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AUTOMATA THEORY

FAST EVALUATION OF THE HURWITZ ZETA FUNCTION AND DIRICHLET L-SERIES[1]

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Based on the FEE method, an algorithm of fast evaluation of the Hurwitz zeta function $\zeta(s,a)$ for integer s and algebraic a is proposed. Fast evaluation of Dirichlet L-series is considered. The evaluation complexity is close to the best possible.

1. Introduction

In [1–7], algorithms for fast evaluation of elementary and higher transcendental functions as well as the classical constants e, π, and the Euler constant γ are presented. They are based on the fast evaluation method for functions of the type of the Siegel E-function (FEE method) with complexity close to the best possible.

Throughout what follows, we assume that the numbers are written in the binary notation.

By the multiplication complexity of two n-digit integer numbers, we mean the number $M(n)$ of elementary (bit) operations required to compute their product.

The number of bit operations required to evaluate a function $y = f(z)$ accurate to 2^{-n} at a point $z = z_0$ of its domain of definition is denoted by $s_f(n)$ and called the evaluation complexity of $f(z)$ at $z = z_0$.

In [1–7], it is proved that the evaluation complexity of the elementary transcendental functions and constants mentioned above using the FEE method as well as the evaluation complexity of higher transcendental functions for algebraic values of arguments is

$$s_f(n) = O(M(n)\log^2 n).$$

The history of fast evaluation dates from A. N. Kolmogorov [8], who posed the problem of bounding $M(n)$ above. In [9–11], algorithms of fast multiplication are presented; details of practical implementation of these algorithms are described in [12]. The first algorithms of fast evaluation of elementary algebraic, elementary transcendental, and some higher transcendental functions are presented in [13–15].

2. Lemma on the representation of the Hurwitz zeta function as a series

In [6], application of the FEE method to the fast evaluation of the Riemann zeta function $\zeta(s)$ for integer values of the argument s is described in detail. For fractional values of s, methods of fast evaluation of $\zeta(s)$ have not yet been found. However, fast evaluation of the Hurwitz zeta function

$$\zeta(s,a) = \sum_{\nu=0}^{\infty} \frac{1}{(\nu+a)^s}, \quad a > 0, \tag{1}$$

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for integer values of s, $s \geq 2$, and algebraic values of a has been proved to be possible. Note that by (1), evaluation of $\zeta(s,a)$ for $a \geq 1$, $a = [a] + \{a\}$, $0 \leq \{a\} < 1$, can easily be reduced to the evaluation of $\zeta(s,a)$ for $0 < a < 1$ (the case of integer values of a is studied in [6]).

Considering evaluation of $\zeta(s,a)$ accurate to 2^{-n}, we assume in what follows that $n \to +\infty$ and s and a are arbitrary fixed numbers, $s \geq 2$, s is even, a is a real algebraic number, $0 < a < 1$.

As in [6], let us first prove two auxiliary lemmas. Their proofs are similar to those of the corresponding lemmas from [6].

Lemma 1. *Let $n_1, n_2, \ldots, n_s$ be nonnegative integers and*

$$P_i = \sum_{\substack{n_1+n_2+\ldots+n_s=i \\ n_1+2n_2+\ldots+sn_s=s}} \frac{s!}{n_1!n_2!\ldots n_s!} \prod_{j=1}^{s} \left(\frac{J_j}{j!}\right)^{n_j}, \tag{2}$$

$$J_j = \int_0^\infty e^{-t} t^{a-1} \log^j t \, dt. \tag{3}$$

Then we have the identity

$$\zeta(s,a) = \frac{(-1)^{s-1}}{(s-1)!} \sum_{i=1}^{s} \frac{(-1)^{i-1}(i-1)! P_i}{(\Gamma(a))^i} - \frac{1}{a^s}. \tag{4}$$

PROOF. By the definition of the Euler gamma function $\Gamma(x)$ (see, e.g., [16, p. 51]), we have

$$\Gamma(x) = \frac{1}{x} e^{-\gamma x} \left(\prod_{\nu=1}^{\infty} \left(1 + \frac{x}{\nu}\right)^{-1} e^{\frac{x}{\nu}} \right), \tag{5}$$

where γ is the Euler constant. From (5), we find

$$-\frac{\Gamma'(x)}{\Gamma(x)} = \gamma + \frac{1}{x} + \sum_{\nu=1}^{\infty} \left(\frac{1}{x+\nu} - \frac{1}{\nu} \right). \tag{6}$$

Taking the $(s-1)$st derivative of (6) with respect to x and then substituting $x = a$, we obtain

$$\frac{d^{s-1}}{dx^{s-1}} \left(-\frac{\Gamma'(x)}{\Gamma(x)} \right) \Bigg|_{x=a} = (-1)^{s-1}(s-1)! \left(\sum_{\nu=1}^{\infty} \frac{1}{(\nu+a)^s} + \frac{1}{a^s} \right). \tag{7}$$

