Non-saturating numerical algorithm for the solution of heat conduction equation

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Non-saturating time discretization, i.e., the one which automatically takes into account the smoothness of the solution of the problem, is considered. As an example, the heat conduction equation is considered, although the approach is applicable to any transient problem such that the discrete operator in space variables possesses a complete set of eigen vectors, and all eigen values are real-valued.

Key words: non-saturating numerical algorithm, time discretization, heat conduction equation.

Introduction. In [1], non-saturating numerical algorithms for the solution of steady-state problems of mathematical physics are considered. In the current work, these results are generalized to transient problems. The non-saturating numerical algorithms were proposed by K. I. Babenko [2] at the beginning of seventies of the past century. Application of these approaches to the solution of mathematical physics problems performed by the author for a number of years proved their high efficiency. However, only the steady-state problems have been considered so far. In the current work, this deficiency is eliminated. As an example, the one-dimensional heat conduction equation is considered; it is shown that problem dimension is not important for the method.

1. Statement of the problem. In a rectangle D = {0 ≤ x ≤ 1, 0 ≤ t ≤ 1 }, consider the heat conduction equation:

(1.1) $$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),\ (x,t) \in D;$$

(1.2) $$u|_{t=0} = u_0(x);$$

(1.3) $$u|_{x=0} = u|_{x=1} = 0.$$

Evidently, we can set $u_0(x) \equiv 0$ without loss of generality.

2. Descretization. In the coordinate *x,* we approximate the function being sought *u(x,t)* by a polynomial; to achieve this, introduce a grid in *x* consisting of *m* nodes:

$$x_\mu = \frac{1}{2}(z_\mu + 1),\ z_\mu = \cos\chi_\mu,\ \chi_\mu = \frac{(2\mu - 1)\pi}{2m},\ \ \mu = 1,2,\ldots,m,$$

and apply the interpolation formula

(2.1) $$q(x) = \sum_{\mu=1}^{m} \frac{T_m(x)(x-1)x q_k}{m\dfrac{(-1)^{\mu-1}}{\sin\chi_\mu}(x_\mu - 1)x_\mu(z - z_\mu)},\ \ q_\mu = q(x_\mu),\ \ z = 2x - 1.$$

The second-order derivative in *x*, involved in Eq. (1.1), is obtained by the differentiation of the interpolating formula (2.1).

In the variable *t*, we choose the grid containing *k* nodes:

$$t_\nu = \frac{1}{2}(z_\nu + 1),\ z_\nu = \cos\chi_\nu,\ \chi_\nu = \frac{(2\nu - 1)\pi}{2k},\ \nu = 1,2,\ldots,k,$$

and also apply polynomial interpolation:

(2.2) $$q(t) = \sum_{\nu=1}^{k} \frac{T_m(t) t q_\nu}{m \dfrac{(-1)^{\nu-1}}{\sin\chi_\nu} t_\nu (z - z_\nu)}.$$

The quantities involved in Eq. (2.2) have been defined above. The first-order derivatives of *u(x,t)* with respect to *t*, involved in the left-hand side of relationships (1.1), are obtained by the differentiation of the interpolation formula (2.2).

Let *A* be the matrix of the discrete operator $-\dfrac{d^2}{dx^2}$, then, denoting $u_{\mu\nu}=u(x_\mu,t_\nu)$, $\mu=1,2,\ldots,m;$ $\nu=1,2,\ldots,k$ we obtain

$$\frac{\partial u(x_\mu,t)}{\partial t} + \sum_{p=1}^{m} A_{\mu p} u(x_p,t) = f(x_\mu,t).$$

Let *B* be the matrix of numerical differentiation with respect to *t* on the interval [0,1]. As a result, we obtain:

(2.3) $$\sum_{q=1}^{k} B_{\nu q} u_{\mu q} + \sum_{p=1}^{m} A_{\mu p} u_{p\nu} = f_{\mu\nu}.$$

