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by

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Separable Aggregation and the Existence of Nash Equilibrium[†]

Nikolai S. Kukushkin[‡]

Abstract

Algebraic conditions for the existence of a Nash equilibrium are studied for two classes of strategic form games: games with public and private objectives and games with decreasing best replies. Considerable evidence is gathered to support the claim that in both cases the crucial factor is the presence of a separable ordering on the set of outcomes, connected with the players' utilities in a certain way. Some technical problems related to the description of separable orderings and to establishing separability are also discussed.
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1. Introduction

It is a usual practice in economic theory to assume everything (or at least as much as possible) smooth and convex. In many cases such assumptions are necessary for reaching clear-cut conclusions, but sometimes they just obscure the real reasons for the existence (or good behavior) of solutions for problems under investigation. Recently this point was raised forcefully by Milgrom and Roberts (1994), who, in particular, pointed out to the importance of monotonicity assumptions.

This paper concerns with purely combinatorial (or algebraic) causes for the existence of a Nash equilibrium, needing no convexity or smoothness. Two classes of strategic form games are considered; the first generalizes the usual model of the voluntary provision of a public good (or a public bad); the second is quite abstract: the strategy sets are ordered and the best replies are decreasing. Besides adding new results to those established earlier, the paper expounds the view that, in both cases, the crucial factor is the presence of separable orderings on $\mathbb{R}^m$ (or its subsets). Unfortunately, the most distinct formulation of the view remains a hypothesis in either case.

It was not without hesitation that I decided to print the results in their current, rather preliminary state. I can put forward three reasons for this decision. First, even the already established results show an unexpected and unexplained similarity between two quite different game-theoretic contexts;

it is difficult not to think that separability can also play an essential role in other situations. Second, there is no grounds to believe that all the remaining open questions could be answered in the foreseeable future. Finally, I have no idea of when I could have another opportunity to relate my results.

Section 2 contains necessary definitions and notations.

Section 3 considers "games with public and private objectives"; the first example of a model of this kind was suggested by Germeier and Vatel' (1974). The most advanced results on the existence of Nash or strong equilibrium in such games are to be found in Kukushkin (1992, 1994b). Theorems 1 and 1' unify these results (in the part concerning Nash equilibrium) and explicitly show the crucial role of separability here. Unfortunately, the necessity of separability for the "persistent" existence of a Nash equilibrium, established under stronger assumptions in Kukushkin (1994b), remains a hypothesis.

Section 4 considers games where strategy sets are (partially) ordered and the best replies are decreasing. It should be noted that in many economic situations the assumption that the best replies are monotonic (i.e. either increasing or decreasing) is quite natural, see Fudenberg and Tirole (1984) and Bulow et al. (1985). However, there is a striking difference between the two cases. The existence of a Nash equilibrium in a game with increasing best replies is guaranteed by Tarski's (1955) theorem (under relatively mild restrictions on the strategy sets). A subclass of such games - games with strategic complementaries, also have nice comparative statics properties, see Topkis (1979), Milgrom and Roberts (1990), Vives (1990), Milgrom and Shannon (1994). On the other hand, not only is the straightforward analogue of Tarski's theorem for decreasing mappings untrue, but the existence of an equilibrium in such a game is by no means a forgone conclusion. As to the comparative statics, I am unaware of any result in this direction (except for duopoly games) and cannot suggest anything myself.

The section is devoted to a sole problem: Under what additional assumptions do decreasing best replies guarantee the existence of a Nash equilibrium? Departing from the fist ever result of this kind, Novshek (1985), refined in Kukushkin (1994a), several new existence theorems are obtained (without any attempt to assess the comparative importance of different sets of assumptions). In each case it turns out that the existence of an equilibrium is guaranteed when there is a separable ordering on the set of outcomes such that each player's preferences are affected by the partners' choices only to the extent that the latter affect the ranking of the outcome. An exact formulation is given to the idea that the existence of an equilibrium should not be expected otherwise, but this statement also remains a hypothesis, except for just one particular case (Theorem 8).

Section 5 may be regarded as an unnecessary technical appendix. The point is that the two necessity results obtained in the area (Kukushkin, 1994b, and Theorem 8 in Section 4) require the continuity assumption and establish additive separability; their proofs are strikingly similar to one another and resemble to some extent the technique used in the classical work on separability, Debreu (1960). Debreu's result was later strengthened by Gorman (1968a), using quite a different approach. Theorem 1 of Section 5 shows that my technique (i.e. an extension of Debreu's one) is enough to advance considerably in the direction of Gorman's result. I believe the Debreu-Gorman theorem is so important that it justifies paying attention to purely technical aspects of feasible proofs.

Theorems, examples, and formulas are numbered in each section separately.

2. General Definitions

An ordering ϑ on a set X is just a reflexive and transitive binary relation on X; the ordering is complete if, for every $x,y \in X$, either $x\vartheta y$ or $y\vartheta x$ (both). The asymmetric part of an ordering ϑ will be denoted ϑ°, i.e. $x\vartheta^{\circ}y$ means $x\vartheta y$ but not $y\vartheta x$. $x \in X' \subseteq X$ is a maximal element of ϑ on X' if no $y \in X'$ satisfies $y\vartheta^{\circ}x$. An ordering ϑ on $\mathbb{R}^m$ (or on a subset of $\mathbb{R}^m$) is called Pareto compatible if $x'\vartheta x''$ whenever x' Pareto dominates x''.

A normal (or strategic) form game is defined by a finite set of players N and, for each $i \in N$, a set of strategies X_i and an ordering $\mathcal{R}_i$ on the set of outcomes $X = \Pi_{i \in N} X_i$ ($x\mathcal{R}_i y$ means that x is better for player i than y). Denoting $X_{-i} = \Pi_{j \neq i} X_j$, we define $R_i(x_{-i})$ (for $x_{-i} \in X_{-i}$) as the set $\{x_i \in X_i \mid (x_i, x_{-i})$ is a maximal element of $\mathcal{R}_i$ on $\{(x_i, x_{-i}) \mid x_i \in X_i\}\}$. Viewed as a correspondence $X_{-i} \to X_i$, R_i is called the best reply correspondence (for player i); generally speaking, it may have empty values for some (or even all) $x_{-i} \in X_{-i}$.

An outcome $x^{\circ} \in X$ is a Nash equilibrium if $x_i^{\circ} \in R(x_{-i}^{\circ})$ for all $i \in N$.

It should be noted that we allow the players to have incomplete preference orderings (cf. Aumann, 1962, 1964; Rozen, 1976). This fact does not mean that I attach any particular importance to such models; just there is no need to assume the completeness. By the way, the above definition gives a rather weak interpretation of the concept of Nash equilibrium for this case.

Now I introduce an abstract definition playing a crucial part throughout all the paper. Suppose there is a finite set N and an ordering ϑ on $X = \Pi_{i \in N} X_i$; suppose also that $M \subseteq N$ and ϑ_M is an ordering on $X_M = \Pi_{i \in M} X_i$. We say that ϑ_M is a separable projection of ϑ on X_M if, for every $x^M, y^M \in X_M$ and $x^{-M} \in X_{-M}$, $\langle x^M, x^{-M} \rangle \vartheta \langle y^M, x^{-M} \rangle$ if and only if $x^M \vartheta_M y^M$. Note that if ϑ is complete, then ϑ_M is also complete.

The separability property emerges naturally in a wide variety of situations and was studied by many authors. In particular, the most popular aggregation rules in social choice theory - utilitarian and leximin ones, see e.g. d'Aspremont and Gevers (1977) or Deschamps and Gevers (1978) - have separable projections on all coordinate subspaces. Leximax and fixed-order lexicographies have the same property.

