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Model of the households' adaptive behavior

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1 Introduction

Most of the dynamical models of the economic systems take into account the fact that the same amount of money is valued not equally at different moments of time. To describe such preferences, the concept of pure time preference is used. Let us denote the pure time preference rate by δ so today's consumption is $e^{\delta t}$ times more preferrable than the same consumption after time t.

In this work we will consider behavior of the households. At the moment of time t, given some idea about the future income and prices, the households are maximizing discounted utility

$$\int_t^{\infty} e^{-\delta(x-t)} u(c(x)) dx \tag{1.1}$$

where $c(x)$ is future consumption flow, $u(\cdot)$ is a concave utility function. At time t the actual consumption is given by the initial point of the optimal plan — the solution of (1.1), and future income and prices are continiously re-estimated in accordance with current economic situation by the continuous revision (sliding planning) procedure [2],[3]).

It is usual to treat pure time preference as an internal characteristic of the consumer. However, in this case it is natural to expect existence of the households with different pure time preferences. Unfortunately, the models considering interaction of the households with different time preference rates appear to be not very attractive because of difficulties with reaching equilibrium. In [4] an example is provided, when there is no equilibrium at all; in [1] the existing equilibrium is non-stationary. Moreover, for some values of the pure time preference rates the problem (1.1) is stated incorrectly and thus can not define behavior of the household.

All these results show that the values of pure time preference rates can not be set arbitrarily. Therefore, a mechanism to describe intertemporal changing of pure time preference rates is needed to provide their mutual coherence. This work is examining one of such mechanisms.

From this moment we consider pure time preference rate not as a constant internal parameter, but as an element of the household's strategy. The household's behavior now is following: during some interval of time consumer is collecting information about the results of his activity and comparing them with the others' results according to some criterion common for the whole society. After that he is changing the pure time preference rate in order to imitate the most succesful agent.

We consider the economy with three types of assets: a uniform consumption good, money, and securities (the general description of such economy see in [5]). Using the supply and demand functions obtained in [1], we are looking for the equilibrium states at each time t. It is shown in [1] that the dynamical equilibrium in the system with different pure time preference rates is non-stationary. We are considering the process resulting in convergence of the pure time preference rates to the unique value and establishement of the stationary equilibrium in which values of wealth and consumption by the households are growing with the same rate which equals the growth rate of the economy. During the adaptation process redistribution of wealth among consumers takes place. The main result of the work is that it is possible to define the final wealth distribution given the initial one.

2 The model

The key subject of our research is the household. The household can either consume or save. The key characteristic of the household is its wealth i. e. the sum of money and securities owned by the household. The securities are valued accordingly to the current market price. One more parameter characterising the state of the consumer is the pure time preference rate. The households are enumerated by indices $\xi, \eta \in \Xi$, where Ξ is a finite or countable set. Denote the wealth of the household ξ at the moment t as $w_\xi(t)$, and pure time preference rate as $\delta_\xi(t)$. The overall wealth $\sum_{\xi\in\Xi} w_\xi$ is assumed to be finite. The set of the pure time preference coefficiients is assumed to be bounded:

$$0 \leq \inf_{\xi\in\Xi} \delta_\xi < \sup_{\xi\in\Xi} \delta_\xi < \infty$$

The economy consists of the households, producers and the government. Producers are supplying the market with product, part of which is allocated for the investment and the other part is consumed by the government and the households. The producers are issuing the securities and sell them to the households according to the market value. The producers are spending their profits on paying dividends on securities. The government also issues the liabilities — the money.

