A universal construction generating potential games

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Abstract

Strategic games are considered where each player's total utility is the sum of local utilities obtained from the use of certain "facilities." All players using a facility obtain the same utility therefrom, which may depend on the identities of users and on their behavior. If a "trimness" condition is satisfied by every facility, then the game admits an exact potential; conversely, if a facility is not trim, adding it to a potential game may destroy that property. In both congestion games and games with structured utilities, all facilities are trim. Under additional assumptions the potential attains its maximum, which is a Nash equilibrium of the game.

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1 Introduction

When Monderer and Shapley (1996) introduced the notion of a *potential game*, the main example they had in mind were Rosenthal's (1973) *congestion games*. Their Theorems 3.1 and 3.2 showed that a finite game admits an exact potential if and only if it can be represented as a congestion game (the sufficiency part was implicit in Rosenthal's reasoning). An alternative, more transparent proof was given in Voorneveld et al. (1999, Theorem 3.3).

Kukushkin (2007) introduced *games with structured utilities*, in a sense, "dual" to congestion games; the players there do not choose *which* facilities to use, only *how* to use facilities from a fixed list. The idea of such a structure of utility functions can be traced back to Germeier and Vatel' (1974), although the local utilities in that paper were aggregated with the minimum function. Theorem 5 from Kukushkin (2007) showed that a strategic game admits an exact potential if and only if it can be represented as a game with structured utilities.

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Thus, two different, but somewhat similar, constructions generating potential games were considered in Kukushkin (2007); one universal for finite potential games, the other for all of them. The possibility to combine both constructions into one was not discussed, actually, was overlooked altogether. Somewhat later, Le Breton and Weber (2011) produced a construction generating potential games, which can rightfully be described as a combination of those universal constructions although not universal itself.

The main objective of this paper is to develop the most general universal construction of this type, which would generate potential games and include all those previous constructions as particular cases. We start with defining a general class of strategic games where the players are able to choose both which facilities to use and how to use them. Those games need not admit potentials of any kind, nor possess Nash equilibria. Then we formulate conditions ensuring that such a game admits an exact potential; naturally, they are satisfied for both congestion games and games with structured utilities, as well as games of Le Breton and Weber. Since those conditions are formulated independently for every facility, a necessity result becomes obtainable: if a facility does not satisfy them, adding it to a potential game may destroy that property.

Harks et al. (2011) found two classes of potential games, which are similar to congestion games, but lack two key features of the latter, viz. anonymity and commonality: different players may affect the same facility differently and derive different utilities therefrom. A straightforward modification in the style of Kukushkin (2007, Section 4) shows that such games, as well their more general analogs, are also generated by our construction.

An important (probably, the most important) reason to be interested in potentials of strategic games is their connection with the existence of (pure-strategy) Nash equilibrium. From this viewpoint, however, it is not enough for a game to admit a potential: that potential should admit a maximizer. Since the strategy sets in our games may be infinite, the latter cannot be taken for granted. Le Breton and Weber (2011) showed, even in a much less general case, that straightforward assumptions such as the compactness of each strategy set and continuity of every local utility function are not enough. Modifying their approach, we formulate a set of assumptions ensuring that the maximum of the potential is attained and hence a Nash equilibrium does exist.

Our basic construction is described in the following section. In Section 3, the key definitions of a trim facility and a trim game are given; Theorem 1 asserts the presence of an exact potential in every trim game. Theorem 2 in Section 4 shows kind of necessity of trimness for this property.

In Section 5, the question of when the potential attains its maximum is addressed. We formulate a list of assumptions ensuring the upper semicontinuity of the potential, and hence the existence of a Nash equilibrium (Theorem 3). The proof of the theorem is in Section 6.

Section 7 demonstrates that the Le Breton–Weber construction is, indeed, a particular case of ours. In Section 8, we show that every game from each class considered by Harks et al. (2011) can be naturally represented as one from our class. Section 9 summarizes the message of the paper.

2 Basic definitions

A *strategic game* Γ is defined by a finite set N of *players*, and, for each $i \in N$, a set X_i of *strategies* and a real-valued *utility function* u_i on the set $X_N := \prod_{i \in N} X_i$ of *strategy profiles*. We denote $\mathcal{N} := 2^N \setminus \{\emptyset\}$ and $X_I := \prod_{i \in I} X_i$ for each $I \in \mathcal{N}$. Given $i, j \in N$, we use notation X_{-i} instead of $X_{N \setminus \{i\}}$ and X_{-ij} instead of $X_{N \setminus \{i,j\}}$.

A function $P : X_N \to \mathbb{R}$ is an *exact potential* of Γ (Monderer and Shapley, 1996) if

$$u_i(y_N) - u_i(x_N) = P(y_N) - P(x_N) \tag{1}$$

whenever $i \in N$, $y_N, x_N \in X_N$, and $y_{-i} = x_{-i}$. If $x_N^0 \in X_N$ maximizes P over X_N, then, obviously, x_N^0 is a Nash equilibrium.

A *game with* (*additive*) *common local utilities* (a *CLU game*) may have an arbitrary finite set N of players and arbitrary sets of strategies X_i ($i \in N$), whereas the utilities are defined by the following construction. First of all, there is a set A of *facilities*; we denote $\mathcal{B}$ the set of all (nonempty) finite subsets of A. For each $i \in N$, there is a mapping $\mathrm{B}_i \colon X_i \to \mathcal{B}$ describing what facilities player i uses having chosen x_i. Every strategy profile x_N determines *local utilities* at all facilities $\alpha \in \mathrm{A}$; each player's *total utility* is the sum of local utilities over chosen facilities. The exact definitions need plenty of notations.

For every $\alpha \in \mathrm{A}$, we denote $I_\alpha^- := \{i \in N \mid \alpha \in \bigcap_{x_i \in X_i} \mathrm{B}_i(x_i)\}$ and $I_\alpha^+ := \{i \in N \mid \alpha \in \bigcup_{x_i \in X_i} \mathrm{B}_i(x_i)\}$; without restricting generality, we may assume $I_\alpha^+ \neq \emptyset$. For each $i \in I_\alpha^+$, we denote $X_i^\alpha := \{x_i \in X_i \mid \alpha \in \mathrm{B}_i(x_i)\}$; if $i \in I_\alpha^-$, then $X_i^\alpha = X_i$. Then we set $\mathcal{I}_\alpha := \{I \in \mathcal{N} \mid I_\alpha^- \subseteq I \subseteq I_\alpha^+\}$ and $\Xi_\alpha := \{\langle I, x_I \rangle \mid I \in \mathcal{I}_\alpha \ \&\ x_I \in X_I^\alpha\}$. The *local utility function* at $\alpha \in \mathrm{A}$ is $\varphi_\alpha \colon \Xi_\alpha \to \mathbb{R}$. For every $\alpha \in \mathrm{A}$ and $x_N \in X_N$, we denote $I(\alpha, x_N) := \{i \in N \mid \alpha \in \mathrm{B}_i(x_i)\}$; obviously, $I_\alpha^- \subseteq I(\alpha, x_N) \subseteq I_\alpha^+$. The *total utility function* of each player i is

$$u_i(x_N) := \sum_{\alpha \in \mathrm{B}_i(x_i)} \varphi_\alpha(I(\alpha, x_N), x_{I(\alpha, x_N)}). \tag{2}$$

Both games with structured utilities and congestion games are CLU games. In the former case, for each $i \in N$, the set $\mathrm{B}_i(x_i)$ is the same for all $x_i \in X_i$; hence $I(\alpha, x_N)$ does not depend on the second argument and hence the first argument of φ_α can be dropped. In the latter case, $X_i \subseteq \mathcal{B}$ for each $i \in N$, each B_i is an identity mapping, and hence the second argument of φ_α can be dropped; besides, φ_α only depends on $\#I$. Note that A is finite in both cases, which is not required generally.

