Aggregation and Acyclicity in Strategic Games

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December 23, 2007

Abstract

Connections between aggregation in preferences and the acyclicity of improvements in strategic games are studied in two distinct contexts. The first are games with common intermediate objectives; the second, games with ordered strategy sets where each player's best responses are increasing in an aggregate of the partners' strategies. A new result on the necessity of additivity is proven for R.W. Rosenthal's congestion games. New sufficient conditions are obtained for the acyclicity of best response dynamics under monotonicity assumptions. The existence of a monotone selection from every ascending correspondence to a chain is proven. *Journal of Economic Literature* Classification Number: C 72.

Key words: Improvement dynamics; Acyclicity; Separable aggregation; Polylinear aggregation

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1 Introduction

A.A. Cournot addressed the question of whether individual myopic adaption may (or must) lead to an equilibrium long before the term "game theory" came into use. Similar questions were raised now and then in various contexts (see, e.g., Topkis, 1979; Bernheim, 1984; Moulin 1984; Vives, 1990; Milgrom and Roberts, 1991; Kandori and Rob, 1995).

A more systematic approach to unilateral improvement dynamics was originated by Monderer and Shapely (1996). Milchtaich (1996) observed similarity with the case of best response improvements. Kukushkin (1999, 2000, 2003) suggested the language of binary relations and developed transfinite techniques. The crucial question can be formulated as: Is the individual (best response) improvement relation in a given game [Ω-]acyclic? A positive answer implies that every individual (best response) improvement path, if continued whenever possible, ends at a Nash equilibrium

This paper is essentially an extension of Kukushkin (2006): conditions for the acyclicity of improvement relations in strategic games are developed along the same lines.

Concerning "games with common intermediate objectives," a single result on the necessity of additivity, Theorem 1 in Section 2, is presented. The result was motivated by Propositions 4.1 and 5.1 from Kukushkin (2004b); it shows that a "universal separable ordering" on a finite set must be "additive lexicography." From a technical viewpoint, the theorem is distinct from the results on the necessity of additivity for the existence of Nash equilibrium in Kukushkin (2004b, 2006, 2007).

In principle, Theorem 1 can be viewed as a discrete analogue of the famous Debreu–Gorman Theorem (Debreu, 1960; Gorman, 1968; see also Wakker, 1989), although it certainly cannot claim anything approaching the importance of the latter.

The main bulk of this paper is about games with ordered strategy sets where each player's best responses are increasing in an aggregate of the partners' strategies. The class includes games of both strategic complements and substitutes (Bulow et al., 1985), which properties are found in many important economic models (Tirole, 1988; Fudenberg and Tirole, 1991; Topkis, 1998). As is well known, the existence of a Nash equilibrium in a game with strategic complements can be derived from Tarski's (1955) fixed point theorem; however, the latter does not ensure acyclicity. In the case of strategic substitutes, acyclicity is virtually the only reason for Nash equilibrium existence (in the absence of convexity).

In Section 3, basic notions such as a system of reactions, an iteration path (cycle), and Ω-acyclicity are reproduced. The section also contains a few ways to extend a (pre)order to nonempty subsets; they are needed to define monotonicity of multi-valued reactions.

In Section 4, a separable ordering is assumed on the space of strategy profiles; in the first two subsections, it is continuous. Theorems 2 and 3 generalize, to multi-valued reactions, Theorems 6.1 and 6.2, respectively, from Kukushkin (2000). Theorem 4 is about strategic complements with discontinuous, to be more precise, lexicographic, aggregation. There are just three players and the reactions are single-valued; to the best of my knowledge, this is the first sufficient condition for Ω-acyclicity without the continuity of aggregation (apart from Theorem 6 of Kukushkin (2003), which was about an increasing *endomorphism* rather than a system of reactions).

Section 5 explores the possibilities opened by a very interesting technical trick due to Huang (2002) and Dubey et al. (2006). Theorem 5 proves Ω-acyclicity of a "system of reactions with reciprocal quasi-polylinear aggregates." The most important economic interpretation is a game with positive linear externalities, see Example 5.3.

It is instructive to compare the Huang–Dubey–Haimanko–Zapechelnyuk trick with Novshek's (1985) construction (see also Kukushkin, 1994) used in the proofs of Theorem 6.2 from Kukushkin (2000), Theorem 2 from Kukushkin (2004a), and Theorem 3 from this paper. Both defy explanation; both produce "almost" the same result for decreasing reactions under additive aggregation (by the way, the two potentials in this case seem not to be related to each other in any way). They are logically independent in the sense that there is a situation where one works but the other does not: generally, neither separable, nor linear, aggregation need be additive. However, if one takes into account the relative importance of the domain of applicability of either approach, the former appears a clear winner (so far).

The last Section 6 is about a purely mathematical problem of the existence of monotone selections from correspondences. The situation here is shown to be much more complicated than was asserted in Milgrom and Shannon (1994). Theorem 6 proves the existence of a monotone selection from every ascending correspondence to a chain. It implies, in particular, that the existence of a Nash equilibrium in Theorems 3 and 5 is retained under weaker monotonicity conditions; actually, we have "restricted acyclicity" in those situations, which is more than the mere existence of an equilibrium, cf. Kukushkin (2004a, Sections 6 and 7.7).

2 Universal separable orderings on a finite set

We start with a modification of a concept introduced in Kukushkin (2004b, Section 4.1). A *universal separable ordering* on $V \subseteq \mathbb{R}$ is a sequence of orderings, i.e., reflexive, transitive, and complete relations, $\succeq^m$ on V^m ($m = 1, 2, \dots$) such that

1. $\succeq^1$ is the standard order $\geq$ on V induced from $\mathbb{R}$;

2. for every permutation σ of $\{1, \dots, m\}$,

$$\langle v_1, \dots, v_m \rangle \sim^m \langle v_{\sigma(1)}, \dots, v_{\sigma(m)} \rangle$$

(symmetry);

3. for every $m' > m \geq 1$, every $\langle v_1, \dots, v_{m'} \rangle \in V^{m'}$, and every $\langle v'_1, \dots, v'_m \rangle \in V^m$,

$$\langle v_1, \dots, v_m, v_{m+1}, \dots, v_{m'} \rangle \succeq^{m'} \langle v'_1, \dots, v'_m, v_{m+1}, \dots, v_{m'} \rangle \iff \langle v_1, \dots, v_m \rangle \succeq^m \langle v'_1, \dots, v'_m \rangle$$

(separability).

Theorem 1. *For every finite $V \subseteq \mathbb{R}$ and every universal separable ordering on V, there is a natural number $n \geq 1$ and a strictly increasing mapping $\mu\colon V \to \mathbb{R}^n$ such that*

$$\langle v'_1, \dots, v'_m \rangle \succeq^m \langle v_1, \dots, v_m \rangle \iff \sum_{s=1}^{m} \mu(v'_s) \geq_{\mathrm{Lex}} \sum_{s=1}^{m} \mu(v_s) \tag{2.1}$$

for every $m \geq 1$, where $\geq_{\mathrm{Lex}}$ denotes the lexicographic order on $\mathbb{R}^n$: first the first coordinate matters, then the second, etc.

Remark. There is a small discrepancy with Kukushkin (2004b): here we assume completeness, which was not needed there. It is unclear whether a preorder may satisfy the above conditions without being complete.

Proof. An *interval* $[v, v']$ is a pair of $v, v' \in V$. An interval $[v, v']$ is *positive* if $v' \geq v$. A formal sum $\sum_{s=1}^{m} [v_s, v'_s]$ is called positive if $\langle v'_1, \dots, v'_m \rangle \succeq^m \langle v_1, \dots, v_m \rangle$; by separability, the sum of positive intervals is positive as well. Since the same interval may be repeated several times, we also have a notion of a positive combination $\sum_{s=1}^{m} k_s [v_s, v'_s]$ with nonnegative integer k_s. Assuming $-[v_s, v'_s] = [v'_s, v_s]$, we extend the notion to negative k_s as well. The separability of the original ordering implies that a formal sum of two positive combinations is also positive.

Now we consider the free Abelian group generated by all positive intervals and define $I' \succeq I \leftrightharpoons [I' - I$ is positive$]$ for all members I' and I of the group (with zero positive by definition). Clearly, $\succeq$ is an ordering consistent with addition ($I \succeq 0 \iff I' + I \succeq I'$); we define $\succ$ and $\sim$ as its asymmetric and symmetric components, respectively. It is worth noting that

$$[v, v'] + [v', v''] \sim [v, v''] \tag{2.2}$$

by symmetry, and $I' \succeq I \iff mI' \succeq mI$ for any $m > 0$ by separability.

We call $[v, v']$ an *elementary interval* if $v' > v$ and there is no $v'' \in V$ such that $v' > v'' > v$. We denote $\mathbb{Q}$ the field of rational numbers and $\mathfrak{Q}$ the set of all formal combinations $\sum_{s=1}^{m} r_s [v_s, v'_s]$ of elementary intervals with rational coefficients. Clearly, $\mathfrak{Q}$ is a vector space over $\mathbb{Q}$; since V is finite, $\mathfrak{Q}$ is finite-dimensional. Our ordering is defined on combinations with integer coefficients; we extend it to the whole $\mathfrak{Q}$ by $\sum_{s=1}^{m'} r'_s [v''_s, v'''_s] \succeq \sum_{s=1}^{m} r_s [v_s, v'_s] \leftrightharpoons \sum_{s=1}^{m'} k \cdot r'_s [v''_s, v'''_s] \succeq \sum_{s=1}^{m} k \cdot r_s [v_s, v'_s]$, where $k > 0$ is an integer such that all coefficients $k \cdot r'_s$ and $k \cdot r_s$ are integer (it does not matter which particular k is chosen for the comparison). The extended ordering is still consistent with addition; besides, $rI \succ 0$ whenever $I \succ 0$ and $r > 0$ ($r \in \mathbb{Q}$).

Let $I', I \in \mathfrak{Q}$ and $I \succ 0$; we say that I' *is not Archimedean dominated by* I, $I' \not\ll I$, if there is an integer k such that $kI' \succ I$. For $I \prec 0$, we define $I' \not\ll I \leftrightharpoons \exists k\,[kI' \succ -I]$. Adding $I \not\ll 0$ by definition for all $I \in \mathfrak{Q}$, we obtain an ordering; its asymmetric and symmetric components are denoted $\gg$ and $\approx$, respectively. When $I' \approx I$, we say that I' and I *have the same Archimedean rank.*

Whenever $I_0 \succ 0$ and $I_0 \not\ll I \succeq 0$, we define

$$I/I_0 = \sup\{r \in \mathbb{Q} \mid \; I \succeq rI_0\} \in \mathbb{R}$$

(an attempt to apply the definition to $I \gg I_0$ would lead to $I/I_0 = +\infty$). When $I \prec 0$, we define $I/I_0 = -[(-I)/I_0] = \inf\{r \in \mathbb{Q} \mid rI_0 \succeq I\}$.

Lemma 1.1. *Let $I, I', I_0 \in \mathfrak{Q}$, $I_0 \succ 0$, $I_0 \not\ll I'$, $I_0 \not\ll I$, and $r \in \mathbb{Q}$. Then*

$$(I' + I)/I_0 = (I'/I_0) + (I/I_0);$$

$$(rI)/I_0 = r(I/I_0);$$

$$I_0 \gg I \iff I/I_0 = 0.$$

Proof. The proof consists of rather tedious checks. Let $I' \succ 0$ and $I \succ 0$; then for every $r \in \mathbb{Q}$ such that $r < (I'/I_0) + (I/I_0)$, we can find $r_1, r_2 \in \mathbb{Q}$ such that $r_1 + r_2 = r$, $r_1 < I'/I_0$, and $r_2 < I/I_0$. By definition, $I' \succ r_1 I_0$ and $I \succ r_2 I_0$, hence $(I' + I) \succ rI_0$; since r was arbitrary, $(I' + I)/I_0 \geq (I'/I_0) + (I/I_0)$. Conversely, for every $r \in \mathbb{Q}$ such that $r > (I'/I_0) + (I/I_0)$, we can find $r_1, r_2 \in \mathbb{Q}$ such that $r_1 + r_2 = r$, $r_1 > I'/I_0$, and $r_2 > I/I_0$. By definition, $I' \prec r_1 I_0$ and $I \prec r_2 I_0$, hence $(I' + I) \prec rI_0$; since r was arbitrary, $(I' + I)/I_0 \leq (I'/I_0) + (I/I_0)$.

Turning to negative intervals, it is enough to consider $I' \succ 0$, $I \succ 0$, and $I' - I \succ 0$; then for every $r \in \mathbb{Q}$ such that $r < (I'/I_0) - (I/I_0)$, we can find $r_1, r_2 \in \mathbb{Q}$ such that $r_1 - r_2 = r$, $r_1 < I'/I_0$, and $r_2 > I/I_0$. By definition, $I' \succ r_1 I_0$ and $I \prec r_2 I_0$, hence $(I' - I) \succ rI_0$; since r was arbitrary, $(I' - I)/I_0 \geq (I'/I_0) - (I/I_0)$. The converse inequality is obtained in a similar way.

Checking the second equality, we may assume $I \succ 0$ and $r > 0$; then $rI \succeq rr'I_0 \iff I \succeq r'I_0$.

As to the last equivalence, it is again sufficient to consider $I \succ 0$. If $nI \succeq I_0$, then $I/I_0 \geq 1/n$. Conversely, if $I/I_0 > 0$, then $I \succeq rI_0$ for every $r \in \mathbb{Q}$ such that $0 < r < I/I_0$, hence $(1/r)I \succeq I_0$, hence $I \not\ll I_0$. □

Let $\mathcal{I}$ be a finite subset of $\mathfrak{Q}$; we denote $\mathfrak{Q}(\mathcal{I})$ the vector subspace of $\mathfrak{Q}$ (over $\mathbb{Q}$) generated by $\mathcal{I}$.

Lemma 1.2. *For every finite subset $\mathcal{I} \neq \emptyset$ of $\mathfrak{Q}$, there is a natural number n and a mapping $\lambda\colon \mathfrak{Q}(\mathcal{I}) \to \mathbb{R}^n$ such that λ is linear over $\mathbb{Q}$ and*

$$\forall I', I \in \mathfrak{Q}(\mathcal{I})\,[I' \succeq I \iff \lambda(I') \geq_{\mathrm{Lex}} \lambda(I)].$$

Proof. We argue by induction in $\#\mathcal{I}$; when it is 1, the statement is obvious. Picking $I^+ \in \mathcal{I}$ with a maximal Archimedean rank, we denote $I_0 = I^+$ if $I^+ \succ 0$ and $I_0 = -I^+$ otherwise. By Lemma 1.1, $I_0 \not\succcurlyeq I$ for every $I \in \mathfrak{Q}(\mathcal{I})$; we denote $q(I) = I/I_0$. By the same lemma, $q\colon \mathfrak{Q}(\mathcal{I}) \to \mathbb{R}$ is linear over $\mathbb{Q}$; since $q(I_0) = 1$, the kernel of q, $K = \{I \in \mathfrak{Q}(\mathcal{I}) \mid q(I) = 0\}$, is a proper vector subspace of $\mathfrak{Q}(\mathcal{I})$. By the induction hypothesis, there is a linear operator $\lambda'\colon K \to \mathbb{R}^n$ representing $\succeq$ on K.

Now we fix a projection $p\colon \mathfrak{Q}(\mathcal{I}) \to K$, i.e., a linear operator such that $p(I) = I$ whenever $I \in K$, and define $\lambda\colon \mathfrak{Q}(\mathcal{I}) \to \mathbb{R}^{n+1}$ by $\lambda(I) = \langle q(I), \lambda'(p(I))\rangle$ for every $I \in \mathfrak{Q}(\mathcal{I})$. Checking that λ represents $\succeq$ on $\mathfrak{Q}(\mathcal{I})$ is straightforward: if $q(I') > q(I)$, then obviously $I' \succ I$; if $q(I') = q(I)$, then $(I' - I) \in K$, hence $\lambda(I') \geq_{\text{Lex}} \lambda(I) \iff \lambda'(I') \geq_{\text{Lex}} \lambda'(I) \iff I' \succeq I$. □

Since the total number of elementary intervals is finite, Lemma 1.2 implies the existence of a λ representing $\succeq$ on the whole $\mathfrak{Q}$. Let $V = \{v^0, v^1, \ldots, v^{\bar{m}}\}$ with $v^s < v^{s+1}$ for every relevant s. We define $\varkappa\colon V \to \mathfrak{Q}$ by $\varkappa(v^0) = 0$ and $\varkappa(v^k) = \sum_{s=0}^{k-1}[v^s, v^{s+1}]$, and $\mu\colon V \to \mathbb{R}^n$ as $\mu = \lambda \circ \varkappa$. By the definition of a positive sum of intervals, $\langle v'_1, \ldots, v'_m\rangle \succeq^m \langle v_1, \ldots, v_m\rangle$ if and only if $\sum_{s=1}^{m}[v_s, v'_s] \succeq 0$; by (2.2), $[v_s, v'_s] \sim \big(\varkappa(v'_s) - \varkappa(v_s)\big)$. Now Lemma 1.2 implies (2.1). □

As was shown in Kukushkin (2004b, Proposition 4.1), an arbitrary universal separable ordering can successfully replace additive aggregation in Rosenthal's (1973) congestion games. Very formally speaking, we have thus obtained a generalization. On the other hand, in a particular game only a finite number of $\succeq^m$ can be relevant and the lexicographic ordering on $\mathbb{R}^n$ obviously admits a scalar additive representation on every finite subset. The application of the necessity part of Theorem 1 to a single congestion game is prevented by the assumption in the theorem that $\succeq^m$ was defined (and well-behaved) for *all* $m \geq 1$. Moreover, universal separable orderings are needed for Rosenthal's *proof* to remain intact; their necessity for the existence of Nash equilibrium, or even for the acyclicity of individual improvements, is by no means obvious. All that is true with respect to Proposition 5.1 from Kukushkin (2004b) as well, but here the necessity of separability was disproved by Example 3.1 of Kukushkin (2006). It should be noted that the example contains a typo corrected in the pdf version.