In his seminal paper Debreu (1960) showed that every continuous, Pareto compatible ordering on $\mathbb{R}^m$ ($m>2$) which has a separable projection on every two-dimensional subspace can be represented by the sum of rescaled coordinates (i.e. is utilitarian in a somewhat generalized sense). In fact, Debreu's result was formulated for orderings on more general sets but, as he showed himself, the general case is easily reduced to that of a rectangular subset of $\mathbb{R}^m$. The same statement was earlier reported by Fleming (1952), but he assumed smoothness and treated infinitesimals rather freely. (The result of Harsanyi (1955), also establishing conditions for additive separability, apparently has no connection with Debreu's theorem either technically or conceptually). In another seminal paper, Gorman (1968a) strengthened the theorem considerably, weakening the separability assumption; see also the edifying discussion between Gorman (1971a,b) and Vind (1971a,b, 1974).

Debreu's approach was based on a geometric theorem due to W. Blaschke; later Fishburn (1970, Chapter 5) replaced the reference to Blaschke's theorem with a direct inductive reasoning. Curiously, the proof of additive separability (in a particular context) developed in Kukushkin (1994b, Appendix) (without any knowledge of the Debreu-Gorman theorem, I must confess) resembles that of Fishburn (1970, Theorem 5.4). From the geometrical viewpoint, it could be interpreted as extending a version of Blaschke's theorem to a bunch of planes (unfortunately, I am no expert in either geometry or German to say whether Blaschke himself ever considered such an extension). The proof is recalled in the following several times; in principle, the reader may substitute "as in Fishburn (1970)" for "as in Kukushkin (1994b)" in each case, although this would make the reference a bit less relevant.

3. Games with Public and Private Objectives

A game with public and private objectives is a strategic form game of the following type. There is a finite set of players N (without restricting generality, we assume $0 \notin N$) and, for each $i \in N$, a compact strategy set X_i, a continuous function $f_i: X_i \to \mathbb{R}$, and a Pareto compatible ordering ρ_i on $\mathbb{R}^2 = \mathbb{R}_0 \times \mathbb{R}_i$; there is also a continuous function $f_0: X \to \mathbb{R}$. Now the preferences of each player i on the set of outcomes X are these:

$$x \mathcal{R}_i y \text{ if and only if } (f_0(x), f_i(x_i)) \; \rho_i \; (f_0(y), f_i(y_i)). \quad (1)$$

There is no real necessity to demand that ρ_i should be defined on the whole $\mathbb{R}^2$; strictly speaking, it would be sufficient to have it on the set $(f_0 \times f_i)(X)$. However, for the interpretation of some results to follow, it is convenient to perceive ρ_i as a stable characteristic of player i who may find himself in different situations, playing different games. $\mathbb{R}^2$ could be replaced with an open rectangular subset if only games with $(f_0 \times f_i)(X)$ included in the subset are to be considered.

Throughout this section we will denote M the set $\{0\} \cup N$, i.e. the set of all (public and private) objective functions in the model under consideration.

Theorem 1. Suppose each strategy set X_i ($i \in N$) is finite and there exists an ordering ϑ on $\mathbb{R}^M$ having separable projections ϑ_i on all planes $\mathbb{R}_0 \times \mathbb{R}_i$ such that $a \rho_i^{\circ} b$ implies $a \vartheta_i^{\circ} b$ for all $i \in N$, $(a,b) \in \mathbb{R}_0 \times \mathbb{R}_i$. Then the game has a Nash equilibrium.

The proof is quite straightforward: consider the mapping $f: X \to \mathbb{R}^M$ defined by $f(x) = (f_0(x), \langle f_i(x_i) \rangle_{i \in N})$ and let $f(x^{\circ})$ be a maximal point of ϑ on $f(X)$. If x° is not a Nash equilibrium, then there exist $i \in N$ and $y \in X$ such that $y \mathcal{R}_i^{\circ} x^{\circ}$ and $y_j = x_j^{\circ}$ for all $j \neq i$. Therefore, $(f_0(y), f_i(y_i))\ \vartheta_i^{\circ}\ (f_0(x^{\circ}), f_i(x_i^{\circ}))$, hence (by separability) $f(y) \vartheta f(x^{\circ})$, contradicting the choice of x°.

We call an ordering ϑ on $\mathbb{R}^M$ regular if ϑ has a maximal element on every compact subset of $\mathbb{R}^M$.

Theorem 1'. Assume everything as in Theorem 1 with the exception that each X_i is compact and ordering ϑ is regular; then the game has a Nash equilibrium.

The proof is virtually the same.

As particular cases of Theorem 1' could be listed the main result of Kukushkin (1992) in its sufficiency part (and restricted to Nash equilibrium), and the theorem (sufficiency part) and Proposition 3 of Kukushkin (1994b). The last case may deserve an explanation; first of all, I remind that every ordering ρ_i on $\mathbb{R}^2$ was there defined by the function

$$F^{\circ}(a,b) = \begin{cases} a \cdot b, & \text{if } a>0 \text{ and } b>0, \\ \min\{a,b\}, & \text{otherwise.} \end{cases}$$

Now for any $\langle a_i \rangle_{i \in M} \in \mathbb{R}^M$, define $\langle b_i \rangle_{i \in M}$ as follows: if $a_i \leq 0$, then $b_i = a_i$, otherwise, b_i is the product of all positive a_i's; now the leximin ordering on b's induces an ordering ϑ on a's, which has a separable projection ϑ_i on every X_{-i} and the ordering defined by F° is a coarsening of ϑ_i.

Theorems 1 and 1' are also applicable to players with lexicographical preferences (leximin, leximax, or fixed-order lexicography). The result of Kukushkin (1992) does not cover these situations. It should be noted also that the proof of the latter result is not completely analogous to that of Theorem 1(′). Consider the following example.

Example. Let $N=\{1,2\}$, $X_1=X_2=\{1,2\}$, $f_1(x_1)\equiv 2$, $f_2(x_2)=x_2$, $f_0(1,1)=1$, $f_0(2,2)=2$, $f_0(x_1,x_2)=0$ if $x_1\neq x_2$, $u_i(x) = \min\{f_0(x), f_i(x_i)\}$. Thus we have the following bimatrix game:

$$\begin{array}{cc} (1,1) & (0,0) \\ (0,0) & (2,2) \end{array}$$

Obviously, there are two Nash equilibria, only one of which is strong, i.e. Pareto optimal. The proof from Kukushkin (1992) selects the strong equilibrium (Leximin in the utility space), while the proof of Theorem 1 may select either of them because in the space $\mathbb{R}^M$ they produce equivalent vectors (1,2,2).

Another (and rather peculiar) example is produced by the Pareto ordering as ϑ: it is regular and its separable projection on any subspace is the Pareto ordering over that subspace.

Hypothesis. Suppose there is a finite set N and, for each $i\in N$, a Pareto compatible ordering ρ_i on $\mathbb{R}^2=\mathbb{R}_0\times\mathbb{R}_i$; suppose also that for any collection of finite sets X_i $(i\in N)$ and any functions $f_i\colon X_i \to \mathbb{R}$ and $f_0\colon X \to \mathbb{R}$, the game with strategy sets X_i and preference relations $\mathcal{R}_i$ defined by (1) has a Nash equilibrium. Then there exists an ordering ϑ on $\mathbb{R}^M$ having separable projections ϑ_i on all planes $\mathbb{R}_0\times\mathbb{R}_i$ such that $x\rho_i^{\circ}y$ implies $x\vartheta_i^{\circ}y$ for all $i\in N$, $x,y\in\mathbb{R}_0\times\mathbb{R}_i$.

Remark. The hypothesis is formulated as a converse to Theorem 1 rather than Theorem 1′ because I have no plausible hypothesis concerning the regularity property (in fact, I even have no example of a separable ordering, say, on $\mathbb{R}^3$ which is not regular).

Particular cases of this hypothesis are proven in Kukushkin (1992, 1994b): necessity parts of the main theorems. If the hypothesis were proved in its full generality, then the latter result would follow directly from the Debreu-Gorman theorem; interestingly, its homogeneous version can be proved with the same technique as used by Debreu (1960). I formulate this version as a separate result here.