3 Trim facilities and games

For every $\alpha \in \mathrm{A}$, we denote $n^-(\alpha) := \min_{I \in \mathcal{I}_\alpha} \#I = \max\{1, \#I_\alpha^-\}$.

We call a facility $\alpha \in \mathrm{A}$ *trim* if there is a real-valued function $\psi_\alpha(m)$ defined for integer m between $n^-(\alpha)$ and $\#I_\alpha^+ - 1$ such that

$$\varphi_\alpha(I, x_I) = \psi_\alpha(\#I) \tag{3}$$

whenever $I \in \mathcal{I}_\alpha$, $I \neq I_\alpha^+$, and $x_I \in X_I^\alpha$.

In other words: whenever a trim facility is not used by all potential users, neither the identities of the users, nor their strategies matter, only the number of users. A term like "quasi-Rosenthal facility" might be justified, but it seems too cumbersome.

The property can also be defined as invariance of the local utility function to certain manipulations with its arguments.

Proposition 1. *A facility $\alpha \in \mathrm{A}$ is trim if and only if these two conditions are satisfied:*

1) whenever $i \notin J \subset N$, $I_\alpha^+ \neq J \cup \{i\} \in \mathcal{I}_\alpha$, $x_i, y_i \in X_i^\alpha$, and $x_J \in X_J^\alpha$, there holds

$$\varphi_\alpha(J \cup \{i\}, (x_J, x_i)) = \varphi_\alpha(J \cup \{i\}, (x_J, y_i)); \tag{4a}$$

2) whenever $J \subset N$ and $i, j \in N \setminus J$ are such that $i \neq j$, $J \cup \{i\} \in \mathcal{I}_\alpha \ni J \cup \{j\}$, and $x_{J\cup\{i,j\}} \in X^\alpha_{J\cup\{i,j\}}$, there holds

$$\varphi_\alpha(J \cup \{i\}, x_{J\cup\{i\}}) = \varphi_\alpha(J \cup \{j\}, x_{J\cup\{j\}}) \tag{4b}$$

($J = \emptyset$ is allowed in both conditions, in which case the term x_J should be just ignored).

Proof. The implication "only if" is obvious; let $\alpha \in \mathrm{A}$ satisfy both conditions (4). If $I_\alpha^+ = I_\alpha^-$, then $\mathcal{I}_\alpha = \{I_\alpha^+\}$ and hence (3) is not required for any $I \in \mathcal{I}_\alpha$. Therefore we may assume that $I_\alpha^- \subset I_\alpha^+$.

Whenever $I \in \mathcal{I}_\alpha$, $I \neq I_\alpha^+$, and $x_I, y_I \in X_I^\alpha$, we can, picking, one by one, $i \in I$ and replacing x_i with y_i, obtain, by (4a), that $\varphi_\alpha(I, x_I) = \varphi_\alpha(I, y_I)$, i.e., the choice of strategies does not matter indeed.

Let us show the irrelevance of the identities of users. If $I_\alpha^- \neq \emptyset$, we define $\psi(\#I_\alpha^-) := \varphi_\alpha(I_\alpha^-, x_{I_\alpha^-})$, which does not depend on $x_{I_\alpha^-} \in X^\alpha_{I_\alpha^-}$ by the argument of the preceding paragraph. There is no other $I \in \mathcal{I}_\alpha$ with the same $\#I$. If $I_\alpha^- = \emptyset$, we set $\psi(1) := \varphi_\alpha(\{i\}, x_i)$, which does not depend on $i \in I_\alpha^+$ by (4b) with $J = \emptyset$, or on $x_i \in X_i^\alpha$ by the argument of the preceding paragraph again.

Finally, supposing that $I, J \in \mathcal{I}_\alpha$, $n^-(\alpha) < \#I = \#J < \#I_\alpha^+$, $x_I \in X_I^\alpha$ and $y_J \in X_J^\alpha$, we have to prove that $\varphi_\alpha(I, x_I) = \varphi_\alpha(J, y_J)$. Obviously, there is a one-to-one correspondence between $J \setminus I := \{j_1, \dots, j_k\}$ and $I \setminus J := \{i_1, \dots, i_k\}$. Consecutively applying (4b), we obtain:

$$\varphi_\alpha(I, x_I) = \varphi_\alpha((I \cap J) \cup \{j_1, i_2, \dots, i_k\}, (x_{(I\cap J)\cup\{i_2,\dots,i_k\}}, y_{j_1})) = \\ \varphi_\alpha((I \cap J) \cup \{j_1, j_2, i_3, \dots, i_k\}, (x_{(I\cap J)\cup\{i_3,\dots,i_k\}}, y_{\{j_1,j_2\}})) = \dots = \varphi_\alpha(J, y_J).$$

Now we can set $\psi(m) := \varphi_\alpha(I, x_I)$ for an arbitrary $I \in \mathcal{I}_\alpha$ with $\#I = m$ and an arbitrary $x_I \in X_I^\alpha$, and have (3) satisfied. $\square$

We call a CLU game *trim* if so is every facility. It is instructive to check that both congestion games and games with structured utilities are trim. In the first case, (3) holds for all $I \in \mathcal{I}_\alpha$, even for $I = I_\alpha^+$; in the second case, conversely, $I_\alpha^- = I_\alpha^+$ for each facility α and hence (3) is not required at all.

Theorem 1. *Every trim CLU game admits an exact potential.*

Proof. Given $x_N \in X_N$, we denote $\mathrm{A}(x_N) := \{\alpha \in \mathrm{A} \mid I(\alpha, x_N) \neq \emptyset\}$ and $\mathrm{A}^+(x_N) := \{\alpha \in \mathrm{A} \mid \#I(\alpha, x_N) > n^-(\alpha)\}$ $[\subseteq \mathrm{A}(x_N)]$; since N and each $\mathrm{B}_i(x_i)$ are finite, $\mathrm{A}(x_N)$ is finite too. Now we define our potential function in this way:

$$P(x_N) := \sum_{\alpha \in \mathrm{A}(x_N)} \varphi_\alpha(I(\alpha, x_N), x_{I(\alpha,x_N)}) + \sum_{\alpha \in \mathrm{A}^+(x_N)} \sum_{m=n^-(\alpha)}^{\#I(\alpha,x_N)-1} \psi_\alpha(m). \tag{5}$$

Given $i \in N$ and $x_{-i} \in X_{-i}$, we denote $I_{-i}(\alpha, x_{-i}) := \{j \in N \setminus \{i\} \mid \alpha \in \mathrm{B}_j(x_j)\}$, $\mathrm{A}_{-i}(x_{-i}) := \{\alpha \in \mathrm{A} \mid I_{-i}(\alpha, x_{-i}) \neq \emptyset\}$ and $\mathrm{A}_{-i}^+(x_{-i}) := \{\alpha \in \mathrm{A} \mid \#I_{-i}(\alpha, x_{-i}) > n^-(\alpha)\}$ $[\subseteq \mathrm{A}_{-i}(x_{-i})]$. Then we define these auxiliary functions $Q_{-i} \colon X_{-i} \to \mathbb{R}$ $(i \in N)$:

$$Q_{-i}(x_{-i}) := \sum_{\alpha \in \mathrm{A}_{-i}(x_{-i})} \varphi_\alpha(I_{-i}(\alpha, x_{-i}), x_{I_{-i}(\alpha,x_{-i})}) + \sum_{\alpha \in \mathrm{A}_{-i}^+(x_N)} \sum_{m=n^-(\alpha)}^{\#I_{-i}(\alpha,x_{-i})-1} \psi_\alpha(m). \tag{6}$$

Once we show that

$$P(x_N) = u_i(x_N) + Q_{-i}(x_{-i}) \tag{7}$$

for all $i \in N$ and $x_N \in X_N$, Theorem 2.1 of Voorneveld et al. (1999) will imply that P is an exact potential.