In order to determine demand on the assets, the households are solving continuous revision (sliding planning) problem (1.1) at every moment of time. The precise formulation of the problem is provided in [1], and it is shown there, that given the price p and market rate $\hat{r}$ demand on the consumption good c_ξ, on the money m_ξ , and on the securities s_ξ, accordingly, is

$$\begin{aligned} pc_\xi &= m_\xi/\tau \\ m_\xi &= \mu_\xi w_\xi \\ s_\xi &= (1-\mu_\xi)w_\xi \end{aligned} \tag{2.1}$$

where τ is the velocity of the money circulation, p is the price index, nd

$$\mu_\xi = \frac{\hat{r} - \nu_\xi}{\hat{r} + \frac{1}{\tau}} \tag{2.2}$$

is the optimal share of money for household ξ. The term ν_ξ here denotes the rate of the wealth growth. Accordingly to [1], it is given by formula

$$\nu_\xi = i + \gamma + \frac{\hat{r} - i - \gamma - \delta_\xi}{\beta} \tag{2.3}$$

where $i = \dot{p}/p$ is inflation rate, γ is the growth rate of the economy, $\hat{r}$ is the effective payment for the unit of the security , it includes both dividends and security prices changing, β is the rate of relative risk aversion (6,7]).

As it is shown in [1], under the assumptions above there is an equilibrium state in the economy. It will be changing in time due to growth and wealth redistribution. The evolution of the system is given by the following equations:

1. The households' behavior: the continuous revision (sliding planning) procedure
$$\dot{w}_\xi = \nu_\xi w_\xi \tag{2.4}$$

2. The gross demand on consumption good, money, and securities
$$pC = \sum_\xi \mu_\xi w_\xi/\tau \tag{2.5}$$

$$M = \sum_\xi \mu_\xi w_\xi \tag{2.6}$$

$$S = \sum_\xi (1 - \mu_\xi) w_\xi \tag{2.7}$$

3. The supply of the consumption good, money, and securities

$$= Y(1 - \pi - b\gamma) \tag{2.8}$$

$$\dot{M} = \pi p Y. \tag{2.9}$$

$$\dot{S} - \frac{\dot{\sigma}}{\sigma} S = b\gamma p Y \tag{2.10}$$

Here π is share of government's consumption in GDP , b is the capital intencity of investment

, σ is the market price of securities.

4. The equations that are closing the system

$$\dot{Y} = \gamma Y \quad \dot{p} = ip \tag{2.11}$$

Initial conditions $w_\xi(0) = w_{\xi 0}$, $\delta_\xi(0) = \delta_{\xi 0}$, $Y(0) = Y_0$, $p(0) = p_0$, $\sigma(0) = \sigma_0$ and constant parameter π, b, γ, τ are known.

It is shown in [1] that in equilibrium

$$\hat{r} = i + \gamma + <\delta> + \frac{\beta<\delta>}{2(\beta - 1)} \left(\sqrt{1 + \frac{4(\beta - 1)D\delta}{\beta^2(<\delta>)^2}} - 1 \right) \tag{2.12}$$

The parenthes $<>$ here denote average for the distribution with dencity $w_\xi / \sum_\xi w_\xi$, and $D\delta = <\delta^2> - <\delta>^2$ is the variance of the pure time preference rate.

The behavior of the household is described in [1] by two individual parameters: relative risk aversion β and pure time preference δ. Both these parameters have analogous impact on supply-demand relationship (2.2). We assume the parameter β to be fixed and equal for all the households, while pure time preference rates are variable and different for all the consumers.

3 The adaptation

Suppose that at every moment of time it is possible to find a household $\xi^*(t)$ which is considered by all the households as the most succesful.The value of the pure time preference of this household $\delta^*(t)$ is a target for all the other households. They are changing their pure time preference rates according to equation:

$$\dot{\delta}_\xi = \frac{\delta^*(t) - \delta_\xi}{T} \tag{3.1}$$

Here T is the characteristic adaptation time.

For example, it is possible to consider the household which consumption is maximal as the most prosperous. For this case the simulations were prepared. The figure 1 shows the evolution of pure time preference rates in time. They are preserving their ordering and converge to some value $\bar{\delta}$. During the adaptation process, the wealth redistribution takes place. However, for $t \to \infty$ the wealth distribution approaches the final one $\bar{w}_\xi$.

The proof of the existence of the final wealth distribution and determining of it are the main results of this work.