Enumerate the grid nodes by a single rowwise index (i.e. the fastest changing is the first index $I \to (\mu,\nu)=(\nu-1)m+\mu$). Then, we obtain a discrete problem:

(2.4) $$(B \otimes I_m + I_k \otimes A)u = f,$$

where *B* is the $k\times k$ matrix of differentiation with respect to *t; A* is the $m \times m$ matrix of second-order differentiation with respect to *x;* I_m, and I_k are unity matrices. Represent *A* in the form:

$$A = \sum_p \lambda_p h_p,\ h_p^2 = h_p,\ h_p h_l = 0, p \neq l \Rightarrow \sum_p h_p = I_m \Rightarrow$$

$$B \otimes \sum_p h_p + I_k \otimes (\sum_p \lambda_p h_p) = \sum_p (B + \lambda_p I_k) \otimes h_p \Rightarrow$$

(2.5) $$(B \otimes I_m + I_k \otimes A)^{-1} = \sum_p (B + \lambda_p I_k)^{-1} \otimes h_p$$ (see. [1]).

Thus, the solution of the discrete problem (2.3) is obtained by multiplying the matrix (2.5) by the right-hand vector. Note that to build the inverse of matrix (2.3), it is sufficient to invert *m* matrices of size $k\times k$, where *k* is the number of nodes in the time approximation. Note also that in the derivation we did not assume any specific features of matrix *A,* i.e., matrix *A* can two-dimensional, three-dimensional, or have arbitrary dimension according to the problem being solved. It is only required that the matrix possess a complete system of eigen vectors, and all eigen values be real-valued.

3. Numerical example. As a numerical example, consider the problem (1.1) – (1.3) with the right-hand side: $f(x,t)=(\cos t+\pi^2 \sin t)\sin\pi x$, then the solution is $u(x,t)=\sin t \sin\pi x$. The results calculated on 5×5 and 10×10 grids are presented below:

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M =                  5  K =                  5
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Exact solution
    0.63594E-01    0.49945E+00    0.82800E+00    0.49945E+00    0.63594E-01
    0.54768E-01    0.43013E+00    0.71309E+00    0.43013E+00    0.54768E-01
    0.36822E-01    0.28919E+00    0.47943E+00    0.28919E+00    0.36822E-01
    0.15718E-01    0.12345E+00    0.20465E+00    0.12345E+00    0.15718E-01
    0.18794E-02    0.14760E-01    0.24469E-01    0.14760E-01    0.18794E-02
Approximate solution
    0.63586E-01    0.49947E+00    0.82808E+00    0.49947E+00    0.63586E-01
    0.54762E-01    0.43015E+00    0.71316E+00    0.43015E+00    0.54762E-01
    0.36817E-01    0.28920E+00    0.47947E+00    0.28920E+00    0.36817E-01
    0.15716E-01    0.12345E+00    0.20467E+00    0.12345E+00    0.15716E-01
    0.18791E-02    0.14760E-01    0.24470E-01    0.14760E-01    0.18791E-02
 Matrix norm of discrete problem
BNORM =   0.181397574207915
 Norm of residual
RNORM =   8.498969703851778E-005