Whenever $\alpha \notin \mathrm{B}_i(x_i)$, we have $I_{-i}(\alpha, x_{-i}) = I(\alpha, x_N)$; therefore, this α brings the same contribution to $Q_{-i}(x_{-i})$ as to $P(x_N)$, while no contribution at all to $u_i(x_N)$. For every $\alpha \in \mathrm{B}_i(x_i)$, we have $I_{-i}(\alpha, x_{-i}) = I(\alpha, x_N) \setminus \{i\}$ and hence $\#I_{-i}(\alpha, x_{-i}) = \#I(\alpha, x_N) - 1$. If $I(\alpha, x_N) = \{i\}$, then this α brings to $u_i(x_N)$ the same contribution, $\varphi_\alpha(\{i\}, x_i)$, as to $P(x_N)$, while no contribution at all to $Q_{-i}(x_{-i})$. If $I(\alpha, x_N) = \{i, j\}$, then this α contributes $\varphi_\alpha(\{i, j\}, (x_i, x_j))$ to $u_i(x_N)$, contributes $\varphi_\alpha(\{j\}, x_j)$ to $Q_{-i}(x_{-i})$, and contributes $\varphi_\alpha(\{i, j\}, (x_i, x_j)) + \psi_\alpha(1)$ to $P(x_N)$. Since α is trim, we have $\varphi_\alpha(\{j\}, x_j) = \psi_\alpha(1)$ and hence total contributions coincide again. Finally, if $\#I(\alpha, x_N) > 2$, we argue virtually in the same way as in the previous case of $\#I(\alpha, x_N) = 2$. Equality (7) being satisfied, Theorem 1 is proven. □

Remark. In the case of a game with structured utilities, the second sum in (5) disappears since $\mathcal{I}_\alpha = \{I_\alpha^+\}$ and hence $\#I_\alpha^+ = n^-(\alpha)$ and $\mathrm{A}^+(x_N) = \emptyset$. Thus, the potential defined by (5) coincides with that defined in the proof of sufficiency in Theorem 4 from Kukushkin (2007). In the case of a congestion game, $\varphi_\alpha(I(\alpha, x_N), x_{I(\alpha,x_N)}) = \psi_\alpha(\#I(\alpha, x_N))$ and hence the potential defined by (5) coincides with Rosenthal's potential.

4 Necessity of trimness

Let a finite set N of players be fixed. An *autonomous facility* α is defined by two subsets $I_\alpha^- \subseteq I_\alpha^+ \in \mathcal{N}$ [I_α^- may be empty], a set X_i^α of relevant strategies for each $i \in I_\alpha^+$, and a local utility function $\varphi_\alpha \colon \Xi_\alpha \to \mathbb{R}$, where $\mathcal{I}_\alpha := \{I \in \mathcal{N} \mid I_\alpha^- \subseteq I \subseteq I_\alpha^+\}$ and $\Xi_\alpha := \{\langle I, x_I^\alpha \rangle \mid I \in \mathcal{I}_\alpha \ \&\ x_I^\alpha \in X_I^\alpha\}$, exactly as in Section 2. We call an autonomous facility α *trim* if it satisfies the same condition (3).

Let α be an autonomous facility, and let Γ be a CLU game with the same set N, a set A such that $\alpha \notin \mathrm{A}$, and $X_i \cap X_i^\alpha = \emptyset$ for each $i \in N$. A CLU game Γ^* is called an *extension of* Γ *with* α if these conditions are satisfied: $N^* = N$; $\mathrm{A}^* = \mathrm{A} \cup \{\alpha\}$; $X_i^* = X_i$ when $i \in N \setminus I_\alpha^+$; $X_i^* \subseteq X_i^\alpha \cup (X_i^\alpha \times X_i)$ when $i \in I_\alpha^-$; $X_i^* \subseteq X_i^\alpha \cup (X_i^\alpha \times X_i) \cup X_i$ when $i \in I_\alpha^+ \setminus I_\alpha^-$; $\mathrm{B}_i^*(x_i) = \mathrm{B}_i(x_i)$ for all $x_i \in X_i$; $\mathrm{B}_i^*(x_i^\alpha) = \{\alpha\}$ for all $x_i^\alpha \in X_i^\alpha$; $\mathrm{B}_i^*(\langle x_i^\alpha, x_i \rangle) = \{\alpha\} \cup \mathrm{B}_i(x_i)$ for all $x_i^\alpha \in X_i^\alpha$ and $x_i \in X_i$; whenever $J \subseteq I \in \mathcal{I}_\alpha$, $x_I^\alpha \in X_I^\alpha$, $x_j^* = x_j^\alpha$ for all $j \in J$, and $x_i^* = \langle x_i^\alpha, x_i \rangle$ for all $i \in I \setminus J$, there holds $\varphi_\alpha^*(I, x_I^*) = \varphi_\alpha(I, x_I^\alpha)$; whenever $\beta \in \mathrm{A}$, $J \subseteq I \in \mathcal{I}_\beta$, $x_I^\beta \in X_I^\beta$, $x_j^* = x_j^\beta$ for all $j \in J$, and $x_i^* = \langle x_i^\alpha, x_i^\beta \rangle$ for all $i \in I \setminus J$, there holds $\varphi_\beta^*(I, x_I^*) = \varphi_\beta(I, x_I^\beta)$.

Remark. It is important to note that there may be various extensions of the same game Γ with the same facility α. Two other features of the definition are also worth mentioning. First, we allow some strategies available in Γ to become unavailable in Γ^*, and some strategies from X_i^α (for $i \in I_\alpha^+$) may also be unavailable. Second, each player $i \in I_\alpha^+$ may have an option of choosing the "new" facility α, forgetting the "old" game Γ altogether. A straightforward modification of our definition would dispense with either feature or both; Theorem 2 would remain correct since neither is invoked in the proof.

Theorem 2. *For every autonomous facility α the following statements are equivalent:*

1. *α is trim.*
2. *Whenever Γ^* is an extension with α of a trim CLU game Γ, Γ^* admits an exact potential.*
3. *Whenever Γ^* is an extension with α of a congestion game Γ, Γ^* admits an exact potential.*
4. *Whenever Γ^* is an extension with α of a game with structured utilities Γ, Γ^* admits an exact potential.*

Proof. The implication Statement 1 $\Rightarrow$ Statement 2 immediately follows from Theorem 1; the implications Statement 2 $\Rightarrow$ Statement 3 and Statement 2 $\Rightarrow$ Statement 4 are trivial. We only have to show the implications Statement 3 $\Rightarrow$ Statement 1 and Statement 4 $\Rightarrow$ Statement 1; so let Statement 3 hold.