3 Systems of monotonic reactions

3.1 Iteration paths

A natural generalization of best response correspondences in a strategic game is a "system of reactions" (Kukushkin, 2000). Virtually the same object was called an "abstract game" by Vives (1990); however, he focussed attention on an endomorphism, the Cartesian product of all reactions.

A *system of reactions* $\mathcal{S}$ is defined by a finite set of players N, and sets X_i and mappings $\mathcal{R}_i\colon X_{-i} \to 2^{X_i} \setminus \{\emptyset\}$ for all $i \in N$. A point $x^0 \in X_N = \prod_{i\in N} X_i$ is called a *fixed point* of $\mathcal{S}$ if $x_i^0 \in \mathcal{R}_i(x_{-i}^0)$ for all $i \in N$. With every system $\mathcal{S}$, one can associate binary relations on X_N:

$$y_N \rhd_i^{\mathcal{S}} x_N \leftrightharpoons [y_{-i} = x_{-i} \ \&\ x_i \notin \mathcal{R}_i(x_{-i}) \ni y_i]; \tag{3.1a}$$

$$y_N \rhd^{\mathcal{S}} x_N \leftrightharpoons \exists i \in N\, [y_N \rhd_i^{S} x_N]. \tag{3.1b}$$

Clearly, $x_N \in X_N$ is a maximizer for $\rhd^{\mathcal{S}}$ if and only if x_N is a fixed point of $\mathcal{S}$. Here we are interested not so much in the existence of fixed points as in what happens when the iteration of $\mathcal{R}_i$'s is combined with picking limit points. I reproduce basic formal constructions (Kukushkin, 2000, 2003).

A linearly ordered set is *well ordered* if every subset contains a least point; Natanson (1974, Chapter XIV) can be used as a reference textbook. Considering a well ordered set Σ, we will denote 0 the least point of the whole Σ, and $\beta + 1$, for $\beta \in \Sigma$, the least point exceeding β (the latter exists unless $\beta = \max \Sigma$). A point $\beta \in \Sigma \setminus \{0\}$ is called *isolated* if $\beta = \beta' + 1$ for some $\beta' \in \Sigma$; otherwise, β is called a *limit point.* Thus, we have a partition $\Sigma = \{0\} \cup \Sigma_{\text{iso}} \cup \Sigma_{\text{lim}}$. Whenever $\beta, \beta' \in \Sigma$ and $\beta \le \beta'$, we denote $[\beta, \beta'] = \{\gamma \in \Sigma \mid \beta \le \gamma \le \beta'\}$.

We always assume that each X_i, hence X too, is a metric space. An *iteration path* for $\mathcal{S}$ is a mapping $\pi_N\colon \Sigma \to X_N$, where Σ is a countable well ordered set, satisfying these two conditions:

$$\pi_N(\beta + 1) \rhd^{\mathcal{S}} \pi_N(\beta) \text{ whenever } \beta, \beta + 1 \in \Sigma; \tag{3.2a}$$

whenever $\beta^\omega \in \Sigma_{\text{lim}}$, there exists a sequence $\{\beta^k\}_{k\in\mathbb{N}}$ in Σ for which

$$\beta^{k+1} > \beta^k \text{ for all } k \in \mathbb{N},\ \beta^\omega = \sup_k \beta^k, \text{ and } \pi_N(\beta^\omega) = \lim_{k\to\infty} \pi_N(\beta^k). \tag{3.2b}$$

An iteration path π_N is *narrow* if

$$\pi_N(\beta^\omega) = \lim_{k\to\infty} \pi_N(\beta^k)$$

whenever $\beta^\omega \in \Sigma_{\text{lim}}$ and a sequence $\{\beta^k\}_k$ in Σ is such that

$$\beta^\omega = \sup_k \beta^k \text{ and } \beta^{k+1} > \beta^k \text{ for all } k. \tag{3.2c}$$

In a general iteration path, *limit points* are taken at appropriate steps; if the path is narrow, they are *limits.*

An *iteration cycle* is an iteration path π_N such that $\pi_N(\alpha) = \pi_N(\beta)$ for some $\alpha > \beta \in \Sigma$. Deleting from Σ all $\gamma < \beta$ and $\gamma > \alpha$, we can assume $\beta = 0$ and $\Sigma = [0, \alpha]$. A system $\mathcal{S}$ is called Ω-*acyclic* if it admits no iteration cycle. By Theorem 2 of Kukushkin (2003), an Ω-acyclic system of reactions with compact sets X_i has a fixed point ([2.1] $\Rightarrow$ [2.6]) and every iteration path "eventually" reaches one of them ([2.1] $\Rightarrow$ [2.4]).

An ω-*potential* of $\mathcal{S}$ is a strict order $\succ$ on X_N which is ω-*transitive*,

$$\left[x^\omega = \lim_{k\to\infty} x^k \;\&\; \forall k \in \mathbb{N}[x^{k+1} \succ x^k]\right] \Rightarrow x^\omega \succ x^0, \tag{3.3a}$$

and satisfies

$$y_N \rhd^{\mathcal{S}} x_N \Rightarrow y_N \succ x_N. \tag{3.3b}$$

By Theorem 2 ([2.1] $\iff$ [2.2]) of Kukushkin (2003), $\mathcal{S}$ is Ω-acyclic if and only if it admits an ω-potential (3.3).

3.2 Monotonicity conditions

A reflexive and transitive binary relation is called a *preorder*; with every preorder $\succeq$, strict orders $\succ$ and $\prec$, as well as an equivalence relation $\sim$, are naturally associated. As usual, we call a set endowed with a partial order a *poset*; a set with a preorder will be called a *proset*. A complete preorder is called an *ordering*.

Given a set X, we denote $\mathfrak{B}_X = 2^X \setminus \{\emptyset\}$. If X is a proset, there are various ways to extend the preorder to $\mathfrak{B}_X$. Quite a few of them are used in the following.

Let X be a proset and $Y, Z \in \mathfrak{B}_X$. We define

$$Y \succeq^{\sup} Z \leftrightharpoons \forall z \in Z\, \exists y \in Y\, [y \succeq z]; \tag{3.4a}$$

$$Y \succeq^{\inf} Z \leftrightharpoons \forall y \in Y\, \exists z \in Z\, [y \succeq z]; \tag{3.4b}$$

$$Y \succeq^{*} Z \leftrightharpoons \forall y \in Y \setminus Z\, \forall x, x' \in Y \cap Z\, \forall z \in Z \setminus Y\, [y \succ x \sim x' \succ z]. \tag{3.4c}$$

All the three relations are preorders on $\mathfrak{B}_X$; if $\succeq$ is an ordering on X, then both $\succeq^{\sup}$ and $\succeq^{\inf}$ are orderings too. Loosely speaking, $\succeq^{\sup}$ compares suprema of subsets of X, while $\succeq^{\inf}$ compares infima. Clearly, X itself is a greatest point in $\mathfrak{B}_X$ for $\succeq^{\sup}$ and a least for $\succeq^{\inf}$. It is easy to see that

$$Y \succ^{\sup} Z \iff \exists y \in Y\, \forall z \in Z\, [y \succ z]; \tag{3.4d}$$

and

$$Y \succ^{\inf} Z \iff \exists z \in Z\, \forall y \in Y\, [y \succ z]. \tag{3.4e}$$

The preorder $\succeq^{*}$ is antisymmetric, i.e., a *partial order.* If $\succeq$ itself is a partial order, (3.4c) can be simplified to

$$Y \succeq^{*} Z \iff \forall y \in Y\, \forall z \in Z\, [y \succeq z]. \tag{3.4f}$$

When X is a lattice (of most interest for us are just chains), Veinott's order (Topkis, 1978) can be defined:

$$Y \succeq^{\mathrm{V}} Z \leftrightharpoons \forall y \in Y\, \forall z \in Z\, [y \vee z \in Y \;\&\; y \wedge z \in Z]. \tag{3.4g}$$

The relation $\succeq^{\mathrm{V}}$ is antisymmetric and transitive on $\mathfrak{B}_X$, hence its reflexive closure is a partial order. It is easy to see that

$$Y \succeq^{*} Z \Rightarrow Y \succeq^{\mathrm{V}} Z \Rightarrow [Y \succeq^{\sup} Z \;\&\; Y \succeq^{\inf} Z]$$

whenever X is a lattice and $Y, Z \in \mathfrak{B}_X$.

Let S and Y be two prosets. A mapping $f\colon S \to Y$ is *increasing* if $y \sim x \Rightarrow f(y) = f(x)$ and $y \succeq x \Rightarrow f(y) \succeq f(x)$; f is *decreasing* if $y \sim x \Rightarrow f(x) = f(y)$ and $y \succeq x \Rightarrow f(x) \succeq f(y)$.

When $Y = \mathfrak{B}_X$, where X is a proset, every (pre)order (3.4) generates two versions of monotonicity. For instance, a correspondence $R\colon S \to \mathfrak{B}_X$, where X is a lattice, is called *ascending* (Topkis, 1978, 1998) if it satisfies the condition

$$\forall s', s \in S\, [s' \succeq s \Rightarrow R(s') \succeq^{\mathrm{V}} R(s)].$$

4 Separable aggregation

Let $\succeq$ be a binary relation on $X_N = X_1 \times X_2$. A relation $\succeq_1$ on X_1 is a *separable projection* of $\succeq$ to X_1 (along X_2) if

$$(x_1', x_2) \succeq (x_1, x_2) \iff x_1' \succeq_1 x_1$$

for all $x_1', x_1 \in X_1$ and $x_2 \in X_2$. Usually X_2 is clear from the context and not mentioned at all. Obviously, a separable projection "inherits" all properties inherited by the restrictions to subsets (as being a preorder, strict order, ordering, etc.). If $X_N = X_1 \times X_2 \times X_3$ and $\succeq$ admits separable projections to both $X_1 \times X_2$ and $X_2 \times X_3$, then $\succeq$ also admits a separable projection to X_2 (Gorman, 1968).

In this section, we consider systems of reactions with *separable aggregation*. This means that there is an ordering $\succeq$ on X_N, which admits a separable projection to each X_{-i}. Therefore, $\succeq$ admits a separable projection to each X_i as well; we assume throughout that it is a linear order. To simplify notation, we use the same symbols, $\succeq$, $\succ$ and $\sim$, for all separable projections. In the two following subsections, we assume that $\succeq$ is *continuous*, i.e., both upper and lower contours, $\{y_N \in X_N \mid y_N \succ x_N\}$ and $\{y_N \in X_N \mid x_N \succ y_N\}$, are open for every $x_N \in X_N$. This implies that all projections of $\succeq$ are continuous too; virtually without restricting generality we assume $X_i \subseteq \mathbb{R}$.

When dealing with a system of reactions, we employ shortened notations $\mathcal{B}_N$ instead of $\mathfrak{B}_{X_N}$ and $\mathcal{B}_i$ instead of $\mathfrak{B}_{X_i}$.

4.1 Increasing reactions

Given a preorder $\succeq$ on X_{-i}, we call a mapping $\mathcal{R}_i\colon X_{-i} \to \mathcal{B}_i$ *increasing* (w.r.t. $\succeq$) if

$$\forall x_{-i}', x_{-i} \in X_{-i}\, [x_{-i}' \succeq x_{-i} \Rightarrow \mathcal{R}_i(x_{-i}') \succeq^{\mathrm{V}} \mathcal{R}_i(x_{-i})], \tag{4.1}$$

where $\succeq^{\mathrm{V}}$ is defined by (3.4g). Since $\succeq^{\mathrm{V}}$ is antisymmetric, (4.1) implies that $\mathcal{R}_i(x_{-i}') = \mathcal{R}_i(x_{-i})$ whenever $x_{-i}' \sim x_{-i}$.

Theorem 2. *Let a system of reactions $\mathcal{S}$ be defined by a finite set N, compact $X_i \subset \mathbb{R}$ and $\mathcal{R}_i \colon X_{-i} \to \mathcal{B}_i$ such that every value $\mathcal{R}_i(x_{-i})$ is closed. Let there be a continuous ordering $\succeq$ on X_N admitting a separable projection to each X_{-i} such that the separable projection of $\succeq$ to each X_i coincides with the natural order $\geq$ and each $\mathcal{R}_i$ is increasing in the sense of* (4.1)*. Then $\mathcal{S}$ is Ω-acyclic.*

Proof. For every $V_N = \langle V_i \rangle_{i \in N} \in \mathcal{B}_N^N$ and $Y \in \mathcal{B}_N$, we denote $I^+(V_N, Y) = \{i \in N \mid V_i \succeq^{\inf} Y\}$, $I^-(V_N, Y) = \{i \in N \mid Y \succeq^{\sup} V_i\}$, $n^+(V_N, Y) = \# I^+(V_N, Y)$, and $n^-(V_N, Y) = \# I^-(V_N, Y)$. Then we define lexicographic orderings $\succeq^{\mathrm{Linf}}$ and $\succeq^{\mathrm{Lsup}}$ on $\mathcal{B}_N^N$:

$$V'_N \succeq^{\mathrm{Linf}} V_N \rightleftharpoons \forall Y \in \mathcal{B}_N \left[n^+(V_N, Y) > n^+(V'_N, Y) \Rightarrow \exists Z \in \mathcal{B}_N \left[Z \succ^{\inf} Y \ \&\ n^+(V'_N, Z) > n^+(V_N, Z) \right] \right]; \quad (4.2a)$$

$$V'_N \succeq^{\mathrm{Lsup}} V_N \rightleftharpoons \forall Y \in \mathcal{B}_N \left[n^-(V_N, Y) > n^-(V'_N, Y) \Rightarrow \exists Z \in \mathcal{B}_N \left[Y \succ^{\sup} Z \ \&\ n^-(V'_N, Z) > n^-(V_N, Z) \right] \right]. \quad (4.2b)$$

For every $i \in N$, $x_i \in X_i$, and $x_N \in X_N$, we define:

$$S_i^+(x_i) = \{x_{-i} \in X_{-i} \mid x_i \in \mathcal{R}_i(x_{-i}) \ \text{ or } \ \forall y_i \in \mathcal{R}_i(x_{-i})[y_i \geq x_i]\};$$

$$\tau_i^+(x_i) = \{x_i\} \times S_i^+(x_i) \subseteq X_N; \quad N^+(x_N) = \{i \in N \mid \tau_i^+(x_i) \succ^{\inf} \{x_N\}\};$$

$$\lambda_i^+(x_N) = \begin{cases} \tau_i^+(x_i), & \text{if } i \in N^+(x_N); \\ X_N, & \text{else;} \end{cases} \quad \lambda_N^+(x_N) = \langle \lambda_i^+(x_N) \rangle_{i \in N} \in \mathcal{B}_N^N;$$

$$S_i^-(x_i) = \{x_{-i} \in X_{-i} \mid x_i \in \mathcal{R}_i(x_{-i}) \ \text{ or } \ \forall y_i \in \mathcal{R}_i(x_{-i})[y_i \leq x_i]\};$$

$$\tau_i^-(x_i) = \{x_i\} \times S_i^-(x_i) \subseteq X_N; \quad N^-(x_N) = \{i \in N \mid \{x_N\} \succ^{\sup} \tau_i^-(x_i)\};$$

$$\lambda_i^-(x_N) = \begin{cases} \tau_i^-(x_i), & \text{if } i \in N^-(x_N); \\ X_N, & \text{else;} \end{cases} \quad \lambda_N^-(x_N) = \langle \lambda_i^-(x_N) \rangle_{i \in N} \in \mathcal{B}_N^N;$$

$$y_N \succ\!\!\succ x_N \rightleftharpoons \big[[\lambda_N^+(x_N) \succ^{\mathrm{Linf}} \lambda_N^+(y_N) \ \&\ \lambda_N^-(y_N) \succeq^{\mathrm{Lsup}} \lambda_N^-(x_N)] \ \text{ or } \ [\lambda_N^+(x_N) \succeq^{\mathrm{Linf}} \lambda_N^+(y_N) \ \&\ \lambda_N^-(y_N) \succ^{\mathrm{Lsup}} \lambda_N^-(x_N)] \big]. \quad (4.3)$$

We denote $X_i^0 = \operatorname{cl} \bigcup_{x_{-i} \in X_{-i}} \mathcal{R}_i(x_{-i})$ for each $i \in N$ and $N^0(x_N) = \{i \in N \mid x_i \in X_i^0\}$ for every $x_N \in X_N$. Finally, we define

$$y_N \succ\!\!\succ\!\!\succ x_N \rightleftharpoons \left[N^0(x_N) \subset N^0(y_N) \ \text{ or } \ [N^0(x_N) = N^0(y_N) = M \ \&\ y_{-M} = x_{-M} \ \&\ y_N \succ\!\!\succ x_N] \right]. \quad (4.4)$$

Clearly, both $\succ\!\!\succ$ and $\succ\!\!\succ\!\!\succ$ are irreflexive and transitive.

Lemma 2.1. *If $i \in N$ and $x_i \in X_i^0$, then $S_i^-(x_i) \cup S_i^+(x_i) = X_{-i}$.*

Proof. Suppose the contrary: there are $x_i' < x_i < x_i''$ and $x_{-i} \in X_{-i}$ such that $x_i' \in \mathcal{R}_i(x_{-i})$ and $x_i'' \in \mathcal{R}_i(x_{-i})$, but $x_i \notin \mathcal{R}_i(x_{-i})$. Since $\mathcal{R}_i(x_{-i})$ is closed, there is $\varepsilon > 0$ such that $[x_i - \varepsilon, x_i + \varepsilon] \cap \mathcal{R}_i(x_{-i}) = \emptyset$, hence $x_i' < x_i - \varepsilon$ and $x_i + \varepsilon < x_i''$. Since $x_i \in X_i^0$, there is $y_N \in X_N$ such that $y_i \in \mathcal{R}_i(y_{-i})$ and $y_i[x_i - \varepsilon, x_i + \varepsilon]$, hence $y_{-i} \not\sim x_{-i}$. Assuming $x_{-i} \succ y_{-i}$, we have $\mathcal{R}_i(x_{-i}) \succeq^{\vee} \mathcal{R}_i(y_{-i})$ by (4.1), hence $y_i = y_i \vee x_i' \in \mathcal{R}_i(x_{-i})$, contradicting the choice of y_i and ε. If $y_{-i} \succ x_{-i}$, then, similarly, $y_i = y_i \wedge x_i'' \in \mathcal{R}_i(x_{-i})$. ☐

Remark. The statement is invalid for $x_i \in X_i \setminus X_i^0$. Lemma 2.1 itself would be wrong without our closed values assumption.