Using the formula for the derivative of a composite function (see, e.g., [17, pp. 116–117]), we find

$$\begin{aligned} &\frac{d^{s-1}}{dx^{s-1}} \left(-\frac{\Gamma'(x)}{\Gamma(x)} \right) \Bigg|_{x=a} \\ &= \sum_{i=1}^{s} \frac{(-1)^{i-1}(i-1)!}{(\Gamma(a))^i} \sum_{\substack{n_1+n_2+\ldots+n_s=i \\ n_1+2n_2+\ldots+sn_s=s \\ n_1,n_2,\ldots,n_s \geq 0 \text{ are integer}}} \frac{s!}{n_1!n_2!\ldots n_s!} \prod_{j=1}^{s} \left(\frac{1}{j!} \frac{d^j}{dx^j} \Gamma(x) \Big|_{x=a} \right)^{n_j}. \end{aligned} \tag{8}$$

Now, let us use the integral representation of $\Gamma(x)$ (see, e.g., [16, p. 53]):

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt, \quad \mathrm{Re}(x-1) > -1. \tag{9}$$

Taking the jth derivative with respect to x under the integral sign in (9) and setting then $x = a$, we obtain

$$\frac{d^j}{dx^j} \Gamma(x) \Bigg|_{x=a} = \int_0^\infty e^{-t} t^{a-1} \log^j t \, dt. \tag{10}$$

To upper estimate the sum A_j'' defined by (15), represent it as a sum of two summands

$$A_j'' = \sum_{k=r+1}^{\infty} \frac{(-1)^k}{k!} \int_0^1 t^{k+a-1} \log^j t \, dt + \sum_{k=r+1}^{\infty} \frac{(-1)^k}{k!} \int_1^n t^{k+a-1} \log^j t \, dt. \tag{18}$$

Each of the two summands in (18) is a series with alternating signs whose terms are monotone decreasing in absolute value and tend to zero. Taking into account that $0 < a < 1$, we can upper estimate these terms and thus obtain from (18) the following estimate for A_j'':

$$A_j'' \le \frac{1}{(r+1)!} \left(\left| \int_0^1 t^{r+a} \log^j t \, dt \right| + \left| \int_1^n t^{r+a} \log^j t \, dt \right| \right) \le \frac{1}{(r+2)!}(1 + n^{r+2} \log^s n). \tag{19}$$

It follows from (12) and (13) that J_j can be represented as

$$J_j = A_j' + \theta_j, \tag{20}$$

where the sum A_j' is defined by (14), and for θ_j, $1 \le j \le s$, $n \ge 2s \log 2s$, $r \ge n$, we have from (16), (17), and (19) the estimate

$$|\theta_j| \le \frac{5}{3} e^{-n} \log^s n + \frac{1}{(r+2)!}(1 + n^{r+2} \log^s n). \tag{21}$$

Taking into account that $\frac{1}{r!} \le \left(\frac{e}{r}\right)^r$ and setting $r \ge 3n$, we obtain from (21) for θ_j that

$$|\theta_j| \le 2e^{-n} \log^s n. \tag{22}$$

For s satisfying (11), we obtain from (22) the estimate

$$|\theta_j| \le 2^{-n-1}. \tag{23}$$

Consider the sum A_j' defined by (14). Let us replace the integrals in (14) by their values:

$$\int_0^n t^{k+a-1} \log^j t \, dt = \frac{n^{k+a}}{k+a} \sum_{m=0}^{j} (-1)^m \frac{j! \log^{j-m} n}{(j-m)!(k+a)^m}.$$

It follows from (20) and (23) that in order to evaluate the integral

$$J_j = \int_0^\infty e^{-t} t^{a-1} \log^j t \, dt, \quad 0 < a < 1,$$

accurate to 2^{-n}, it suffices to evaluate with the same accuracy the sum

$$A_j = \sum_{k=0}^{r} \frac{(-1)^k}{k!} \frac{n^{k+a}}{k+a} \sum_{m=0}^{j} (-1)^m \frac{j! \log^{j-m} n}{(j-m)!(k+a)^m} \tag{24}$$

for

$$r \ge 3n, \quad n \ge 2s \log 2s, \quad s \ge 2, \quad 1 \le j \le s. \tag{25}$$

Let us rewrite (24) in the form

$$A_j' = n^a \sum_{m=0}^{j} (-1)^m \frac{j! \log^{j-m} n}{(j-m)!} S_m, \tag{26}$$

From (1)–(3) and (7)–(10), the validity of (4) follows.

3. Lemma on the fast evaluation of an integral of a special kind

Let $s_J(n)$ be the evaluation complexity of the integral J_j defined by (3) for a natural parameter j, $j = 1, 2, \ldots, s$. Then the following lemma holds.

Lemma 2.

$$s_J(n) = O(M(n)\log^2 n).$$

PROOF. We assume that

$$n \geq 2s\log 2s, \quad s \geq 2. \tag{11}$$

Represent the integral (3) as a sum of two integrals

$$J_j = A_j + B_j, \tag{12}$$

where

$$A_j = \int_0^n e^{-t}t^{a-1}\log^{j-1}t\,dt,$$

$$B_j = \int_n^\infty e^{-t}t^{a-1}\log^{j-1}t\,dt.$$

Taking a Taylor series expansion of e^{-t} in powers of t, $0 \leq t \leq n$, let us represent A_j as a sum

$$A_j = A'_j + A''_j, \tag{13}$$

where

$$A'_j = \sum_{k=0}^{r}\frac{(-1)^k}{k!}\int_0^n t^{k+a-1}\log^j t\,dt, \tag{14}$$

$$A''_j = \sum_{k=r+1}^{\infty}\frac{(-1)^k}{k!}\int_0^n t^{k+a-1}\log^j t\,dt, \tag{15}$$

$r \geq n$, r is a natural number.

