Optimization of hexahedral and polyhedral grids

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Optimization of computational grids via node movement can be formulated as a construction of optimal one-to-one mapping between nonregular manifolds. Minimization of polyconvex functionals similar to those arising in the framework of hyperelasticity theory allows to obtain provably quasi-isometric mappings between multidimensional nonregular manifolds. Such a minimization can be used not only for construction of piecewise-affine mappings between polyhedral manifolds, but also for more general classes of piecewise-smooth homeomorphisms, including piecewise polylinear maps and Bernstein-Bezier polynomial maps.

Polyconvexity of variational principle leads to the set of important properties of resulting minimization problems. In particular, it is proven that one can introduce special types of quadrature rules for approximation of polyconvex functional, such that the minimizer of the discrete problem is piecewise smooth homeomorphism. This result allows to construct nondegenerate deformations of grids consisting of linear triangles and bilinear quadrilaterals, quadratic triangles in 2D, as well as tetrahedra, prisms, pyramids, hexahedra in 3D, and Bernstein-Bezier polynomial maps using triangular, quadrilateral and hexahedral control nets.

For the same set of grids it is proven local convexity of the minimization problem. In particular, grid functionals are convex with respect to unidirectional grid node displacements (of course this includes the case when all grid nodes but one are fixed).

Polyconvex grid functionals can be used for optimization of polyhedral grids. This grids consist of generalized polyhedra where edges are straight segments but faces are not assumed flat. We show that local convexity property holds in this case as well. Optimization of polyhedral grids presents serious difficulties due to the fact that optimality principle for generalized polyhedra is not clearly formulated. Here it is suggested to minimize deviation from regular and semiregular polyhedra. Unfortunately it is not possible to introduce simple deviation measure in 3D. However, one can consider generalized polyhedron as a three-dimensional polyhedral manifold glued from tetrahedra. In general this manifold cannot be embedded into Cartesian 3D space. However one can compute intrinsic 3D curvature of this manifold, in particular one can compute total variation of Regge action. One can easily construct target generalized polyhedron with the same connectivity as initial generalized polyhedron but with minimal total variation of curvature (absolute curvature). Thus optimal generalized polyhedron is the image of target polyhedron under quasi-isometric map with minimal equivalence constant. Optimal polyhedron in 3D is just "flattening" of optimal generalized polyhedron. The resulting grid optimization procedure is quite simple and essentially coincides with that for standard types of grids. However it possesses quite nice property: regular polyhedra (5 Platonic bodies), semiregular polyhedra (13 Archimedean bodies) as well as infinite sequence of regular prisms and antiprisms are exact minimizers of resulting grid functional.