M =          10 K =          10
Exact solution
    0.16208E-01    0.14279E+00    0.37214E+00    0.63389E+00    0.81295E+00
    0.81295E+00    0.63389E+00    0.37214E+00    0.14279E+00    0.16208E-01
    0.15679E-01    0.13814E+00    0.36000E+00    0.61322E+00    0.78644E+00
    0.78644E+00    0.61322E+00    0.36000E+00    0.13814E+00    0.15679E-01
    0.14573E-01    0.12840E+00    0.33462E+00    0.56998E+00    0.73098E+00
    0.73098E+00    0.56998E+00    0.33462E+00    0.12840E+00    0.14573E-01
    0.12853E-01    0.11323E+00    0.29511E+00    0.50267E+00    0.64466E+00
    0.64466E+00    0.50267E+00    0.29511E+00    0.11323E+00    0.12853E-01
    0.10569E-01    0.93113E-01    0.24267E+00    0.41335E+00    0.53011E+00
    0.53011E+00    0.41335E+00    0.24267E+00    0.93113E-01    0.10569E-01
    0.79167E-02    0.69748E-01    0.18177E+00    0.30963E+00    0.39709E+00
    0.39709E+00    0.30963E+00    0.18177E+00    0.69748E-01    0.79167E-02
    0.52140E-02    0.45937E-01    0.11972E+00    0.20392E+00    0.26153E+00
    0.26153E+00    0.20392E+00    0.11972E+00    0.45937E-01    0.52140E-02
    0.28219E-02    0.24861E-01    0.64792E-01    0.11037E+00    0.14154E+00
    0.14154E+00    0.11037E+00    0.64792E-01    0.24861E-01    0.28219E-02
    0.10533E-02    0.92801E-02    0.24185E-01    0.41197E-01    0.52834E-01
    0.52834E-01    0.41197E-01    0.24185E-01    0.92801E-02    0.10533E-02
    0.11904E-03    0.10488E-02    0.27333E-02    0.46557E-02    0.59709E-02
    0.59709E-02    0.46557E-02    0.27333E-02    0.10488E-02    0.11904E-03
Approximate solution
    0.16208E-01    0.14279E+00    0.37214E+00    0.63389E+00    0.81295E+00
    0.81295E+00    0.63389E+00    0.37214E+00    0.14279E+00    0.16208E-01
    0.15679E-01    0.13814E+00    0.36000E+00    0.61322E+00    0.78644E+00
    0.78644E+00    0.61322E+00    0.36000E+00    0.13814E+00    0.15679E-01
    0.14573E-01    0.12840E+00    0.33462E+00    0.56998E+00    0.73098E+00
    0.73098E+00    0.56998E+00    0.33462E+00    0.12840E+00    0.14573E-01
    0.12853E-01    0.11323E+00    0.29511E+00    0.50267E+00    0.64466E+00
    0.64466E+00    0.50267E+00    0.29511E+00    0.11323E+00    0.12853E-01
    0.10569E-01    0.93113E-01    0.24267E+00    0.41335E+00    0.53011E+00
    0.53011E+00    0.41335E+00    0.24267E+00    0.93113E-01    0.10569E-01
    0.79167E-02    0.69748E-01    0.18177E+00    0.30963E+00    0.39709E+00
    0.39709E+00    0.30963E+00    0.18177E+00    0.69748E-01    0.79167E-02
    0.52140E-02    0.45937E-01    0.11972E+00    0.20392E+00    0.26153E+00
    0.26153E+00    0.20392E+00    0.11972E+00    0.45937E-01    0.52140E-02
    0.28219E-02    0.24861E-01    0.64792E-01    0.11037E+00    0.14154E+00
    0.14154E+00    0.11037E+00    0.64792E-01    0.24861E-01    0.28219E-02
    0.10533E-02    0.92801E-02    0.24185E-01    0.41197E-01    0.52834E-01
    0.52834E-01    0.41197E-01    0.24185E-01    0.92801E-02    0.10533E-02
    0.11904E-03    0.10488E-02    0.27333E-02    0.46557E-02    0.59709E-02
    0.59709E-02    0.46557E-02    0.27333E-02    0.10488E-02    0.11904E-03
 Matrix norm of discrete problem
BNORM =   0.156331288248615
 Norm of residual
RNORM =   1.533128068942347E-010
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Note. The error estimate of the method is trivial. We note here only the qualitative features. In the above derivation, polynomial interpolation of the solution was used. It is known [2], that algorithms constructed in this way do not suffer from saturation, i.e., adjust automatically to the smoothness of the problem solution. For stability of the method, only the norm of the inverse matrix of the discrete problem is essential. The calculations performed show that its value is below 1.

Conclusion. Full versions of programs can be obtained by contacting the author by e-mail: ***algazinsd@mail.ru*** or by writing to Institute for Problems in Mechanics RAS, ave. Vernadskogo 101, bldg. 1, Moscow, 119526, Russia.

References

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