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Discrete Extrinsic Curvatures and Approximation of Surfaces by Polar Polyhedra[1]

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Abstract—Duality principle for approximation of geometrical objects (also known as Eu-doxus exhaustion method) was extended and perfected by Archimedes in his famous tractate "Measurement of circle". The main idea of the approximation method by Archimedes is to construct a sequence of pairs of inscribed and circumscribed polygons (polyhedra) which approximate curvilinear convex body. This sequence allows to approximate length of curve, as well as area and volume of the bodies and to obtain error estimates for approximation. In this work it is shown that a sequence of pairs of locally polar polyhedra allows to construct piecewise-affine approximation to spherical Gauss map, to construct convergent point-wise approximations to mean and Gauss curvature, as well as to obtain natural discretizations of bending energies. The Suggested approach can be applied to nonconvex surfaces and in the case of multiple dimensions.

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Key words: polar polyhedra, discrete curvatures, DC surfaces (representable as a difference of convex functions), bending energy.

1. LEGENDRE TRANSFORM AND POLAR POLYHEDRA

Let us consider twice continuously differentiable function $u(x)$ of d real variables

$$u = u(x) = u(x_1, \dots, x_d)$$

and introduce new variables $p = p_1, \dots, p_d$ using relation

$$p_i = \partial u / \partial x_i. \tag{1}$$

Suppose that the Hessian matrix of the function u is not degenerate. Then using Eq. (1) one can express at least locally x_i as functions of $p_1, \dots, p_d$.

Let us define new function u^* via

$$u^* = x^{\mathrm{T}} p - u(x). \tag{2}$$

Substituting function $x(p)$ into (2) one can derive

$$u^* = u^*(p_1, \dots, p_d).$$

Let us express variation of function u^* via variations of variables p_i.

$$\delta u^* = \Sigma \frac{\partial u^*}{\partial p_i} \delta p_i = \Sigma (x_i \delta p_i + p_i \delta x_i) - \delta u$$

$$= \Sigma \left(x_i \delta p_i + \left(p_i - \frac{\partial u}{\partial x_i} \right) \delta x_i \right)$$

For completion of this expression one should express variations of x via variation of p. This task is greatly simplified by the fact that due to Eq. (1) coefficients multiplying δx_i are equal to zero. Hence we derive

$$x_i = \partial u^* / \partial p_i. \tag{3}$$

[1]The article is published in the original.

This equality illustrates remarkable duality of the Legendre transform which can be expressed by the following diagram [1]

	Primal system	Dual system
Variables	$x_1, \ldots, x_d$	$p_1, \ldots, p_d$
Functions	$u = u(x_1, \ldots, x_d)$	$u^* = u^*(p_1, \ldots, p_d)$.

Transform

$$
\begin{array}{ll}
p_i = \dfrac{\partial u}{\partial x_i} & x_i = \dfrac{\partial u^*}{\partial p_i} \\
u^* = x^{\mathrm{T}} p - u(x) & u = x^{\mathrm{T}} p - u^*(p) \\
u^* = u^*(p_1, \ldots, p_d) & u = u(x_1, \ldots, x_d) \\
H_{ij} = \dfrac{\partial^2 u}{\partial x_i x_j}, \quad H = (H^*)^{-1} & H^*_{ij} = \dfrac{\partial^2 u^*}{\partial p_i p_j}, \quad H^* = H^{-1}
\end{array}
\tag{4}
$$

Thus, new variables are partial derivatives of primal function with respect to primal variables, ans vice versa. Hessian matrices for primal and dual functions are mutually inverse. One can see that transform defined by (4) is symmetric.

Of course, above considerations are not rigorous, because, for example, one cannot guarantee that nonlinear systems of Eqs. (1), (3) are uniquely solvable.

Suppose now that function $u(x)$ is strictly convex. Then equality $u^*(p) = x^{\mathrm{T}}p - u(x)$ where x is expressed as a function of p using equality $p_i = \partial u/\partial x_i$ can be obtained as the solution of the maximization problem

$$u^*(p) = \max_x \{x^{\mathrm{T}} p - u(x)\}.$$

Since function $x^{\mathrm{T}}p - u(x)$ is strictly concave, its maximum is attained in a single stationary point $p_i = \partial u/\partial x_i$. These arguments explain the idea of rigorous formulation of the Legendre transform suggested by Fenchel [18].