Theorem 2. Suppose there is a continuous, strictly increasing (in both arguments) function F: $\mathbb{R}^2 \to \mathbb{R}$ such that for any finite set N, any collection of finite sets X_i ($i \in N$), and any functions f_i: $X_i \to \mathbb{R}$ and f_0: $X \to \mathbb{R}$, the game with strategy sets X_i and utilities $u_i(x) = F(f_0(x), f_i(x_i))$ has a Nash equilibrium. Then there exist continuous, strictly increasing functions λ, μ_0, μ: $\mathbb{R} \to \mathbb{R}$ such that

$$F(a,b) = \lambda(\mu_0(a) + \mu(b)) \quad \text{for all } (a,b) \in \mathbb{R}^2.$$

Lemma 4 (b) from Kukushkin (1994b, Appendix), in projection to the case under consideration effectively says that if we have $a_0 < b_0 < c_0$ and $a_1 < b_1 < c_1$ such that $F(a_0,b_1) = F(b_0,a_1)$ and $F(a_0,c_1) = F(b_0,b_1) = F(c_0,a_1)$, then $F(b_0,c_1) = F(c_0,b_1)$. And this is Blaschke's condition for the existence of a homeomorphic transformation of the plane preserving the lines parallel to the coordinate axes and converting the indifference curves of F into straight lines (parallel to one another) - exactly the same condition as used by Debreu (1960) for similar purposes.

It would be interesting to obtain a general formulation (with separable orderings) for the main result of Kukushkin

(1992), showing the existence of a strong equilibrium among players with minimum- or maximum-like utilities in a game with public and private objectives. Unfortunately, my thoughts on the subject are too preliminary to be written here; the above example shows some of the emerging difficulties. I can only state that no connection with the properties of similar aggregation rules studied in the literature (see e.g. Moulin 1988; Thomson and Lensberg, 1989) has been established as yet.

4. Games with Decreasing Best Replies

4.1. Preliminaries

We will call a function f increasing if $x \le y$ implies $f(x) \le f(y)$ and decreasing if $x \le y$ implies $f(x) \ge f(y)$.

Throughout the section, we assume that in the game under consideration each best reply correspondence has non-empty values for all x_{-i}. In fact we will study fixed points rather than Nash equilibria; the basic model may be described as follows: There is a finite set (of players) N, and for each $i \in N$ there is a (partially) ordered set X_i and a correspondence $R_i\colon X_{-i} \to X_i$, allowing a decreasing single-valued selection $r_i\colon X_{-i} \to X_i$. A fixed point for the system of $\langle R_i \rangle_{i \in N}$ is a point $x^o \in X = \Pi_{i \in N} X_i$ such that

$$x_i^{\,o} \in R_i(x_{-i}^{\,o}) \qquad \text{for all } i \in N.$$

In many cases the correspondences R_i may be replaced with their (arbitrary) decreasing single-valued selections r_i, in which case the definition of a fixed point becomes this:

$$x_i^{\,o} = r_i(x_{-i}^{\,o}) \qquad \text{for all } i \in N. \tag{1}$$

When one is interested in fixed points of a system of increasing mappings, Tarski's (1955) theorem gives a general existence result: if all X_i's are complete lattices and r_i's increasing, then (1) is always satisfied for some x°. However, the situation with decreasing mappings is more complicated as the straightforward analogue of Tarski's theorem for such mappings is not true (consider e.g. the set $X=\{0,1\}$ and the mapping $r(0)=1$, $r(1)=0$). Nonetheless, Tarski's theorem could sometimes be applied to systems of decreasing mappings too.

The first result of this kind was obtained by Vives (1990): If $\#N=2$ and each X_i is a complete lattice then a fixed point always exists. To see that, let us turn one of the sets, say X_2, upside down; now both mappings are increasing and Tarski's theorem implies the existence of a fixed point (since (1) contains no inequality, our trick with turning X_2 does no harm). There is no obvious way to extend the trick (attributed by Milgrom and Roberts, 1990, to X. Vives) to $\#N>2$; moreover, the corresponding statement is just not true.

Example 1. Let $N=\{1,2,3\}$, $X_i=\{0,1\}$ $(i \in N)$ and $r_1(x_2,x_3)=1-x_2$, $r_2(x_1,x_3)=1-x_3$, $r_3(x_1,x_2)=1-x_1$. It is easy to see that no x° satisfies (1): we should have $x_1^{\circ}=1-x_2^{\circ}=x_3^{\circ}=1-x_1^{\circ}$.

4.2. "Novshek-style" Results

The first ever result on the existence of a Nash equilibrium with decreasing best replies is apparently due to Novshek (1985). He showed (although his proof is open to criticism concerning its mathematical rigor) that decreasing best replies in the Cournot model imply the existence of an equilibrium without any further assumptions (it should be noted that this property holds in a "great majority" of models considered in the literature). Novshek's proof was refined in Kukushkin (1994a), the main result of which paper sounds as follows:

Theorem 1. Suppose N is a finite set and for each $i \in N$ there is a compact set X_i of reals and an upper hemi-continuous correspondence $R_i\colon S_i \to X_i$, where $S_i = \sum_{j \in N \setminus \{i\}} X_j$, allowing of a decreasing single-valued selection. Then there exists a point $x^{\circ} \in X = \prod_{i \in N} X_i$ such that

$$x_i^{\circ} \in R_i(\textstyle\sum_{j \neq i} x_j^{\circ}) \qquad \text{for all } i \in N.$$

The key point in the proof is the lemma stating that the theorem is valid if every X_i is the set of integers satisfying the inequalities $0 \le x_i \le m_i$; let us denote such a set $\mathrm{Int}[0,m_i]$. The lemma is proved by a backward induction process, which in fact somewhat simplifies that developed by Novshek for continuous variables. Then it can easily be seen that the theorem remains true if every X_i is a subset of $\mathrm{Int}[0,m_i]$; finally, an integer approximation proves the theorem for arbitrary compact X_i's (here the continuity of addition and assumed upper hemi-continuity of each R_i are essential).

If all X_i's are finite, every mapping has a closed graph, so there is no need to consider correspondences (however, the continuity of addition remains essential).

Theorem 1'. Suppose N is a finite set and for each $i \in N$ there is a finite set X_i of reals and a decreasing function $r_i\colon S_i \to X_i$, where $S_i = \sum_{j \in N \setminus \{i\}} X_j$. Then there exists a point

$x^{\circ} \in X = \Pi_{i \in N} X_i$ such that

$$x_i^{\circ} = r_i(\Sigma_{j \neq i} x_j^{\circ}) \qquad \text{for all } i \in N. \qquad (2)$$

If the finiteness assumption in Theorem 1′ is replaced with compactness (or compactness in order interval topology), the proof through approximation as in Kukushkin (1994a) becomes obviously invalid. However, I know no example disproving such a statement and even strongly suspect that in fact it is true (I had spent some time under the impression that I had a proof to the effect before recognizing its invalidity).

As was noted in Kukushkin (1994a), the difference between the situations covered by Theorems 1 and 1′ and the general case lies in the "limited interdependence" property: the partners' choices enter every player's utility only through their aggregate, the sum. It is important for deeper understanding of the theorems to note that their straightforward extension to the multi-dimensional case is untrue.

Example 2. Consider the three mappings $r_i \colon \mathbb{R}^3 \to \mathbb{R}^3$ $(i=1,2,3)$:

$$r_1(s_1,s_2,s_3) = \begin{cases} (1,0,0), & s_2 \le 1/2, \\ (0,0,0), & s_2 > 1/2, \end{cases}$$

$$r_2(s_1,s_2,s_3) = \begin{cases} (0,1,0), & s_3 \le 1/2, \\ (0,0,0), & s_3 > 1/2, \end{cases}$$

$$r_3(s_1,s_2,s_3) = \begin{cases} (0,0,1), & s_1 \le 1/2, \\ (0,0,0), & s_1 > 1/2. \end{cases}$$

The ranges of the mappings are $X_1 = \{(0,0,0),\ (1,0,0)\}$, $X_2 =$

$\{(0,0,0),\ (0,1,0)\}$, $X_3 = \{(0,0,0),\ (0,0,1)\}$.

