Riemannian Metric of Harmonic Parameterization of Geodesic Quadrangles and 2-D Regular Quasi-Isometric Grids

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Consider a problem of generating a 2-D structured boundary-fitting rectangular grid in a curvilinear quadrangle $\mathcal{D}$ with angles $\alpha_{-1}, \ldots, \alpha_2$ and a conformal module $\mathcal{M}$.

The problem can be solved by constructing a mapping $X(\xi, \eta)$, $Y(\xi, \eta)$ from the unit square $\mathcal{R}$ onto $\mathcal{D}$. We propose to use a special class of mappings with nonzero and bounded Jacobian – μ-quasi-isometric. The mapping from $\mathcal{R}$ onto $\mathcal{D}$ is called *μ-quasi-isometric* if there exists $\mu > 0$ such that on $\mathcal{R}$ the Jacobi matrix' singular values are bounded above and below by numbers μ and $1/\mu$. It is clear that μ-quasi-isometric mapping will be also μ^2-quasi-conformal.

The mapping we seek is proved to be the unique solution to the following BVP: given a quasi-isometric correspondence between $\partial\mathcal{R}$ and $\partial\mathcal{D}$, extend it inside $\mathcal{R}$, without loss of the property of being a quasi-isometric mapping, as a solution to the following Beltrami system:

$$\sqrt{g_{11}g_{22} - g_{12}^2}\, X_\xi = -g_{12}\, Y_\xi + g_{11}\, Y_\eta, \qquad \sqrt{g_{11}g_{22} - g_{12}^2}\, X_\eta = -g_{22}\, Y_\xi + g_{12}\, Y_\eta. \tag{1}$$

In order to make the solution of this BVP a quasi-isometric mapping, we have to introduce a special class of coefficients g_{jk} with an additional unknown parameter r. The strict boundaries for r and the sought value of r can be found together with the mapping itself as described in Chumakov and Chumakov [1]. During the iterative process of solving the BVP, if needed, we are able to change the given boundary correspondence as well, which can lead to any desired distribution of boundary points of the grid thus providing a tool for the grid adaptation. In this work, we consider a special case of "free" boundary conditions i.e., the boundary points distribution is determined in the process of solving the BVP.

A Beltrami system with a smooth g_{jk} has a simple geometric meaning. Solution $X(\xi, \eta)$, $Y(\xi, \eta)$ of the Beltrami system (1) generates a metric

$$ds^2 = G_{11} d\xi^2 + 2G_{12} d\xi d\eta + G_{22} d\eta^2,$$

such that $G_{jk} = \lambda\, g_{jk}$ with $\lambda = \lambda(\xi, \eta) \neq 0$, where coefficients G_{jk} are defined by

$$G_{11} = X_\xi^2 + Y_\xi^2, \quad G_{12} = X_\xi X_\eta + Y_\xi Y_\eta, \quad G_{22} = Y_\eta^2 + Y_\eta^2.$$

This means that metrics defined by G_{jk} and g_{jk} are conformally equivalent.

The present paper is aimed to obtain a simple representative g_{jk} from the class of conformally equivalent metrics on $\mathcal{R}$ generated by harmonic mappings of $\mathcal{R}$ onto $\mathcal{P}$ where $\mathcal{P}$ is a geodesic quadrangle with given angles on a surface of constant curvature. Using such sort of g_{jk} tends to generate a grid with uniform spacing.

References

1. **Chumakov G.A. and Chumakov S.G. (1998)** A method for the 2-D quasi-isometric regular grid generation. *J. Comput. Phys.* **143**, 1-28.