Definition 1. (Legendre stransform). Consider function $u : \mathbb{R}^d \longrightarrow \overline{\mathbb{R}}$ with closed epigraph. Legendre transform of u is given by relation

$$u^*(x^*) = \sup_{x \in \mathbb{R}^d} \{x^{\mathrm{T}} x^* - u(x)\}, \tag{5}$$

where function $u^*(x^*)$ is called dual function (or polar function).

The following theorem shows when the generalized Legendre transform formulation can be reduced to the original one.

Theorem 1. (see [2]). (a) *Let function $u \in C^1(\mathbb{R}^d)$ be convex and finite. Then*

$$u(x) + u^*(\nabla u(x)) = x^{\mathrm{T}} \nabla u(x);$$

(b) *If function u is strictly convex and if*

$$\lim_{|x| \to \infty} \frac{u(x)}{|x|} = +\infty,$$

then $u^* \in C^1(\mathbb{R}^d)$. Moreover, if $u \in C^1(\mathbb{R}^d)$ and

$$u(x) + u^*(x^*) = x^{\mathrm{T}} x^*,$$

then

$$x^* = \nabla u(x), \quad x = \nabla u^*(x^*).$$

Geometric interpretation of Legendre transform. In order to explain geometric meaning of the Legendre transform, one has to recall the relation of polarity in projective geometry [20]. Consider point $p \in \mathbb{R}^d$ and

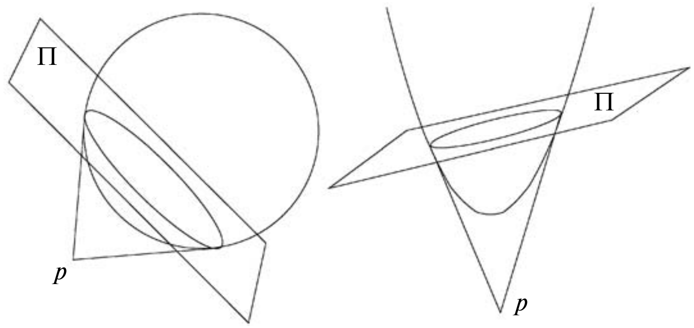

Fig. 1. Polar correpondence between point p and plane Π.

nondegenerate $(d-1)$-dimensional quadric surface P in $\mathbb{R}^d$. Suppose that one can draw from point p rays touching surface P which is shown in Fig. 1.

If quadric is defined by generic equation

$$\phi(x) = 0, \quad \phi(x) = \frac{1}{2}x^{\mathrm{T}}Ax + b^{\mathrm{T}}x + c,$$

where A is $d \times d$ martix, then any touching point is the solution of the system

$$(y-p)^{\mathrm{T}}\nabla\phi(y) = 0,$$
$$\phi(y) = 0$$

or

$$y^{\mathrm{T}}Ay + b^{\mathrm{T}}y - p^{\mathrm{T}}Ay - p^{\mathrm{T}}b = 0,$$
$$y^{\mathrm{T}}Ay + 2b^{\mathrm{T}}y + 2c = 0.$$

Thus the set of all touching points is the intersection of the surface $\phi(x) = 0$ and the plane Π

$$\Pi = \{x : x^{\mathrm{T}}(Ap + b) + p^{\mathrm{T}}b + 2c = 0\}. \tag{6}$$

As a result one-to-one correspondence between point p and plane Π is established.

Consider relation (6) for sphere and circular paraboloid.

Definition 2. Point p and $(d-1)$-dimensional plane $\Pi = \{x : x^{\mathrm{T}}p = 1\}$ are polar with respect to unit sphere.

Definition 3. Point p and $(d-1)$-dimensional plane $\Pi = \{x : x^{\mathrm{T}}p = r^2\}$ are polar with respect to sphere with radius r.