All the three functions are decreasing, but no point $x^\circ = \langle x_1^\circ, x_2^\circ, x_3^\circ\rangle$ from the product of X_i's satisfies (2): if e.g. $x_1^\circ=(0,0,0)$, we would have $x_3^\circ=(0,0,1)$, hence $x_2^\circ=(0,0,0)$, hence $x_1^\circ=(1,0,0)$ (essentially, this example is equivalent to the previous one). No fixed point will emerge if we replace each X_i e.g. with its convex hull; the functions can be slightly modified so that they become strictly decreasing in every argument.

In fact, there is no "limited interdependence" in the example since knowing the sum of x_i's allows one to know each of them separately.

There may be different opinions about the exact message of Example 2. I will interpret it as showing that the sum in Theorems 1 and 1' should be understood as a means to define a complete ordering on X_{-i} rather than just an algebraic operation applicable to multi-dimensional objects. This interpretation is supported by the following simple result.

Theorem 2. Suppose N is a finite set and for each $i\in N$ there is a partially ordered, finite set X_i, an increasing real-valued function f_i on X_i, and a decreasing function $r_i\colon S_i \to X_i$, where $S_i = \sum_{j\in N\setminus\{i\}} f_j(X_j)$ $(\subseteq\mathbb{R})$. Then there exists a point $x^\circ \in X = \Pi_{i\in N} X_i$ such that

$$x_i^\circ = r_i(\textstyle\sum_{j\neq i} f_j(x_j^\circ)) \qquad \text{for all } i\in N.$$

It is sufficient to apply Theorem 1' to the sets $Y_i=f_i(X_i)\subseteq\mathbb{R}$ and the mappings $q_i=f_i\circ r_i$. Having a fixed point y°, $y_i^\circ=f_i\circ r_i(\sum_{j\neq i} y_j^\circ)$, we denote $x_i^\circ=r_i(\sum_{j\neq i} y_j^\circ)$; now $f_i(x_i^\circ)=y_i^\circ$, so $r_i(\sum_{j\neq i} f_j(x_j^o)) = r_i(\sum_{j\neq i} y_j^o) = x_i^o$.

The leximax ordering turns out to be almost as good, from our current viewpoint, as the utilitarian one, defined by the sum. We will write $x' \mathcal{L} x''$ (for $x', x'' \in \mathbb{R}^m$) whenever x' dominates x'' in the leximax sense or both are equivalent in the same sense. A mapping $r: \mathbb{R}^m \to \mathbb{R}$ is called leximax decreasing if $x' \mathcal{L} x''$ implies $r_i(x') \leq r_i(x'')$.

Theorem 3. Suppose N is a finite set and for each $i \in N$ there is a finite set $X_i \subseteq \mathbb{R}$ and a leximax decreasing mapping $r_i: X_{-i} \to X_i$. Then there exists a point $x^o \in X = \Pi_{i\in N} X_i$ such that

$$x_i^o = r_i(x_{-i}^o) \qquad \text{for all } i \in N. \tag{1}$$

Denoting $\mathbb{N}$ the set of all non-negative integers, we define $f: \mathbb{N} \to \mathbb{N}$ by $f(k) = (n-1)^k - 1$, where $n = \#N$, $k \in \mathbb{N}$, and $g: \cup_{i\in N} X_i \to \mathbb{N}$ as the rank function ($g(x)=0$ for the smallest of all x_i (all $i \in N$), $g(x)=1$ for the second smallest, etc.). The following equivalence holds for all $x', x'' \in \mathbb{N}^{n-1}$ because $f(k) > (n-1)\cdot f(k-1)$ for any $k \geq 1$:

$$\textstyle\sum_{j=1}^{n-1} f(x_j') \geq \sum_{j=1}^{n-1} f(x_j'') \quad \text{if and only if } x' \mathcal{L} x''. \tag{3}$$

For each $i \in N$ we define $Y_i = f \circ g(X_i) \subseteq \mathbb{N}$ and $S_i = \sum_{j\neq i} Y_i$ $(\subseteq \mathbb{N})$; since r_i is leximax decreasing, it easily follows from (3) that there exists a decreasing function $h_i: S_i \to X_i$ such that

$$r_i(x_{-i}) = h_i(\textstyle\sum_{j\neq i} (f \circ g(x_j))) \quad \text{for all } x_{-i} \in X_{-i}. \tag{4}$$

Now it follows from Theorem 1' that the system of sets Y_i and functions $q_i = f \circ g \circ h_i: S_i \to Y_i$ has a fixed point y^o such that $y_i^o = q_i(\sum_{j\neq i} y_j^o)$ $(i \in N)$. Denoting $x_i^o = g^{-1} \circ f^{-1}(y_i^o)$ for each $i \in N$, we have $f \circ g(x_i^o) = f \circ g \circ h_i(\sum_{j\neq i} f \circ g(x_j^o))$. Since both f and

g are monomorphisms, from (4) we obtain $x_i^\circ = r_i(x_{-i}^\circ)$.

Remark 1. A particular case of leximax decreasing function is a function depending only on $\max_{j\in N\setminus\{i\}} x_j$. Since the maximum function is continuous, the complete analogue of Theorem 1 for this case is valid. Interestingly, I have another proof of this statement, based directly on Tarski's theorem rather than on Novshek's induction process, but I can see no point in actually writing it down here. For general leximax decreasing functions, the validity of a similar statement (even if the closed graph is assumed) remains unclear.

Remark 2. By replacing each X_i with $-X_i$, we obtain the exact analogue of Theorem 3 for mappings decreasing w.r.t. leximin ordering and the analogue of Theorem 1 for mappings depending on the minimum (as in the above remark).

Remark 3. The extensions of Theorem 3 described in previous remarks are obviously applicable to games with utilities of the form $u_i(x_i, \max_{j\neq i} x_j)$ or of the form $u_i(x_i, \min_{j\neq i} x_j)$ (provided the best replies exist, have closed graphs, and are decreasing). Such utilities really emerge in some public good models, see e.g. Hirshleifer (1983).

Quite a similar approach works for the function F° considered in Kukushkin (1994b):

$$F^\circ(\langle x_i \rangle_{i\in N}) = \begin{cases} \Pi\, x_i, & \text{if all } x_i > 0, \\ \min x_i, & \text{otherwise} \end{cases}$$

(note that the definition is equally meaningful for any number of arguments).

Theorem 4. Suppose N is a finite set and for each $i \in N$ there is a compact set X_i of reals and an upper hemi-continuous correspondence $R_i: S_i \to X_i$, where $S_i = F^{\circ}(\Pi_{j \in N \setminus \{i\}} X_j)$, allowing of a decreasing single-valued selection. Then there exists a point $x^{\circ} \in X = \Pi_{i \in N} X_i$ such that

$$x_i^{\circ} \in R_i(F^{\circ}(\langle x_j^{\circ} \rangle_{j \neq i})) \qquad \text{for all } i \in N.$$

The proof follows the same lines as in Theorem 3 and in Remark 1 after it. First we consider finite models where positive x_i's form a geometric progression (with the same multiplier for all $i \in N$) while negative x_i's may be assumed integer. Every such model is isomorphic to a model satisfying the assumptions of Theorem 1': the isomorphism is established by applying a logarithmic function to positive x_i's and a function similar to $f(\cdot)$ from the proof of Theorem 3 to negative x_i's; therefore, it has a fixed point. Every model with compact X_i's can be approximated with such models just as in Proposition 3 of Kukushkin (1994a).

Certainly, analogues of Theorem 2 for dependencies considered in Theorems 3 and 4 (and in the remarks following the former) are also valid; in the following, I will restrict myself to the case $X_i \subseteq \mathbb{R}$ without explicitly mentioning the possibility of such an extension every time.