Claim 2.1. *Let $i, j \in I_\alpha^+$, $i \notin I \in \mathcal{I}_\alpha$, $j \in I$, $x_I^\alpha \in X_I^\alpha$ and $y_j^\alpha \in X_j^\alpha$. Then $\varphi_\alpha(I, x_I^\alpha) = \varphi_\alpha(I, (x_{I \setminus \{j\}}^\alpha, y_j^\alpha))$.*

Proof of Claim 2.1. Let us consider a congestion game Γ with the same set N of players, a singleton set of facilities $\mathrm{A} := \{\beta\}$, and a singleton set of strategies $X_h := \{x_h^\beta\}$ with $\mathrm{B}_h(x_h^\beta) := \{\beta\}$ for each $h \in N$, and an arbitrary constant (in lieu of a function) $\varphi_\beta(N, x_N^\beta) = \psi_\beta(\#N)$. We define an extension Γ^* of Γ with α by: $N^* := N$; $\mathrm{A}^* := \{\alpha, \beta\}$; $X_h^* := \{x_h^\beta\}$ for each $h \in N \setminus I_\alpha^+$; $X_h^* := X_h^\alpha \cup \{x_h^\beta\}$ for each $h \in I_\alpha^+ \setminus I_\alpha^-$; $X_h^* := X_h^\alpha$ for each $h \in I_\alpha^-$; $\mathrm{B}_h^*(x_h^\beta) = \{\beta\}$; $\mathrm{B}_h^*(x_h^\alpha) = \mathrm{A}^*$ for each $x_h^\alpha \in X_h^\alpha$; $\varphi_\alpha^*(J, x_J^\alpha) = \varphi_\alpha(J, x_J^\alpha)$ for every $J \in \mathcal{I}_\alpha$ and $x_J^\alpha \in X_J^\alpha$; $\varphi_\beta^*(N, x_N^\beta) = \varphi_\beta(N, x_N^\beta)$.

Remark. To avoid too cumbersome notations, we allowed a small discrepancy with the general definition of an extension of a CLU game. Strictly speaking, the strategy set of each player $h \in I_\alpha^-$ in Γ^* is $X_h^\alpha \times X_h$, which can be identified with X_h^α because of an obvious one-to-one correspondence $\langle x_h^\alpha, x_h^\beta \rangle \leftrightarrow x_h^\alpha$. The same correspondence allows us to identify $(X_h^\alpha \times X_h) \cup X_h$ with $X_h^\alpha \cup X_h$ for each $h \in I_\alpha^+ \setminus I_\alpha^-$.

Since we assumed Statement 3 to hold, Γ^* admits an exact potential; hence so does every subgame. As was noted by Monderer and Shapley (1996, Theorem 2.8), it is enough to consider 2×2 subgames. We leave players i and j with two strategies each: $\{x_i^\alpha, x_i^\beta\}$ and $\{x_j^\alpha, y_j^\alpha\}$, respectively, fixing strategies for all other players: x_h^α for $h \in I$ and x_h^β for $h \notin I$. Note that $h \notin I_\alpha^-$ whenever $h \notin I$, and hence $x_h^\beta \in X_h^*$; in particular, $x_i^\beta \in X_i^*$. Introducing auxiliary notations, $u^\beta := \varphi_\beta(N, x_N^\beta)$, $v_x^\alpha := \varphi_\alpha(I \cup \{i\}, x_{I\cup\{i\}}^\alpha)$, $v_y^\alpha := \varphi_\alpha(I \cup \{i\}, (x_{I\cup\{i\}\setminus\{j\}}^\alpha, y_j^\alpha))$, $u_x^\alpha := \varphi_\alpha(I, x_I^\alpha)$, and $u_y^\alpha := \varphi_\alpha(I, (x_{I\setminus\{j\}}^\alpha, y_j^\alpha))$, we obtain the following matrix of the resulting subgame:

$$\begin{array}{cc} & \begin{array}{cc} \quad x_j^\alpha & \qquad\qquad y_j^\alpha \end{array} \\ \begin{array}{c} x_i^\alpha \\ x_i^\beta \end{array} & \begin{bmatrix} \langle v_x^\alpha + u^\beta, v_x^\alpha + u^\beta \rangle & \langle v_y^\alpha + u^\beta, v_y^\alpha + u^\beta \rangle \\ \langle u^\beta, u_x^\alpha + u^\beta \rangle & \langle u^\beta, u_y^\alpha + u^\beta \rangle \end{bmatrix} \end{array}.$$

Straightforward calculations show that $P(x_i^\beta, y_j^\alpha, x_{-ij}) - P(x_i^\beta, x_j^\alpha, x_{-ij}) = [P(x_i^\alpha, x_j^\alpha, x_{-ij}) - P(x_i^\beta, x_j^\alpha, x_{-ij})] + [P(x_i^\alpha, y_j^\alpha, x_{-ij}) - P(x_i^\beta, x_j^\alpha, x_{-ij})] + [P(x_i^\beta, y_j^\alpha, x_{-ij}) - P(x_i^\alpha, y_j^\alpha, x_{-ij})] = v_x^\alpha + (v_y^\alpha - v_x^\alpha) - v_y^\alpha = 0$. Therefore, $\varphi_\alpha(I, x_I^\alpha) = u_j(x_i^\beta, y_j^\alpha, x_{-ij}) = u_j(x_i^\beta, x_j^\alpha, x_{-ij}) = \varphi_\alpha(I, (x_{I\setminus\{j\}}^\alpha, y_j^\alpha))$. In other words, (4a) is established. □

Claim 2.2. *Let $i, j \in I \in \mathcal{I}_\alpha$, $I_\alpha^- \subseteq I \setminus \{i, j\}$, and $x_I^\alpha \in X_I^\alpha$. Then $\varphi_\alpha(I\setminus\{i\}, x_{I\setminus\{i\}}^\alpha) = \varphi_\alpha(I\setminus\{j\}, x_{I\setminus\{j\}}^\alpha)$.*

Proof of Claim 2.2. We consider the same congestion game Γ used in the proof of Claim 2.1 and the same extension Γ^* of Γ with α. This time, we consider a 2×2 subgame where players i and j have two strategies each: $\{x_i^\alpha, x_i^\beta\}$ and $\{x_j^\alpha, x_j^\beta\}$, respectively, while the strategies of all other players are fixed: x_h^α for $h \in I$ and x_h^β for $h \notin I$.

Again, this subgame must admit an exact potential. Introducing auxiliary notations, $u^\beta := \varphi_\beta(N, x_N^\beta)$, $u^\alpha := \varphi_\alpha(I, x_I^\alpha)$, $v_i^\alpha := \varphi_\alpha(I \setminus \{i\}, x_{I\setminus\{i\}}^\alpha)$ and $v_j^\alpha := \varphi_\alpha(I \setminus \{j\}, x_{I\setminus\{j\}}^\alpha)$, we obtain the following matrix:

$$\begin{array}{ccc} & x_j^\alpha & x_j^\beta \\ \begin{matrix} x_i^\alpha \\ x_i^\beta \end{matrix} & \multicolumn{2}{c}{\begin{bmatrix} \langle u^\alpha + u^\beta, u^\alpha + u^\beta \rangle & \langle v_j^\alpha + u^\beta, u^\beta \rangle \\ \langle u^\beta, v_i^\alpha + u^\beta \rangle & \langle u^\beta, u^\beta \rangle \end{bmatrix}} \end{array}.$$

Now we have $0 = [P(x_i^\alpha, x_j^\alpha, x_{-ij}) - P(x_i^\beta, x_j^\alpha, x_{-ij})] + [P(x_i^\alpha, x_j^\beta, x_{-ij}) - P(x_i^\alpha, x_j^\alpha, x_{-ij})] + [P(x_i^\beta, x_j^\beta, x_{-ij}) - P(x_i^\alpha, x_j^\beta, x_{-ij})] + [P(x_i^\beta, x_j^\alpha, x_{-ij}) - P(x_i^\beta, x_j^\beta, x_{-ij})] = u^\alpha - u^\alpha - v_j^\alpha + v_i^\alpha = v_i^\alpha - v_j^\alpha$. Therefore, $\varphi_\alpha(I \setminus \{i\}, x^\alpha_{I \setminus \{i\}}) = v_i^\alpha = v_j^\alpha = \varphi_\alpha(I \setminus \{j\}, x^\alpha_{I \setminus \{j\}})$. In other words, (4b) is established. □

A reference to Proposition 1 completes the proof of the implication Statement 3 ⇒ Statement 1.