Definition 4. $p \in \mathbb{R}^{d+1}$ and d-dimensional plane

$$\Pi = \left\{x : -\sum_{i=1}^{d} p_i x_i + p_{d+1} + x_{d+1} = 0\right\} \tag{7}$$

are polar with respect to circular paraboloid,

$$x_{d+1} = \frac{1}{2}\sum_{i=1}^{d} x_i^2. \tag{8}$$

Formula (6) was derived assuming that point p is placed outside quadric, as shown in Fig. 1. In this case plane Π intersects the quadric. When point p is placed inside quadric, then one have to consider all planes passing through p. These planes intersect quadric along certain curves. Tangent planes to quadric at the intersection curve define a cone. With variation of plane, the summit of the resulting cone sweeps precisely the plane polar to p.

Geometrization of the Legendre transform is formulated in the following theorem due to W. Frenchel.

Theorem 2 (see [18]). *Let $u(x) : \mathbb{R}^d \longrightarrow \overline{\mathbb{R}}$ be convex function with closed epigraph, not equal to $+\infty$ and not taking value $-\infty$. Then epigraph of function $u^*(x^*)$ is intersection of halfspaces lying above planes polar with respect to paraboloid* (8) *to the point of the graph of* u.

Let $u(x)$ be smooth strictly convex function. Point x, $u(x)$ on the graph of u has the polar plane $\{(x^*, h) : h = x^{\mathrm{T}}x^* - u(x)\}$. Infinitesimal displacement dx leads to new point $x + dx$, $u(x) + \nabla u(x)^{\mathrm{T}}dx$ and new polar plane $\{(x^*, h) : h = x^{\mathrm{T}}x^* + dx^{\mathrm{T}}x^* - u(x) - dx^{\mathrm{T}}\nabla u(x)\}$. Intersection of two d-dimensional planes is $(d-1)$-dimensional plane defined by $dx^{\mathrm{T}}x^* = dx^{\mathrm{T}}\nabla u(x)$. Since differential dx is arbitrary, one obtains

$$x^* = \nabla u(x). \tag{9}$$

Thus in full accordance with original formulation of the Lagrange transform, the plane $\{(x^*, h) : h = x^{\mathrm{T}}x^* - u(x)\}$ touches the graph of u^* at the point $x^* = \nabla u(x)$ and $u^*(x^*) = h(x)$, where x is expressed via x^* from Eq. (9).

2. DELAUNAY AND VORONOI PARTITIONINGS AND POLAR £ POLYHEDRA

Consider finite point system $\{p_1, \ldots, p_n\}$, $p_i \in \mathbb{R}^d$. Let us remind that the Delaunay partitioning is normal partitioning of the Euclidian space into convex polyhedra where each polyhedron is the convex envelope of the points from $\mathscr{E}$ lying on the surface of a certain ball, which is empty, i.e. it does not contain any points from $\mathscr{E}$ beside vertices of the polyhedron. Dual partitioning to the Delaunay partitioning is called Voronoi partitioning (or Voronoi diagram) and consists of convex polyhedra V_i being the set of the points which are closer to a given point $p_i \in \mathscr{E}$ compared to any other point $p_j \in \mathscr{E}$, $i \neq j$.

It is well known that the problem of constructing Delaunay and Voronoi partitionings in $\mathbb{R}^d$ can be reduced to construction of convex envelopes in $\mathbb{R}^{d+1}$. For simplicity consider a finite point set $\mathscr{E}$ on the plane $\mathbb{R}^2$. Consider paraboloid of revolution

$$x_3 = \frac{1}{2}(x_1^2 + x_2^2)$$

and lift the points from $\mathscr{E}$ on the surface of paraboloid, i.e. for a point $a \in \mathscr{E}$ we define $(a^l)^{\mathrm{T}} = \left(a^{\mathrm{T}}\ \frac{1}{2}|a|^2\right)$.

The lifted point set is denoted by $\mathscr{E}_l$.