4.3. On Sufficiency in General

Now let us consider the question: what unites all the results of the previous subsection? In my view, it is the separability property.

Hypothesis 1. Suppose there is a finite set N and, for each $i\in N$, there is a finite set X_i of reals and a mapping $r_i\colon X_{-i} \to X_i$; suppose also that there is a complete, Pareto compatible ordering ϑ on X ($\subseteq\mathbb{R}^N$), having separable projections on all sets X_{-i}; suppose finally that each r_i is decreasing w.r.t. the projection ϑ_{-i} (i.e. $x'\vartheta_{-i}x''$ implies $r_i(x')\le r_i(x'')$). Then there exists a fixed point satisfying (1).

Remark. The assumption that ϑ is complete is essential: otherwise Pareto dominance would be acceptable and the hypothesis would be disproved by Example 1 of this section (Subsection 4.1). Thus we see a technical difference with Theorem 1 of Section 3.

It is easy to see that Theorems 1′, 3, and 4 (reduced to finite sets) are indeed particular cases of this statement. Unfortunately, I am unable to prove Hypothesis 1 in its full generality. The main difficulty lies in my inability to describe all separable orderings on finite sets. As a kind of inductive support for the hypothesis, I end up this subsection with a couple of examples not covered by Novshek's scheme. For simplicity, we will restrict ourselves to the case $\#N=3$.

Theorem 5. Suppose there are given three sets $X_i\subseteq\mathbb{R}$ ($i=1,2,3$) compact in the order interval topology; suppose there are three functions $r_i\colon X_{-i} \to X_i$ such that $r_3(x_1,x_2)$ is decreasing in both arguments (not necessarily strictly), $r_1(x_2,x_3)$ is lexicographically decreasing in the sense that $r_1(x_2',x_3') \le r_1(x_2'',x_3'')$ if $x_2'>x_2''$, or $x_2'=x_2''$ and $x_3'>x_3''$, and $r_2(x_1,x_3)$ is lexicographically decreasing in the same sense (first x_1 matters and only then x_3). Then there exists a fixed point satisfying

$$x_i^{\circ} = r_i(x_{-i}^{\circ}) \qquad\qquad \text{for all } i \in N. \qquad (1)$$

Given x_2, we have two decreasing mappings, $r_1(x_2,\cdot)$ and $r_3(\cdot,x_2)$, between X_1 and X_3. By Tarski's theorem (with turning one of the sets upside down), there exists a fixed point $x_1=q_1(x_2)$, $x_3=q_3^{(2)}(x_2)$ such that $q_1(x_2)=r_1(x_2,q_3^{(2)}(x_2))$, $q_3^{(2)}(x_2)=r_3(q_1(x_2),x_2)$. Now $x_2'<x_2''$ implies $r_1(x_2',x_3') \geq r_1(x_2'',x_3'')$ for any x_3',x_3''; therefore, $q_1(\cdot)$ is decreasing. Similarly, there exist (for each $x_1 \in X_1$) $q_2(x_1) \in X_2$ and $q_3^{(1)}(x_1)$ such that $q_2(x_1)=r_2(x_1,q_3^{(1)}(x_1))$ and $q_3^{(1)}(x_1)=r_3(x_1,q_2(x_1))$; $q_2(\cdot)$ is also decreasing. Applying Tarski's theorem to the pair $q_1(\cdot)$, $q_2(\cdot)$, we obtain x_1°, x_2° such that $x_1^{\circ}=q_1(x_2^{\circ})$ and $x_2^{\circ}=q_2(x_1^{\circ})$. Define $x_3^{\circ} = q_3^{(1)}(x_1^{\circ}) = r_3(x_1^{\circ},x_2^{\circ}) = q_3^{(2)}(x_2^{\circ})$. Obviously, x_1°, x_2°, x_3° constitute the fixed point needed.

Theorem 6. All the assumptions are the same as in Theorem 5 with the following exception: $r_1(\cdot,\cdot)$ and $r_2(\cdot,\cdot)$ are lexicographically decreasing in the reversed sense, i.e. r_1 reacts to x_3 first and then to x_2, and r_2 reacts to x_3 and then to x_1. (There seems to be no formal equivalence between the two situations). Then there exists a fixed point satisfying (1).

For each x_3, there exist $q_1(x_3) \in X_1$ and $q_2(x_3) \in X_2$ such that $q_1(x_3)=r_1(q_2(x_3),x_3)$ and $q_2(x_3)=r_2(q_1(x_3),x_3)$ (by Tarski's theorem applied to a duopoly with decreasing best replies). Since $x_3'<x_3''$ implies $r_1(x_2',x_3') \geq r_1(x_2'',x_3'')$ for all $x_2',x_2'' \in X_2$, $q_1(\cdot)$ is decreasing; similarly, $q_2(\cdot)$ is decreasing. Now we have two decreasing mappings $r_3\colon X_{-3} \to X_3$ and $q_1 \times q_2\colon X_3 \to X_{-3}$; Tarski's theorem implies the existence of x°

such that $x_3^o=r_3(x_1^o,x_2^o)$, $x_1^o=q_1(x_3^o)$, and $x_2^o=q_2(x_3^o)$; obviously, this is just what we need.

To connect Theorems 5 and 6 with Hypothesis 1, appropriate separable orderings should be produced. In the case of Theorem 5, I suggest lexicographical maximization of the following functions (listed in the order of decreasing importance): $<-r_3(x_1,x_2),\ x_1,\ x_2,\ x_3>$. For Theorem 6, the same functions should be lexicographically maximized, but in a different order of importance: $<x_3,\ -r_3(x_1,x_2),\ x_1,\ x_2>$. (Were the function $r_3(\cdot,\cdot)$ supposed strictly decreasing, the terms x_1 and x_2 could be omitted in both cases).

Theorem 7. All the assumptions are the same as in Theorem 5 with the following exception: there exists a level x_3^* such that, whenever $x_3 \ge x_3^*$, $r_1(x_2,x_3)$ and $r_2(x_1,x_3)$ are lexicographically decreasing in the sense of Theorem 6, while for $x_3<x_3^*$, they are decreasing in the sense of Theorem 5 (to preserve monotonicity, we also require $r_i(x_j',x_3') \ge r_i(x_j'',x_3'')$, whenever $x_3'<x_3^* \le x_3''$ for all $i,j = 1,2$, $i \ne j$, $x_j',x_j'' \in X_j$). Then there exists a fixed point satisfying (1).

Denote $x_1^*=r_1(\min X_2,x_3^*)$, $x_2^*=r_2(\min X_1,x_3^*)$; by the assumptions, $x_3 \ge x_3^*$ implies $r_i(x_j,x_3) \le x_i^*$, and $x_3<x_3^*$ implies $r_i(x_j,x_3) \ge x_i^*$ (for all $i,j \in \{1,2\}$, $i \ne j$, $x_j \in X_j$).

Suppose $x_3^* \le r_3(x_1^*,x_2^*)$ and denote $X_3^*=\{x_3 \mid x_3 \ge x_3^*\}$, $X_i^* = \{x_i \mid x_i \le x_i^*\}$ $(i=1,2)$. It can easily be checked that r_3 maps $X_1^* \times X_2^*$ into X_3^*, while each r_i $(i=1,2)$ maps $X_j^* \times X_3^*$ into X_i^*. Now for r_i's $(i \in N)$ restricted to X_i^*'s the assumptions of Theorem 6 are fulfilled, hence the existence of a fixed point.

Now suppose $x_3^*>r_3(x_1^*,x_2^*)=x_3^{**}$ and denote $X_3^*=\{x_3 \mid$

$x_3 \le x_3^{**}\}$, $X_i^* = \{x_i \mid x_i \ge x_i^*\}$ $(i=1,2)$; again each r_i maps X_{-i}^* into X_i $(i \in N)$. This time for r_i's $(i \in N)$ restricted to X_i^*'s, the assumptions of Theorem 5 are fulfilled; the theorem is proved.