Let us upper estimate B_j and A''_j. Since $0 < a < 1$, we have

$$B_j < B'_j, \quad B'_j = \int_n^\infty e^{-t}\log^j t\,dt. \tag{16}$$

Taking account of (11), let us upper estimate the integral B'_j, $1 \leq j \leq s$, integrating it by parts and successively passing to the estimates

$$B'_j = -e^{-t}\log^j t\Big|_n^\infty + j\int_n^\infty e^{-t}t^{-1}\log^{j-1}t\,dt \leq \ldots$$

$$\ldots \leq e^{-n}\log^j n\frac{1-\left(\dfrac{j}{n\log n}\right)^j}{1-\dfrac{j}{n\log n}} \leq \frac{5}{3}e^{-n}\log^s n. \tag{17}$$

where

$$S_m = \sum_{k=0}^{r} \frac{(-1)^k}{k!} \frac{n^k}{(k+a)^{m+1}}. \tag{27}$$

Let us evaluate S_m using the FEE process.

4. Continuation of the proof of Lemma 2 for the case of rational a

Assume first that a is a rational number, $a = \frac{a_1}{a_2}$, $(a_1, a_2) = 1$. Let us rewrite (27) in the form

$$S_m = a_2^{m+1} S'_m, \tag{28}$$

where

$$S'_m = \sum_{k=0}^{r} (-1)^k \frac{1}{k!} \frac{n^k}{(a_2 k + a_1)^{m+1}}, \tag{29}$$

and evaluate the sum S'_m. Take

$$r + 1 = 2^q \quad (q \geq 1, \quad 2^{q-1} < 3n < 2^q) \tag{30}$$

terms of the series (29). Let the numbers $S_{r+1-\nu}(0)$, $\nu = 0, 1, \ldots, r$, be defined by the equalities

$$S_{r+1-\nu}(0) = (-1)^{r-\nu} \frac{n^{r-\nu}}{(r-\nu)!(a_2(r-\nu)+a_1)^{m+1}}.$$

By the definition of S'_m, we have

$$S'_m = S_1(0) + S_2(0) + \ldots + S_{r+1}(0).$$

Let us evaluate S'_m in q steps of the FEE process described in [3–6] in detail. Namely, successively joining at each step the summands S'_m in pairs and taking the common multiplier out of the parentheses, we compute at each step only the values of the expressions inside the parentheses (these values are integer). The evaluation process for the sum S'_m defined by (29) is quite similar to that described in detail in [6] for the corresponding sum related to the series and integral concerned with the Riemann zeta function. Therefore, we do not elaborate on this, noting only that the evaluation complexity of S'_m is estimated as in [6] and amounts at the ith step to

$$O\left(\sum_{\tau=1}^{i} M(2^\tau \log r)\right) + O(M(2^i m \log(nr)))$$

operations. Summing up the number of operations over all steps i, $1 \leq i \leq q$, and adding to this the number

$$O(r \log r M(\log r) + M(r \log r) + M(2^q m \log r))$$

of operations required at the final step for dividing the obtained integer number by the integer

$$r!\left[(a_2 r + a_1)(a_2(r-1)+a_1)\ldots(a_2+a_1)a_1\right]^{m+1},$$

we find from (28)–(30) that the evaluation of S_m requires

$$O(M(n)\log^2 n) \tag{31}$$

operations.

5. Continuation of the proof of Lemma 2 for the case of algebraic a. Theorem on the fast evaluation of the Hurwitz zeta function

nsider now evaluation of the sum S_m defined by (27) for a real algebraic value of a. We assume that ven both in an explicit form and by a polynomial with integer coefficients such that a is its root such a polynomial, one can quickly find the value of its root using, for instance, Newton's method Note that the algorithm presented below can also be applied for fast evaluation of S_m for rational a s case, a is an algebraic number of degree 1).

a be a real algebraic number of degree $t \geq 1$ and $h(x)$ be a polynomial with integer coefficients such is its root, i.e.,

$$h(x) = h_t x^t + h_{t-1}x^{t-1} + \ldots + h_1 x + h_0, \quad h_t, h_{t-1}, \ldots, h_1, h_0 \text{ are integer}, \quad t \geq 1, \tag{32}$$

$$h(a) = 0. \tag{33}$$

that $0 < a < 1$. Let us evaluate the sum S_m taking into account that a satisfies Eq. (33). As in the hm presented above, take $r+1 = 2^q$ ($q \geq 1$, $2^{q-1} < 3n < 2^q$) terms of the series (27) and let the rs $S_{r+1-\nu}(0)$, $\nu = 0, 1, \ldots, r$, be defined by the equalities

