# Concerning a Caratéodory – Suvorov Theorem on Kernel Convergence

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We consider questions which are connected with homeomorphic maps $f : D \to \Delta$ of a class $W_n^1(D)$ of domains $D \subset \mathbf{R}^n$. This theory was constructed by J. Lelong - Ferrand [1], G.D. Suvorov [2], their learners and successors. We surpass to these themes in connection with newest applications of the theory to questions of the surface triangulation and grid generation in domains of $\mathbf{R}^n$ (for example, see S.A. Ivanenko [3], G.P. Prokopov [4] etc).

A key to the applications is a theorem of Caratéodory type on connections between a kernel convergence of domains and a convergence of conformal maps. The far-reaching generalization of the Caratéodory theorem has been found by G.D. Suvorov for metric spaces. G.D. Suvorov is also the author of the theory of prime ends of domain sequences which converge to a kernel.

For further development of different aspects of the Caratéodory – Suvorov theory see articles of I.S. Ovchinnikov, B.P. Kufarev, V.P. Luferenko, V.I. Kruglikov, V.I. Kruglikov – V.M. Miklyukov, S.R. Nasyrov, V.I. Ryazanov, A.P. Karmazin.

This paper is an attempt to make results in the modern way in connection with some new problems of indicated applications. The central ideas of our approach are a spreading of the theory for the case of $W_n^1$-homeomorphisms of surfaces in $\mathbf{R}^n$ (including an anisotropic effects) and the change of boundedness of Dirichlét integrals for the maps $T : D \to \Delta$ and $T^{-1} : \Delta \to D$ with boundedness of a functional in the form of

$$I(T; D, \Delta) \equiv \int\limits_D \lambda^n(x,T)\, d\mathcal{H}^n + \int\limits_D Q_T^n(x)\, d\mathcal{H}^n < \infty\,,$$

where homeomorphisms $y = T(x) : D \to \mathbf{R}^n$ of subdomains $D \subset \mathbf{R}^n$ satisfy the following properties

$$T \in W_n^1(D)\,, \quad T^{-1} \in W_n^1(\Delta)\,, \quad \Delta = T(D)\,,$$

and

$$\lambda(x,T) = \frac{1}{\sqrt{n}} \left( \sum_{i=1}^n |\nabla T_i|^2 \right)^{\frac{1}{2}}\,, \quad Q_T(x) = \frac{\lambda^n(x,T)}{|J(x,T)|}\,.$$

We recall that *a kernel of a domain sequence* $\{D_n\}$ (with respect to a point $O$) is called a domain $D_0$, $O \in D_0$, which is the union of all domains $D_0'$, $O \in D_0'$, with properties: every point $x \in D_0'$ belongs to all $D_n'$ for sufficiently great $n = n(x)$ together with some neighborhood in $\mathbb{R}^n$.

A sequence $\{D_n\}$ *converges to the kernel* $D_0$ with respect to a point $O$, if every its subsequence $\{D_{n_k}\}$ $(k = 1, 2, \ldots)$ has $D_0$ as its kernel.

The following statement is the main result for subdomains of $\mathbf{R}^n$.

**Theorem.** *Let* $\{D_n\}$ $(n = 1, 2, \ldots)$ *be a sequence of domains in* $\mathbb{R}^n$*, each of which is homeomorphic to the unit ball, is contained in a fixed ball with the center* $O$ *and contains a fixed ball with the center at this point. Let* $\{\Delta_n\}$ *be a sequence of domains with analogous properties with respect to a fixed point* $O'$.

*Let* $\{T_n\}$, $T_n(O) = O'$, *be a sequence of homeomorphic maps of the first sequence onto the second sequence,* $T_n(D_n) = \Delta_n$ *and*

$$I(T_n; D_n, \Delta_n) \le k < \infty, \quad k = \mathrm{const} < \infty, \quad \forall\, n = 1, 2, \ldots\,.$$

*Then there exists a subsequence* $\{T_{n_k} : D_{n_k} \to \Delta_{n_k}\}$ $(k = 1, 2, \ldots)$ *of* $\{T_n : D_n \to \Delta_n\}$ *such that* *i) the sequences $\{D_{n_k}\}$ and $\{\Delta_{n_k}\}$ converge to the kernel $D_0$ and $\Delta_0$ with respect to points $O$ and $O'$ respectively ;*

*ii) the sequences $\{T_{n_k}\}$ and $\{T_{n_k}^{-1}\}$ converge uniformly inside of $D_0$ and $\Delta_0$ respectively to the homeomorphic maps $T_0 : D_0 \to \Delta_0$ and $T_0^{-1} : \Delta_0 \to D_0$;*

*iii) the maps $T_0$, $T_0^{-1}$ belong to $W_n^1$ in domains $D_0$ and $\Delta_0$ respectively , moreover $I(T_0; D_0, \Delta_0) \leq k$.*

**Remarks.** 1) The quantity $I(T; D, \Delta)$ contains two integral terms. It is not difficult to indicate examples, which show that absence of one of them does not permit to conclude neither on homeomorphism of the limit mapping $y = T(x)$ nor on correctness of the relation $T(D) = \Delta$. In particular, the limit mapping $T$ can map $D$ onto a part of $\Delta$ or map of a part of $D$ onto all domain $\Delta$.

2) By the well know Liouville theorem, the set of the conformal maps in $\mathbf{R}^n$, $n \geq 3$, becomes exhausted with transformations of similarity and inversions with respect to spheres. The proximity of $I(T; D, \Delta)$ to the sum of volumes of $D$ and $\Delta$ characterizes a proximity of $T$ to a conformal mappings.

3) Cases of normings different from $T_n(O) = O'$, $n = 1, 2, \ldots$ are considered.

With a view to consider domains $D_n$ and $\Delta_n$ with infinity volumes, there study functionals $I(T_n; D_n, \Delta_n)$ in spherical metrics.

4) There are new conditions of nonexistence of homeomorphic $W_n^1$-maps from a domain onto other domain for $n \geq 3$.

5) As an example of applications, we offer an algorithm of a numerical structure of a conformal mapping from long and narrow rectangles onto an unit disc, and calculations of corresponding special functions. If the ratio of the rectangle length to its width is sufficiently large then they are known as "stagnation zones" (see [5] section 7.1), in which maps are almost invariable and in which we should choose exceptionally scarce nodes of the network. In particular, this algorithm makes use of a minimization of $I(T; D, \Delta)$ for the best choice of nodes of the network.

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