The proof of the implication Statement 4 ⇒ Statement 1 is now straightforward: the congestion game Γ used in the proofs of Claims 2.2 and 2.1 can as well be perceived as a game with structured utilities. Theorem 2 is proven. □

Theorem 2 takes it for granted that the players sum up their local utilities. Actually, the necessity (in a sense) of addition was showed in Kukushkin (2007): If the players may aggregate local utilities with arbitrary (continuous and strictly increasing) functions, then the existence of an exact potential is ensured regardless of other characteristics of the game only if the players sum up local utilities; that statement remains valid when attention is restricted to congestion games (Theorem 2 of Kukushkin, 2007), or to games with structured utilities (Theorem 4).

Strictly speaking, those theorems do not exclude the possibility that the aggregation of local utilities with some other, *non-strictly* increasing functions might also ensure the existence of an exact potential, but there is no reason to expect anything interesting here. On the other hand, the minimum aggregation, as envisaged by Germeier and Vatel' (1974), ensures the acyclicity of coalitional improvements and hence the existence of a strong Nash equilibrium (Harks et al., 2013; Kukushkin, 2017).

5 The existence of Nash equilibrium

To ensure that the potential P attains a maximum, some additional assumptions are needed. The simplest approach would be to have X_N compact and P upper semi continuous. As was noted by Le Breton and Weber (2011) even in a much less general case, a certain degree of subtlety is required, however, since even the continuity of every φ_α does not imply the upper semicontinuity of the potential.

Assumption 1. *The set of facilities* A *and each strategy set* X_i *are metric spaces; each mapping* B_i *is continuous in the Hausdorff metric on the target; for every* $\alpha \in \mathrm{A}$ *and* $I \in \mathcal{I}_\alpha$*, the function* $\varphi_\alpha(I, \cdot) \colon X_I \to \mathbb{R}$ *is upper semicontinuous.*

Henceforth, we assume each set X_I ($I \in \mathcal{N}$) to be endowed with the maximum metrics. For each $i \in N$ and $m \in \mathbb{N}$, we denote $X_i^m := \{x_i \in X_i \mid \#\mathrm{B}_i(x_i) = m\}$.

Assumption 2. *For each $i \in N$ and $m \in \mathbb{N}$, either $X_i^m = \emptyset$ or X_i^m is a compact subset of X_i.*

Assumption 3. *For each $i \in N$, $X_i^m \neq \emptyset$ only for a finite number of $m \in \mathbb{N}$.*

Assumptions 1 – 3 have a technical implication useful in the following.

Lemma 1. *Let $i \in N$, $x_i^k \in X_i$ for all $k \in \mathbb{N}$, and $x_i^k \to x_i^\omega \in X_i$; let open neighborhoods O_α of $\alpha \in \mathrm{B}_i(x_i^\omega)$ be such that $O_\alpha \cap O_\beta = \emptyset$ whenever $\alpha \neq \beta$. Then $\#\mathrm{B}_i(x_i^k) = \#\mathrm{B}_i(x_i^\omega)$ and $\#(\mathrm{B}_i(x_i^k) \cap O_\alpha) = 1$ for all $\alpha \in \mathrm{B}_i(x_i^\omega)$ and all $k \in \mathbb{N}$ large enough.*

Proof. By Assumption 3, there is a finite number of possible values of $\#\mathrm{B}_i(x_i^k)$; therefore, we must have $x_i^k \in X_i^m$ for some $m \in \mathbb{N}$, $m \neq 0$, and an infinite number of $k \in \mathbb{N}$. Since X_i^m is compact by Assumption 2, and hence closed in X_i, we have $x_i^\omega \in X_i^m$ too. It follows immediately that such an m must be unique, i.e., $x_i^k \in X_i^m$ for all $k \in \mathbb{N}$ large enough.

By Assumption 1, we have $\mathrm{B}_i(x_i^k) \to \mathrm{B}_i(x_i^\omega)$. Therefore, for each $k \in \mathbb{N}$ large enough and for each $\alpha \in \mathrm{B}_i(x_i^\omega)$, there is $\alpha^k \in \mathrm{B}_i(x_i^k) \cap O_\alpha$. Since $O_\alpha \cap O_\beta = \emptyset$ whenever $\alpha \neq \beta$, and $\#\mathrm{B}_i(x_i^k) = \#\mathrm{B}_i(x_i^\omega)$, we must have $\#(\mathrm{B}_i(x_i^k) \cap O_\alpha) = 1$ indeed. □

Our final assumption combines some sorts of upper semicontinuity (of φ_α "in α") and monotonicity (of φ_α "in I").

Assumption 4. *For every $\alpha \in \mathrm{A}$, $I \in \mathcal{I}_\alpha$, and $\varepsilon > 0$, there is $\delta > 0$ such that:*

$$\varphi_\alpha(I, x_I) > \varphi_\beta(J, y_J) - \varepsilon \tag{8}$$

whenever $\beta \in \mathrm{A} \setminus \{\alpha\}$, $J \in \mathcal{I}_\beta$, $x_I \in X_I^\alpha$, $y_J \in X_J^\beta$, $J \subseteq I$, and the distance between α and β in A, as well as between x_J and y_J in X_J, is less than δ.

If A is finite as, e.g., in a game with structured utilities or in a congestion game, then Assumption 4 holds vacuously since a $\delta > 0$ smaller than the minimal distance between $\alpha \neq \beta$ can be chosen.

Theorem 3. *Every trim CLU game satisfying Assumptions 1, 2, 3, and 4 possesses a (pure strategy) Nash equilibrium.*

The proof is deferred to Section 6.

It is impossible to argue that the assumptions imposed in Theorem 3 are *necessary* in a proper sense. After all, neither upper semicontinuity, nor compactness are necessary for a function to attain its maximum. Nonetheless, dropping any one of them makes the theorem wrong. There is no need to discuss Assumption 1, but for the three others, appropriate counterexamples follow. In Examples 1 and 2, even one-player games suffice.

Example 1. Let us consider a "congestion game with an infinite set of facilities," where $N := \{1\}$, $\mathrm{A} := [0,1]$, $X_1 := \{\{0\}\} \cup \{\{1/2^m, 1/2^{m+1}\}\}_{m\in\mathbb{N}}$, $\mathrm{B}_1(x_1) := \{x_1\}$ for every $x_1 \in X_1$, and $\psi_\alpha(1) := 1-\alpha$ for every $\alpha \in [0,1]$. All assumptions of Theorem 3 except Assumption 2 are satisfied, X_1 is compact (in the Hausdorff metrics), but X_1^2 is not. And there is no Nash equilibrium, i.e., maximum of u_1: $\sup_{x_1\in X_1^2} u_1(x_1) = 2$, whereas $u_1(x_1) < 2$ for every $x_1 \in X_1$.