$$S_{r+1-\nu}(0) = (-1)^{r-\nu} \frac{n^{r-\nu}}{(r-\nu)!(r-\nu+a)^{m+1}}.$$

definition of S_m, we have

$$S_m = S_1(0) + S_2(0) + \ldots + S_{r+1}(0). \tag{34}$$

luation of S_m is done in q steps as follows. Successively joining the summands S_m in (34) in pairs ing the common multiplier out of the parentheses, we have at the first step

$$S_m = S_1(1) + S_2(1) + \ldots + S_{r_1}(1), \quad r_1 = \frac{r+1}{2},$$

$$\text{1)} = S_{r-2\nu}(0) + S_{r-2\nu-1}(0) = \frac{(-1)^{r-2\nu} n^{r-2\nu}}{(r-2\nu)!(r-2\nu+a)^{m+1}} + \frac{(-1)^{r-2\nu-1} n^{r-2\nu-1}}{(r-2\nu-1)!(r-2\nu-1+a)^{m+1}}$$

$$= (-1)^{r-2\nu-1} \frac{n^{r-2\nu-1}\beta_{r_1-\nu}(1)}{(r-2\nu)!},$$

$$\text{1)} = -\frac{n}{(r-2\nu+a)^{m+1}} + \frac{r-2\nu}{(r-2\nu-1+a)^{m+1}}$$

$$= \frac{(r-2\nu)(r-2\nu+a)^{m+1} - n(r-2\nu-1+a)^{m+1}}{(r-2\nu-1+a)^{m+1}(r-2\nu+a)^{m+1}}. \tag{35}$$

t $(-1)^{r-2\nu-1} = 1$, hence, for r satisfying condition (30), the numerator of (35) is

$$(r-2\nu)(r-2\nu+a)^{m+1} - n(r-2\nu-1+a)^{m+1} > 0.$$

g the parentheses in (35), represent $\beta_{r_1-\nu}(1)$ as a fraction whose numerator and denominator are ials in a of degrees $m+1$ and $2m+2$ respectively:

$$\beta_{r_1-\nu}(1) = \frac{\alpha_{r_1-\nu}(1)}{\delta_{r_1-\nu}(1)}, \tag{36}$$

$$\alpha_{r_1-\nu}(1) = \sum_{k=0}^{m+1} A_{r_1-\nu}(k,1)a^k, \qquad \delta_{r_1-\nu}(1) = \sum_{\ell=0}^{2m+2} D_{r_1-\nu}(\ell,1)a^\ell, \tag{37}$$

$$A_{r_1-\nu}(k,1) = \binom{m+1}{k}\left((r-2\nu)^{m+2-k} - n(r-2\nu-1)^{m+1-k}\right), \tag{38}$$

$$D_{r_1-\nu}(\ell,1) = \sum_{\substack{\ell_1+\ell_2=\ell \\ 0\leq\ell_1,\ell_2\leq m+1}} \binom{m+1}{\ell_1}\binom{m+1}{\ell_2}(r-2\nu-1)^{m+1-\ell_1}(r-2\nu)^{m+1-\ell_2}. \tag{39}$$

Recall that if r satisfies (30), we have in (38)

$$(r-2\nu)^{m+2-k}-n(r-2\nu-1)^{m+1-k}>0.$$

At the first step, we compute, according to (38) and (39), the values

$$\begin{aligned}
&A_{r_1-\nu}(k,1), \qquad k=0,1,2,\ldots,m+1, \quad \nu=0,1,2,\ldots,r_1-1, \quad r_1=\frac{r+1}{2},\\
&D_{r_1-\nu}(\ell,1), \qquad \ell=0,1,2,\ldots,2m+2, \quad \nu=0,1,2,\ldots,r_1-1, \quad r_1=\frac{r+1}{2},
\end{aligned}$$

which are integer.

At the second step, we have

$$\begin{aligned}
S_m &= S_1(2)+S_2(2)+\ldots+S_{r_2}(2), \qquad r_2=2^{-1}r_1=2^{-2}(r+1),\\
S_{r_2-\nu}(2) &= S_{r_1-2\nu}(1)+S_{r_1-2\nu-1}(1) = n^{r-2^2\nu-3}\frac{\beta_{r_2-\nu}(2)}{(r-2^2\nu)!},\\
\beta_{r_2-\nu}(2) &= n^2\beta_{r_1-2\nu}(1)+(r-2^2\nu)(r-2^2\nu-1)\beta_{r_1-2\nu-1}(1).
\end{aligned}$$