Let us consider lower convex envelope of the points from $\mathscr{E}_l$ which constitutes the graph of the piecewise linear function $x_3 = u_D(x_1, x_2)$. Function u_D is assigned value $+\infty$ beyond convex envelope of $\mathscr{E}$. It can be easily shown, that projection of the faces of resulting polyhedral surface on the plane $x_3 = 0$ is nothing else but the Delaunay partitioning. Intersection of upper halfspaces above tangent planes to paraboloid at the points from $\mathscr{E}_l$ constitutes epigraph of the convex piecewise linear function $u_V(x_1, x_2)$ which is called the Voronoi generatrice (see [4]). Projection of the faces of the graph of this function onto plane $x_3 = 0$ is precisely the Voronoi partitioning. Function u_{VD} can be obtained from u_D using Legendre transform and vice versa. These statements hold in multiple dimensions as well.

One can consider lifting to elliptic paraboloid as well, which is shown in Fig. 2, since this case reduces to the case of circular paraboloid via affine map.

It is clear that polyhedra inscribed into paraboloid and circumscribed around paraboloid are special cases of general polar polyhedra. Hence one can consider inexact lifting procedure when lifted points are not placed on the graph of paraboloid. In this case one can construct two polyhedral surfaces as well: lower convex envelope P_h of the set of lifted points $\mathscr{E}_l$ and another convex polyhedral surface P_h^* being polar to P_h with respect to the paraboloid. In this case projection of faces of P_h onto plane $x_3 = 0$ constitutes weighted Delaunay partitioning, while projection of faces of P_h^* makes up radical partitioning or power diagram (see [5, 6]).

Figure 3 illustrates the concept of polarity with respect to sphere in 2D and 3D.

Polarity with respect to sphere implies certain orthogonality properties. Consider d-dimensional case. If two polyhedra are polar with respect to sphere and origin lies inside both polyhedra then each k-dimensional face f of polar polyhedron Q^* is orthogonal to $(d-k)$-dimensional plane passing through origin and

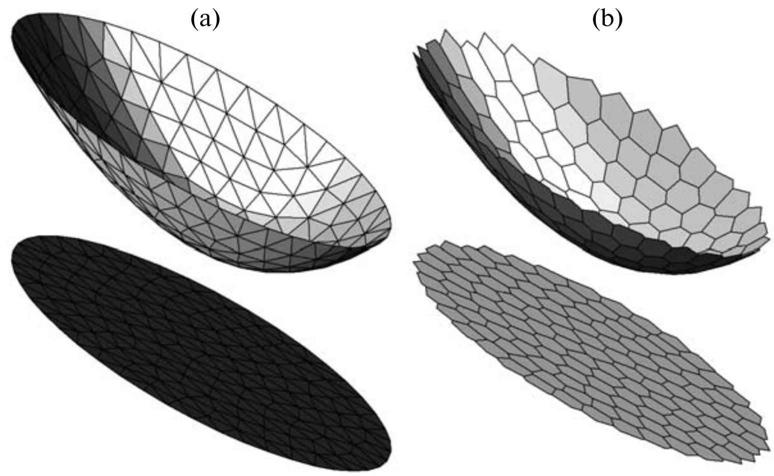

Fig. 2. Polarity with respect to elliptic paraboloid and Delaunay/Voronoi partitionings.

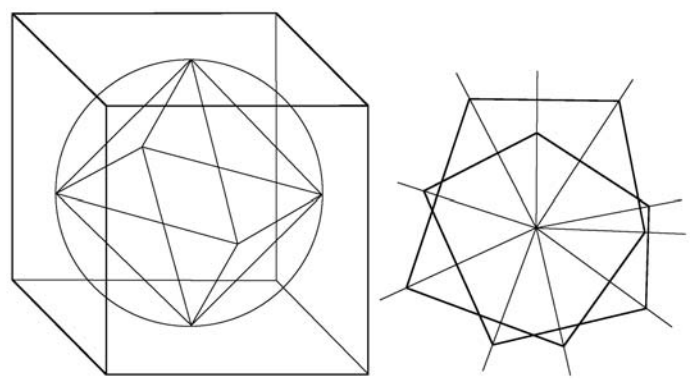

Fig. 3. Polar polyhedra and polygones.

$(d-k-1)$-dimensional face e of primal polyhedron Q dual to face f. Simple orthogonality proof can be found in [19].