Lemma 2.2. *If $y_N \rhd^{\mathcal{S}} x_N$, then $y_N \ggcurly x_N$.*

Proof. Let $y_N \rhd_i^{\mathcal{S}} x_N$. Clearly, $y_i \in X_i^0$, hence $N^0(x_N) \subseteq N^0(y_N)$. If $x_i \notin X_i^0$, the inclusion is strict, hence the first disjunctive term in (4.4) holds. Assuming $x_i \in X_i^0$, we consider two alternatives.

Let $y_i < x_i$. Then $x_{-i} \notin S_i^+(x_i)$, hence $i \in N^+(x_N)$; on the other hand, $x_{-i} = y_{-i} \in S_i^+(y_i)$, hence $i \notin N^+(y_N)$, hence $\lambda_i^+(x_N) \succ^{\inf} \lambda_i^+(y_N) = X_N$. Since $y_j = x_j$ for $j \neq i$, we have $I^+(\lambda_N^+(y_N), Y) = I^+(\lambda_N^+(x_N), Y)$ whenever $Y \succ^{\inf} \tau_i^+(x_i)$, whereas $I^+(\lambda_N^+(y_N), \tau_i^+(x_i)) = I^+(\lambda_N^+(x_N), \tau_i^+(x_i)) \setminus \{i\}$. Therefore, the first disjunctive term of (4.2a) applies, producing $\lambda_N^+(x_N) \succ^{\mathrm{Linf}} \lambda_N^+(y_N)$.

Since $x_i \in X_i^0$ and $x_{-i} \notin S_i^+(x_i)$, Lemma 2.1 implies $x_{-i} \in S_i^-(x_i)$, hence $i \notin N^-(x_N)$. Since $x_N \succ y_N$ and $y_j = x_j$ for $j \neq i$, we have $N^-(x_N) \supseteq N^-(y_N)$, hence $\lambda_N^-(y_N) \succeq^{\mathrm{Lsup}} \lambda_N^-(x_N)$. Now the first disjunctive term in (4.3) applies, producing $y_N \gtrdot x_N$, hence $y_N \ggcurly x_N$.

The case of $y_i > x_i$ is treated dually. ☐

Lemma 2.3. *The relation $\ggcurly$ is ω-transitive.*

Proof. Let $x_N^k \to x_N^\omega$ and $x_N^{k+1} \ggcurly x_N^k$ for all $k \in \mathbb{N}$. Since each X_i^0 is closed, we have $N^0(x_N^0) \subseteq N^0(x_N^\omega)$ by (4.4). If the inclusion is strict, the first disjunctive term in (4.4) ensures that $x_N^\omega \ggcurly x_N^0$. Otherwise, we have $N^0(x_N^k) = N^0(x_N^\omega) = M$ for all $k \in \mathbb{N}$, hence $x_{-M}^0 = x_{-M}^\omega$.

Step 2.3.1. *If $i \in N$, $x_i', x_i \in X_i^0$, and $x_i' > x_i$, then*

$$S_i^+(x_i') \subseteq S_i^+(x_i); \quad S_i^-(x_i) \subseteq S_i^-(x_i'); \tag{4.5a}$$

$$\tau_i^+(x_i') \succ^{\inf} \tau_i^+(x_i); \quad \tau_i^-(x_i') \succ^{\sup} \tau_i^-(x_i). \tag{4.5b}$$

Proof. Both inclusions in (4.5a) immediately follow from Lemma 2.1. Each relation in (4.5b) immediately follows from the appropriate relation in (4.5a). ☐

Since N is finite while $\mathbb{N}$ infinite, we may assume that $N^+(x^k_N) = N^+$ and $N^-(x^k_N) = N^-$ for all $k \in \mathbb{N}$; moreover, we may assume that the (pre)orders induced on N by $\tau^+_{(\cdot)}(x^k_N)$ and $\succeq^{\inf}$, as well as by $\tau^-_{(\cdot)}(x^k_N)$ and $\succeq^{\sup}$, also do not depend on k. Finally, we assume that N is partitioned into $N = N^\uparrow \cup N^= \cup N^\downarrow$, where $N^\uparrow = \{i \in N \mid \forall k \in \mathbb{N}\,[x^{k+1}_i > x^k_i]\}$, $N^= = \{i \in N \mid \forall k \in \mathbb{N}\,[x^{k+1}_i = x^k_i]\}$, $N^\downarrow = \{i \in N \mid \forall k \in \mathbb{N}\,[x^{k+1}_i < x^k_i]\}$. Clearly, $N^\uparrow \cup N^\downarrow \subseteq M$, while $x^\omega_i = x^0_i$ for each $i \in N^=$, hence $N^=$ may be ignored.

Step 2.3.2. *If $i \in N^\uparrow$, then $S^+_i(x^\omega_i) = \bigcap_{k\in\mathbb{N}} S^+_i(x^k_i)$. If $i \in N^\downarrow$, then $S^-_i(x^\omega_i) = \bigcap_{k\in\mathbb{N}} S^-_i(x^k_i)$.*

Proof. If $x_{-i} \in S^+_i(x^\omega_i)$ and $k \in \mathbb{N}$, then either $x^k_i \in \mathcal{R}_i(x_{-i})$ or $x_{-i} \notin S^-_i(x^k_i)$. In the first case, we have $x_{-i} \in S^+_i(x^k_i)$ by definition; in the second, by Lemma 2.1. If $x_{-i} \notin S^+_i(x^\omega_i)$, then $x^\omega_i \notin \mathcal{R}_i(x_{-i}) \ni y_i < x^\omega_i$. Since $\mathcal{R}_i(x_{-i})$ is closed, we have $x^k_i \notin \mathcal{R}_i(x_{-i}) \ni y_i < x^k_i$ for all $k \in \mathbb{N}$ large enough; therefore, $x_{-i} \notin S^+_i(x^k_i)$.

The second statement is proven dually. □

Step 2.3.3. *Let $Y, Y' \in \mathcal{B}_N$, $i \in N^\uparrow$, and $Y \succ^{\inf} Y' \succeq^{\inf} \tau^+_i(x^k_i)$ for each $k \in \mathbb{N}$. Then $Y \succeq^{\inf} \tau^+_i(x^\omega_i)$.*

Proof. Suppose to the contrary that $\tau^+_i(x^\omega_i) \succ^{\inf} Y$; by (3.4e), there is $t \in Y$ such that $(x^\omega_i, x_{-i}) \succ t$ for every $x_{-i} \in S^+_i(x^\omega_i)$. By (3.4e) and (3.4b), there are $t' \in Y'$ and $z^k_{-i} \in S^+_i(x^k_i)$ for each $k \in \mathbb{N}$ such that $t \succ t' \succeq (x^k_i, z^k_{-i})$. Since X_{-i} is compact, we may assume $z^k_{-i} \to z^\omega_{-i}$, hence $t \succ t' \succeq (x^\omega_i, z^\omega_{-i})$, hence $z^\omega_{-i} \notin S^+_i(x^\omega_i)$. Without restricting generality, we may assume that either $z^{k+1}_{-i} \succeq z^k_{-i}$ for each $k \in \mathbb{N}$ or $z^k_{-i} \succ z^{k+1}_{-i}$ for each $k \in \mathbb{N}$. In the first case, we would have $z^\omega_{-i} = \sup_k z^k_{-i} \in \bigcap_{k\in\mathbb{N}} S^+_i(x^k_i) = S^+_i(x^\omega_i)$ by Step 2.3.2. In the second, $z^k_{-i} \in S^+_i(x^h_i)$ for all $k, h \in \mathbb{N}$, hence $z^k_{-i} \in S^+_i(x^\omega_i)$, hence $(x^\omega_i, z^k_{-i}) \succ t$ for each $k \in \mathbb{N}$, which is incompatible with $t \succ (x^\omega_i, z^\omega_{-i})$ since $\succeq$ is continuous. □

Remark. Without the continuity assumption, the statement would be just wrong. Moreover, it would be wrong without Y' in the assumptions.

Step 2.3.4. *Let $Y, Y' \in \mathcal{B}_N$, $i \in N^\downarrow$, and $\tau^-_i(x^k_i) \succeq^{\sup} Y' \succ^{\sup} Y$ for each $k \in \mathbb{N}$. Then $\tau^-_i(x^\omega_i) \succeq^{\sup} Y$.*

The proof is dual to that of Step 2.3.3.

To complete the proof of Lemma 2.3, we consider a couple of alternatives. Let $N^+ \not\subseteq N^=$. Then there must be $j \in N^+ \cap N^\downarrow$ such that $\tau^+_j(x^k_j) \succ^{\inf} \tau^+_i(x^k_i)$ for each $i \in N^+ \cap N^\uparrow$ and $k \in \mathbb{N}$. Step 2.3.3 with $Y = \tau^+_j(x^1_j)$ and $Y' = \tau^+_j(x^2_j)$ immediately gives us $\tau^+_j(x^0_j) \succ^{\inf} \tau^+_j(x^1_j) \succeq^{\inf} \tau^+_i(x^\omega_i)$ for each $i \in N^+ \cap N^\uparrow$. Therefore, $n^+(\lambda^+_N(x^0_N), \tau^+_j(x^0_j)) > n^+(\lambda^+_N(x^\omega_N), \tau^+_j(x^0_j))$ while $n^+(\lambda^+_N(x^0_N), Y) = n^+(\lambda^+_N(x^\omega_N), Y)$ for every $Y \in \mathcal{B}_N$ such that $Y \succ^{\inf} \tau^+_j(x^0_j)$. By (4.2a), we have $\lambda^+_N(x^0_N) \succ^{\mathrm{Linf}} \lambda^+_N(x^\omega_N)$.

Dually, if $N^- \not\subseteq N^=$, then $\lambda^-_N(x^\omega_N) \succ^{\mathrm{Lsup}} \lambda^-_N(x^0_N)$.

If $N^+ \subseteq N^=$, then, obviously, $\lambda^+_N(x^k_N) = \lambda^+_N(x^0_N)$ for all $k \in \mathbb{N}$. If $N^+(x^\omega_N) = N^+$, then $\lambda^+_N(x^0_N) = \lambda^+_N(x^\omega_N)$; otherwise (which is *not* impossible), $N^+(x^\omega_N) \subset N^+$, hence

$\lambda^+_N(x^0_N) \succeq^{\mathrm{Linf}} \lambda^+_N(x^\omega_N)$ because $\tau^+_i(x^0_N) \succeq^{\inf} X_N$ for each $i \in N$. Dually, if $N^- \subseteq N^=$, then $\lambda^-_N(x^\omega_N) \succeq^{\mathrm{Lsup}} \lambda^-_N(x^0_N)$. Both inclusions simultaneously, i.e., $N^+ \cup N^- \subseteq N^=$, would contradict the assumption $x^{k+1}_N \not\succ x^k_N$. Therefore, $x^\omega_N \not\succ x^0_N$, hence $x^\omega_N \ggg x^0_N$. □

Lemmas 2.2 and 2.3 mean that $\ggg$ is an ω-potential for $\mathcal{S}$. □

4.2 Decreasing reactions

Given a preorder $\succeq$ on X_{-i}, we call a mapping $\mathcal{R}_i\colon X_{-i} \to \mathcal{B}_i$ *decreasing* (w.r.t. $\succeq$) if

$$\forall x'_{-i}, x_{-i} \in X_{-i}\,[x'_{-i} \succeq x_{-i} \Rightarrow \mathcal{R}_i(x_{-i}) \succeq^* \mathcal{R}_i(x'_{-i})], \tag{4.6}$$

where $\succeq^*$ is defined by (3.4c). Since $\succeq^*$ is antisymmetric, (4.6) implies that $\mathcal{R}_i(x'_{-i}) = \mathcal{R}_i(x_{-i})$ whenever $x'_{-i} \sim x_{-i}$. A subset $Y \subseteq \mathbb{R}$ is *upper closed* if whenever $y \notin Y$, there is $\varepsilon > 0$ such that $[y - \varepsilon, y] \cap Y = \emptyset$.

Theorem 3. *Let a system of reactions $\mathcal{S}$ be defined by a finite set N, compact $X_i \subset \mathbb{R}$ and $\mathcal{R}_i\colon X_{-i} \to \mathcal{B}_i$ such that every value $\mathcal{R}_i(x_{-i})$ is upper closed. Let there be a continuous ordering $\succeq$ on X_N admitting a separable projection to each X_{-i} such that the separable projection of $\succeq$ to each X_i coincides with the natural order $\geq$ and each $\mathcal{R}_i$ is decreasing in the sense of* (4.6)*. Then $\mathcal{S}$ is Ω-acyclic.*

Proof. For each $i \in N$, $x_i \in X_i$, and $t \in X_N$, we define:

$$G_i = \bigl\{z_N \in X_N \mid z_i \in \mathcal{R}_i(z_{-i}) \text{ or } \exists \langle z^k_N\rangle_{k\in\mathbb{N}}\bigl[\, z^k_N \to z_N \,\&\, \forall k \in \mathbb{N}\,[z^k_i \in \mathcal{R}_i(z^k_{-i}) \,\&\, z^{k+1}_N \succ z^k_N]\,\bigr]\bigr\}, \tag{4.7}$$

"the upper closure of the graph of $\mathcal{R}_i$";

$$\Xi_i(x_i, t) = \{z_N \in G_i \mid x_i \geq z_i \,\&\, z_N \succeq t\};$$

$$\xi_i(x_i, t) = \{z_i \in X_i \mid \exists z_{-i} \in X_{-i}[(z_{-i}, z_i) \in \Xi_i(x_i, t)]\}.$$

Lemma 3.1. *These statements hold for every $i \in N$, $x_i, x'_i, x''_i \in X_i$, and $t, t', t'' \in X_N$:*

$$\{x_i\} \succeq^{\sup} \xi_i(x_i, t); \tag{4.8a}$$

$$x''_i \geq x'_i \Rightarrow \xi_i(x''_i, t) \succeq^{\sup} \xi_i(x'_i, t); \tag{4.8b}$$

$$t'' \succeq t' \Rightarrow \xi_i(x_i, t') \succeq^{\sup} \xi_i(x_i, t''); \tag{4.8c}$$

$$\xi_i(x''_i, t) \succ^{\sup} \xi_i(x'_i, t) \Rightarrow \xi_i(x''_i, t) \succ^{\sup} \{x'_i\}; \tag{4.8d}$$

$$[\xi_i(x''_i, t') \succeq^{\sup} \xi_i(x'_i, t') \,\&\, t'' \succeq t'] \Rightarrow \xi_i(x''_i, t'') \succeq^{\sup} \xi_i(x'_i, t''). \tag{4.8e}$$

Proof. The statement (4.8a) immediately follows from the definition; (4.8b), from $\Xi_i(x_i', t) \subseteq \Xi_i(x_i'', t)$; (4.8c), from $\Xi_i(x_i, t'') \subseteq \Xi_i(x_i, t')$. The left hand side of (4.8d) implies the existence of $z_N'' \in \Xi_i(x_i'', t)$ such that $z_i'' > z_i'$ for each $z_N' \in \Xi_i(x_i', t)$; therefore, $z_i'' > x_i'$ because we would have $z_N'' \in \Xi_i(x_i', t)$ otherwise; therefore, $\xi_i(x_i'', t) \succ^{\sup} \{x_i'\}$. The negation of the right hand side of (4.8e) implies, by (4.8d), the existence of $z_N' \in \Xi_i(x_i', t'') \subseteq \Xi_i(x_i', t')$ such that $z_i' > x_i''$; therefore, $z_i'' \geq z_i'$ is impossible for any $z_N'' \in \Xi_i(x_i'', t')$, which contradicts the first relation in the left hand side of (4.8e). $\square$

For each $x_N \in X_N$, we define binary relations on X_N:

$$y_N \gg' x_N \leftrightharpoons \exists t^* \in X_N \ \big[t^* \succeq y_N \ \& \ t^* \succeq x_N \ \& \ \forall i \in N \, [\xi_i(y_i, t^*) \succeq^{\sup} \xi_i(x_i, t^*)] \ \& \\ \exists i \in N \, [\xi_i(y_i, t^*) \succ^{\sup} \xi_i(x_i, t^*)]\big]; \quad (4.9)$$

$$y_N \gg'' x_N \leftrightharpoons \big[x_N \succ y_N \ \& \ \forall i \in N \, [\xi_i(y_i, x_N) \succeq^{\sup} \xi_i(x_i, x_N)]\big];$$
$$y_N \gg x_N \leftrightharpoons \big[y_N \gg' x_N \ \text{ or } \ y_N \gg'' x_N\big].$$

The relation $\gg$ is obviously irreflexive.