Example 2. Let us consider a "congestion game with an infinite set of facilities," where $N := \{1\}$, $\mathrm{A} := [0,1]$, $X_1 := \{\{0\}\} \cup \{\{1/2^m - k/[(m+1)2^{m+1}]\}_{k=0,\dots,m}\}_{m\in\mathbb{N}}$, $\mathrm{B}_1(x_1) := \{x_1\}$ for every $x_1 \in X_1$, and $\psi_\alpha(1) := 1-\alpha$ for every $\alpha \in [0,1]$. All assumptions of Theorem 3 except Assumption 3 are satisfied, X_1 is compact, as well as each X_1^m ($m \in \mathbb{N}$), which is actually a singleton. And again, there is no Nash equilibrium, i.e., maximum of u_1, since $\sup_{x_1\in X_1} u_1(x_1) = +\infty$.

Example 3. Let us consider a "congestion game with an infinite set of facilities," where $N := \{1,2\}$, $\mathrm{A} := X_1 := X_2 := [0,1]$, $\mathrm{B}_i(x_i) := \{x_i\}$ for every $x_i \in X_i$, $\psi_\alpha(2) := 1-\alpha$ and $\psi_\alpha(1) := 2-\alpha$ for every $\alpha \in [0,1]$. All assumptions of Theorem 3 except Assumption 4 are satisfied, but there is no Nash equilibrium. Let $(x_1,x_2) \in X_N$ and $i \in N$. If $x_i \neq 0$, then player i is better off slightly decreasing x_i. On the other hand, $(\{0\},\{0\})$ is not an equilibrium either, because each player will be better off choosing any $x_i \in]0,1[$.

6 Proof of Theorem 3

As was hinted at the start of Section 5, our strategy is to show that P defined by (5) is upper semicontinuous on a compact X_N. Then any strategy profile which maximizes P will be a Nash equilibrium.

The compactness of X_N immediately follows from Assumptions 2 and 3. Let $x_N^k \to x_N^\omega \in X_N$; we have to show that

$$P(x_N^\omega) \geq \limsup_{k\to\infty} P(x_N^k).$$

Since $\mathrm{A}(x_N^\omega)$ is finite, there is an open neighborhood O_α of each $\alpha \in \mathrm{A}(x_N^\omega)$ such that $O_\alpha \cap O_\beta = \emptyset$ whenever $\alpha \neq \beta \in \mathrm{A}(x_N^\omega)$. Now Lemma 1 applies; therefore, we may, without restricting generality, assume that $\mathrm{B}_i(x_i^k) \cap O_\alpha = \{\beta_i^k\}$ for each $\alpha \in \mathrm{A}(x_N^\omega)$, $i \in I(\alpha, x_N^\omega)$, and $k \in \mathbb{N}$ [we should have written $\beta_i^k(\alpha)$, but β_i^k related to different α's will never be considered simultaneously]. Note that $\beta_i^k \to \alpha$ for each $i \in N$. Since there is a finite number of possible values of $I(\beta_i^k, x_N^k)$, we may, without restricting generality, assume that, given $i \in I(\alpha, x_N^\omega)$, the set $I(\beta_i^k, x_N^k)$ is the same for all k. Similarly, we may assume that $I(\alpha, x_N^\omega)$ is partitioned into $\bar{I}(\alpha, x_N^\omega) := \{i \in I(\alpha, x_N^\omega) \mid \forall k \in \mathbb{N}\,[\beta_i^k = \alpha]\}$ $[= \{i \in I(\alpha, x_N^\omega) \mid \forall k \in \mathbb{N}\,[\alpha \in \mathrm{B}_i(x_i^k)]\}]$ and $\tilde{I}(\alpha, x_N^\omega) := \{i \in I(\alpha, x_N^\omega) \mid \forall k \in \mathbb{N}\,[\beta_i^k \neq \alpha]\}$ $[= \{i \in I(\alpha, x_N^\omega) \mid \forall k \in \mathbb{N}\,[\alpha \notin \mathrm{B}_i(x_i^k)]\}]$.

Now we are ready to analyze and compare the right-hand side of (5) for x_N^k,

$$\sum_{\alpha \in \mathrm{A}(x_N^k)} \varphi_\alpha(I(\alpha, x_N^k), x^k_{I(\alpha, x_N^k)}) + \sum_{\alpha \in \mathrm{A}^+(x_N^k)} \sum_{m=n^-(\alpha)}^{\#I(\alpha, x_N^k)-1} \psi_\alpha(m), \tag{9a}$$

and for x_N^ω,

$$\sum_{\alpha \in \mathrm{A}(x_N^\omega)} \varphi_\alpha(I(\alpha, x_N^\omega), x_{I(\alpha, x_N^\omega)}) + \sum_{\alpha \in \mathrm{A}^+(x_N^\omega)} \sum_{m=n^-(\alpha)}^{\#I(\alpha, x_N^\omega)-1} \psi_\alpha(m). \tag{9b}$$

If $\tilde{I}(\alpha, x_N^\omega) = \emptyset$ and hence $I(\alpha, x_N^\omega) = I(\alpha, x_N^k) =: I$ for each k, then this α contributes

$$\varphi_\alpha(I, x_I^k) + \Big[\sum_{m=n^-(\alpha)}^{\#I-1} \psi_\alpha(m)\Big]$$

to (9a) [the term in square brackets disappears if $\#I = n^-(\alpha)$] and

$$\varphi_\alpha(I, x_I^\omega) + \Big[\sum_{m=n^-(\alpha)}^{\#I-1} \psi_\alpha(m)\Big]$$

to (9b); since $\varphi_\alpha(I, \cdot)$ is upper semicontinuous by Assumption 1, there is no problem with this α.

If $\tilde{I}(\alpha, x_N^\omega) \neq \emptyset$, the analysis is more complicated. For brevity, we denote $I := I(\alpha, x_N^\omega)$, $\bar{I} := \bar{I}(\alpha, x_N^\omega)$, $\tilde{I} := \tilde{I}(\alpha, x_N^\omega)$, and $I(i) := I(\beta_i^k, x_N^k)$ for each $i \in \tilde{I}$; as was noted above, $I(i)$ does not depend on k. Now the contribution of this α to (9a) is

$$\varphi_\alpha(\bar{I}, x_{\bar{I}}^k) + \Big[\sum_{m=n^-(\alpha)}^{\#\bar{I}-1} \psi_\alpha(m)\Big] + \sum_{i \in \tilde{I}} \frac{1}{\#I(i)} \Big(\varphi_{\beta_i^k}(I(i), x^k_{I(i)}) + \Big[\sum_{m=1}^{\#I(i)-1} \psi_{\beta_i^k}(m)\Big]\Big). \tag{10a}$$

[The terms in square brackets disappear if, respectively, $\#\bar{I} = n^-(\alpha)$ or $\#I(i) = 1$; we divide the rightmost sum in (10a) by $\#I(i)$ to compensate for multiple counting of the same terms.]