3. DISCRETE CURVATURES AND SURFACES OF BOUNDED CURVATURE

One of the hard problems of modern geometry is approximation of nonregular surfaces by polyhedra allowing to approximate curvature of the surface in a certain generalized sense. In the sense of intrinsic metric (based on the distance along surface) this problem was solved in the works of A.D. Alexandrov and his school [8]. Alexandrov introduced concept of the curvature of polyhedral manifolds and developed theory of "good" approximation of manifolds of bounded curvature by polyhedral manifolds implying weak convergence of curvature. However these results are not sufficient to establish "good" convergence in the sense of extrinsic metric. The class of surfaces being manifolds of bounded curvatures in the intrinsic senses well defined: they are called surfaces of bounded curvature (see [9, 11–13]).

Extrinsic curvatures for polyhedra can be introduced using integral relations. Gauss–Bonnet theorem allows to assign to the vertex of polyhedron curvature which is equal to angular excess of its conical neighborhoods (see [8, 19]). Balance equations for vector mean curvature can be used to derive discrete mean curvature for polyhedra and to construct discrete approximation to the Laplace–Beltrami operator (see [7]). To this end one can also use variation of surface area and its relation with the sweep volume (see [17]). In [15] with each region on the surface it is associated a tensor which in the smooth case is the average of curvature tensor over this region. For polyhedral domain the same value provides weakly convergent estimator of the curvature tensor. In [16] it is shown that if a sequence of polyhedral surfaces converges to a regular surface in Hausdorff distance, then the following conditions are equivalent: a) convergence of normal fields, b) convergence of metric tensors, c) convergence of area, d) convergence of Laplace–Beltrami operators in inverse Sobolev norms.

3.1. Properties of DC Surface, Short Review

DC surface (short for surface representable as a difference of the convex functions) is a surface which can be at least locally represented as a graph of the function

$$x_3 = f(x_1, \ldots, x_2), \quad f(x) = g(x) - h(x)$$

and $g(x)$ and $h(x)$ are convex functions. A.D. Alexandrov [24] has shown that any DC surface can be approximated by a sequence of DC surfaces $f_m(x) = g_m(x) - h_m(x)$, where $h_m \longrightarrow h$, $g_m \longrightarrow g$ are convex functions which can be chosen as analytical or polyhedral ones. Such a convergence is called a *strong* one. In 2D case strong convergence implies uniform convergence of intrinsic metric of the surfaces of graphs f_m to f.

For each point on the DC surface one can define tangent cone, the same cone is tangent in the sense of intrinsic metric. Neighborhood of each point on DC surface is almost isometric to the surface tangent cone at this point.

Each DC surface is manifold of bounded curvature and can be approximated proportionally by the sequence of polyhedral manifolds in the sense of intrinsic metric.

In 1d case complete characterization of DC surfaces (curves) is well known: each curve with bounded variation of turn can be locally represented as a difference of convex functions and vice versa. It is very well known fact (the Jordan theorem) that any function of bounded variation can be represented as a difference of two monotone functions. Integrating this difference once one obtains difference of convex functions. Unfortunately precise characterization of DC functions in multiple dimensions is not available. A.D. Alexandrov has proved that any polyhedral surface where neighborhood of each vertex admits one to one projection onto certain plane is the DC surface. In [10] it was shown that the surface which can be touched at each point by a ball of fixed radius are DC surface. Each twice continuously differentiable surface is DC as well.

DC surfaces inherit many nice properties of the convex surfaces [24]. In particular, the number of conical points on the DC surface is at most countable. The set of points where tangent cone becomes dihedral angle (sharp edges) represents at most countable set of rectifiable curves. If at a certain point p of the DC surface M it is defined a tangent plane, and a sequence of points $p_k \in M$ converges to p, then the sequence of tangent cones at p_k converges to tangent plane at p.

At each point p of the DC surface one can compute one-sided derivative in arbitrary direction, this derivative can be approximated by secant and convergence of secant to derivative is uniform with respect to angle around point p. One can assume that the DC surface is lipschitz continuous and differentiable almost everywhere. Moreover, it is twice differentiable almost everywhere and for almost every point p of the DC surface one can define tangent paraboloid

$$f(x) = \frac{1}{2}x^{\mathrm{T}}Ax + o(|x|^2), \tag{10}$$

where entries of matrix $A(p)$ are bounded. In formula (10) it is assumed that the origin of the coordinate frame is placed at p and tangent plane is defined as $x_3 = 0$.