Lemma 3.2. *The relations $\gg$, $\gg'$, and $\gg''$ are transitive.*

Proof. Whenever $y_N \gg' x_N$, we pick $\tau^*(y_N, x_N) \in X_N$ suited for the role of t^* in (4.9); for $y_N \gg'' x_N$, we denote $\tau^*(y_N, x_N) = x_N$. Having $z_N \gg y_N \gg x_N$, we define $t^* = \tau^*(z_N, y_N)$ if $\tau^*(z_N, y_N) \succeq \tau^*(y_N, x_N)$, and $t^* = \tau^*(y_N, x_N)$ otherwise; by Lemma 3.1, (4.8e), $\xi_i(z_i, t^*) \succeq^{\sup} \xi_i(y_i, t^*) \succeq^{\sup} \xi_i(x_i, t^*)$ for all $i \in N$. If t^* can be associated with a $\gg'$ relation (which holds if $z_N \gg' y_N \gg' x_N$), then at least one of the relations for at least one $i \in N$ is strict, hence $\xi_i(z_i, t^*) \succ^{\sup} \xi_i(x_i, t^*)$, hence $z_N \gg' x_N$. Otherwise (which holds if $z_N \gg'' y_N \gg'' x_N$), $y_N \gg'' x_N$, $t^* = x_N \succ y_N$, and $x_N \succeq \tau^*(z_N, y_N) \succeq z_N$; therefore, $z_N \gg'' x_N$ because $\xi_i(z_i, x_N) \succeq^{\sup} \xi_i(x_i, x_N)$ for all $i \in N$. $\square$

Lemma 3.3. *Let $i \in N$, $t \in X_N$, $x_i, x_i^\omega, x_i^k \in X_i$ $(k \in \mathbb{N})$, $x_i^k \to x_i^\omega$, and $\xi_i(x_i^k, t) \succeq^{\sup} \xi_i(x_i, t)$ for all $k \in \mathbb{N}$. Then $\xi_i(x_i^\omega, t) \succeq^{\sup} \xi_i(x_i, t)$.*

Proof. Otherwise, there would exist $z_N \in \Xi_i(x_i, t)$ such that $z_i > x_i^\omega$; therefore, $z_i > x_i^k$ for k large enough, which contradicts the condition $\xi_i(x_i^k, t) \succeq^{\sup} \xi_i(x_i, t)$. $\square$

Lemma 3.4. *The relation $\gg$ is ω-transitive.*

Proof. Let $x_N^k \to x_N^\omega$ and $x_N^{k+1} \gg x_N^k$ for all $k \in \mathbb{N}$. Since both $\gg'$ and $\gg''$ are transitive by Lemma 3.2, we may, without restricting generality, assume that either $x_N^{k+1} \gg' x_N^k$ for all $k \in \mathbb{N}$, or $x_N^{k+1} \gg'' x_N^k$ for all $k \in \mathbb{N}$. In the second case, we have $x_N^{k+1} \prec x_N^k$ for all k, hence $x_N^\omega \prec x_N^0$; besides, $\xi_i(x_i^\omega, x_N^0) \succeq^{\sup} \xi_i(x_i^0, x_N^0)$ for every $i \in N$ by Lemma 3.3. Therefore, $x_N^\omega \gg'' x_N^0$.

Assuming $x_N^{k+1} \gg' x_N^k$ for all $k \in \mathbb{N}$, we can pick $t^k \in X_N$ suited for the role of t^* in (4.9). Since N is finite, we may, without restricting generality, assume that $\xi_i(x_i^{k+1}, t^k) \succ^{\sup} \xi_i(x_i^k, t^k)$ for an $i \in N$ and all $k \in \mathbb{N}$. Now we consider two alternatives.

If the sequence $\langle t^k \rangle_k$ contains a greatest (w.r.t. $\succeq$) element, we may, without restricting generality, assume that $t^0 \succeq t^k$ for each $k \in \mathbb{N}$. Then we have $\xi_j(x_j^k, t^0) \succeq^{\sup} \xi_j(x_j^1, t^0)$ for all $j \in N$ and $k \geq 1$ by (4.8e). Therefore, $\xi_j(x_j^\omega, t^0) \succeq^{\sup} \xi_j(x_j^1, t^0) \succeq^{\sup} \xi_j(x_j^0, t^0)$ by Lemma 3.3. For $j = i$, the last relation is strict by the definition of t^0, hence $\xi_i(x_i^\omega, t^0) \succ^{\sup} \xi_i(x_i^0, t^0)$. Therefore, $x_N^\omega \ggg' x_N^0$.

Otherwise, we may assume that $t^{k+1} \succ t^k$ for each k and $t^k \to t^\omega$; then $t^\omega \succ t^k$ for each k. By (4.8e), we have $\xi_j(x_j^k, t^\omega) \succeq^{\sup} \xi_j(x_j^0, t^\omega)$; hence by Lemma 3.3, $\xi_j(x_j^\omega, t^\omega) \succeq^{\sup} \xi_j(x_j^0, t^\omega)$, for each $j \in N$. We only have to show that the last relation is strict for $j = i$. By (4.8d), we may pick $z_N^k \in \xi_i(x_i^{k+1}, t^k)$ such that $z_i^k > x_i^k$; without restricting generality, we may assume $z_N^k \to z_N^\omega$. Clearly, $z_N^\omega \in G_i$, $z_i^\omega = x_i^\omega$, and $z_N^\omega \succeq t^\omega \succeq x_N^\omega$; therefore, $z_N^\omega \in \xi_i(x_i^\omega, t^\omega) \succ^{\sup} \xi_i(x_i^0, t^\omega)$, hence $x_N^\omega \ggg' x_N^0$. □

Lemma 3.5. *If $y_N \rhd^{\mathcal{S}} x_N$, then $y_N \ggg x_N$.*

Proof. Let $y_N \rhd_i^{\mathcal{S}} x_N$; we consider two alternatives.

Let $y_i > x_i$; then $y_N \succ x_N$ by separability, hence $y_N \in \Xi_i(y_i, y_N)$, hence $y_i \in \xi_i(y_i, y_N) \succ^{\sup} \xi_i(x_i, y_N)$. On the other hand, for $j \neq i$, we have $\xi_j(y_j, y_N) = \xi_j(x_j, y_N)$ since $y_j = x_j$. Thus, $y_N \ggg' x_N$ (with "$t^* = y_N$").

Now let $y_i < x_i$; then $x_N \succ y_N$. To prove that $y_N \ggg'' x_N$, it is sufficient to show that $\xi_j(y_j, x_N) \succeq^{\sup} \xi_j(x_j, x_N)$ for all $j \in N$. For every $j \neq i$, we have $\xi_j(y_j, x_N) = \xi_j(x_j, x_N)$ since $y_j = x_j$. By the monotonicity of $\mathcal{R}_i$, we have $z_i \leq y_i$ whenever $z_i \in \mathcal{R}_i(z_{-i})$ and $z_{-i} \succ x_{-i}$. Let $\xi_i(x_i, x_N) \succ^{\sup} \xi_i(y_i, x_N)$; then, by (4.8d), there is $z_N \in G_i$ such that $y_i < z_i \leq x_i$ and $z_N \succeq x_N$, hence $z_{-i} \succeq x_{-i}$. Clearly, $z_i \in \mathcal{R}_i(z_{-i})$ is impossible, hence there must be a sequence $\langle z_N^k \rangle_{k \in \mathbb{N}}$ as in (4.7) with strictly decreasing z_{-i}^k, i.e. there must hold $z_i^k \in \mathcal{R}_i(z_{-i}^k)$, $y_i < z_i^k < z_i$, and $z_{-i}^k \succ z_{-i}$; but this is impossible. □

Lemmas 3.4 and 3.5 immediately imply that $\ggg$ is an ω-potential for $\mathcal{S}$. Theorem 3 is proved. □

Example 5.1 below shows that our rather strong monotonicity condition cannot be weakened; therefore, the difference between (4.1) and (4.6) reflects some underlying reality. As to the closed values assumptions in both Theorems 2 and 3, it is very difficult to believe that they are indispensable; Theorem 5 below shows that they are not needed for additive aggregation. On the other hand, a relatively simple description of an ω-potential could, indeed, be impossible without the assumptions.

4.3 Lexicographic aggregation

Theorem 6 of Kukushkin (2003) shows that a proset may have the property that every increasing endomorphism is Ω-acyclic without the preorder being continuous; in particular, lexicographic orderings will do. For systems of increasing reactions (even with two players) the situation is more complicated; no characterization result like that Theorem 6 has yet been obtained. The following example shows that a system of two increasing reactions with chains in a lexicographic order as strategy sets need not be Ω-acyclic.

To the end of this subsection, we only consider single-valued reactions, which we denote $r_i\colon X_{-i} \to X_i$.

Example 4.1. Let $N = \{0,1\}$, $X_1 = X_2 = [0,1] \times [0,1]$ with a lexicographical order

$$(y_i^1, y_i^2) \succeq_i (x_i^1, x_i^2) \leftrightharpoons \big[y_i^1 > x_i^1 \ \text{ or } \ [y_i^1 = x_i^1 \ \&\ y_i^2 \geq x_i^2]\big],$$

and reactions be as follows: $r_1(0, x_2^2) = (x_2^2, 1)$ if $0 \leq x_2^2 < 1$, $r_1(0,1) = (1,0)$, and $r_1(x_2^1, x_2^2) = (1, x_2^1)$ if $0 < x_2^1 \leq 1$ and $0 \leq x_2^2 \leq 1$; $r_2(x_1^1, x_1^2) = (0, (x_1^1+1)/2)$ if $0 \leq x_1^1 < 1$ and $0 \leq x_1^2 \leq 1$, $r_2(1,0) = (0,1)$, and $r_2(1, x_1^2) = (x_1^2/2, 0)$ if $0 < x_1^2 \leq 1$. It is easily checked that both reactions are increasing.

We define an iteration path recursively. First, $\pi_N(0) = \langle (1,0), (0,0) \rangle$; then, $\pi_1(2k+1) = r_1(\pi_2(2k)) = \pi_1(2k+2)$, $\pi_2(2k+1) = \pi_2(2k)$, and $\pi_2(2k+2) = r_2(\pi_1(2k+1))$ for all $k \in \mathbb{N}$. It is easily checked that $\pi_N(2k+1) = \langle (1 - 1/2^k, 1), (0, 1 - 1/2^k) \rangle$ while $\pi_N(2k+2) = \langle (1 - 1/2^k, 1), (0, 1 - 1/2^{k+1}) \rangle$; therefore, $\pi_N(k)$ converges to $\pi_N(\omega) = \langle (1,1), (0,1) \rangle$. Then we define $\pi_1(\omega + 2k + 1) = \pi_1(\omega + 2k)$, $\pi_1(\omega + 2k + 2) = r_1(\pi_2(\omega + 2k + 1))$, and $\pi_2(\omega + 2k + 1) = r_2(\pi_1(\omega + 2k)) = \pi_2(\omega + 2k + 2)$ for all $k \in \mathbb{N}$. It is easily checked that $\pi_N(\omega + 2k) = \langle (1, 1/2^k), (1/2^k, 0) \rangle$ while $\pi_N(\omega + 2k + 1) = \langle (1, 1/2^k), (1/2^{k+1}, 0) \rangle$; therefore, $\pi_N(\omega + k)$ converges to $\pi_N(\omega + \omega) = \langle (1,0), (0,0) \rangle = \pi_N(0)$.

Theorem 4. *Let a system of reactions $\mathcal{S}$ be defined by $N = \{1,2,3\}$, compact $X_i \subset \mathbb{R}$ and $r_i\colon X_{-i} \to X_i$ increasing w.r.t. the separable projection to X_{-i} of the lexicographic order $\geq_{\mathrm{Lex}}$ on $\mathbb{R}^3 \supset X_N$. Then $\mathcal{S}$ is Ω-acyclic.*

Proof. We start with auxiliary notions and statements. Let $x_N \in X_N$ and $i \in N$; we say that x_i *is supported at* x_N, and denote the fact $i \in N^+(x_N)$, if $r_i(x_{-i}) \geq x_i$. If $N^+(x_N) = N$, we say that x_N is *completely supported.* The following assertions are checked easily:

$$y_N \rhd_i^{\mathcal{S}} x_N \Rightarrow i \in N^+(y_N); \tag{4.10a}$$

$$[y_N \rhd_i^{\mathcal{S}} x_N \ \&\ i \in N^+(x_N)] \Rightarrow y_i > x_i; \tag{4.10b}$$

$$[y_N \rhd^{\mathcal{S}} x_N \ \&\ N^+(x_N) = N] \Rightarrow N^+(y_N) = N. \tag{4.10c}$$

Let π_N be an iteration path for $\mathcal{S}$, defined on $\Sigma = [0, \alpha]$.

Lemma 4.1. *Whenever $\beta' > \beta$ and $\pi_N(\beta)$ is completely supported, there hold:*

$$N^+(\pi_N(\beta')) = N; \tag{4.11a}$$

$$\forall i \in N\, [\pi_i(\beta') \geq \pi_i(\beta)]; \tag{4.11b}$$

$$\pi_N \textit{ is narrow on } [\beta, \alpha]; \tag{4.11c}$$

$$\exists i \in N\, [\pi_i(\beta') > \pi_i(\beta)]. \tag{4.11d}$$

Proof. Supposing the contrary, we denote β^* the least $\beta' \in \Sigma$ for which there is $\beta < \beta'$ such that $\pi_N(\beta)$ is completely supported while at least one of (4.11a), (4.11b), or (4.11c) is violated. If $\beta^* \in \Sigma_{\text{iso}}$, then (4.11a) holds by (4.10c); (4.11b), as well as (4.11d), holds by (4.10b); and (4.11c) holds because $\beta^* \in \Sigma_{\text{iso}}$. Therefore, we must have $\beta^* \in \Sigma_{\text{lim}}$. Then (4.11b) for all $\beta < \beta' < \beta^*$ immediately implies

$$\pi_i(\beta^*) = \sup_{\beta' < \beta^*} \pi_i(\beta') \tag{4.12}$$

for each $i \in N$, hence (4.11b) and (4.11c) hold for $\beta' = \beta^*$. If (4.11a) is violated at $\beta' = \beta^*$, we pick $i \in N$ such that $\pi_i(\beta^*) > r_i(\pi_{-i}(\beta^*))$. By (4.12), there is $\beta' < \beta^*$ such that $\pi_i(\beta') > r_i(\pi_{-i}(\beta^*))$; since $\pi_{-i}(\beta^*) \geq \pi_{-i}(\beta')$, we have a contradiction with (4.11a) for β'.

Finally, if (4.11a), (4.11b), and (4.11c) hold for all $\beta, \beta' \in \Sigma$, then (4.11d) holds because of (4.10b) applied to $\pi_N(\beta+1)$ and $\pi_N(\beta)$, and (4.11b) applied to $\beta+1$ and β'. $\square$

Lemma 4.1 immediately implies that no iteration cycle can pass through a completely supported profile. In the rest of the proof, we assume that π_N is an iteration cycle, i.e., $\alpha > 0$ and $\pi_N(\alpha) = \pi_N(0)$, and derive the existence of $\beta \in \Sigma$ for which $\pi_N(\beta)$ is completely supported. This will constitute a contradiction proving the theorem.

We denote $Y = \text{cl}\{\pi_N(\beta)\}_{\beta \in \Sigma} \subseteq X_N$. Since Y is compact, there is a (unique) maximum of $\geq_{\text{Lex}}$ on Y; we denote it M_N. Without restricting generality, we may assume that for each $i \in N$ there is $\beta \in \Sigma$ such that $\pi_N(\beta+1) \rhd_i^{\mathcal{S}} \pi_N(\beta)$: otherwise, we would have a system of two increasing reactions on continuous chains where everything is crystal clear. A consequence of the assumption is that, whenever $i \in N$ and $v_i < M_i$, there is $\beta \in \Sigma$ such that $v_i < \pi_i(\beta) = r_i(\pi_{-i}(\beta))$.

Lemma 4.2. *M_N is completely supported.*

Proof. Supposing $r_i(M_{-i}) < M_i$ for an $i \in N$, we consider two alternatives. If there is $\beta \in \Sigma$ such that $M_i \leq \pi_i(\beta) = r_i(\pi_{-i}(\beta))$, then we obviously have $\pi_{-i}(\beta) >_{\text{Lex}} M_{-i}$, hence $\pi_N(\beta) >_{\text{Lex}} M_N$, which contradicts the definition of M_N. Otherwise, we can pick an infinite sequence $\langle \beta^k \rangle_{k \in \mathbb{N}}$ such that $\pi_i(\beta^0) > r_i(M_{-i})$, $\pi_i(\beta^k) = r_i(\pi_{-i}(\beta^k))$ and $\pi_i(\beta^{k+1}) > \max\{\pi_i(\beta^k), M_i - 1/k\}$ for each $k \in \mathbb{N}$. Then $\pi_{-i}(\beta^0) >_{\text{Lex}} M_{-i}$, and $\pi_{-i}(\beta^{k+1}) >_{\text{Lex}} \pi_{-i}(\beta^k)$ for each $k \in \mathbb{N}$. Picking an arbitrary limit point y_N of the sequence $\langle \pi_N(\beta^k) \rangle_{k \in \mathbb{N}}$, we obviously have $y_i = M_i$ and $y_{-i} >_{\text{Lex}} M_{-i}$, hence $y_N >_{\text{Lex}} M_N$, which again contradicts the definition of M_N because $y_N \in Y$. $\square$

Remark. For $i = 3$, the infinite sequence is superfluous: just β^0 is enough.

Lemma 4.3. *There is either $\beta \in \Sigma$ for which $\pi_N(\beta) = M_N$, or an infinite sequence β^k such that $\beta^{k+1} > \beta^k$ and $\pi_N(\beta^k) \to M_N$.*

Proof. Let $M_N \neq \pi_N(\beta)$ for any $\beta \in \Sigma$. Then there must be an infinite sequence of $\gamma^h \in \Sigma$ such that $\pi_N(\gamma^h) \to M_N$. We pick $\varkappa(0) \in \mathbb{N}$ such that $\gamma^{\varkappa(0)} = \min_{h \in \mathbb{N}} \gamma^h$. Then we recursively, for $k = 0, 1, \ldots$, define $\mathrm{B}^k = \{\gamma^h \mid h > \varkappa(k) \;\&\; \gamma^h > \gamma^{\varkappa(k)}\}$ [$\mathrm{B}^k \neq \emptyset$ because

$M_N \neq \pi_N(\gamma^h)$ for any $h \in \mathbb{N}$] and pick $\varkappa(k+1)$ such that $\gamma^{\varkappa(k+1)} = \min \mathrm{B}^k$. Finally, we define $\beta^k = \gamma^{\varkappa(k)}$; now $\beta^{k+1} > \beta^k$ by definition, and $\beta^k \to M_N$ because $\langle \beta^k \rangle_k$ is a subsequence of $\langle \gamma^h \rangle_h$. □

If the first alternative in Lemma 4.3 holds, then Lemma 4.2 applies; therefore, we may assume that Σ is infinite and the second alternative in Lemma 4.3 holds. Since we can start the cycle anyplace, we may assume that $\sup_k \beta^k = \alpha$.