A similar treatment can be given to the reversed case where (for $i=1,2$) the ordering from Theorem 5 is used for big x_3's and that from Theorem 6 for small ones. The underlying separable orderings on $\mathbb{R}^3$ are constructed in a straightforward way.

Most likely, the "right" proof of Hypothesis 1 should consist in applying Tarski's theorem to auxiliary mappings (somewhat similar to the proofs of Theorems 5 - 7 but in a more complicated way), in which case Novshek's scheme, with all its mysterious beauty, will become superfluous; alas, the progress of knowledge often demands sacrifices! On the other hand, it is also possible that some orderings ϑ may be treated with Tarski's theorem, while others really require Novshek's scheme (Remark 1 after Theorem 3 hints that some overlapping is possible). In the latter case the problem of how to relax the closed graph assumption in Theorem 1 (and in its analogues) becomes quite intriguing.

4.4. On Necessity

Hypothesis 2. Suppose there is a finite set N and, for each $i \in N$, a Pareto compatible ordering η_i on $\mathbb{R}^{N \setminus \{i\}}$ such that a fixed point (1) exists for any collection of finite sets X_i of reals and mappings $r_i\colon X_{-i} \to X_i$ whenever each r_i is decreasing w.r.t. ordering η_i. Then there must exist a complete ordering ϑ

on $\mathbb{R}^N$ having separable projection ϑ_{-i} on each $\mathbb{R}^{N\setminus\{i\}}$ such that $x\vartheta_{-i}y$ implies $x\eta_i y$ for all $i\in N$, $x,y\in\mathbb{R}^{N\setminus\{i\}}$.

The statement is trivially true for $\#N=2$. I can only prove it for $\#N=3$ under stronger assumptions on η_i's.

Theorem 8. Suppose $\#N=3$ and there are three continuous, strictly increasing functions $s_i\colon \mathbb{R}^{N\setminus\{i\}} \to \mathbb{R}$ having the following property: For any finite sets $X_i\subseteq\mathbb{R}$ and any mappings $r_i\colon X_{-i} \to X_i$ such that $r_i=q_i\circ s_i$, where $q_i\colon \mathbb{R}\to\mathbb{R}$ is decreasing, there exists a fixed point satisfying (1). Then there exist strictly increasing continuous functions $\lambda_i,\mu_i\colon \mathbb{R}\to\mathbb{R}$ $(i\in N)$ such that $s_i(x_j,x_k)=\lambda_i(\mu_j(x_j)+\mu_k(x_k))$ (here and later on we adopt the convention $i,j,k\in N$, $i\neq j\neq k\neq i$; the arguments of the functions s_i will always be accompanied with subscripts, so we may not bother about their correct order).

The proof is somewhat similar to that of Kukushkin (1994b, Appendix), but the current situation is much easier to handle (not only because we assumed $n=3$). We will consider three copies of the real line, $\mathbb{R}_i$ $(i\in N)$ simultaneously; we say that a closed interval $I_i=[a_i,b_i]\subseteq\mathbb{R}_i$ matches an interval $I_j=[a_j,b_j]\subseteq\mathbb{R}_j$ iff $s_k(a_i,b_j)=s_k(b_i,a_j)$, and denote this $I_i\simeq I_j$.

Lemma 1. If $I_i\simeq I_j$ and $I_j\simeq I_k$, then $I_i\simeq I_k$.

Suppose the contrary; let, for example,

$$s_j(a_i,b_k)>s_j(b_i,a_k). \tag{5}$$

Taking into account the monotonicity and continuity, we may slightly decrease first b_k and then b_j so that (5) continue to hold and besides

$$s_i(a_j,b_k)<s_i(b_j,a_k).$$
$$s_k(a_i,b_j)<s_k(b_i,a_j).$$

Define $X_i=\{a_i,b_i\}$, $X_j=\{a_j,b_j\}$, $X_k=\{a_k,b_k\}$,

$$q_i(t) = \begin{cases} b_i, & \text{if } t\le s_i(a_j,b_k), \\ a_i, & \text{otherwise}, \end{cases}$$

$$q_j(t) = \begin{cases} b_j, & \text{if } t\le s_j(a_k,b_i), \\ a_j, & \text{otherwise}, \end{cases}$$

$$q_k(t) = \begin{cases} b_k, & \text{if } t\le s_k(a_i,b_j), \\ a_k, & \text{otherwise}; \end{cases}$$

then we have

$$r_i(x_j,x_k)=q_i\circ s_i(x_j,x_k) = \begin{cases} b_i, & x_j=a_j, \\ a_i, & x_j=b_j, \end{cases}$$

$$r_j(x_i,x_k) = \begin{cases} b_j, & x_k=a_k, \\ a_i, & x_k=b_k, \end{cases} \qquad r_k(x_i,x_j) = \begin{cases} b_k, & x_i=a_i, \\ a_k, & x_i=b_i. \end{cases}$$

It is easy to see that the situation is essentially equivalent to that of Example 1 and has no fixed point for the same reason. The lemma is proved.

We call an interval I_i short if there exist I_j, I_k such that $I_j\simeq I_i\simeq I_k$; short intervals obviously exist. If $I_i', I_i''\subseteq \mathbb{R}_i$ are short, we say that $I_i'\simeq I_i''$ if there exists I_j $(j\neq i)$ such that $I_i'\simeq I_j\simeq I_i''$.

Lemma 2. The relation $\simeq$ is an equivalence on the set of all short intervals (from all the three axes put together).

The only thing worth proving is transitivity. If the three intervals belong to different axes, Lemma 1 immediately applies. Suppose $I_i'\simeq I_i''\simeq I_i'''$; we have to show $I_i'\simeq I_i'''$. By definition, we have

$I_i'\simeq I_j\simeq I_i''$ and $I_i''\simeq I_k\simeq I_i'''$; (6)

note that j and k here may coincide, in which case we have to move a bit longer. Let us start with this case: $I_i'\simeq I_j'\simeq I_i''$ and $I_i''\simeq I_j''\simeq I_i'''$. Since I_j'' is short, there must exist I_k (this time certainly $i\neq j\neq k\neq i$!) such that $I_j''\simeq I_k$; by Lemma 1 we have $I_i'''\simeq I_k$ and so we return to (6) with $j\neq k$. Now Lemma 1 implies $I_j\simeq I_k$, hence $I_j\simeq I_i'''$, hence $I_i'\simeq I_i'''$. The proof for the case when two intervals belong to the same axis and the third to a different one is even simple.

The rest of the proof literally follows the proof from the Appendix of Kukushkin (1994b) with several simplifications: There is no need to distinguish between ψ and φ_i; all $i\in N$ are perfectly symmetric. The analogue of Lemma 1 (of Kukushkin, 1994b) remains obviously valid; Lemmas 3 and 4 transform into an obvious statement of the existence of a unique middle for every short interval (c_i is a middle for $[a_i,b_i]$ if $[a_i,c_i]\simeq[c_i,b_i]$); Lemma 5 becomes completely superfluous. The main induction process of the proof goes virtually the same way.

As was noted in the previous section, this approach can be interpreted as an extension of Blaschke's theorem to a bunch of planes and is thus comparable with that of Debreu (1960). Most likely, the necessity to assume $\#N=3$ in Theorem 8 follows from the fact that my technique is essentially two-dimensional (though it can be applied to objects of a more complex geometrical nature than a bunch of planes, see the next section). It should be noted that Debreu's problem was quite multidimensional but he used Blaschke's theorem only to start an induction process; I have been unable to do anything similar in the current context. I am also unable to apply in this case Gorman's technique, which seems more powerful.

5. A Version of the Debreu-Gorman Theorem

The following result is, in fact, an intermediate version of the theorem: stronger than that of Debreu (1960) but apparently weaker than that of Gorman (1968a). As noted in the Introduction, it is related here since it shows the limits to which the approach of Kukushkin (1994b) can be stretched.