4. SPHERICAL MAPPING AND EXTRINSIC CURVATURE

Let M denote sufficiently smooth 2D surface in $\mathbb{R}^3$ (in a sense that it admits thrice continuously differentiable nondegenerate local parameterization) $x^{\mathrm{T}}(\xi) = (x_1(\xi_1, \xi_2)\ \ x_2(\xi_1, \xi_2)\ \ x_3(\xi_1, \xi_2))$. Tangential basis vectors s_i at the point p of M are given by $s_i = \partial x/\partial \xi_i$. Denoting by $\nu(p)$ unit normal to M at p, one can consider representation of the vectors $\partial\nu/\partial\xi_i$ using s_1, s_2, ν as a local basis:

$$\begin{aligned} \partial\nu/\partial\xi_1 &= -a_{11}s_1 - a_{21}s_2 + \beta_1\nu, \\ \partial\nu/\partial\xi_2 &= -a_{12}s_1 - a_{22}s_2 + \beta_2\nu. \end{aligned} \tag{11}$$

One can easily show that $\beta_i = 0$. Matrix $A = \{a_{ij}\}$ is called *shape operator matrix* or *curvature tensor*.

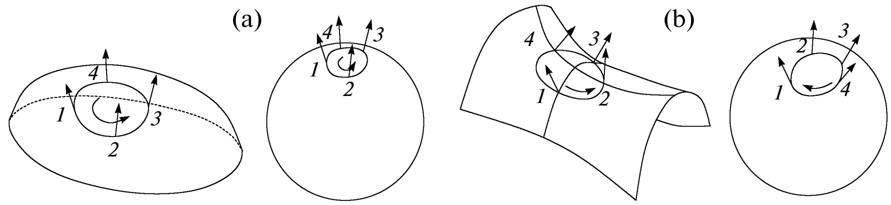

Fig. 4. Spherical mapping for convex surface (a) and for concave surface (b).

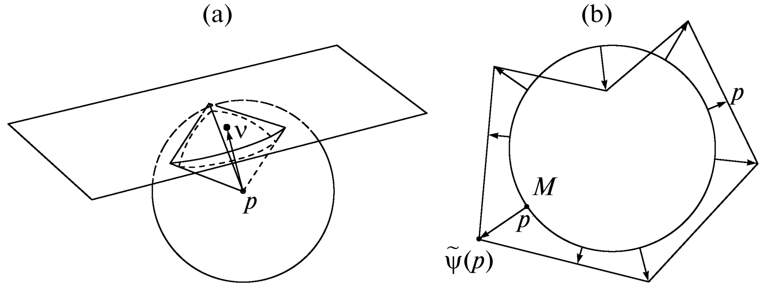

Fig. 5. (a) Spherical and normal images, (b) normal graph over the surface.

Let us remind that *spherical map* or *Gauss map* (see [23]) ϕ identifies with each point p of regular surface M a point $b = \nu(p)$ on a unit sphere $\mathbb{S}^2$. Matrix A is nothing else but the jacobian matrix of the spherical map ϕ.

If neighborhood of a point p lies on strictly convex surface then spherical mapping locally preserves orientation, while for strictly saddle surface it reverses orientation.

The set of points $\phi(p)$ for all $p \in Q$, where Q is subset of M is called *spherical image* of Q and if denoted below by $\psi(Q)$.

One can compute intersection point q between plane, passing through point $b = \nu(p)$ and orthogonal to $\nu(p)$ and a ray going through $\nu(p')$, where point p' belongs to some neighborhood of p. Mapping $p' \longrightarrow q$ is called *normal map* and defines *normal image* of neighborhood of p, as illustrated in Fig. 5a. Normal mapping is distance extending compared to spherical mapping. Normal and spherical mappings coincide at the point $p = p'$. It is said that polyhedral surface P_h is normal graph over M if projection $\tilde{\psi}$ of surface M onto P_h along normals to M is homemorphism, which is shown in Fig. 5b.