The contribution of the same α to (9b) is

$$\varphi_\alpha(I, x_I^\omega) + \Big[\sum_{m=n^-(\alpha)}^{\#\bar{I}-1} \psi_\alpha(m) + \sum_{\#\bar{I}}^{\#I-1} \psi_\alpha(m)\Big]. \tag{10b}$$

Taking into account Assumption 4 and the fact that $\tilde{I}(\alpha, x_N^\omega) \subseteq I(\alpha, x_N^\omega)$, we see that the upper limit of (10a) cannot be greater than (10b).

The upper semicontinuity of P is proven, and so is the theorem.

7 Le Breton–Weber construction

First, we reproduce the construction in somewhat streamlined notations. All strategy sets X_i are compact subsets of a Euclidean space $\mathbb{R}^T$; $X = \bigcup_{i\in N} X_i$. Given a strategy profile $x_N \in X_N$ and $x \in X$, we denote $n(x, x_N)$ the number of players with $x_i = x$. The payoff $U_i(x_N)$ of player i is the sum of three terms ("taste component," "local social interaction component," and "global social interaction component"):

$$U_i(x_N) = V_i(x_i) + \sum_{j\in N\setminus\{i\}} W_i^j(x_i, x_j) + H(x_i, n(x_i, x_N)). \tag{11}$$

Three substantial assumptions are made: (1) each function V_i, W_i^j, and $H(\cdot, m)$ ($m \in \mathbb{N}$) is upper semi-continuous; (2) $W_i^j(x_i, x_j) = W_j^i(x_j, x_i)$ for every $i, j \in N$, every $x_i \in X_i$ and every $x_j \in X_j$; (3) $H(x, \cdot)$ is increasing for all $x \in X$. Under those assumptions, Le Breton and Weber (2011) showed that the following function is an upper semi-continuous exact potential:

$$P(x_N) = \sum_{i\in N} V_i(x_i) + 1/2 \sum_{i\in N} \sum_{j\in N\setminus\{i\}} W_i^j(x_i, x_j) + \sum_{x\in X:\, n(x,x_N)>0} \sum_{m=1}^{n(x,x_N)} H(x, m). \tag{12}$$

Given a Le Breton–Weber game Γ, we denote $\mathcal{N}_2$ the set of all unordered pairs in N, i.e., subsets of cardinality 2. Then we define a CLU game Γ^* by $\mathrm{A}^* := N \cup \mathcal{N}_2 \cup X$; $\mathrm{B}_i^*(x_i) := \{i\} \cup \{\{i, j\}\}_{j\in N\setminus\{i\}} \cup \{x_i\}$; $\varphi_i^*(x_i) := V_i(x_i)$; $\varphi_{\{i,j\}}^*(x_i, x_j) := W_i^j(x_i, x_j)$; $\psi_x^*(m) := H(x, m)$.

Proposition 2. *For every Le Breton–Weber game Γ, the CLU game Γ^* just defined is trim and isomorphic to Γ. Assumptions 1–4 are satisfied for Γ^*. Moreover, the exact potential (5) for Γ^* coincides with potential (12) for Γ.*

A straightforward proof is omitted.

Remark. Actually, Assumptions 1–4 were developed as a generalization of the assumptions of Le Breton and Weber (2011).

8 Player-specific local utilities

Congestion games with player-specific local utilities are a natural generalization of Rosenthal's (1973) model. Typically, one cannot expect the existence of an equilibrium, to say nothing of an exact potential, in such games. Nonetheless, there are results on the existence of a Nash equilibrium (Milchtaich, 1996) or even a strong Nash one (Konishi, et al., 1997) in some particular cases. Sometimes, even an exact potential exists; the most advanced results to this effect are due to Harks et al. (2011). In this section, we show that our construction generates all potential games discovered in that paper and some more.

Harks et al. (2011) started with "*weighted congestion games*," where the strategy sets are $X_i \subseteq \mathcal{B}$, as in proper congestion games, B_i's are identity mappings, each player is characterized by a "demand" $d_i \in \mathbb{R}$, and the (player-specific) local utility function at each facility $\alpha \in \mathrm{A}$ is $d_i \cdot p_\alpha(\sum_{j \in I(\alpha,x_N)} d_j)$; in other words, each player's load is multiplied by the unit cost depending on the total load on the facility. Note that local utilities become common if each player's utility is divided by d_i; however, the trimness condition need not hold.

Then (on pp. 66–69), two extensions of the model are considered, where local utilities are not common in an essential way. In the first, "*weighted congestion games with facility-dependent demands*," the demand of each player is additionally parameterized by the facility, so the local utility function of player i at facility $\alpha \in \mathrm{A}$ is $d_i^\alpha \cdot p_\alpha(\sum_{j \in I(\alpha,x_N)} d_j^\alpha)$. In the second, "*weighted congestion games with elastic demands*," the demand of each player is uniform over facilities, but may be chosen by the player from a feasible set, $d_i \in D_i$, so the local utility function of player i at facility $\alpha \in \mathrm{A}$ is again $d_i \cdot p_\alpha(\sum_{j \in I(\alpha,x_N)} d_j)$, but the strategy sets are $X_i \times D_i$. Every weighted congestion game obviously belongs to both extended classes.

Generally, a game from either class need not admit an exact potential; however, it does so in the case of affine local unit cost functions, $p_\alpha(d) = b_\alpha + a_\alpha \cdot d$. In the following, we assume that affine local cost functions are included in all the three above definitions, although Harks et al. (2011) did not do that.

We consider an even more general model, which simultaneously includes both extensions as particular cases. There is a finite set N of *players* and an arbitrary set A of *facilities*; we denote X the set of $x = \langle x^\alpha \rangle_{\alpha \in \mathrm{A}} \in \mathbb{R}^{\mathrm{A}}$ such that $\mathrm{B}(x) := \{\alpha \in \mathrm{A} \mid x^\alpha \neq 0\}$ is finite. For each player $i \in N$, there is a *strategy set* $X_i \subseteq X$ and a function $F_i \colon X_i \to \mathbb{R}$; for every $\alpha \in \mathrm{A}$, there are constants $a_\alpha, b_\alpha \in \mathbb{R}$. Given a strategy profile $x_N \in X_N$, the *local utility* obtained by player i from a facility α is

$$\varphi_i^\alpha(x_N^\alpha) := x_i^\alpha \cdot \big(b_\alpha + a_\alpha \cdot \sum_{j \in N} x_j^\alpha\big). \tag{13}$$

The *total utility function* is

$$u_i(x_N) := F_i(x_i) + \sum_{\alpha \in \mathrm{B}(x_i)} \varphi_i^\alpha(x_N^\alpha). \tag{14}$$

We may say that the players belonging to the set $I(\alpha, x_N) := \{i \in N \mid x_i^\alpha \neq 0\,[\equiv \alpha \in \mathrm{B}(x_i)]\}$ have chosen facility α. Then we notice that $\varphi_i^\alpha(x_N^\alpha) = 0$ whenever $i \notin I(\alpha, x_N)$; similarly, $\varphi_i^\alpha(x_N^\alpha)$ only depends on $x^\alpha_{I(\alpha,x_N)}$, so we could write $\varphi_i^\alpha(x^\alpha_{I(\alpha,x_N)})$ in the left hand side of (13) and the right hand side of (14). Now we see that (14) can be viewed as a generalization of (2) where different players may extract different local utilities from the same facility. By an analogy with Harks et al. (2011), we may call such models *generalized weighted congestion games with controllable demands* (*and affine local unit cost functions*), or, for brevity, *games with controllable demands* (*CD games*).