Theorem 1. Suppose N is a finite set and ϑ is an ordering on $\mathbb{R}^N$ defined by a continuous, strictly increasing in each argument, function F (thus ϑ is Pareto compatible). Let G be a graph with vertices $i \in N$ such that i and j are connected with an edge if and only if ϑ has a separable projection on $\mathbb{R}_i \times \mathbb{R}_j$. Then the equality:

$$F(\langle x_i \rangle_{i \in N}) = \lambda(\textstyle\sum_{i \in N} (\mu_i(x_i)))$$

(for some continuous and strictly increasing functions λ, μ_i: $\mathbb{R} \to \mathbb{R}$) holds if and only if G is connected.

Remark 1. A resemblance with the formulation of the main result of Gorman (1968b) is obvious; I cannot explain the fact quite convincingly.

Remark 2. The theorem implies that if G is connected, it is complete.

The necessity is quite obvious. I start with a three-dimensional version of the sufficiency part of the result, which requires less cumbersome notations.

Theorem 2. Let $N=\{1,2,3\}$; suppose that an ordering ϑ on $\mathbb{R}^N$ is defined by a continuous, strictly increasing in each argument, function F and has separable projections on $\mathbb{R}_1\times\mathbb{R}_2$ and $\mathbb{R}_2\times\mathbb{R}_3$. Then the equality:

$$F(x_1,x_2,x_3) = \lambda(\mu_1(x) + \mu_2(x) + \mu_3(x))$$

holds, for some appropriate continuous and strictly increasing functions λ, μ_i: $\mathbb{R} \to \mathbb{R}$, for all $(x_1,x_2,x_3)\in\mathbb{R}^N$.

The proof goes along the same lines as in Kukushkin (1994b). For $I_1=[a_1,b_1]\subseteq\mathbb{R}_1$ and $I_2=[a_2,b_2]\subseteq\mathbb{R}_2$, we say that $I_1\simeq I_2$ if $F(a_1,b_2,x_3)=F(b_1,a_2,x_3)$ (x_3 is unessential, according to our assumptions); similarly is defined $I_2\simeq I_3$. For I_1 and I_3, we will only use the equivalence relation $I_1\simeq(x_2)I_3$, given $x_2\in\mathbb{R}_2$, meaning $F(a_1,x_2,b_3)=F(b_1,x_2,a_3)$.

Lemma 1. If $I_1=[a_1,b_1] \simeq I_2=[a_2,b_2]$, and $I_2 \simeq I_3=[a_3,b_3]$, then $I_1\simeq(a_2)I_3$ and $I_1\simeq(b_2)I_3$.

Indeed, $I_1\simeq I_2$ implies $F(b_1,a_2,a_3)=F(a_1,b_2,a_3)$, while $I_2\simeq I_3$ implies $F(a_1,b_2,a_3)=F(a_1,a_2,b_3)$, which means just $I_1\simeq(a_2)I_3$. To prove the second statement, we write $I_1\simeq I_2$ as $F(a_1,b_2,b_3)= F(b_1,a_2,b_3)$, and $I_2\simeq I_3$ as $F(b_1,a_2,b_3)=F(b_1,b_2,a_3)$.

Lemma 2. If $I_1\simeq I_2$ and $[I_1\simeq(a_2)I_3$ or $I_1\simeq(b_2)I_3]$, then $I_2\simeq I_3$.

$I_1\simeq I_2$ implies $F(a_1,b_2,a_3)=F(b_1,a_2,a_3)$, while $I_1\simeq(a_2)I_3$ implies $F(b_1,a_2,a_3)=F(a_1,a_2,b_3)$. The second statement is proved quite similarly.

Now we choose $I_1^\circ\simeq I_2^\circ\simeq I_3^\circ$ and denote their ends $\mu_i^{-1}(0)$, $\mu_i^{-1}(1)$ $(i\in N)$. Then the usual process takes place: if there exists $x_2\in\mathbb{R}_2$ such that $I_1^\circ\simeq[\mu_2^{-1}(1),x_2]$, we define $\mu_2^{-1}(2)=x_2$; similarly are defined $\mu_2^{-1}(\pm n)$ for natural n. Finally, we define $\mu_1^{-1}(\pm n)$ and $\mu_3^{-1}(\pm n)$, using I_2° as a yardstick.

Lemma 3. For both i=1,3 and all integers n and m, $[\mu_i^{-1}(n),\mu_i^{-1}(n+1)]\simeq[\mu_2^{-1}(m),\mu_2^{-1}(m+1)]$, provided both sides are well defined. Similarly, $[\mu_1^{-1}(n), \mu_1^{-1}(n+1)] \simeq(\mu_2^{-1}(k)) [\mu_3^{-1}(m), \mu_3^{-1}(m+1)]$ for all integers n, m, and k.

The proof goes by induction, using Lemmas 1 and 2. I demonstrate just the first step: $I_1^\circ \simeq [\mu_2^{-1}(1), \mu_2^{-1}(2)]$ by definition, $I_1^\circ \simeq (\mu_2^{-1}(1)) I_3^\circ$ by Lemma 1; therefore, by Lemma 2, $[\mu_2^{-1}(1), \mu_2^{-1}(2)] \simeq I_3^\circ$ and so on.

Lemma 4. If both sides are well defined, then

$$F(\mu_1^{-1}(m_1), \mu_2^{-1}(m_2), \mu_3^{-1}(m_3)) = F(\mu_1^{-1}(n_1), \mu_2^{-1}(n_2), \mu_3^{-1}(n_3))$$

if and only if $m_1+m_2+m_3 = n_1+n_2+n_3$.

Is derived from Lemma 3 by induction.

To complete the process of defining $\mu_i^{-1}(\cdot)$, we only need the existence of the middles for short intervals on the axes.

Lemma 5. For each $I_i=[a_i, b_i]$ such that $I_1 \simeq I_2 \simeq I_3$, there exist a unique collection of $c_i \in I_i$ such that:

(1) $[a_1, c_1] \simeq [a_2, c_2]$,

(2) $[c_1, b_1] \simeq [a_2, c_2]$,

(3) $[a_1, c_1] \simeq [c_2, b_2]$,

(4) $[c_1, b_1] \simeq [c_2, b_2]$,

(5) $[a_3, c_3] \simeq [a_2, c_2]$,

(6) $[c_3, b_3] \simeq [a_2, c_2]$,

(7) $[a_3, c_3] \simeq [c_2, b_2]$,

(8) $[c_3, b_3] \simeq [c_2, b_2]$.

First of all, choose c_1 and c_2 so that (1) - (3) be satisfied; this is quite similar to the proof of Lemma 3 in Kukushkin (1994b): for each $x_1 \in I_1$ we define $e(x_1)$ by the equivalence $[a_1, x_1] \simeq [a_2, e(x_1)]$, find a solution for the equation $F(x_1, e(x_1), a_3) = F(a_1, b_2, a_3)$ and denote $c_2 = e(c_1)$. Then we choose c_3 co that (5) be satisfied. Now, (1), (5), and Lemma 1 imply $[a_1, c_1] \simeq (c_2)\ [a_3, c_3]$, and this with (3) and Lemma 2 imply (7), hence (6) too. Further, (2), (5), and Lemma 1 imply $[c_1, b_1] \simeq (c_2)\ [a_3, c_3]$, and this together with (7) and Lemma 2 produce (4). Finally, (1), (6), and Lemma 1 imply $[a_1, c_1] \simeq (c_2)$

$[c_3,b_3]$, hence (using (3) and Lemma 2), we have (8).

It should be noted that Lemmas 5 and 1 immediately imply that $I_1'\simeq(x_2)I_3'$, where one may substitute either $[a_1,c_1]$ or $[c_1,b_1]$ for I_1', either $[a_3,c_3]$ or $[c_3,b_3]$ for I_3' and any of a_2, c_2, or b_2 for x_2.

Now Lemma 5 allows us to define $\mu_i^{-1}(1/2)$ and, using Lemmas 1 and 2, extend each $\mu_i^{-1}(\cdot)$ to all the "halves"; then we proceed to quarters, etc. without any difference with Kukushkin (1994b). Theorem 2 may be regarded as proven.