Note that the final wealth distribution does *not* depend on principle in according to which the best value for pure time preference is defined. Moreover, no matter if there exists a household with the pure time preference rate $\delta^*(t)$. The only essential coditions are

- the value of $\delta^*(t)$ is defined;
- the value of $\delta^*(t)$ belongs to the set of observed pure time preference rates;

$$\delta^*(t) \in \left[\inf_\xi \delta_\xi(t), \sup_\xi \delta_\xi(t)\right]$$

- the characteristic adaptation times T for all households are equal.

We will first prove that the adaptation process leads to the unique value of the pure time preference rate, and the ordering of the rates does not change in time. In the next section we will find the final wealth distribution.

Lemma 1. The pure time preference of the household ξ is greater than that of the houshold η at time t

$$\delta_\xi(t) > \delta_\eta(t) \tag{3.2}$$

if and only if

$$\delta_\xi(0) > \delta_\eta(0) \tag{3.3}$$

Proof. Subtracting the adaptation equations (3.1) for the households ξ and η we obtain:

$$\dot{\delta}_\xi - \dot{\delta}_\eta = \frac{\delta_\eta - \delta_\xi}{T}$$

Then integrate this equation:

$$\delta_\eta(t) - \delta_\xi(t) = (\delta_\eta(0) - \delta_\xi(0))e^{-t/T} \tag{3.4}$$

Therefore, inequalities (3.2) and (3.3) are equivalent.

Lemma 2. As a result of the adaptation process, pure time preference rates are converging for time intervals approximately equal to T to the same value, so that for every ξ the value

$$\bar{\delta} = \lim_{t\to\infty} \delta_\xi(t) \ ,$$

exists which is independent on ξ.

Proof. Consider the set of closed intervals

$$\Delta_t = \left[\inf_\xi \delta_\xi(t), \sup_\xi \delta_\xi(t)\right]$$

It follows from (3.1) that the left edge of the closed interval Δ_t is shifting right, and the right edge of it is shifting left so the intervals are embedded:

$$\Delta_{t'} \in \Delta_t$$

for all $t' > t$. From (3.4) and Lemma 1 follows that Δ_t is decreasing exponentialy with relaxation time T. Using the theorem about contracting intervals, we find that all the intervals have unique common point $\bar{\delta}$. By definition, $\delta_\xi(t) \in \Delta_t$, therefore δ_ξ tends to $\bar{\delta}$, q. e. d.

4 The final wealth distribution

lemma 1 shows that δ_ξ induces Ξ the full order relation $\succ_\delta$, in the set of households which is invariant in time: $\xi \succ_\delta \eta$ if and only if $\delta_\xi > \delta_\eta$. However, one more function exists for which the ordering does not depend on time. It is defined as follows:

$$R_\xi(t) = w_\xi(t)e^{-\frac{\delta_\xi(t)T}{\beta}} \tag{4.1}$$

Lemma 3. The household ξ surpasses the household η in accordance with criteron R at time t $R_\xi(t) > R_\eta(t)$ if and only if the household ξ surpases the household η by R at time $t = 0$ $R_\xi(0) > R_\eta(0)$, and the ratio $R_{\xi\eta} = R_\xi(t)/R_\eta(t)$ is independent on time.

Proof. Integrating (2.4) we obtain:

$$w_\xi(t) = w_\xi(0)e^{\int_0^t \nu_\xi(t')dt'}$$

Substituting for $\nu_\xi(t')$ from (2.3) and $\delta_\xi(t') = \delta^*(t') - T\dot{\delta}_\xi(t')$ from (3.1), we have

$$w_\xi(t) = w_\xi(0)e^{\frac{\delta_\xi(t)-\delta_\xi(0)}{\beta}T}e^{\int_0^t \nu^*(t')dt'}$$

Therefore

$$\frac{R_\xi(t)}{R_\xi(0)} = e^{\int_0^t \nu^*(t')dt'} \tag{4.2}$$

is independent on ξ and $\frac{R_\xi(t)}{R_\xi(0)} = \frac{R_\eta(t)}{R_\eta(0)}$. So, $\frac{R_\xi(t)}{R_\eta(t)} = \frac{R_\xi(0)}{R_\eta(0)}$, q. e. d.