For any $i \in N$, $\beta \in \Sigma$, and $v_i \in \mathbb{R}$, we denote

$$P_i(\beta, v_i) = \{\gamma \in \Sigma \mid \gamma > \beta \;\&\; \pi_i(\gamma) > v_i\}.$$

Whenever $P_i(\beta, v_i) \neq \emptyset$, we define $\tau^i(\beta, v_i) = \min P_i(\beta, v_i)$; the minimum exists because Σ is well ordered.

Lemma 4.4. *Whenever $\beta \in \Sigma$ and $\pi_1(\beta) < v_1 < M_1$, there hold:*

$$P_1(\beta, v_1) \neq \emptyset; \tag{4.13a}$$
$$\pi_1(\tau^1(\beta, v_1)) = r_1(\pi_{-1}(\tau^1(\beta, v_1))); \tag{4.13b}$$
$$i \in N \setminus N^+(\pi_N(\tau^1(\beta, v_1))) \Rightarrow \big[\pi_i(\cdot) \textit{ is a constant on } [\beta, \tau^1(\beta, v_1)]\big]; \tag{4.13c}$$
$$N^+(\pi_N(\beta)) \subseteq N^+(\pi_N(\tau^1(\beta, v_1))); \tag{4.13d}$$
$$N^+(\pi_N(\beta)) = N^+(\pi_N(\tau^1(\beta, v_1))) \Rightarrow \big[N^+(\pi_N(\cdot)) \textit{ is a constant on } [\beta, \tau^1(\beta, v_1)]\big]; \tag{4.13e}$$
$$N^+(\pi_N(\beta)) = N^+(\pi_N(\tau^1(\beta, v_1))) \Rightarrow \forall i \in N \big[\pi_i(\cdot) \textit{ is increasing on } [\beta, \tau^1(\beta, v_1)]\big]. \tag{4.13f}$$

Proof. (4.13a) and (4.13b) are obvious. To prove (4.13c), we suppose to the contrary that $r_i(\pi_{-i}(\tau^1(\beta, v_1))) < \pi_i(\tau^1(\beta, v_1)) \neq \pi_i(\beta')$ and $\beta \leq \beta' < \tau^1(\beta, v_1)$. We denote $\mathrm{B} = \{\gamma \in [\beta', \tau^1(\beta, v_1)] \mid \pi_i(\beta') \neq \pi_i(\gamma) > r_i(\pi_{-i}(\tau^1(\beta, v_1)))\} \ni \tau^1(\beta, v_1)$ and $\beta^* = \min \mathrm{B}$. Clearly, $\pi_i(\beta^*) = r_i(\pi_{-i}(\beta^*))$, hence $\pi_{-i}(\beta^*) >_{\mathrm{Lex}} \pi_{-i}(\tau^1(\beta, v_1))$, hence $\pi_1(\beta^*) \geq \pi_1(\tau^1(\beta, v_1)) > \pi_1(\beta)$, hence $\beta^* \in P_1(\beta)$, hence $\beta^* = \tau^1(\beta, v_1)$, hence $\pi_i(\beta^*) > r_i(\pi_{-i}(\beta^*))$: a contradiction.

If $i \in N \setminus N^+(\pi_N(\tau^1(\beta, v_1)))$, then $\pi_i(\tau^1(\beta, v_1)) > r_i(\pi_{-i}(\tau^1(\beta, v_1))) \geq r_i(\pi_{-i}(\beta))$ because $\pi_1(\tau^1(\beta, v_1)) > \pi_1(\beta)$; therefore, $i \in N \setminus N^+(\pi_N(\beta))$. Thus, (4.13d) holds. Now both (4.13e) and (4.13f) follow from Lemma 4.1 applied to the reduced system with $N^+(\pi_N(\beta))$ as the set of players. □

Now we define a sequence β_1^k in Σ recursively. First, we define $\beta_1^0 = \min\{\beta \in \Sigma \mid \pi_1(\beta) < M_1\}$. Whenever β_1^k is defined and $\pi_1(\beta_1^k) < M_1$, we define $\beta_1^{k+1} = \tau^1(\beta_1^k, M_1/2 + \pi_1(\beta_1^k)/2)$. If β_1^k is defined and $\pi_1(\beta_1^k) = M_1$, we stop the process and denote $\alpha_1 = \beta_1^k$. Finally, if β_1^k is defined for all $k \in \mathbb{N}$, we denote $\alpha_1 = \sup_{k \in \mathbb{N}} \beta_1^k$; clearly, $\sup_{k \in \mathbb{N}} \pi_1(\beta_1^k) = M_1$ in this case.

Lemma 4.5. *There hold $\pi_1(\alpha_1) = M_1$ and $1 \in N^+(\pi_N(\alpha_1))$.*

Proof. If α_1 was reached at a finite step, both statements are obvious. Let β_1^k be defined for all $k \in \mathbb{N}$, hence $\alpha_1 = \sup_{k \in \mathbb{N}} \beta_1^k$. By (4.13d), we have $N^+(\pi_N(\beta_1^k)) = N^*$ for all $k \geq \bar{k}$; by (4.13b), $1 \in N^*$; by (4.13c), $\pi_i(\cdot)$ is a constant on $[\beta_1^{\bar{k}}, \alpha_1[$, hence a constant on $[\beta_1^{\bar{k}}, \alpha_1]$, for each $i \in N \setminus N^*$. Now Lemma 4.1, applied to the reduced system with N^* as the set of players, gives us both statements. □

If $\alpha_1 = \alpha$, then $\pi_N(\alpha_1) = M_N$ by (4.11c) in the reduced system, hence Lemma 4.2 applies; let $\alpha_1 < \alpha$. If $\pi_N(\alpha_1)$ is completely supported, we are home. Let $i \in N \setminus N^+(\pi_N(\alpha_1)) \neq \emptyset$; we denote $j = 5 - i$.

A. Let $\pi_i(\alpha_1) < M_i$; then $\beta^* = \tau^i(\alpha_1, \pi_i(\alpha_1))$ is well defined. We have $\pi_i(\beta^*) = r_i(\pi_{-i}(\beta^*))$ as in (4.13b), hence $i \in N^+(\pi_N(\beta^*))$, $\pi_1(\beta^*) = M_1$, and $\pi_j(\beta^*) > \pi_j(\alpha_1)$; further, $1 \in N^+(\pi_N(\beta^*))$ because $\pi_{-1}(\beta^*) >_{\text{Lex}} \pi_{-1}(\alpha_1)$; finally, $j \in N^+(\pi_N(\beta^*))$ as in (4.13c) because $\pi_j(\cdot)$ is *not* a constant on $[\alpha_1, \beta^*]$. Thus, $\pi_N(\beta^*)$ is completely supported.

B. Now let $\pi_i(\alpha_1) \geq M_i$; then $\pi_j(\alpha_1) < M_j$ by the definition of M_N, hence $j \in N^+(\pi_N(\alpha_1))$ by Lemma 4.2. It is convenient to consider the cases $i = 2$ and $i = 3$ separately.

B1. Let $i = 2$; then $\pi_2(\alpha_1) = M_2 > r_2(\pi_{-2}(\alpha_1))$, hence $\beta_2^0 = \tau^2(\alpha_1, r_2(\pi_{-2}(\alpha_1))$ is well defined. We have $\pi_2(\beta_2^0) = r_2(\pi_{-2}(\beta_2^0))$ as in (4.13b), hence $2 \in N^+(\pi_N(\beta_2^0))$, $\pi_1(\beta_2^0) = M_1$ and $\pi_3(\beta_2^0) > \pi_3(\alpha_1)$; moreover, $3 \in N^+(\pi_N(\beta_2^0))$ as in (4.13c) because $\pi_3(\cdot)$ is *not* a constant on $[\alpha_1, \beta_2^0]$. If $1 \in N^+(\pi_N(\beta_2^0))$, then $\pi_N(\beta_2^0)$ is completely supported and we are home.

Let $1 \notin N^+(\pi_N(\beta_2^0))$. Then $\pi_1(\cdot)$ is a constant (M_1) on $[\alpha_1, \beta_2^0]$ as in (4.13c). Now we define a sequence β_2^k in Σ recursively, similar to β_1^k above. Whenever β_2^k is defined and $\pi_2(\beta_2^k) < M_2$, we define $\beta_2^{k+1} = \tau^2(\beta_2^k, M_2/2 + \pi_2(\beta_2^k)/2)$. As in the case of β_1^k, we always have $\{2, 3\} \subseteq N^+(\pi_N(\beta_2^k))$. If $1 \in N^+(\pi_N(\beta_2^k))$, we are home; otherwise $\pi_1(\cdot)$ is a constant (M_1) on $[\alpha_1, \beta_2^k]$ as in (4.13c). Once we have reached $\pi_2(\alpha_2) = M_2$, with either $\alpha_2 = \beta_2^k$ for some $k \in \mathbb{N}$, or $\alpha_2 = \sup_{k \in \mathbb{N}} \beta_2^k$, we *do* have $1 \in N^+(\pi_N(\alpha_2))$ because $\pi_{-1}(\alpha_2) >_{\text{Lex}} \pi_{-1}(\alpha_1)$.

B2. Let $i = 3$; then we consider two alternatives.

B2a. Let $P_3(\alpha_1, r_3(\pi_{-3}(\alpha_1))) \neq \emptyset$, hence $\beta^* = \tau^i(\alpha_1, r_3(\pi_{-3}(\alpha_1))$ is well defined. Then $\pi_N(\beta^*)$ is completely supported for exactly the same reasons as in **A** above: $\pi_3(\beta^*) = r_3(\pi_{-3}(\beta^*))$ as in (4.13b), hence $3 \in N^+(\pi_N(\beta^*))$, $\pi_1(\beta^*) = M_1$ and $\pi_2(\beta^*) > \pi_2(\alpha_1)$; further, $1 \in N^+(\pi_N(\beta^*))$ because $\pi_{-1}(\beta^*) >_{\text{Lex}} \pi_{-1}(\alpha_1)$, while $2 \in N^+(\pi_N(\beta^*))$ as in (4.13c) because $\pi_2(\cdot)$ is *not* a constant on $[\alpha_1, \beta^*]$.

B2b. Finally, let $P_3(\alpha_1, r_3(\pi_{-3}(\alpha_1))) = \emptyset$; then $\pi_3(\alpha) \leq r_3(\pi_{-3}(\alpha_1))$, hence $r_3(\pi_{-3}(\alpha_1)) \geq M_3$. Since $\pi_3(0) = \pi_3(\alpha)$, we have $P_3(0, r_3(\pi_{-3}(\alpha_1))) \ni \alpha_1$, hence $\beta^* = \tau^3(0, r_3(\pi_{-3}(\alpha_1)))$ is well defined and $\pi_3(\beta^*) = r_3(\pi_{-i}(\beta^*))$. Again, $\pi_1(\beta^*) = M_1$ and $\pi_2(\beta^*) > \pi_2(\alpha_1)$; again, $1 \in N^+(\pi_N(\beta^*))$ because $\pi_{-1}(\beta^*) >_{\text{Lex}} \pi_{-1}(\alpha_1)$. Finally, $2 \in N^+(\pi_N(\beta^*))$ because $\pi_3(\beta^*) > M_3$. Thus, $\pi_N(\beta^*)$ is completely supported. □

Most likely, similar arguments work for $\#N > 3$ (most auxiliary statements are valid for arbitrary finite N) and for multi-valued reactions (under an appropriate interpretation of monotonicity); probably, simple lexicography can be replaced with, say, leximax. The prospects for a similar approach to arbitrary separable aggregation are unclear. Even more intriguing is the question of whether this proof can be made applicable to *decreasing* reactions, in which case a new fixed point theorem would be obtained.

5 Reciprocal polylinear aggregates

5.1 Formulation

A *system of reactions with reciprocal quasi-polylinear aggregates* (an *RQPLA system*) is characterized by these assumptions: each X_i is simultaneously a proset and a metric space; there is a continuous and strictly increasing mapping $\nu_i \colon X_i \to \mathbb{R}$ for each $i \in N$; $\mathcal{R}_i = R_i \circ \sigma_i$ for every $i \in N$, where

$$\sigma_i(x_{-i}) = \sum_{h=1}^{n-1} \sum_{\substack{j_1,\dots,j_h \in N\setminus\{i\} \\ j_{h'} \neq j_{h''} (h' \neq h'')}} \alpha^{(h)}_{ij_1\dots j_h} \times \nu_{j_1}(x_{j_1}) \times \cdots \times \nu_{j_h}(x_{j_h}), \tag{5.1}$$

and $R_i \colon S_i \to \mathcal{B}_i$ for $S_i = \sigma_i(X_{-i}) \subseteq \mathbb{R}$; each $\alpha^{(h)}_{i_0 i_1 \dots i_h}$ is invariant under all permutations of $i_0, i_1, \dots, i_h$.

For each $i \in N$, we also impose a monotonicity condition:

$$\forall s_i', s_i \in S_i \, [s_i' > s_i \Rightarrow R_i(s_i') \succeq^* R_i(s_i)], \tag{5.2}$$

where $\succeq^*$ is defined by (3.4c).

Theorem 5. *Every RQPLA system where every mapping R_i satisfies* (5.2) *is Ω-acyclic.*

Remark. There is no requirement on concord between topology and preorder on each X_i beyond the existence of a continuous and strictly increasing function.

The proof is deferred to Subsection 5.2.

Condition (5.2) is rather strong. For instance, if X_i is a lattice, it implies that R_i is ascending, whereas the converse implication does not hold even if X_i is a chain. However, there seems to be no way to relax the condition while preserving the theorem.

Example 5.1. Let $N = \{1,2,3\}$, $X_1 = \{0,1,2,3,4\}$, $X_2 = \{0,1,2,3,4,5\}$, $X_3 = \{0,1\}$, $\sigma_i(x_{-i}) = -x_j - x_k$, hence $S_1 = \{0,-1,\dots,-6\}$, $S_2 = \{0,-1,\dots,-5\}$, and $S_3 = \{0,-1,\dots,-9\}$. Let $R_1(-6) = \{0\}$, $R_1(s_1) = \{1,2,3\}$ for $-5 \le s_1 \le -1$, $R_1(0) = \{4\}$, $R_2(s_2) = \{0\}$ for $s_2 < -2$, $R_2(s_2) = \{5\}$ for $s_2 \ge -2$, $R_3(s_3) = \{0\}$ for $s_3 < -4$, and $R_3(s_3) = \{1\}$ for $s_3 \ge -4$. Condition (5.1) holds with identity mappings as ν_i; (5.2) holds for $i = 2$ and $i = 3$, but not for $i = 1$ although R_1 is ascending. There is an iteration cycle:

$$\begin{array}{ccccccc}
(3,0,0) & \xrightarrow{1} & (4,0,0) & \xrightarrow{3} & (4,0,1) & \xrightarrow{1} & (1,0,1) \\
\uparrow 2 & & & & & & \downarrow 2 \\
(3,5,0) & \xleftarrow{1} & (0,5,0) & \xleftarrow{3} & (0,5,1) & \xleftarrow{1} & (1,5,1)
\end{array}.$$

The theorem also becomes wrong if $\nu_i \colon X_i \to \mathbb{R}$ are not *strictly* increasing.

Example 5.2. Let $N = \{1, 2, 3\}$ and, for each $i \in N$, $X_i = \{0, 1, 2, 3\}$, $\nu_i(0) = \nu_i(1) = 0$, $\nu_i(2) = \nu_i(3) = 1$, $\sigma_i(x_{-i}) = -\nu_j(x_j) - \nu_k(x_k)$ (hence $S_i = \{-2, -1, 0\}$), $R_i(-2) = \{0\}$, $R_i(-1) = \{1, 2\}$, and $R_i(0) = \{3\}$. Both (5.1) and (5.2) are satisfied. However, there is an iteration cycle:

$$\begin{array}{ccccccccccc}
(2,3,1) & \xrightarrow{3} & (2,3,0) & \xrightarrow{2} & (2,1,0) & \xrightarrow{1} & (3,1,0) & \xrightarrow{3} & (3,1,2) & \xrightarrow{2} & (3,0,2) \\
\uparrow{\scriptstyle 1} & & & & & & & & & & \downarrow{\scriptstyle 1} \\
(0,3,1) & \xleftarrow{2} & (0,2,1) & \xleftarrow{3} & (0,2,3) & \xleftarrow{1} & (1,2,3) & \xleftarrow{2} & (1,0,3) & \xleftarrow{3} & (1,0,2)
\end{array}.$$

Theorem 5 admits a straightforward application to the best responses in a strategic game where each utility function satisfies $u_i(x_N) = U_i(\sigma_i(x_{-i}), x_i)$ for all $i \in N$ and $x_N \in X_N$, where each $\sigma_i \colon X_{-i} \to \mathbb{R}$ is defined by (5.1) with strictly increasing mappings $\nu_i \colon X_i \to \mathbb{R}$ and each $\alpha^{(h)}_{i_0 i_1 \dots i_h}$ invariant under all permutations of $i_0, i_1, \dots, i_h$. The standard argument (Milgrom and Shannon, 1994; Topkis, 1998) shows that the following *strict single crossing* condition is sufficient for (5.2):

$$[x'_i > x_i \;\&\; s'_i > s_i \;\&\; U_i(s_i, x'_i) \geq U_i(s_i, x_i)] \Rightarrow U_i(s'_i, x'_i) > U_i(s'_i, x_i) \tag{5.3}$$

for all $i \in N$, $x'_i, x_i \in X_i$, and $s'_i, s_i \in S_i$. The condition (5.3) cannot be called either strategic substitutes or strategic complements because $s_i = \sigma_i(x_{-i})$ can be either decreasing or increasing in each x_j, depending on α's and perhaps on the other players' choices; "*strategic supplements*" may be an appropriate term.