Turning to Theorem 1, we first delete some edges in G so that the remaining graph constitute a tree (the connectedness assumption allows us to do so). The simple equivalence relation $I_i\simeq I_j$, meaning $F(a_i,b_j,x_{-ij})=F(b_i,a_j,x_{-ij})$ for all x_{-ij}, is well defined for $(i,j)\in G$; in the general case, we will consider the relation $I_i\simeq(x_K)I_j$, where $K\subseteq N\setminus\{i,j\}$ and $x_K\in X^K$ are given, meaning $F(a_i,b_j,x_K,x_{-ijK}) = F(b_i,a_j,x_K,x_{-ijK})$ for all x_{-ijK} (in fact, each pair i, j requires just one set K to be considered but it depends on the choice of the initial vertex).

Lemma 1. If $(i,j)\in G$, $(j,k)\in G$, and $I_i\simeq I_j\simeq I_k$ $(I_j=[a_j,b_j])$, then $I_i\simeq(a_j)I_k$ and $I_i\simeq(b_j)I_k$.

Lemma 2. If $(i,j)\in G$, $I_i\simeq I_j$, and $I_j\simeq(x_K)I_k$, then $I_i\simeq(<x_K,a_j>)I_k$ and $I_i\simeq(<x_K,b_j>)I_k$.

Both proofs are quite similar to those of Lemmas 1 and 2 from Theorem 2.

To start the induction process, we need a collection of I_i°, $i\in N$, such that $I_i^\circ\simeq I_j^\circ$ whenever $(i,j)\in G$; such a collection really exists: we may go along the tree "backwards", from the terminal vertices, finding, for each $i\in N$, an interval I_i that matches some intervals for all succeeding vertices. Lemmas 1 and 2 allow us to define points $\mu_i^{-1}(m)$ for $i \in N$, m integer such that $F(<\mu_i^{-1}(m_i)>_{i \in N}) = F(<\mu_i^{-1}(n_i)>_{i \in N})$ if and only if $m_1+m_2+m_3 = n_1+n_2+n_3$ (provided both sides are well defined).

Then we prove the exact analogue of Lemma 5 from Theorem 2 and finish the proof in the same way (each induction process should start at the initial vertex and go along the tree).

Gorman (1968a) derived additive separability from Aczel's (1966) theorem on associative functions. Hopefully, there should exist a purely geometrical interpretation of Aczel's results - cf. Artin (1957) where e.g. the validity of Pappus' diagram is shown to be equivalent to the commutative law in the underlying field (p. 73-74); such an interpretation would finally clarify the relationship between Debreu's and Gorman's approaches.

On the other hand, to advance towards proving the above hypothesis in full generality, one has to abandon the continuity assumption, thus leaving a very small chance that even Gorman's technique could be of much help. Note that the ordering underlying the main result of Kukushkin (1992) has separable projections on each $\mathbb{R}_0 \times \mathbb{R}_i$ $(i \in N)$ but not on $\mathbb{R}_i \times \mathbb{R}_j$, in sharp contrast with the principal feature of Gorman's (1968a) result, captured in Theorems 1 and 2 above. The usual argument that everything is continuous in economics is not convincing in our context: the minimum- or maximum-like functions are continuous, but their relations with the separability property can only be established through leximin or leximax orderings, which are utterly discontinuous.

References

Aczel J. (1966) Lectures on Functional Equations and their Applications. Academic Press, New York.

Artin E. (1957) Geometric Algebra. Interscience Publishers, New York.

d'Aspremont C. and L. Gevers (1977) Equity and the Informational Basis of Collective Choice. Review Econ. Stud. 44, 199-209.

Aumann R.J. (1962) Utility Theory without the Completeness Axiom. Econometrica 30, 445-462.

Aumann R.J. (1964) Utility Theory without the Completeness Axiom: A Correction. Econometrica 32, 210-212.

Bulow J.I., J.D. Geanakoplos, and P.D. Klemperer (1985) Multimarket Oligopoly: Strategic Substitutes and Complements. J. Polit. Economy 93, 488-511.

Debreu G. (1960) Topological Methods in Cardinal Utility. In: Mathematical Methods in Social Sciences. Stanford University Press. P.16-26.

Deschamps R. and L. Gevers (1978) Leximin and Utilitarian Rules: A Joint Characterization. J. Econ. Theory 17, 143-163.

Fishburn P.C. (1970) Utility Theory for Decision Making. J. Wiley & Sons, New York.

Fleming M. (1952) A Cardinal Concept of Welfare. Quart. J. Econ. 46, 366-384.

Fudenberg D. and J. Tirole (1984) The Fat-Cat Effect, The Puppy-Dog Ploy and the Lean and Hungry Look. American Economic Review 74, 361-366.

Germeier Yu.B., I.A. Vatel' (1974) On Games with a Hierarchical Vector of Interests. Izv. Akad. Nauk SSSR. Tekhn. Kibernet., N.3, P.54-69 (in Russian; English transl. in Engrg. Cybernetics, V.12).

Gorman W.M. (1968a) The Structure of Utility Functions. Rev. Econ. Studies 35, 367-390.

Gorman W.M. (1968b) Conditions for Additive Separability. Econometrica 36, 605-609.

Gorman W.M. (1971a) Apologia for a Lemma. Rev. Econ. Studies 38, 114.

Gorman W.M. (1971b) Clontarf Revisited. Rev. Econ. Studies 38, 116.

Harsanyi J.C. (1955) Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility. J. Polit. Econ. 63, 309-321.

Hirshleifer J. (1983) From Weakest-Link to Best-Shot: The Voluntary Provision of Public Goods. Public Choice 41, 371-386.

Kukushkin N.S. (1992) On Existence of Stable and Efficient Outcomes in Games with Public and Private Objectives. Int. J. Game Theory, V.20, N.3, P.295-303.

Kukushkin N.S. (1994a) A Fixed-Point Theorem for Decreasing Mappings. Economics Letters 46, 23-26.

Kukushkin N.S. (1994b) A Condition for the Existence of a Nash Equilibrium in Games with Public and Private Objectives. Games Econ. Behavior 7, 177-192.

Milgrom P. and J. Roberts (1990) Rationalizability, learning, and equilibrium in games with strategic complementarities. Econometrica 58, 1255-1277.

Milgrom P. and J. Roberts (1994) Comparing Equilibria. American Economic Review 84, 441-459.

Milgrom P. and C. Shannon (1994) Monotone Comparative Statics. Econometrica 62, 157-180.

Moulin H. (1988) Axiomatic of Cooperative Decision Making. Cambridge University Press, Cambridge - New York - Sydney.

Novshek W. (1985) On the Existence of Cournot Equilibrium. Rev. Econ. Studies 52, 85-98.

Rozen V.V. (1976) Mixed Extensions of Games with Ordered Outcomes. Zh. Vychisl. Mat. i Mat. Fiz. 16, 1436-1450 (in Russian; English transl. in USSR Comput. Math. and Math. Phys. V.16, N6 52-67).

Tarski A. (1955) A Lattice-Theoretical Fixpoint Theorem and its applications. Pacific Journal of Mathematics 5, 285-309.

Thomson W. and Lensberg T. (1989) Axiomatic Theory of Bargaining with a Variable Number of Agents. Cambridge University Press, Cambridge - New York - Sydney.

Topkis D.M. (1979) Equilibrium Points in Nonzero-Sum n-Person Submodular Games. SIAM J. Contr. Optimization 17, 773-787.

Vind K. (1971a) Note on "The Structure of Utility Functions". Rev. Econ. Studies 38, 113.

Vind K. (1971b) Comment. Rev. Econ. Studies 38, 115.

Vind K. (1974) A Note on "A Four-Flagged Lemma". Rev. Econ. Studies 41, 571.

Vives X. (1990) Nash Equilibrium with Strategic Complementarities. J. Mathem. Econ. 19, 305-321.

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