Now we introduce three special subsets of X: $Y^{\mathrm{w}}(d) := \{x \in X \mid \forall \alpha \in \mathrm{A}\,[x^\alpha \in \{0, d\}]\}$ ($d \in \mathbb{R}$), $Y^{\mathrm{fd}}(d^{\mathrm{A}}) := \{x \in X \mid \forall \alpha \in \mathrm{A}\,[x^\alpha \in \{0, d^\alpha\}]\}$ ($d^{\mathrm{A}} \in \mathbb{R}^{\mathrm{A}}$), and $Y^{\mathrm{e}} := \bigcup_{d \in \mathbb{R}} Y^{\mathrm{w}}(d)\, [= \{x \in X \mid \forall \alpha, \beta \in \mathrm{A}\,[(x^\alpha - x^\beta)x^\alpha x^\beta = 0]\}\,]$.

Proposition 3. *A CD game is a weighted congestion game if and only if* A *is finite,* $F_i(x_i) = 0$ *for all* $i \in N$ *and* $x_i \in X_i$*, and there is* $d_N \in \mathbb{R}^N$ *such that* $X_i \subseteq Y^{\mathrm{w}}(d_i)$ *for each* $i \in N$*. A CD game is a weighted congestion game with facility-dependent demands if and only if* A *is finite,* $F_i(x_i) = 0$ *for all* $i \in N$ *and* $x_i \in X_i$*, and there is* $d_N^{\mathrm{A}} \in \mathbb{R}^{N \times \mathrm{A}}$ *such that* $X_i \subseteq Y^{\mathrm{fd}}(d_i^{\mathrm{A}})$ *for each* $i \in N$*. A CD game is a weighted congestion game with elastic demands if and only if* A *is finite,* $F_i(x_i) = 0$ *for all* $i \in N$ *and* $x_i \in X_i$*, and* $X_i \subseteq Y^{\mathrm{e}}$ *for each* $i \in N$*.*

A straightforward proof is omitted.

As an example of a CD game not covered by Harks et al. (2011), assume that A is the set of edges of a network. A strategy of each player is a flow through the network with given source and destination nodes, satisfying Kirchhoff's law at each intermediate node. The cost of pushing a unit of flow through an edge α is affine in the total load: $-b_\alpha - a_\alpha \sum_{i \in N} x_i^\alpha$. The function F_i is the gain obtained from the total flow. Under this interpretation, (14) is an adequate description of the payoff to player i. The natural assumptions are $a_\alpha, b_\alpha \le 0$ and $x_i^\alpha \ge 0$ for all i and α. Upper restrictions on x_i^α can be added; moreover, there may be arbitrary restrictions on x_i^α as well, e.g., they may be all integer.

Given a CD game Γ, we define, in a simple and natural way, a trim CLU game Γ^* which is isomorphic to Γ. There are the same players with the same strategy sets, $N^* := N$ and $X_i^* := X_i$ for each $i \in N$. The set of facilities is modified: $\mathrm{A}^* := N \cup (\mathrm{A} \times N) \cup (\mathrm{A} \times \mathcal{N}_2)$, where, as in Section 7, $\mathcal{N}_2$ is the set of all unordered pairs in N, i.e., subsets of cardinality 2. Given $i \in N$ and $x_i \in X_i$, we define $\mathrm{B}_i^*(x_i) := \{i\} \cup \{\{(\alpha, i)\} \cup \{(\alpha, \{i, j\})\}_{j \in N \setminus \{i\}}\}_{\alpha \in \mathrm{B}(x_i)}$; thus, $I_i^+ = I_{(\alpha,i)}^+ = \{i\}$ and $I_{(\alpha,\{i,j\})}^+ = \{i, j\}$ for all $\alpha \in \mathrm{A}$ and $i, j \in N$, $i \ne j$. The local utilities are defined in this way: $\varphi_i^*(\{i\}, x_i) := F_i(x_i)$; $\varphi_{(\alpha,i)}^*(\{i\}, x_i^\alpha) := x_i^\alpha \cdot (b_\alpha + a_\alpha \cdot x_i^\alpha)$; $\varphi_{(\alpha,\{i,j\})}^*(I, x_I^\alpha) := a_\alpha \cdot x_i^\alpha \cdot x_j^\alpha$ if $I = \{i, j\}$, while $\varphi_{(\alpha,\{i,j\})}^*(I, x_I^\alpha) := 0$ if $I \ne \{i, j\}$.

Proposition 4. *For every CD game* Γ*, the CLU game* Γ^* *just defined is trim and isomorphic to* Γ*.*

A straightforward proof is omitted.

Remark. If A is finite, we could set $\mathrm{B}_i^*(x_i) := \{i\} \cup \{\{(\alpha, i)\} \cup \{(\alpha, \{i, j\})\}_{j \in N \setminus \{i\}}\}_{\alpha \in \mathrm{A}}$ for all $i \in N$ and $x_i \in X_i$, in which case Γ^* would be a game with structured utilities. For an infinite A, such a representation is inadmissible without a revision of our basic definitions.

If all strategy sets X_i are finite, then we have the existence of a Nash equilibrium as well. If A is finite, then it will be enough to assume that each X_i is compact (the utility functions are continuous anyway). Otherwise, we may assume that A is a metric space; then each X_i is also a metric space and Assumption 1 holds. We have to impose Assumptions 2 and 3 as they are; as to Assumption 4, it holds if $D_i^\alpha \subset \mathbb{R}_+$ for all $i \in N$ and $\alpha \in \mathrm{A}$, while $b_\alpha, a_\alpha > 0$ for all $\alpha \in \mathrm{A}$. In other words, it is natural in this case to restrict attention to *positive* externalities (Le Breton and Weber, 2011, have already come to the same conclusion in their situation).

Harks et al. (2011) also showed the necessity of affine local unit costs for the guaranteed existence of a potential; it does not seem possible to derive the fact from our Theorem 2. On the other hand, if we drop the idea that the unit costs should be the same for all users of a given facility, then polylinear combinations with symmetric coefficients would be acceptable as well. For instance, consider local utility functions of the form

$$\varphi_i^\alpha(x_N^\alpha) := x_i^\alpha \cdot \Big[c_\alpha^i + \sum_{j \in N} b_\alpha^{ij} \cdot x_j^\alpha \ + \sum_{j,k \in N \setminus \{i\},\, j \neq k} a_\alpha^{ijk} \cdot x_j^\alpha \cdot x_k^\alpha \Big],$$

where coefficients b_α^{ij} and a_α^{ijk} are invariant w.r.t. permutations of the indices from N for every $\alpha \in \mathrm{A}$. Virtually the same argument as above shows that such a game is isomorphic to a trim CLU game and hence admits an exact potential. One may doubt that such cost functions could adequately describe any real-world interrelationships; but an interesting point is that they also emerge in the study of Cournot tâtonnement in aggregative games with monotone best responses (Kukushkin, 2005).

9 Conclusion

Let us summarize our main findings. It is, in principle, possible to allow the players in a congestion game to choose some additional parameters beside the facilities they use (e.g., type of vehicle, load, etc.) without destroying the presence of an exact potential. The "only" restriction is that those additional parameters should not affect the local utility unless *all* players able to use the facility actually show up. Games with structured utilities fit here since each player uses the same list of facilities under every strategy.

This generalization allowed us to include the classes of potential games considered by Le Breton and Weber (2011) and Harks et al. (2011) into the same general scheme; moreover, a wider class of congestion-style games with player-specific local utilities also fits in. It seems quite possible that still other examples could be found as well, but, so far, I have been unable to produce anything specific.

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