It is the value of R_ξ that determinwes the final wealth distribution

$$\bar{w}_\xi = \lim_{t\to\infty} \frac{w_\xi(t)}{\sum_\xi w_\xi(t)}$$

Theorem 1. The final distribution $\bar{w}_\xi(t)$ exists and

$$\frac{\bar{w}_\xi}{\bar{w}_\eta} = \frac{R_\xi(0)}{R_\eta(0)} \tag{4.3}$$

If $R_\xi(0) = R_\eta(0)$, than for t large enough

$$w_\xi(t) > w_\eta(t) \Leftrightarrow \delta_\xi(t) > \delta_\eta(t)$$

Proof. From (4.1), the wealth ratio for households ξ and η at time t equals

$$\frac{w_\xi(t)}{w_\eta(t)} = \frac{R_\xi(t)}{R_\eta(t)}\, e^{\frac{\delta_\xi(t)-\delta_\eta(t)}{\beta}T} \tag{4.4}$$

By Lemma 2 the difference $\delta_\xi - \delta_\eta$ tends to zero, and $e^{\frac{\delta_\xi(t)-\delta_\eta(t)}{\beta}T}$ tends to unity for $t \to \infty$. Therefore for all $\varepsilon > 0$ exist $t(\varepsilon)$ such that

$$\left| e^{\frac{(\delta_\xi(t)-\delta_\eta(t))T}{\beta}} - 1 \right| < \varepsilon \tag{4.5}$$

for all $t > t(\varepsilon)$. Consider $\bar{t} = t\left(\frac{R_{\xi\eta}-1}{R_{\xi\eta}}\right)$. Because of (4.4) and (4.5), for all $t > \bar{t}$ the inequality holds

$$\frac{w_\xi(t)}{w_\eta(t)} > (1 - \frac{R_{\xi\eta} - 1}{R_{\xi\eta}})R_{\xi\eta} = 1$$

Remark. The theorem analogous to the theorem 1 holds not only for the wealth levels $w_\xi(t)$ $w_\eta(t)$, but also for the values of consumption $_\xi(t)$ $_\eta(t)$ So that, if $\delta^*(t)$ is defined by the current consumption $\delta^*(t) = \delta(\xi^*)$, $\xi^* = argmax\ c_\xi(t)$, than it is easy to know which household will be the lider after the interval of time large enough. It will be the household $\xi^* = argmax\ R_\xi(0)$. However, it is unnecessary that it will be the leader at the beginning of the adaptation process. Therefore the value of the final wealth is not easy to calculate. The value of $\bar{\delta}$ depends significantly on how $\delta^*(t)$ is defined.

The theorem 1 shows that the wealth distribution is changing in time in accordance with the lexicographic ordering by two criteria (R, δ) (the indifference curves are shown in Fig.2). Therefore the higher values of R the wealthier are the households. Among the households with equal R the wealthiest will be those for which the values of δ are minimal. Because the lexigographic ordering (R, δ) is independent on time at every moment, given the wealth distribution $w_\xi(t)$ and current values of $\delta_\xi(t)$ it is easy to determine which household will have the highest consumption $c_\xi(t)$ and wealth $w_\xi(t)$ as a result of the adaptation process.

5 Conclusion

We have considered the model of adaptation of pure time preference in the closed economy with three assets and non-uniform households. We were considering that the households are trying to imitate the currently optimal one. Such adaptive behavior leads to the convergence of the pure time preference rates and wealth redistribution. We have shown that the system posses the equifinality property: regardless on the criterion of the determination of the most prosperous household the final distribution is defined solely by the value of $R_\xi = w_\xi \exp -\delta_\xi T/\beta$. The criterion R_ξ shows the existing relationship between wealth and pure time preference rate. The more δ the more wealth is required to ensure the high consumption level in future. The more is β the less wealth w is required to compensate too high level of the pure time preference.

The authors are grateful to professor I. G. Pospelov for the statement of the problem and to all the colleagues from the division of the mathematical modeling of evonomic systems of the Computer Center of the Russian Academy of Science for useful discussions.

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