If $\alpha^{(m)}_{i_0 i_1 \dots i_m} = 0$ for $m > 1$ and $\alpha^{(1)}_{ij} = 1$, we obtain a game with strict strategic complements and additive aggregation; here Theorem 5 gives a result similar to Theorem 2, with stronger monotonicity conditions, but without the closed values assumption. For finite games from the class, the acyclicity of best response improvements was established in Kukushkin (2004a, Theorem 1), under even weaker monotonicity conditions (6.2a). If $\alpha^{(m)}_{i_0 i_1 \dots i_m} = 0$ for $m > 1$, while $\alpha^{(1)}_{ij} = -1$, we obtain a game with strict strategic substitutes and additive aggregation; here, again, Theorem 5 requires a bit less than Theorem 3. For finite games from the class, either theorem is equivalent to Theorem 2 from Kukushkin (2004a).

Example 5.3. Each player owns a small business in an area. The decision problem for each of them is how much lighting, x_i, to provide at her location at night. The higher x_i, the higher expenses; on the other hand, the more light, the lower insurance costs. There is a positive externality effect: each player's lamps add something to the light at other lots. It seems reasonable to assume that insurance costs decrease in $x_i + \sum_{j \neq i} \alpha_{ij} x_j$, where $0 \leq \alpha_{ij} < 1$. Each coefficient α_{ij} depending primarily on the distance between i's and j's locations, the reciprocity condition, $\alpha_{ij} = \alpha_{ji}$, seems natural. If we assume the insurance-cost-reduction effect of light to be subject to strictly diminishing returns, then (5.2) becomes valid, for $\sigma_i(x_{-i}) = -\sum_{j \neq i} \alpha_{ij} x_j$, regardless of the production costs.

Theorem 5 implies that the game possesses a Nash equilibrium and the behavior of best response improvements is nice enough. In particular, if we assume that only a finite number of x_i's are technologically feasible, then every best response improvement path reaches an equilibrium in a finite number of steps. It is impossible to derive either statement from the previous literature.

Example 5.4. The players are music fans living in the same apartment block. Each player chooses the volume x_i of his own music, the others providing a negative externality, noise, $\sum_{j \neq i} \alpha_{ij} x_j$ $(0 \leq \alpha_{ij} < 1)$. It seems reasonable to assume $\alpha_{ij} = \alpha_{ji}$ and that each player's optimal volume increases in the outside noise. The existence of an equilibrium, certainly, follows from Tarski's fixed point theorem, but the acyclicity of best response improvements can only be derived from our Theorem 5.

Remark. Generally, linear aggregates as in Examples 5.3 and 5.4 are *not* separable projections to X_{-i} of the same ordering on X_N.

I am as yet unprepared to produce specific models with more general aggregates allowed by Theorem 5, but such aggregates do not seem redundant. For instance, $\alpha_{ij}^{(1)}$ of different signs could appear in a monopolistic competition model if x_i describes the level of advertising by firm i. It seems natural to expect strategic complementarity, $\alpha_{ij}^{(1)} > 0$, when the products of the two firms are substitutes, strategic substitutability, $\alpha_{ij}^{(1)} < 0$, when the products are complements, and "strategic indifference," $\alpha_{ij}^{(1)} = 0$, when they are independent.

The possibility to include nonlinear terms, in principle, widens the scope of potential applications. It must be admitted, though, that the interpretation of nonlinear aggregation in the style of Examples 5.3 and 5.4 meets with difficulties. For instance, $\alpha_{ijk}^{(2)} \neq 0$ means that the externalities produced at locations j and k interact nonlinearly between themselves when they affect location i; however, similar nonlinear interaction with x_i would be incompatible with (5.1).

5.2 Proof

For every $i \in N$, we denote $W_i = \nu_i(X_i) \subseteq \mathbb{R}$ and define $\nu_N \colon W_N = \prod_{i \in N} W_i \to X_N$ by $\nu_N(x_N) = \langle \nu_i(x_i) \rangle_{i \in N}$. For every $t \in \mathbb{R}$, we denote $\Xi_i(t) = \{s_i \in S_i \mid s_i > t\}$ and define $r_i^+(t) = \inf \bigcup_{s_i \in \Xi_i(t)} \nu_i\big(R_i(s_i)\big)$ if $\Xi_i(t) \neq \emptyset$ and $r_i^+(t) = \sup \bigcup_{s_i \in S_i} \nu_i\big(R_i(s_i)\big)$ otherwise; $r_i^-(t) = \sup_{t' < t} r_i^+(t')$. Condition (5.2) implies that $r_i^-(t) \leq r_i^+(t)$; by definition, $r_i^-(t') \geq r_i^+(t)$ whenever $t' > t$. Therefore, $r_i^+(t) = r_i^-(t)$ for all $t \in \mathbb{R}$ except for a countable subset. We also define $\bar{R}_i(t) = [r_i^-(t), r_i^+(t)]$; clearly, $\nu_i(x_i) \in \bar{R}_i(s_i)$ whenever $x_i \in R_i(s_i)$.

We denote $s_i^{+\infty} = \inf\{t \in \mathbb{R} \mid r_i^+(t) = +\infty\}$ $[= \inf\{t \in \mathbb{R} \mid r_i^-(t) = +\infty\}]$, and $s_i^{-\infty} = \sup\{t \in \mathbb{R} \mid r_i^-(t) = -\infty\}$ $[= \sup\{t \in \mathbb{R} \mid r_i^+(t) = -\infty\}]$. Clearly, $s_i^{-\infty} \leq s_i^{+\infty}$; an equality implies $\#S_i = 1$, i.e., player i does not react to anything and may be deleted. In the following, we assume $s_i^{-\infty} < s_i^{+\infty}$ for all $i \in N$. For every $i \in N$ and $m \in \mathbb{N}$, we define functions $s_i^+(m)$ and $s_i^-(m)$ in a rather complicated way. If $s_i^{+\infty} = +\infty$, we set $s_i^+(m) = \max\{s_i^{-\infty}, m\}$; if $s_i^{+\infty} \in S_i$ (in which case, $s_i^{+\infty} = \max S_i$ and $r_i^+(s_i^{+\infty}) = +\infty$), we set $s_i^+(m) = s_i^{+\infty}$; if $+\infty > s_i^{+\infty} = \sup S_i \notin S_i$, we set $s_i^+(m) = \max\{s_i^{-\infty}, s_i^{+\infty} - (1/m)\}$. Similarly, $s_i^-(m) = \min\{s_i^{+\infty}, -m\}$ whenever $s_i^{-\infty} = -\infty$, $s_i^-(m) = s_i^{-\infty}$ whenever $s_i^{-\infty} \in S_i$, and $s_i^-(m) = \min\{s_i^+(m), s_i^{-\infty} + (1/m)\}$ whenever $-\infty < s_i^{-\infty} = \inf S_i \notin S_i$. Those definitions ensure that $s_i^-(m) \le s_i^+(m)$ and that $-\infty < r_i^+(t) < +\infty$ whenever $s_i^-(m) < t < s_i^+(m)$.

If each X_i is bounded (hence so is each S_i), we have $s_i^{+\infty} = +\infty$ and $s_i^{-\infty} = -\infty$, hence $s_i^+(m) = m$ and $s_i^-(m) = -m$ for all $m \in \mathbb{N}$. In this case, much of the following becomes superfluous: There is $m \in \mathbb{N}$ such that $S_i \subset [-m, m]$ for each $i \in N$; Lemma 5.1 holds for this m and every $i \in N$ and $s_i \in S_i$; instead of a sequence $\langle P^{(m)}(\nu_N(x_N))\rangle_{m\in\mathbb{N}}$, we may consider just one function $P^{(m)}$, which is obviously continuous, hence Lemma 5.2 is redundant. From a "pragmatic" viewpoint, a compactness assumption in Theorem 5 looks natural because otherwise Ω-acyclicity does not imply even the existence of a fixed point. Nonetheless, since the theorem is valid as stated, it seemed worthwhile to develop an appropriate proof. (Generally, a binary relation may be Ω-acyclic on every compact subset of its domain without being Ω-acyclic on the domain itself.)

For every $i \in N$, $w_i \in W_i$, $w_N \in W_N$, and $m \in \mathbb{N}$, we define these functions:

$$F_i^{(m)}(w_i) = \int_{s_i^-(m)}^{s_i^+(m)} \min\{w_i, r_i^+(t)\}\, dt \ - \ s_i^+(m) \cdot w_i; \tag{5.4}$$

$$Q(w_N) = \sum_{h=1}^{n-1} \sum_{\substack{i_0, i_1, \dots, i_h \in N \\ i_{h'} \neq i_{h''} (h' \neq h'')}} \frac{1}{h+1}\, \alpha^{(h)}_{i_0 i_1 \dots i_h} \times w_{i_0} \times w_{i_1} \times \cdots \times w_{i_h}; \tag{5.5}$$

$$P^{(m)}(w_N) = Q(w_N) + \sum_{i\in N} F_i^{(m)}(w_i). \tag{5.6}$$

We denote $P(w_N) = \langle P^{(m)}(w_N)\rangle_{m\in\mathbb{N}}$ and define two binary relations:

$$P(w'_N) \succ^\infty P(w_N) \leftrightharpoons \exists \bar m \in \mathbb{N}\, \forall m > \bar m\, [P^{(m)}(w'_N) > P^{(m)}(w_N)];$$
$$P(w'_N) \sim^\infty P(w_N) \leftrightharpoons \exists \bar m \in \mathbb{N}\, \forall m > \bar m\, [P^{(m)}(w'_N) = P^{(m)}(w_N)].$$

Let $s_i \in S_i$ and $m \in \mathbb{N}$. We say that m *is proper for* s_i if these two conditions hold: either $s_i > s_i^-(m)$ or $s_i = s_i^{-\infty} = s_i^-(m)$; either $s_i < s_i^+(m)$ or $s_i = s_i^{+\infty} = s_i^+(m)$.

Lemma 5.1. *Let $y_N \rhd_i^{\mathcal{S}} x_N$, $\bar s_i = \sigma_i(x_{-i}) = \sigma_i(y_{-i})$, and m be proper for $\bar s_i$. Then $P^{(m)}(\nu_N(y_N)) \ge P^{(m)}(\nu_N(x_N))$; besides, $P^{(m)}(\nu_N(y_N)) = P^{(m)}(\nu_N(x_N))$ if and only if $\nu_i(x_i) \in \bar R_i(\bar s_i)$.*

Proof. For every $w_i \in W_i$, we have

$$F_i^{(m)}(w_i) = \int_{s_i^-(m)}^{\bar s_i} \min\{w_i, r_i^+(t)\}\, dt + \int_{\bar s_i}^{s_i^+(m)} \min\{w_i, r_i^+(t)\}\, dt \ - \ s_i^+(m)\cdot w_i =$$
$$\int_{s_i^-(m)}^{\bar s_i} \min\{w_i, r_i^+(t)\}\, dt + \int_{\bar s_i}^{s_i^+(m)} \min\{r_i^+(t) - w_i, 0\}\, dt \ - \ \bar s_i \cdot w_i; \tag{5.7}$$

therefore,

$$F_i^{(m)}(w_i) = \int_{s_i^-(m)}^{\bar{s}_i} r_i^+(t)\,dt \;-\; \bar{s}_i \cdot w_i \tag{5.8}$$

whenever $w_i \in \bar{R}_i(\bar{s}_i)$. Since $y_i \in R_i(\bar{s}_i)$, (5.8) holds for $w_i = \nu_i(y_i)$. If $w_i < r_i^-(\bar{s}_i)$, which is only possible if $\bar{s}_i > s_i^-(m)$, then

$$F_i^{(m)}(w_i) < \int_{s_i^-(m)}^{\bar{s}_i} r_i^+(t)\,dt \;-\; \bar{s}_i \cdot w_i \tag{5.9}$$

because the first additive term in (5.7) is strictly less than in (5.8), while the second is still zero. If $w_i > r_i^+(\bar{s}_i)$, which is only possible if $\bar{s}_i < s_i^+(m)$, then again (5.9) holds because now the first additive term in (5.7) is the same as in (5.8), but the second is strictly negative.

Combining the terms containing $\nu_i(x_i)$ (respectively, $\nu_i(y_i)$) and taking into account that $x_{-i} = y_{-i}$, we obtain $Q(\nu_N(x_N)) = \bar{s}_i \cdot \nu_i(x_i) + q(x_{-i})$ and $Q(\nu_N(y_N)) = \bar{s}_i \cdot \nu_i(y_i) + q(x_{-i})$. Therefore,

$$P^{(m)}(\nu_N(y_N)) = \int_{s_i^-(m)}^{\bar{s}_i} r_i^+(t)\,dt \,+ q(x_{-i}) + \sum_{j \neq i} F_j^{(m)}(\nu_j(x_j)) \tag{5.10}$$

by (5.8). Simultaneously, (5.8) and (5.9) imply that

$$P^{(m)}(\nu_N(x_N)) \leq \int_{s_i^-(m)}^{\bar{s}_i} r_i^+(t)\,dt \,+ q(x_{-i}) + \sum_{j \neq i} F_j^{(m)}(\nu_j(x_j)) \tag{5.11}$$

with an equality if and only if $\nu_i(x_i) \in \bar{R}_i(\bar{s}_i)$. Now both statements of the lemma immediately follow from (5.10) and (5.11). □

Lemma 5.2. *Let $x_N^k \to x_N^\omega$ and, for each $k \in \mathbb{N}$, there hold $P^{(m)}(\nu_N(x_N^{k+1})) \geq P^{(m)}(\nu_N(x_N^k))$ for all $m \in \mathbb{N}$ except for a finite number of them. Then either $P(\nu_N(x_N^\omega)) \succ^\infty P(\nu_N(x_N^0))$ or $P(\nu_N(x_N^\omega)) \sim^\infty P(\nu_N(x_N^0))$.*

Proof. For each $i \in N$, we denote

$$\bar{W}_i = \operatorname{conv} \bigcup_{s_i \in S_i} \nu_i\big(R_i(s_i)\big);$$

for each $k \in \mathbb{N}$,

$$N^0(k) = \{i \in N \mid \nu_i(x_i^k) \in \bar{W}_i\};$$
$$N^{++}(k) = \{i \in N \mid \forall w_i \in \bar{W}_i\, [\nu_i(x_i^k) > w_i]\};$$
$$N^{--}(k) = \{i \in N \mid \forall w_i \in \bar{W}_i\, [\nu_i(x_i^k) < w_i]\}.$$

Without restricting generality, the argument k in each of the three sets can be dropped. We partition N^0 into three subsets:

$$N^{00} = \{i \in N^0 \mid \nu_i(x_i^\omega) \in \bar{W}_i\};$$

$$N^+ = \{i \in N^0 \mid \forall w_i \in \bar{W}_i\, [\nu_i(x_i^\omega) > w_i]\};$$
$$N^- = \{i \in N^0 \mid \forall w_i \in \bar{W}_i\, [\nu_i(x_i^\omega) < w_i]\}.$$

Note that $s_i^{+\infty} = +\infty$ for each $i \in N^+ \cup N^{++}$ while $s_i^{-\infty} = -\infty$ for each $i \in N^- \cup N^{--}$.

We pick $m^* \in \mathbb{N}$ such that: (1) $s_i^{-\infty} < m^* = s_i^+(m^*)$ for all $i \in N^+ \cup N^{++}$; (2) $s_i^-(m^*) = m^* < s_i^{+\infty}$ for all $i \in N^- \cup N^{--}$; (3) $r_i^-(s_i^-(m^*)) \le \nu_i(x_i^k), \nu_i(x_i^\omega) \le r_i^+(s_i^+(m^*))$ for all $i \in N^{00}$ and all $k \in \mathbb{N}$; (4) $r_i^-(s_i^-(m^*)) \le \nu_i(x_i^k) < \nu_i(x_i^\omega)$ for all $i \in N^+$ and all $k \in \mathbb{N}$; (5) $\nu_i(x_i^\omega) < \nu_i(x_i^k) \le r_i^+(s_i^+(m^*))$ for all $i \in N^-$ and all $k \in \mathbb{N}$. Clearly, the same inequalities hold for all $m > m^*$.