It follows from (36) that $\beta_{r_2-\nu}(2)$ can be represented as a fraction:

$$\beta_{r_2-\nu}(2)=\frac{\alpha_{r_2-\nu}(2)}{\delta_{r_2-\nu}(2)},$$

where

$$\alpha_{r_2-\nu}(2) = n^2\alpha_{r_1-2\nu}(1)\delta_{r_1-2\nu-1}(1)+(r-2^2\nu)(r-2^2\nu-1)\alpha_{r_1-2\nu-1}(1)\delta_{r_1-2\nu}(1), \tag{40}$$

$$\delta_{r_2-\nu}(2) = \delta_{r_1-2\nu}(1)\delta_{r_1-2\nu-1}(1). \tag{41}$$

Taking (37)–(39) into account, let us multiply the polynomials in (40) and (41) and, removing the parentheses, represent $\alpha_{r_2-\nu}(2)$ and $\delta_{r_2-\nu}(2)$ as the following polynomials:

$$\begin{aligned}
\alpha_{r_2-\nu}(2) &= \sum_{k=0}^{3m+3} A_{r_2-\nu}(k,2)a^k,\\
\delta_{r_2-\nu}(2) &= \sum_{\ell=0}^{4m+4} D_{r_2-\nu}(\ell,2)a^\ell,
\end{aligned}$$

where

$$\begin{aligned}
A_{r_2-\nu}(k,2) &= \sum_{\substack{k_1+k_2=k\\ 0\le k_1\le m+1;\ 0\le k_2\le 2m+2}} \big[n^2A_{r_1-2\nu}(k_1,1)D_{r_1-2\nu-1}(k_2,1)\\
&\quad + (r-2^2\nu)(r-2^2\nu-1)A_{r_1-2\nu-1}(k_1,1)D_{r_1-2\nu}(k_2,1)\big], \qquad (42)\\
D_{r_2-\nu}(\ell,2) &= \sum_{\substack{\ell_1+\ell_2=\ell\\ 0\le \ell_1\le 2m+2;\ 0\le \ell_2\le 2m+2}} D_{r_1-2\nu}(\ell_1,1)D_{r_1-2\nu-1}(\ell_2,1). \qquad (43)
\end{aligned}$$

At the second step, we compute the integer values

$$\begin{aligned}
&A_{r_2-\nu}(k,2), \qquad k=0,1,2,\ldots,3m+3, \quad \nu=0,1,2,\ldots,r_2-1, \quad r_2=2^{-2}(r+1),\\
&D_{r_2-\nu}(\ell,2), \qquad \ell=0,1,2,\ldots,4m+4, \quad \nu=0,1,2,\ldots,r_2-1, \quad r_2=2^{-2}(r+1),
\end{aligned}$$

according to (42) and (43). And so forth.

At the ith step (we assume that $1 \le i \le q_0 \le q$, where q_0 is defined by the following two inequalities:

$$2^{q_0-1}(m+1) \le t \le 2^{q_0}(m+1), \tag{44}$$

and, hence, $q_0 \ge 1$ since $t \ge 1$), we have

$$\begin{aligned} S_m &= S_1(i) + S_2(i) + \ldots + S_{r_i}(i), \quad r_i = 2^{-1}r_{i-1} = 2^{-i}(r+1), \\ S_{r_i-\nu}(i) &= S_{r_{i-1}-2\nu}(i-1) + S_{r_{i-1}-2\nu-1}(i-1) = n^{r-2^i\nu-2^i+1}\frac{\beta_{r_i-\nu}(i)}{(r-2^i\nu)!}, \\ \beta_{r_i-\nu}(i) &= n^{2^{i-1}}\beta_{r_{i-1}-2\nu}(i-1) + \frac{(r-2^i\nu)!}{(r-2^i\nu-2^{i-1})!}\beta_{r_{i-1}-2\nu-1}(i-1), \end{aligned}$$

where

$$\beta_{r_i-\nu}(i) = \frac{\alpha_{r_i-\nu}(i)}{\delta_{r_i-\nu}(i)}, \tag{45}$$

$$\alpha_{r_i-\nu}(i) = \sum_{k=0}^{(2^i-1)(m+1)} A_{r_i-\nu}(k,i)a^k, \qquad \delta_{r_i-\nu}(i) = \sum_{\ell=0}^{2^i(m+1)} D_{r_i-\nu}(\ell,i)a^\ell, \tag{46}$$

$$\begin{aligned} A_{r_i-\nu}(k,i) &= \sum_{\substack{k_1+k_2=k \\ 0\le k_1\le(2^{i-1}-1)(m+1) \\ 0\le k_2\le 2^{i-1}(m+1)}} \Big[n^{2^{i-1}}A_{r_{i-1}-2\nu}(k_1,i-1)D_{r_{i-1}-2\nu-1}(k_2,i-1) \\ &\quad + \frac{(r-2^i\nu)!}{(r-2^i\nu-2^{i-1})!}A_{r_{i-1}-2\nu-1}(k_1,i-1)D_{r_{i-1}-2\nu}(k_2,i-1)\Big], \end{aligned} \tag{47}$$

$$D_{r_i-\nu}(\ell,i) = \sum_{\substack{\ell_1+\ell_2=\ell \\ 0\le\ell_1\le 2^{i-1}(m+1) \\ 0\le\ell\le 2^{i-1}(m+1)}} D_{r_{i-1}-2\nu}(\ell_1,i-1)D_{r_{i-1}-2\nu-1}(\ell_2,i-1). \tag{48}$$

