part of the theorem is proven.

5 Concluding remarks

5.1. An alternative interpretation of the “amnesty” may be the appearance of a poorly informed, or reckless, outsider who does not take into account the possibility of a loss. (Spoiler alert.) Readers familiar with the story will note that what happens there is not inconsistent with this interpretation. The game model should be modified in this case, as the “sucker” may emerge at any moment, not necessarily at the end. On the other hand, if we are interested in the maximal effect a given probability of the “amnesty” may produce (as in the theorem above), then the “sucker” must come at the end.

5.2. Since the Devil receives no private signal, P describes his beliefs. In the game defined in Section 4.2, P also describes beliefs of player T .

5.3. There is a drawback to the equilibrium defined in Section 4.2: strategies “always reject” also form a perfect equilibrium in the model and each player gets the same expected utility. If (6a) holds as an equality, then, apparently, nothing could be done about that. Otherwise, we can slightly modify the probabilities in the same model so that “always reject” is no longer a perfect equilibrium, and the equilibrium x_N^0 gives each player a strictly positive expected utility. To be more precise, we may pick $\theta > 1$ such that $p \geq (\theta\pi)^T$, and set $P(0) := (\theta\pi)^T$ and $P(i) := (1 - \theta\pi) \cdot (\theta\pi)^{T-i}$ for every $i \in N$; now the conditional expected utility of player $i \in N$ is 0 if $s_i = \text{No}$ and $(\theta - 1)L > 0$ if $s_i = \text{Yes}$.

5.4. There is some vague similarity with the bubble game of Moinas and Pouget (2013), as well as the absent-minded centipede of Dulleck and Oechssler (1997), but it is rather superficial. In our model there is no doubt among the agents about the nature of the underlying process; it is only small hope that a savior will come at the last moment that *may* keep it going.

5.5. There was no word on mixed strategies so far. Actually, they are already covered by our basic model: player i can employ a mixed strategy if his signal σ_i includes component(s) independent of everything else. Thus, our theorem shows that the use of mixed strategies could not extend the set of (p, π, T) for which an equilibrium is possible; when an equilibrium *is* possible, mixed strategies are not needed.

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