Let $m' > m \ge m^*$ and $k' > k$. If $i \in N^{00}$, we have

$$F_i^{(m')}(\nu_i(x_i^k)) - F_i^{(m)}(\nu_i(x_i^k)) = \int_{s_i^-(m')}^{s_i^-(m)} r_i^+(t)\, dt, \tag{5.12}$$

hence

$$F_i^{(m')}(\nu_i(x_i^{k'})) - F_i^{(m')}(\nu_i(x_i^k)) = F_i^{(m)}(\nu_i(x_i^{k'})) - F_i^{(m)}(\nu_i(x_i^k)). \tag{5.13}$$

Let $i \in N^+ \cup N^{++}$; by the choice of m^*, $s_i^+(m) = m$ and $s_i^+(m') = m'$. Similarly to (5.12), we obtain

$$F_i^{(m')}(\nu_i(x_i^k)) - F_i^{(m)}(\nu_i(x_i^k)) = \int_{s_i^-(m')}^{s_i^-(m)} r_i^+(t)\, dt \;+\; \int_m^{m'} \min\{r_i^+(t) - \nu_i(x_i^k), 0\}\, dt; \tag{5.14}$$

therefore,

$$\begin{aligned}[F_i^{(m')}(\nu_i(x_i^{k'})) - F_i^{(m')}(\nu_i(x_i^k))] - [F_i^{(m)}(\nu_i(x_i^{k'})) - F_i^{(m)}(\nu_i(x_i^k))] = \\ [F_i^{(m')}(\nu_i(x_i^{k'})) - F_i^{(m)}(\nu_i(x_i^{k'}))] - [F_i^{(m')}(\nu_i(x_i^k)) - F_i^{(m)}(\nu_i(x_i^k))] = \\ \int_m^{m'} \left[\min\{r_i^+(t) - \nu_i(x_i^{k'}), 0\} - \min\{r_i^+(t) - \nu_i(x_i^k), 0\}\right] dt.\end{aligned} \tag{5.15}$$

In particular,

$$\begin{aligned}[F_i^{(m')}(\nu_i(x_i^{k+1})) - F_i^{(m')}(\nu_i(x_i^k))] - [F_i^{(m)}(\nu_i(x_i^{k+1})) - F_i^{(m)}(\nu_i(x_i^k))] = \\ (m' - m) \cdot [\nu_i(x_i^k) - \nu_i(x_i^{k+1})]\end{aligned} \tag{5.16}$$

for each $i \in N^{++}$. For each $i \in N^+$, we may, without restricting generality, assume $\nu_i(x_i^k) < \nu_i(x_i^{k+1})$ for all $k \in \mathbb{N}$, hence

$$\begin{aligned}[F_i^{(m')}(\nu_i(x_i^{k'})) - F_i^{(m')}(\nu_i(x_i^k))] - [F_i^{(m)}(\nu_i(x_i^{k'})) - F_i^{(m)}(\nu_i(x_i^k))] = \\ \int_m^{m'} \min\{\max\{r_i^+(t), \nu_i(x_i^k)\} - \nu_i(x_i^{k'}), 0\}\, dt \le 0.\end{aligned} \tag{5.17}$$

Let $i \in N^- \cup N^{--}$; by the choice of m^*, $s_i^-(m) = -m$ and $s_i^-(m') = -m'$. This time, we obtain

$$F_i^{(m')}(\nu_i(x_i^k)) - F_i^{(m)}(\nu_i(x_i^k)) = \int_{-m'}^{-m} \min\{r_i^+(t), \nu_i(x_i^k)\}\, dt; \tag{5.18}$$

therefore,

$$[F_i^{(m')}(\nu_i(x_i^{k'})) - F_i^{(m')}(\nu_i(x_i^k))] - [F_i^{(m)}(\nu_i(x_i^{k'})) - F_i^{(m)}(\nu_i(x_i^k))] = \int_{-m'}^{-m} \left[\min\{r_i^+(t), \nu_i(x_i^{k'})\} - \min\{r_i^+(t), \nu_i(x_i^k)\}\right] dt. \tag{5.19}$$

In particular,

$$[F_i^{(m')}(\nu_i(x_i^{k+1})) - F_i^{(m')}(\nu_i(x_i^k))] - [F_i^{(m)}(\nu_i(x_i^{k+1})) - F_i^{(m)}(\nu_i(x_i^k))] = (m' - m) \cdot [\nu_i(x_i^{k+1}) - \nu_i(x_i^k)] \tag{5.20}$$

for each $i \in N^{--}$. For each $i \in N^-$, we may, without restricting generality, assume $\nu_i(x_i^{k+1}) < \nu_i(x_i^k)$ for all $k \in \mathbb{N}$, hence

$$[F_i^{(m')}(\nu_i(x_i^{k'})) - F_i^{(m')}(\nu_i(x_i^k))] - [F_i^{(m)}(\nu_i(x_i^{k'})) - F_i^{(m)}(\nu_i(x_i^k))] = \int_{-m'}^{-m} \min\{\nu_i(x_i^{k'}) - \min\{r_i^+(t), \nu_i(x_i^k)\}, 0\}\, dt \le 0. \tag{5.21}$$

For each $k \in \mathbb{N}$, we denote

$$\Delta^k = \sum_{i \in N^{++}} [\nu_i(x_i^k) - \nu_i(x_i^{k+1})] + \sum_{i \in N^{--}} [\nu_i(x_i^{k+1}) - \nu_i(x_i^k)];$$

$$\Delta^\omega = \sum_{i \in N^{++}} [\nu_i(x_i^0) - \nu_i(x_i^\omega)] + \sum_{i \in N^{--}} [\nu_i(x_i^\omega) - \nu_i(x_i^0)] = \sum_{k \in \mathbb{N}} \Delta^k.$$

Step 5.2.1. $\Delta^k \ge 0$ *for each* $k \in \mathbb{N}$.

Proof. Fixing $k \in \mathbb{N}$, we pick $\bar{m} \ge m^*$ such that $r_i^+(s_i^+(\bar{m})) \ge \nu_i(x_i^{k+1}) > \nu_i(x_i^k)$ for each $i \in N^+$ and $r_i^+(s_i^-(\bar{m})) \le \nu_i(x_i^{k+1}) < \nu_i(x_i^k)$ for each $i \in N^-$. It is immediately seen from (5.13), (5.17), and (5.21) that

$$F_i^{(m)}(\nu_i(x_i^{k+1})) - F_i^{(m)}(\nu_i(x_i^k)) = F_i^{(\bar{m})}(\nu_i(x_i^{k+1})) - F_i^{(\bar{m})}(\nu_i(x_i^k))$$

for all $m > \bar{m}$ and $i \in N^0$. Taking into account (5.16) and (5.20), we obtain

$$P^{(m)}(\nu_N(x_N^{k+1})) - P^{(m)}(\nu_N(x_N^k)) = P^{(\bar{m})}(\nu_N(x_N^{k+1})) - P^{(\bar{m})}(\nu_N(x_N^k)) + (m - \bar{m}) \cdot \Delta^k.$$

If $\Delta^k < 0$, then $P^{(m)}(\nu_N(x_N^{k+1})) < P^{(m)}(\nu_N(x_N^k))$ for all $m \in \mathbb{N}$ large enough, contradicting the assumptions of the lemma. $\square$

It follows immediately that $\Delta^\omega \geq 0$ as well.

Step 5.2.2. *If $\Delta^\omega > 0$, then $P(\nu_N(x_N^\omega)) \succ^\infty P(\nu_N(x_N^0))$.*

Proof. For each $i \in N^+ \cup N^-$, we pick $\bar{w}_i \in \bar{W}_i$ such that $\nu_i(x_i^0) \leq \bar{w}_i$ for each $i \in N^+$, $\nu_i(x_i^0) \geq \bar{w}_i$ for each $i \in N^-$, and $\sum_{i\in N^+}[\nu_i(x_i^\omega) - \bar{w}_i] + \sum_{i\in N^-}[\bar{w}_i - \nu_i(x_i^\omega)] \leq \Delta^\omega/2$. Then we pick $m^{**} > m^*$ such that $r_i^+(m^{**}) \geq \bar{w}_i$ for each $i \in N^+$, and $r_i^+(-m^{**}) \leq \bar{w}_i$ for each $i \in N^-$. Finally, we pick $\bar{m} \in \mathbb{N}$ such that

$$\bar{m} \geq \max\{m^{**}, m^{**} - 2[P^{(m^{**})}(\nu_N(x_N^\omega)) - P^{(m^{**})}(\nu_N(x_N^0))]/\Delta^\omega\}. \tag{5.22}$$

For each $m > m^{**}$ and $i \in N$, we denote

$$\Delta_i^{(m)} = F_i^{(m)}(\nu_i(x_i^\omega)) - F_i^{(m^{**})}(\nu_i(x_i^\omega)) - F_i^{(m)}(\nu_i(x_i^0)) + F_i^{(m^{**})}(\nu_i(x_i^0)).$$

Clearly,

$$P^{(m)}(\nu_N(x_N^\omega)) - P^{(m)}(\nu_N(x_N^0)) = P^{(m^{**})}(\nu_N(x_N^\omega)) - P^{(m^{**})}(\nu_N(x_N^0)) + \sum_{i\in N}\Delta_i^{(m)}. \tag{5.23}$$

If $i \in N^{00}$, then $\Delta_i^{(m)} = 0$ by (5.13). If $i \in N^+$, then $r_i^+(m^{**}) \geq \bar{w}_i$, hence (5.17) implies $\Delta_i^{(m)} \geq (m^{**} - m)[\nu_i(x_i^\omega) - \bar{w}_i]$; if $i \in N^-$, then $r_i^+(-m^{**}) \leq \bar{w}_i$, hence (5.21) implies $\Delta_i^{(m)} \geq (m^{**} - m)[\bar{w}_i - \nu_i(x_i^\omega)]$. Therefore, $\sum_{i\in N^+\cup N^-}\Delta_i^{(m)} \geq (m^{**} - m)\Delta^\omega/2$. By (5.16) and (5.20), $\sum_{i\in N^{++}\cup N^{--}}\Delta_i^{(m)} = (m - m^{**})\Delta^\omega$. Now $P^{(m)}(\nu_N(x_N^\omega)) - P^{(m)}(\nu_N(x_N^0)) > 0$ immediately follows from (5.23), (5.22), and $m > \bar{m}$. □

Now let us suppose that $\Delta^\omega = 0$, hence $\Delta^k = 0$ for each k, hence players from $N^{++} \cup N^{--}$ can be forgotten about.

Step 5.2.3. *For each $m > \bar{m}$, $P^{(m)}(\nu_N(x_N^\omega)) \geq P^{(m)}(\nu_N(x_N^0))$.*

Proof. Suppose the contrary; since $P^{(m)}(\nu_N(\cdot))$ is continuous, we have $P^{(m)}(\nu_N(x_N^k)) < P^{(m)}(\nu_N(x_N^0))$ for all k large enough. Let $m' > m$; if $i \in N^{00}$, we have $F_i^{(m')}(\nu_i(x_i^k)) - F_i^{(m')}(\nu_i(x_i^0)) = F_i^{(m)}(\nu_i(x_i^k)) - F_i^{(m)}(\nu_i(x_i^0))$ by (5.13); if $i \in N^+ \cup N^-$, then $F_i^{(m')}(\nu_i(x_i^k)) - F_i^{(m')}(\nu_i(x_i^0)) \leq F_i^{(m)}(\nu_i(x_i^k)) - F_i^{(m)}(\nu_i(x_i^0))$ by (5.17) or (5.21). Therefore, $P^{(m')}(\nu_N(x_N^k)) < P^{(m')}(\nu_N(x_N^0))$ for all $m' > m$, which contradicts the assumptions of the lemma. □

Finally, if $N^+ \cup N^- = \emptyset$, then $P^{(m)}(\nu_N(x_N^\omega)) - P^{(m)}(\nu_N(x_N^0))$ is the same for all $m > m^*$ by (5.13); if it is zero, we have $P(\nu_N(x_N^\omega)) \sim^\infty P(\nu_N(x_N^0))$; if it is strictly positive, $P(\nu_N(x_N^\omega)) \succ^\infty P(\nu_N(x_N^0))$. Whenever $i \in N^+ \cup N^-$, $F_i^{(m)}(\nu_i(x_i^\omega)) - F_i^{(m)}(\nu_i(x_i^0))$ strictly decreases in $m > m^*$ by (5.17) or (5.21), hence it must be strictly positive, hence $P(\nu_N(x_N^\omega)) \succ^\infty P(\nu_N(x_N^0))$ again. □

To construct an ω-potential, we lexicographically complement the sequence of functions $P^{(m)}$ with binary relations on each X_i. For each $i \in N$, $y_i, x_i \in X_i$, $x_N \in X_N$, we define:

$$y_i \rhd\!\!\rhd_i x_i \leftrightharpoons \exists \bar{s}_i \in S_i \left[x_i \notin R_i(\bar{s}_i) \ni y_i \;\&\; \nu_i(x_i) \in \bar{R}_i(\bar{s}_i)\right]$$

(in the following, we say "$y_i \rhd\!\!\rhd_i x_i$ holds with $s_i = \bar{s}_i$");

$$y_i \rhd\!\!\rhd_i^{\sim} x_i \leftrightharpoons [y_i \rhd\!\!\rhd_i x_i \;\&\; \nu_i(y_i) = \nu_i(x_i)];$$

$$y_i \rhd\!\!\rhd_i^{+} x_i \leftrightharpoons [y_i \rhd\!\!\rhd_i x_i \;\&\; \nu_i(y_i) \geq \nu_i(x_i)];$$
$$y_i \rhd\!\!\rhd_i^{++} x_i \leftrightharpoons [y_i \rhd\!\!\rhd_i x_i \;\&\; \nu_i(y_i) > \nu_i(x_i)];$$
$$y_i \rhd\!\!\rhd_i^{-} x_i \leftrightharpoons [y_i \rhd\!\!\rhd_i x_i \;\&\; \nu_i(y_i) \leq \nu_i(x_i)];$$
$$y_i \rhd\!\!\rhd_i^{--} x_i \leftrightharpoons [y_i \rhd\!\!\rhd_i x_i \;\&\; \nu_i(y_i) < \nu_i(x_i)];$$

$\mathrel{>\!\!\!\succ}_i^{+}$ and $\mathrel{>\!\!\!\succ}_i^{-}$ are ω-transitive closures of $\rhd\!\!\rhd_i^{+}$ and $\rhd\!\!\rhd_i^{-}$, respectively;

$$y_i \mathrel{>\!\!\!\succ}_i^{++} x_i \leftrightharpoons [y_i \mathrel{>\!\!\!\succ}_i^{+} x_i \;\&\; \nu_i(y_i) > \nu_i(x_i)];$$

$$y_i \mathrel{>\!\!\!\succ}_i^{--} x_i \leftrightharpoons [y_i \mathrel{>\!\!\!\succ}_i^{-} x_i \;\&\; \nu_i(y_i) < \nu_i(x_i)];$$
$$y_i \mathrel{>\!\!\!\succ}_i x_i \leftrightharpoons [y_i \mathrel{>\!\!\!\succ}_i^{++} x_i \text{ or } y_i \mathrel{>\!\!\!\succ}_i^{--} x_i \text{ or } y_i \rhd\!\!\rhd_i^{\sim} x_i];$$
$$y_N \mathrel{>\!\!\!\succ} x_N \leftrightharpoons \forall i \in N\, [y_i = x_i \text{ or } y_i \mathrel{>\!\!\!\succ}_i x_i] \;\&\; \exists i \in N\, [y_i \mathrel{>\!\!\!\succ}_i x_i];$$
$$y_N \mathrel{>\!\!\!>\!\!\!\succ} x_N \leftrightharpoons \left[P(\nu_N(y_N)) \succ^{\infty} P(\nu_N(x_N)) \text{ or } [P(\nu_N(y_N)) \sim^{\infty} P(\nu_N(x_N)) \;\&\; y_N \mathrel{>\!\!\!\succ} x_N]\right].$$

Lemma 5.3. *If $y_N \rhd^{\mathcal{S}} x_N$, then $y_N \mathrel{>\!\!\!>\!\!\!\succ} x_N$.*

Proof. Let $y_N \rhd_i^{\mathcal{S}} x_N$ and $\bar{s}_i = \sigma_i(x_{-i})$. Clearly, all $m \in \mathbb{N}$ are proper for $\bar{s}_i$ except for a finite number of them. If $\nu_i(x_i) \notin \bar{R}_i(\bar{s}_i)$, then $P(\nu_N(y_N)) \succ^{\infty} P(\nu_N(x_N))$ by Lemma 5.1, hence $y_N \mathrel{>\!\!\!>\!\!\!\succ} x_N$. If $\nu_i(x_i) \in \bar{R}_i(\bar{s}_i)$, then $P(\nu_N(y_N)) \sim^{\infty} P(\nu_N(x_N))$, but $y_i \rhd\!\!\rhd_i x_i$, hence $y_i \mathrel{>\!\!\!\succ}_i x_i$; since $y_j = x_j$ for $j \neq i$, we have $y_N \mathrel{>\!\!\!\succ} x_N$, hence $y_N \mathrel{>\!\!\!>\!\!\!\succ} x_N$. □

Lemma 5.4. *Each relation $\mathrel{>\!\!\!\succ}_i$ is ω-transitive.*

Proof.