At the ith step, we compute the integer values

$$\begin{aligned} &A_{r_i-\nu}(k,i), \quad k = 0,1,2,\ldots,(2^i-1)(m+1), \quad \nu = 0,1,2,\ldots,r_i-1, \quad r_i = 2^{-i}(r+1), \\ &D_{r_i-\nu}(\ell,i), \quad \ell = 0,1,2,\ldots,2^i(m+1), \quad \nu = 0,1,2,\ldots,r_i-1, \quad r_i = 2^{-i}(r+1), \end{aligned}$$

according to (47) and (48). After the q_0th step (q_0 is defined by inequalities (44), $1 \le q_0 \le q$), the expression (45) is a fraction whose numerator and denominator are polynomials in a of degrees $(2^{q_0}-1)(m+1)$ and $2^{q_0}(m+1)$ respectively. Before proceeding to the (q_0+1)th step, let us reduce these polynomials modulo the polynomial $h(x)$ with $x = a$. Consider these reductions in more detail.

Let

$$A(x) = \sum_{k=0}^{u} A_k x^k = A_u x^u + A_{u-1}x^{u-1} + \ldots + A_1 x + A_0, \tag{49}$$

$$D(x) = \sum_{\ell=0}^{w} D_\ell x^\ell = D_w x^w + D_{w-1}x^{w-1} + \ldots + D_1 x + D_0, \tag{50}$$

where

$$u = (2^{q_0}-1)(m+1), \qquad w = 2^{q_0}(m+1), \tag{51}$$

$$A_k = A_{r_{q_0}-\nu}(k,q_0), \qquad D_\ell = D_{r_{q_0}-\nu}(\ell,q_0), \tag{52}$$

and let $h(x) = \sum_{i=0}^{t} h_i x^i$ be the polynomial defined by (32) and (33). Recall that (47), (48), and (52) imply that the numbers A_k, $k = 0,1,2,\ldots,u$, and D_ℓ, $\ell = 0,1,2,\ldots,w$, are integer and the numbers

h_i, $i = 0, 1, 2, \ldots, t$, are integer by definition. By definition (33), we have $h(a) = 0$. It follows from (46), (47), and (49)–(52) that

$$A(a) = \alpha_{r_{q_0}-\nu}(q_0), \qquad D(a) = \delta_{r_{q_0}-\nu}(q_0). \tag{53}$$

Let us divide the polynomials $A(x)$ and $D(x)$ by $h(x)$ with remainders $R(x)$ and $Q(x)$ respectively:

$$A(x) = h(x)B(x) + R(x), \tag{54}$$
$$D(x) = h(x)G(x) + Q(x). \tag{55}$$

Thus,

$$\sum_{k=0}^{u} A_k x^k = \sum_{i=0}^{t} h_i x^i \sum_{j=0}^{u-t} B_j x^j + \sum_{k=0}^{t-1} R_k x^k, \tag{56}$$
$$\sum_{\ell=0}^{w} D_\ell x^\ell = \sum_{i=0}^{t} h_i x^i \sum_{j=0}^{w-t} G_j x^j + \sum_{\ell=0}^{t-1} Q_\ell x^\ell. \tag{57}$$

By (56), we have

$$A_k = \sum_{\substack{i+j=k \\ 0\le i\le t;\, 0\le j\le u-t}} h_i B_j, \quad k = u, u-1, u-2, \ldots, t+1, t. \tag{58}$$

Hence,

$$B_{u-t-j} = \frac{1}{h_t}\left(A_{u-j} - \sum_{i=0}^{j-1} B_{u-t-i} h_{t-j+i}\right), \quad j = 0, 1, 2, \ldots, u-t. \tag{59}$$

Similarly,

$$D_\ell = \sum_{\substack{i+j=\ell \\ 0\le i\le t;\, 0\le j\le w-t}} h_i G_j, \quad \ell = w, w-1, w-2, \ldots, t+1, t, \tag{60}$$
$$G_{w-t-j} = \frac{1}{h_t}\left(D_{w-j} - \sum_{i=0}^{j-1} G_{w-t-i} h_{t-j+i}\right), \quad j = 0, 1, 2, \ldots, w-t. \tag{61}$$

Then, for the coefficients R_k and Q_ℓ of the remainders, we obtain from (56) and (58) and, respectively, (57) and (60) that

$$R_k = A_k - \sum_{\substack{i+j=k \\ 0\le i\le t,\, 0\le j\le u-t}} h_i B_j, \quad k = t-1, t-2, \ldots, 1, 0, \tag{62}$$
$$Q_\ell = D_\ell - \sum_{\substack{i+j=\ell \\ 0\le i\le t,\, 0\le j\le w-t}} h_i G_j, \quad \ell = t-1, t-2, \ldots, 1, 0, \tag{63}$$

where the coefficients B_j and G_j are defined by (59) and (61) respectively.

