Multilevel ILU preconditionings for general unsymmetric matrices

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1. The problem of numerical solution of a linear system with a general sparse nonsingular matrix A is still actual due to new problem settings coming from many important applications. For the iterative solution of the linear system $Ax = b$, a preconditioner $C \approx A$ is used to implicitly form the preconditioned system $(AC^{-1})y = b$, an approximate solution of which is obtained using a Krylov subspace method (e.g., BiCGStab [1]). Thus, the k-th iterate x_k of the method is a (nearly) optimum approximation to x over the subspace $\text{span}\{C^{-1}b, C^{-1}AC^{-1}b, \dots, (C^{-1}A)^{k-1}C^{-1}b\}$. Here we consider preconditioning based on an approximation to the coefficient matrix via an Incomplete LU-factorization with pivoting:

$$A = C + E, \qquad C = P_L L U P_R \tag{1}$$

where P_L and P_R are permutation matrices, L and U are lower and upper triangular matrices, respectively, and E is the ILU error matrix.

2. The idea of *multilevel ILU* algorithms for the construction of decomposition (1) can formally be related to the so-called *multigrid* methods. More important, there also exists a relation of the multilevel ILU to the triangular factorization with *complete pivoting.* Note that the latter method can hardly be implemented efficiently as a matrix algorithm (whether dense or sparse) due to a large amount of non-numerical operations involved.

The key component of the multilevel ILU (see [2] and references therein) is a recursive construction of a 2×2-block splitting with a diagonally dominant leading block.

Let the current *reduced matrix* be initialized as $A^{(1)} = A$, and its dimension be $n^{(1)} = n$. At the mth level, the scaled and reordered matrix is constructed:

$$A_{SR}^{(m)} \equiv P_L^{(m)} D_L^{(m)} A^{(m)} D_R^{(m)} P_R^{(m)} = \begin{bmatrix} A_{11}^{(m)} & A_{12}^{(m)} \\ A_{21}^{(m)} & A_{22}^{(m)} \end{bmatrix},$$

where $n^{(m)} = n_1^{(m)} + n^{(m+1)}$ determines the block 2×2-splitting, diagonal matrices $D_L^{(m)}$ and $D_R^{(m)}$ determine row and column norm-balancing scaling, $P_L^{(m)}$ and $P_R^{(m)}$ are permutation matrices, and $A_{11}^{(m)}$ is a diagonal dominant matrix of the order $n_1^{(m)}$. Here we have used the iterative evaluation of the scaling matrices [3] and the two-stage construction of permutation matrices [3, 4]. Therefore, the current reduced matrix satisfies

$$\sum_{j=1}^{n^{(m)}} |(A_{SR}^{(m)})_{ij}| = 1, \qquad \sum_{i=1}^{n^{(m)}} |(A_{SR}^{(m)})_{ij}| \approx 1. \tag{2}$$

and its leading block $A_{11}^{(m)}$ of the order $n_1^{(m)}$ satisfies the diagonal dominance condition

$$\sum_{j \neq i} |(A_{11}^{(m)})_{ij}| \leq (1-\theta)|(A_{11}^{(m)})_{ii}|, \qquad 0 < \theta < 1. \tag{3}$$

where θ is a prescribed parameter. Moreover, the following additional property holds:

$$|(A_{11}^{(m)})_{kk}| \geq |(A_{11}^{(m)})_{ll}|, \qquad k > l. \tag{4}$$

Next we perform the first $n_1^{(m)}$ steps of the Crout type ILU factorization 'by value' [3, 5] over the matrix $A_{SR}^{(m)}$:

$$\begin{bmatrix} A_{11}^{(m)} & A_{12}^{(m)} \\ A_{21}^{(m)} & A_{22}^{(m)} \end{bmatrix} - \begin{bmatrix} E_{11}^{(m)} & E_{12}^{(m)} \\ E_{21}^{(m)} & E_{22}^{(m)} \end{bmatrix} = \begin{bmatrix} L_{11}^{(m)} & 0 \\ L_{21}^{(m)} & I_2 \end{bmatrix} \begin{bmatrix} I_1 & 0 \\ 0 & A^{(m+1)} \end{bmatrix} \begin{bmatrix} U_{11}^{(m)} & U_{12}^{(m)} \\ 0 & I_2 \end{bmatrix},$$

where $E_{kl}^{(m)}$, $k + l < 4$, arise due to ILU truncation and the absolute values of their entries are bounded by the truncation threshold $0 < \tau \ll 1$. The trailing block of the error matrix arises in the equation for the evaluation of the $(m+1)$th reduced matrix,

$$A^{(m+1)} - E_{22}^{(m)} = A_{22}^{(m)} - L_{21}^{(m)} U_{12}^{(m)},$$

which provides for the truncation of entries of $A^{(m+1)}$ sufficiently small by the magnitude.

This construction is applied recursively until $n^{(m+1)} > 0$. At each recursive step, one obtains $n_1^{(m)}$ consecutive rows of sparse upper triangular matrices L_S^T and U_S such as $P_L^T D_L A D_R P_R^T = L_S U_S + E_S$. A proper renumbering and rescaling of the earlier computed entries of L_S and U_S converts the evaluated preconditioner to the form (1).

An interesting observation is that we do not use any sparsity optimization. Indeed, properties (2)-(4) alone can restrict the fill-in for L and U.

3. Numerical testing has been done using a subset of 96 real unsymmetric matrices with $n > 1000$ extracted from the University of Florida sparse matrix collection. Each of these problems could be succesively solved using at least one of the two fixed parameter presets (with $\theta = 0.1$ and $\tau = 0.0004$ or 0.00001). Note that most of these test problems cannot be efficiently solved by the existing preconditioned iterative methods.

References

[1] H. A. van der Vorst. Bi-CGStab: a fast and smoothly converging variant of Bi-CG for the solution of nonsymmetric linear systems. *SIAM J. Sci. Statist. Comput.* **13**, 631–644, 1992.

[2] Y. Saad. Multilevel ILU with reorderings for diagonal dominance. *SIAM Journal on Scientific Computing*, **27**, 1032-1057, 2005.

[3] I. Kaporin, Scaling, Reordering, and Diagonal Pivoting in ILU Preconditionings, *Russian Journal of Numerical Analysis and Mathematical Modelling*, **22**, no.4, pp. 341–375, 2007.

[4] I.E. Kaporin and I.N. Konshin, Recursive scaling, permutation and 2×2- block splitting in ILU preconditionings. In: *2nd International Conference on Matrix Methods and Operator Equations* (July 23 - 27, 2007, Moscow) - Abstracts, pp. 33-34.

[5] N. Li, Y. Saad, and E. Chow, Crout versions of ILU for general sparse matrices. *SIAM J. Sci. Comput.*, **25**, 716–728, 2003.