Step 5.4.1. *If $z_i \rhd\!\!\rhd_i^{\sim} y_i$, then $y_i \rhd\!\!\rhd_i x_i$ is impossible for any $x_i \in X_i$.*

Proof. Let $z_i \rhd\!\!\rhd_i^{\sim} y_i$ hold with $s_i = \bar{s}_i$; then $y_i \notin R_i(\bar{s}_i) \ni z_i$. For $s_i < \bar{s}_i$, $y_i \in R_i(s_i)$ would imply $z_i \succ y_i$ by (5.2), hence $\nu_i(z_i) > \nu_i(y_i)$ since ν_i is strictly increasing. For $s_i > \bar{s}_i$, $y_i \in R_i(s_i)$ would imply $y_i \succ z_i$ by (5.2), hence $\nu_i(y_i) > \nu_i(z_i)$. Therefore, $y_i \notin R_i(s_i)$ for any $s_i \in S_i$. □

Step 5.4.2. *Let $z_i \rhd\!\!\rhd_i^{++} y_i$ hold with $s_i = \bar{s}_i$, and $\nu_i(y_i') \in [\nu_i(y_i), \nu_i(z_i)[$; then $y_i' \rhd\!\!\rhd_i^{--} x_i$ is only possible, for any $x_i \in X_i$, with $s_i = \bar{s}_i$.*

Proof. By definition, $y_i' \rhd\!\rhd_i^{--} x_i$ implies $\nu_i(y_i') < \nu_i(x_i)$. For $s_i > \bar{s}_i$, we have $r_i^-(s_i) \geq r_i^+(\bar{s}_i) \geq \nu_i(z_i) > \nu_i(y_i')$, hence $y_i' \notin R_i(s_i)$; for $s_i < \bar{s}_i$, $r_i^+(s_i) \leq r_i^-(\bar{s}_i) \leq \nu_i(y_i) \leq \nu_i(y_i') < \nu_i(x_i)$, hence $\nu_i(x_i) \notin \bar{R}_i(s_i)$. Therefore, both conditions in the definition of $y_i' \rhd\!\rhd_i x_i$ could only be satisfied when $s_i = \bar{s}_i$. $\square$

Step 5.4.3. *If $z_i \rhd\!\rhd_i^{++} y_i$, then $y_i \rightarrowtail_i^{--} x_i$ is impossible for any $x_i \in X_i$.*

Proof. Supposing the contrary, let $z_i \rhd\!\rhd_i^{++} y_i$ hold with $s_i = \bar{s}_i$. By the previous step, $y_i \rhd\!\rhd_i^{--} x_i$ could only hold with $s_i = \bar{s}_i$ as well; but this is impossible because $y_i \notin R_i(\bar{s}_i)$. Therefore, y_i must be a limit point of an improvement path for $\rhd\!\rhd_i^{-}$, hence there are $y_i^0, y_i^1, y_i^2 \in X_i$ such that $\nu_i(z_i) > \nu_i(y_i^0) > \nu_i(y_i^1) > \nu_i(y_i^2) > \nu_i(y_i)$ and $y_i^2 \rhd\!\rhd_i^{--} y_i^1 \rhd\!\rhd_i^{--} y_i^0$. By Step 5.4.2, both relations must hold with $s_i = \bar{s}_i$, hence $y_i^1 \notin R_i(\bar{s}_i) \ni y_i^1$: a contradiction. $\square$

Step 5.4.4. *If $z_i \rightarrowtail_i^{++} y_i$, then $y_i \rightarrowtail_i^{--} x_i$ is impossible for any $x_i \in X_i$.*

Proof. By definition, $z_i' \rhd\!\rhd_i^{+} y_i$ must hold for some $z_i' \in X_i$ such that $z_i \rightarrowtail_i^{++} z_i'$. If $z_i' \rhd\!\rhd_i^{++} y_i$, the previous step applies; let $z_i' \rhd\!\rhd_i^{\sim} y_i$. There must be $z_i'' \in X_i$ such that $z_i'' \rhd\!\rhd_i^{+} z_i'$; this time, $z_i'' \rhd\!\rhd_i^{\sim} z_i'$ is impossible by Step 5.4.1, hence $z_i'' \rhd\!\rhd_i^{++} z_i'$. Since $\nu_i(z_i') = \nu_i(y_i)$, there holds $z_i'' \rhd\!\rhd_i^{++} y_i$ (with the same s_i as the previous relation), and we are home again. $\square$

Step 5.4.5. *If $z_i \rightarrowtail_i^{--} y_i$, then $y_i \rightarrowtail_i^{++} x_i$ is impossible for any $x_i \in X_i$.*

Proof. The proof is dual to the proof of Steps 5.4.2, 5.4.3, and 5.4.4. $\square$

To finish with the lemma, it is enough, in the light of Steps 5.4.1, 5.4.4, and 5.4.5, to notice that $z_i \rightarrowtail_i^{++} x_i$ ($z_i \rightarrowtail_i^{--} x_i$) whenever $z_i \rightarrowtail_i^{++} y_i \rhd\!\rhd_i^{\sim} x_i$ ($z_i \rightarrowtail_i^{--} y_i \rhd\!\rhd_i^{\sim} x_i$). $\square$

Lemma 5.5. *The relation $\ggg$ is irreflexive and ω-transitive.*

Proof. The irreflexivity is obvious; to check transitivity, it is sufficient to notice that $\succ^{\infty}$, $\sim^{\infty}$, and $\rightarrowtail$ are transitive and that $P(w_N'') \succ^{\infty} P(w_N)$ whenever $P(w_N'') \succ^{\infty} P(w_N') \sim^{\infty} P(w_N)$ or $P(w_N'') \sim^{\infty} P(w_N') \succ^{\infty} P(w_N)$. Let $x_N^k \to x_N^{\omega}$ and $x_N^{k+1} \ggg x_N^k$ for all $k \in \mathbb{N}$. Lemma 5.2 is applicable, producing $P(\nu_N(x_N^{\omega})) \succ^{\infty} P(\nu_N(x_N^0))$ (hence $x_N^{\omega} \ggg x_N^0$) or $P(\nu_N(x_N^{\omega})) \sim^{\infty} P(\nu_N(x_N^0))$. If $P(\nu_N(x_N^{k+1})) \succ^{\infty} P(\nu_N(x_N^k))$ for some k, we may apply Lemma 5.2 to the sequence $x_N^{k+1}, x_N^{k+2}, \ldots$ (which converges to the same x_N^{ω}), and obtain $P(\nu_N(x_N^{\omega})) \succ^{\infty} P(\nu_N(x_N^{k+1}))$ or $P(\nu_N(x_N^{\omega})) \sim^{\infty} P(\nu_N(x_N^{k+1}))$, hence $P(\nu_N(x_N^{\omega})) \succ^{\infty} P(\nu_N(x_N^0))$. If $P(\nu_N(x_N^{k+1})) \sim^{\infty} P(\nu_N(x_N^k))$ for all k, then $x_N^{k+1} \rightarrowtail x_N^k$ for all k, hence $x_N^{\omega} \rightarrowtail x_N^0$ by Lemma 5.4, hence $x_N^{\omega} \ggg x_N^0$ even if $P(\nu_N(x_N^{\omega})) \sim^{\infty} P(\nu_N(x_N^0))$. $\square$

Lemmas 5.5 and 5.3 mean that $\ggg$ is an ω-potential, so Theorem 5 is proved.

6 Monotone selections

Let X and S be posets and $R\colon S \to \mathfrak{B}_X$. Then a *monotone selection* from R is a mapping $r\colon S \to X$ such that (1) $r(s) \in R(s)$ and (2) $s' \geq s \Rightarrow r(s') \geq r(s)$, for all $s, s' \in S$.

The best-known results on the existence of monotone selections (Topkis, 1998) are applicable to ascending correspondences to lattices. Similar results are easily obtained under weaker monotonicity conditions:

$$\forall s', s \in S \left[[s' > s \;\&\; x \in R(s) \;\&\; x' \in R(s')] \Rightarrow x' \vee x \in R(s') \right]; \tag{6.1a}$$

$$\forall s', s \in S \left[[s' > s \;\&\; x \in R(s) \;\&\; x' \in R(s')] \Rightarrow x' \wedge x \in R(s) \right]. \tag{6.1b}$$

Note that the corresponding modifications of Veinott's order $\succeq^{\mathrm{V}}$ need not even be transitive.

Proposition 6.1. *A correspondence R from a poset S to a lattice X admits a monotone selection if it satisfies* (6.1a) *and every $R(s)$ contains a greatest element, or if it satisfies* (6.1b) *and every $R(s)$ contains a least element.*

Proof. In the first case, we define $r(s) = \max R(s)$; in the second, $r(s) = \min R(s)$. □

At a first glance, the existence of a greatest/least point may seem a far-fetched requirement. However, it is satisfied, e.g., if X is a separable metric space, the order is continuous, each $R(s)$ is compact, and either condition (6.1) holds in a strengthened version, with $s' > s$ in the left hand side replaced with $s' \geq s$. On the other hand, the assumption cannot be just dropped.

Example 6.1. Let $X = [0, 1]$ and $R\colon X \to \mathfrak{B}_X$ be this: $R(0) =]0, 1]$; $R(x) = \{x/2\}$ for $x > 0$. Clearly, R satisfies (6.1b), but there is neither monotone selection, nor fixed point.

There is no ground to believe that Proposition 6.1 is the last word on monotone selections. For instance, Milgrom and Shannon (1994) ascribe the following statement to Veinott (1989).

Theorem A2. *Let $\{S_\tau\}$ be a net of nonempty sets that is* weakly ascending*, that is, such that if $\tau' \geq \tau$, and $x \in S_\tau$, $x' \in S_{\tau'}$, then either $x \vee x' \in S_{\tau'}$ or $x \wedge x' \in S_\tau$. Then there exists a monotone selection $\{x(\tau)\}$ from $\{S_\tau\}$.*

Example 6.1 shows that the statement is just wrong. A valid analog can be obtained under much stronger conditions.

Proposition 6.2. *Let X be a complete lattice, S be a poset, and R be a weakly ascending mapping $S \to \mathfrak{B}_X$ such that every $R(s)$ is a* complete sublattice *of X, i.e., whenever $Y \subseteq R(s)$, there hold* $\sup Y \in R(s)$ *and* $\inf Y \in R(s)$. *Then there exists a monotone selection from R.*

Proof. For every $s \in S$, we denote $r^+(s) = \sup R(s) \in R(s)$, $R^+(s) = \{r^+(s')\}_{s' \geq s}$, and $r^0(s) = \inf R^+(s)$. Whenever, $s' \geq s$, we have $R^+(s') \subseteq R^+(s)$, hence $r^0(s') \geq r^0(s)$. Let us show that $r^0(s) \in R(s)$ for every $s \in S$; then r^0 will be the selection needed.

We denote $R^*(s) = \{x \in R(s) \mid r^0(s) \leq x\} \ni r^+(s)$ and $r^*(s) = \inf R^*(s) \in R(s)$. By definition, $r^*(s) \geq r^0(s)$; in the case of an equality, we are home. Suppose the contrary. Then there must be $s' > s$ such that $r^+(s') \not\geq r^*(s)$, hence $r^+(s') \vee r^*(s) > r^+(s')$ and $r^+(s') \wedge r^*(s) < r^*(s)$. The first inequality implies that $r^+(s') \vee r^*(s) \notin R(s')$; therefore, $r^+(s') \wedge r^*(s) \in R(s)$. Now the inequalities $r^+(s') \geq r^0(s)$ and $r^*(s) \geq r^0(s)$ imply that $r^+(s') \wedge r^*(s) \in R^*(s)$, which plainly contradicts the definition of $r^*(s)$. □

The distance from the conditions of Theorem A2 to those of Proposition 6.2 is great; there must be valid statements in between. I was unable to find out exactly what was meant by Veinott or Milgrom and Shannon; the only result of my enquiries can be found at http://www.stanford.edu/∼milgrom/publishedarticles/Kukushkin%20CounterExample.pdf

Without topological restrictions on values $R(s)$, the existence of a monotone selection can be obtained for an ascending correspondence to a chain.

Theorem 6. *Let X be a chain, S be a poset, and R be a mapping $S \to \mathfrak{B}_X$ satisfying both conditions* (6.1); *then there exists a monotone selection from R.*

Proof. We use the Axiom of Choice to the full extent. The set S can be well ordered; to avoid considering two independent orders on the same set, we assume that there is a bijection $\lambda\colon \mathrm{A} \to S$, where A is a well ordered set of the same cardinality as S. We define $r(\lambda(\alpha))$ by (transfinite) induction in $\alpha \in \mathrm{A}$. First, we pick $r(\lambda(0)) \in R(\lambda(0))$ arbitrarily.

Let $r(\lambda(\beta))$ be defined for all $\beta < \alpha$. We define $\mathrm{B}(\alpha) = \{\beta < \alpha \mid r(\lambda(\beta)) \in R(\lambda(\alpha))\}$. If $\mathrm{B}(\alpha) = \emptyset$, we pick $r(\lambda(\alpha)) \in R(\lambda(\alpha))$ arbitrarily. Otherwise, we define $r(\lambda(\alpha)) = r(\lambda(\min \mathrm{B}(\alpha)))$, the minimum existing because A is well ordered. Since there is no possibility that $r(\lambda(\alpha))$ could be left undefined, we obtain $r(\lambda(\alpha))$ for all $\alpha \in \mathrm{A}$ eventually. Clearly, $r(\lambda(\alpha)) \in R(\lambda(\alpha))$ for all $\alpha \in \mathrm{A}$, so we only have to check monotonicity.

Suppose to the contrary that $\lambda(\alpha') < \lambda(\alpha)$ whereas $r(\lambda(\alpha')) > r(\lambda(\alpha))$; the assumption that X is a chain is essential here. By (6.1a), $r(\lambda(\alpha')) \in R(\lambda(\alpha))$; by (6.1b), $r(\lambda(\alpha)) \in R(\lambda(\alpha'))$. Without restricting generality, $\alpha' < \alpha$, hence $\alpha' \in \mathrm{B}(\alpha) \neq \emptyset$. The assumption that $r(\lambda(\alpha')) \neq r(\lambda(\alpha))$ implies that $\min \mathrm{B}(\alpha) = \beta < \alpha'$ and $r(\lambda(\alpha)) = r(\lambda(\beta))$. Now $\beta \in \mathrm{B}(\alpha') \neq \emptyset$, so the assumption that $r(\lambda(\alpha')) \neq r(\lambda(\beta))$ implies that $\min \mathrm{B}(\alpha') = \beta' < \beta$ and $r(\lambda(\alpha')) = r(\lambda(\beta'))$. However, now we have $\beta' \in \mathrm{B}(\alpha)$, hence $\beta \leq \beta'$. The contradiction proves the monotonicity of r. □

Remark. If $S \subseteq \mathbb{R}$, there exists a countable subset order dense in S; then the transfinite induction can be replaced with ordinary one where parameters are natural numbers.

Generally speaking, the assumption that X is a chain is very restrictive; in our context, however, multi-dimensional strategies afford bleak prospects for acyclicity anyway.

Proposition 6.1 and Theorem 6 allow us to prove the existence of a Nash equilibrium under monotonicity conditions milder than (5.3).

Proposition 6.3. *Let a game* Γ *satisfy these assumptions:*

1. *each* X_i *is a compact subset of* $\mathbb{R}$;

2. $u_i(x_N) = U_i(\sigma_i(x_{-i}), x_i)$ *for all* $i \in N$ *and* $x_N \in X_N$, *where each* $\sigma_i \colon X_{-i} \to \mathbb{R}$ *is defined by* (5.1) *with strictly increasing mappings* $\nu_i \colon X_i \to \mathbb{R}$ *and each* $\alpha^{(h)}_{i_0 i_1 \dots i_h}$ *invariant under all permutations of* $i_0, i_1, \dots, i_h$;

3. *each function* U_i *is upper semicontinuous in the second argument;*

4. *each function* U_i *satisfies either*

$$[x'_i \geq x_i \ \&\ s'_i \geq s_i \ \&\ U_i(s_i, x'_i) \geq U_i(s_i, x_i)] \Rightarrow U_i(s'_i, x'_i) \geq U_i(s'_i, x_i), \tag{6.2a}$$

or

$$[x'_i \geq x_i \ \&\ s'_i \geq s_i \ \&\ U_i(s'_i, x_i) \geq U_i(s'_i, x'_i)] \Rightarrow U_i(s_i, x_i) \geq U_i(s_i, x'_i). \tag{6.2b}$$

Then Γ *possesses a Nash equilibrium.*

Proof. Assumptions 1 and 3 imply that each player $i \in N$ has both the greatest and the least best response to every $x_{-i} \in X_{-i}$. As is well known, (6.2a) implies (6.1a) for the best response correspondence, whereas (6.2b) implies (6.1b). Therefore, Theorem 5 can be applied to monotone selections existing by Proposition 6.1. □

Proposition 6.4. *Let a game* Γ *satisfy these assumptions:*

1. *each* X_i *is a compact subset of* $\mathbb{R}$;

2. $u_i(x_N) = U_i(\sigma_i(x_{-i}), x_i)$ *for all* $i \in N$ *and* $x_N \in X_N$, *where each* $\sigma_i \colon X_{-i} \to \mathbb{R}$ *is defined by* (5.1) *with strictly increasing mappings* $\nu_i \colon X_i \to \mathbb{R}$ *and each* $\alpha^{(h)}_{i_0 i_1 \dots i_h}$ *invariant under all permutations of* $i_0, i_1, \dots, i_h$;

3. *for each* $i \in N$ *and* $s_i \in \sigma_i(X_{-i})$, *the preference relation* $y_i \succ_i x_i \leftrightharpoons U_i(s_i, y_i) > U_i(s_i, x_i)$ *is* ω*-transitive;*

4. *each function* U_i *satisfies both conditions* (6.2), *which is Milgrom and Shannon's (1994)* single crossing *condition.*

Then Γ *possesses a Nash equilibrium.*

Proof. Assumptions 1 and 3 imply that each player's best response correspondence has nonempty values. As is well known, (6.2) imply (6.1) for the best response correspondence. Therefore, Theorem 5 can be applied to monotone selections existing by Theorem 6. □

Similar corollaries to Theorem 3 are proven in the same way. On the other hand, no aggregation is needed for the mere existence of Nash equilibrium when the best responses are increasing.

Proposition 6.5. *Let a game* Γ *satisfy these assumptions:*

1. *each* X_i *is a compact subset of* $\mathbb{R}$;
2. *for each* $i \in N$ *and* $x_{-i} \in X_{-i}$, *the preference relation* $y_i \succ_i x_i \leftrightharpoons u_i(y_i, x_{-i}) > u_i(x_i, x_{-i})$ *is* ω*-transitive;*
3. *each function* u_i *satisfies both conditions* (6.2), *where* s_i *is replaced with* x_{-i}.

Then Γ *possesses a Nash equilibrium.*

Proof. Again, each player's best response correspondence has nonempty values by assumptions 1 and 2, and satisfies (6.1) by (6.2). Therefore, Tarski's (1955) fixed point theorem can be applied to the Cartesian product of monotone selections existing by Theorem 6. □

Remark. To the best of my knowledge, the statement cannot be derived from the previous literature.

In Propositions 6.3, 6.4, and, especially, 6.5, the assumption $X_i \subset \mathbb{R}$ is too restrictive. Unfortunately, there is no idea so far about how Theorem 6 could be proven for multi-dimensional X. There is another, even more compelling, reason to be interested in extensions of the theorem. Consider a supermodular game with bounded, but not (semi)continuous in any sense, utilities. We cannot hope for the existence of a Nash equilibrium, but the existence of an ε-equilibrium might be expected to be derivable via the application of Tarski's fixed point theorem to monotone selections. The problem is that the ε-best response correspondence need not be ascending, hence Theorem 6 is inapplicable even in the case of scalar strategies. Each ε-best response correspondence is *weakly* ascending, but this is of no immediate help because "Theorem A2" is wrong while the assumptions of Proposition 6.2 are too strong.

Remark. The conditions of Proposition 6.2 seem to admit no clear interpretation in terms of utility functions.

7 Acknowledgments

Financial support from a Presidential Grant for the State Support of the Leading Scientific Schools (NSh-5379.2006.1), the Russian Foundation for Basic Research (grant 05-01-00942), and the Spanish Ministry of Education (project SEJ2004-00968) is acknowledged.

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