It follows from (33) and (53)–(55) that

$$R(a) = A(a) = \alpha_{r_{q_0}-\nu}(q_0), \qquad Q(a) = D(a) = \delta_{r_{q_0}-\nu}(q_0), \tag{64}$$

where

$$R(x) = \sum_{k=0}^{t-1} R_k x^k,$$
$$Q(x) = \sum_{\ell=0}^{t-1} Q_\ell x^\ell,$$

and the coefficients R_k and Q_ℓ are defined by (62) and (59) and, respectively, (63) and (61) and are not integer in general. It follows from (59) and (61) that the numbers ${h_t}^{u-t+1}R_k$, $k = t-1, t-2, \ldots, 1, 0$, and ${h_t}^{w-t+1}Q_\ell$, $\ell = t-1, t-2, \ldots, 1, 0$, must be integer. Since

$$\beta_{r_{q_0}-\nu}(q_0) = \frac{\alpha_{r_{q_0}-\nu}(q_0)}{\delta_{r_{q_0}-\nu}(q_0)} = \frac{R(a)}{Q(a)} \tag{65}$$

due to (45) and (64), taking (51) into account, let us multiply the numerator and denominator of (65) by ${h_t}^{w-t+1}$. Then we obtain

$$\beta_{r_{q_0}-\nu}(q_0) = \frac{R_1(a)}{Q_1(a)}, \tag{66}$$

where $R_1(a) = \sum_{k=0}^{t-1} R'_k a^k$ and $Q_1(a) = \sum_{\ell=0}^{t-1} Q'_\ell a^\ell$ are polynomials in a of degree $t-1$ with integer coefficients

$$R'_k = {h_t}^{w-t+1}R_k, \qquad Q'_\ell = {h_t}^{w-t+1}Q_\ell, \qquad k = 0,1,2,\ldots,t-1, \quad \ell = 0,1,2,\ldots,t-1. \tag{67}$$

Hence, at the q_0th step (q_0 is defined by (44)), we compute integer numbers R'_k, $k = 0,1,2,\ldots,t-1$, according to (67), (62), (59), and (47) with $i = q_0$ and integer numbers Q'_ℓ, $\ell = 0,1,2,\ldots,t-1$, according to (67), (63), (61), and (48) with $i = q_0$. Then we have the fraction (66) for $\beta_{r_{q_0}-\nu}(q_0),: \nu = 0,1,2,\ldots,r_{q_0}-1$, $r_{q_0} = 2^{-q_0}(r+1)$, and at the (q_0+1)th step we compute the integer coefficients of the polynomials in the numerator and denominator of $\beta_{r_{q_0+1}-\nu}(q_0+1)$, $\nu = 0,1,2,\ldots,r_{q_0+1}-1$, $r_{q_0+1} = 2^{-q_0-1}(r+1)$, according to (47) and (48) with $i = q_0+1$. Then we reduce, as is described above, the numerator and denominator of $\beta_{r_{q_0+1}-\nu}(q_0+1)$ modulo $h(x)$ with $x = a$. Multiplying the numerator and denominator of the reduced fraction by a common factor, we obtain for $\beta_{r_{q_0+1}-\nu}(q_0+1)$ a fraction whose numerator and denominator are polynomials in a of degree $t-1$ with integer coefficients. And so forth. We make the reduction at each step $i = q_0, q_0+1, q_0+2, \ldots, q$. Since the coefficients and the degree of $h(x)$ are absolute constants, these reductions do not worsen the estimate for the complexity of the computations done. At the final qth step, after the reduction we have

$$S_m = S_{r_q}(q) = S_1(q) = \frac{\beta_{r_q}(q)}{r!}, \tag{68}$$

where

$$\beta_{r_q}(q) = \frac{\alpha_{r_q}(q)}{\delta_{r_q}(q)}, \tag{69}$$

$$\alpha_{r_q}(q) = \sum_{k=0}^{t-1} \widetilde{R}_k a^k, \qquad \delta_{r_q}(q) = \sum_{\ell=0}^{t-1} \widetilde{Q}_\ell a^\ell, \tag{70}$$

$\widetilde{R}_k$ and $\widetilde{Q}_\ell$ are integer.

To compute $\alpha_{r_q}(q)$ and $\delta_{r_q}(q)$ according to (70) using the coefficients $\widetilde{R}_k$ and $\widetilde{Q}_\ell$ already computed and taking into account that a is an algebraic number, $t = \text{const}$, and also to compute the fraction $\beta_{r_q}(q)$ according to (69),

$$O(M(n)) \tag{71}$$

operations are required. To compute (68), taking (30) into account,

$$O(M(\log n) n \log n) \tag{72}$$

operations are required. It follows from (71), (72), and (31) that the evaluation of S_m for a real algebraic a requires

$$O(M(n) \log^2 n)$$

operations.

Consider the sum (26). In (26), computation of the value of $\log^{j-m} n$, $0 \le m \le j$, $1 \le j \le s$, requires $O(M(\log n)\log^2 n)$ operations. Computation of $n^a = e^{a\log n}$, where a is an algebraic number, requires $O(M(n)\log^2 n)$ operations.

As in [6], to evaluate the sum

$$S = \sum_{m=0}^{j} (-1)^m \frac{j! \log^{j-m} n}{(j-m)!} S_m \tag{73}$$

accurate to 2^{-n}, it suffices to compute S_m and $\log^{j-m} n$ with accuracy to 2^{-3n}. Estimating the absolute value of S, we find from (73) and (26) that to compute the sum A'_j accurate to 2^{-n}, it suffices to compute S and n^a with accuracy to $2^{-(a+2)n}$, $0 < a < 1$.

