Parallelohedra: the Voronoi Problem and the Minkowskii Theorems

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One of the earliest important applications of the modern discrete geometry is crystallography. The fundamental domain for a translational symmetry group of a crystal is a parallelohedron. A parallelohedron is a convex polyhedron that tiles space with its parallel copies by a face-to face way. A face-to-face tiling of space by parallelohedra

While 3-dimensional parallelohedra (introduced by E. Fedorov in 1889) play role in crystallography, d-dimensional parallelohedra play role in the geometry of numbers, in particular, in the theory of positive quadratic forms.

H. Minkowski has proved two celebrated theorems on parallelohedra (1897). One theorem says on necessity of three conditions for parallelohedra (central symmetricity of a parallehedra, central symmetricity of all their hyperfaces, and 4- or 6-facets belts). Another theorem gives B. Venkov (1954) proved that the necessary properties of parallelohedra in Minkowski's theorem 1 are also sufficient conditions. Indeed, the Minkowski theorems are true for "non-face-to-face parallelohedra" as well (see N. Dolbilin, 2003).

G. Voronoi has developed a very deep theory of an important kind (1908) of parallelohedra, presently called Voronoi parallelohedra. Voronoi parallelohedra are Voronoi domains for integer point lattices. Voronoi himself had been thinking that, likely in low dimensions d=2 and 3, any parallelohedron is affinally equivalent to some Voronoi parallelohedron. If it is the case, the constructive Voronoi theory of Voronoi parallelohedra can be applied to all the rest parallelohedra.

Therefore, Voronoi stated a corresponding conjecture on the affine equivalence of arbitrary parallelohedra to some Voronoi parallelohedra. On this way Voronoi proved his prominent theorem on affine equivalence of any primitive parallelohedra. By definition, primitive paralleohedra are such that at any vertex the minimally possible number (i.e. d+1 one for a given dimension d) of tiles meet at any vertex of a tiling.

30 years later a striking progress on this conjecture was been obtained by O. Zhitomirski (1937) who proved the Voronoi conjecture for all paralleohedra except for those which are non-primitive just in some (d-2)-faces. Thus, due to Minkowski theorem after the Zhitomirski theorem the Voronoi conjecture stands unsolved just for such parallelohedra that meet four at some of their (d-2)-dimensional faces.

The Voronoi problem gave rise to a problem of existence of a tiling dual to a tiling by parallelohedra. By the dual tiling of a tiling T by parallelohedra one means a tiling, if exists, by the following polyhedra. Given a vertex v of a parallelohedra in a tiling T and let D(v) denote a convex hull of center points of all

parallelohedra meeting at the vertex v. A face-to-face tiling D, if exists, of space by convex polyhedra D(v), where v runs over all the vertices of T.
In case of a Voronoi parallelohedra the dual tiling is trivially exists: the dual to a Voronoi tilings is a well-defined Delone tiling for a point lattice. But the existence problem in non-Voronoi parallelohedra's case gets non-trivial.

Given arbitrary dimension d, all dual faces of (d-2)-faces (Minkowski) and (d-3)-faces (B.Delone) are known.
d=4: B. Delone proved that all 4-parallelohedra are affinally equivalent to Voronoi ones and, consequently, any tiling of 4-dimensional space by parallelohedra has its dual.

Since the problem of dual is open, it makes sense to investigate a slightly weaker problem of the fan of a tiling at k-dimensional face, $0<k<d-1$. A fan of T at k-face is trace of the T on a sufficiently small sphere centered at a relatively interior point. If a dual tiling exists, the fan of k-face is a (d-k)-face dual
to this k-face.

In the talk the Voronoi problem will be discussed, particularly, in the context of the problem of dual and a fan in touch to a recent advance in Minkowski's theorems obtained by the author (2007).
The author introduced a concept of maximal common face (m.c.f.) and, by means of it, managed to modify
both theorems by Minkowski theorems. Now last point of Minkowski's theorem 1 can be obtained by means of m.c.f. Minkowski's theorem 2 easily follows from the "index theorem" (Dolbilin, 2007). We will show how the use of m.c.f. predetrmines the structure of fans. In particular, a fan for m.c.f of dimension k is a (d-k)-polyhedron whose each (d-k-1)-face has one and only one more (d-k-1)-face having nothing in common with the first one.