The estimates given above imply that the evaluation of the integral $J_j = \int\limits_0^\infty e^{-t} t^{a-1} \log^j t \, dt$, where a is a real algebraic number, $0 < a < 1$, requires

$$O(M(n)\log^2 n)$$

operations. Thus, the following theorem holds.

Theorem 1. *For the evaluation complexity of the Hurwitz zeta function $\zeta = \zeta(s,a)$ for any natural $s = k$, $k \ge 2$, and any real algebraic a, we have the estimate*

$$s_\zeta(n) = O(M(n)\log^2 n).$$

6. Theorem on the evaluation complexity for Dirichlet series

A corollary of Theorem 1 is Theorem 2 on the evaluation complexity of the Dirichlet L-series $L(s,\chi)$ for any natural $s = k$, $k \ge 2$.

Let m be an integer number, $m \ge 2$, and $\chi(\ell)$ be any Dirichlet character modulo m. Then, for $\operatorname{Re} s > 1$, we have

$$L(s,\chi) = \sum_{k=1}^{\infty} \frac{\chi(k)}{k^s} = \sum_{\ell=1}^{m}\sum_{k=0}^{\infty} \frac{\chi(km+\ell)}{(km+\ell)^s} = \sum_{\ell=1}^{m} \frac{\chi(\ell)}{m^s} \sum_{k=0}^{\infty} \frac{1}{\left(k+\frac{\ell}{m}\right)^s} = \frac{1}{m^s}\sum_{\ell=1}^{m} \chi(l)\zeta\left(s,\frac{\ell}{m}\right). \tag{74}$$

Let us use the constructive definition of $\chi(l)$ (see, e.g., [18, p. 106]).

Let $m = 2^\alpha p_1^{\alpha_1} \dots p_r^{\alpha_r}$ be a canonical decomposition of m. Let the numbers c, c_0, and c_ν be such that

$$\begin{aligned} c = c_0 = 1 \qquad & \text{if} \quad \alpha = 0 \quad \text{or} \quad \alpha = 1, \\ c = 2,\ c_0 = 2^{\alpha-2} \qquad & \text{if} \quad \alpha \ge 2, \\ c_\nu = \varphi(p_\nu^{\alpha_\nu}) = p_\nu^{\alpha_\nu} - p_\nu^{\alpha_\nu - 1}, \quad & \nu = 1, 2, \dots, r. \end{aligned}$$

Let also g_ν be the smallest primitive root modulo $p_\nu^{\alpha_\nu}$, $\nu = 1, 2, \dots, r$. Let $\gamma, \gamma_0, \gamma_1, \dots, \gamma_r$ be the index system of a number ℓ modulo m, $1 \le \ell < m$, $\gcd(\ell, m) = 1$, i.e., for the numbers $\gamma, \gamma_0, \gamma_1, \dots, \gamma_r$ we have the congruences

$$\begin{aligned} \ell &= (-1)^\gamma 5^{\gamma_0} \pmod{2^\alpha}, \\ \ell &= g_1^{\gamma_1} \pmod{p_1^{\alpha_1}}, \\ &\dots\dots\dots\dots\dots\dots \\ \ell &= g_r^{\gamma_r} \pmod{p_r^{\alpha_r}}. \end{aligned} \tag{75}$$

Let the numbers $R, R_0, R_1, \dots, R_r$ be any roots of the equations

$$R^c = 1, \qquad R^{c_0} = 1, \qquad R^{c_1} = 1, \qquad \dots, \qquad R^{c_r} = 1. \tag{76}$$

Note that the roots of the equation $R^\lambda = 1$ are the numbers

$$e^{2\pi i \frac{\nu}{\lambda}}, \quad \nu = 0, 1, 2, \ldots, \lambda - 1. \tag{77}$$

Then

$$\chi(\ell) = \begin{cases} R^\gamma R_0^{\gamma_0} R_1^{\gamma_1} \ldots R_r^{\gamma_r} & \text{if } \gcd(\ell, m) = 1, \\ 0 & \text{if } \gcd(\ell, m) > 1. \end{cases} \tag{78}$$

The function $\chi(\ell)$ thus defined for each integer ℓ is called the Dirichlet character modulo m.

If one knows the numbers $\gamma, \gamma_0, \gamma_1, \ldots, \gamma_r$ from (75), one can evaluate any Dirichlet character $\chi(\ell)$, $\gcd(\ell, m) = 1$, according to (76)–(78) accurate to 2^{-n} in

$$O(M(n) \log^2 n) \tag{79}$$

operations (fast evaluation of $\exp(z)$ for any complex argument z is described in detail in [3]). It follows from (74) that the number of operations sufficient to compute $L(s, \chi)$ accurate to 2^{-n} is also given by (79).

Theorem 2. *For the evaluation complexity of the Dirichlet L-series $L(s, \chi)$ for any natural $s = k$, $k \geq 2$, and any Dirichlet character $\chi(\ell)$ modulo m, $m \geq 2$, m being an integer, we have the estimate*

$$s_L(n) = O(M(n) \log^2 n).$$

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