Extrinsic Gauss curvature K is the limit of the following ratio (specific curvature)

$$\frac{\text{area}\,\psi(Q)}{\text{area}\,Q} \longrightarrow K$$

assuming that $\text{diam}\,Q \longrightarrow 0$ and $\text{area}\,B/\text{area}\,Q \leqslant C$, where C is a constant and B is the smallest circle on the surface M, containing Q. If DC surface is smooth then its specific curvature is bounded. Intrinsic curvature of domain $Q \subset M$ can be defined via total angle of ∂Q using the Gauss–Bonnet theorem. Extrinsic curvature of Q is just area of the spherical image $\psi(Q)$. For sufficiently smooth surface intrinsic and extrinsic curvatures coincide which was established by Gauss is his famous Theorema Egregium (remarkable theorem see [23]).

For nonregular surfaces in general Theorema Egregium does not hold. However for DC surfaces intrinsic curvature coincides with extrinsic one in the following sense (see [24]): suppose that the boundary of $Q \subset M$ is simple closed curve γ, not passing through the conical points of the surface. Suppose that spherical image of this curve is certain, generally self-intersecting curve γ_s on the unit sphere. Then the curve γ_s can be partitioned into a finite set of simple closed curves with different orientations. One can assign to interior of each such curve a signed area where sign depends on the orientation of the bounding curve. Area of the spherical image of Q is computed as a sum of these signed areas and is equal to the intrinsic curvature of Q.

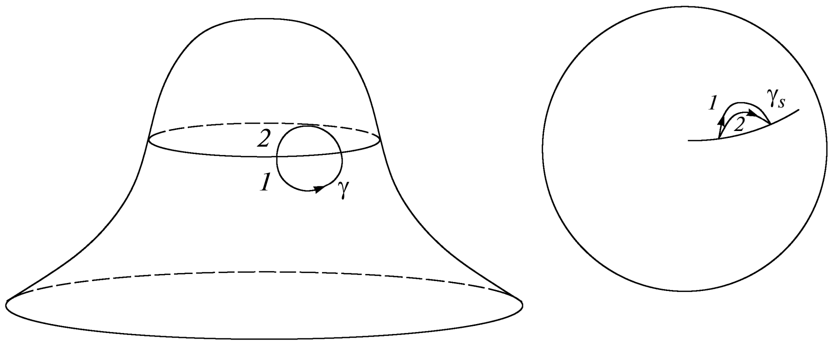

Fig. 6. Multivalued spherical mapping near zero Gauss curvature line.

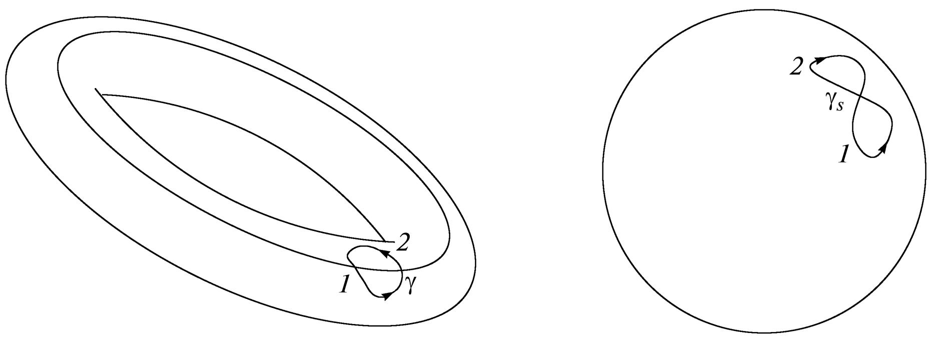

Fig. 7. Multivalued spherical mapping near zero Gauss curvature line.

One should note that absolute extrinsic curvature for DC surfaces can exceed absolute intrinsic curvature.

Multivalued behaviour of spherical mapping is illustrated in Fig. 6 and Fig. 7. In these examples surfaces are infinitely smooth and simple curve γ intersects line $K = 0$ hence its spherical image can be self-intersecting.

5. SPHERICAL AND NORMAL IMAGE FOR POLYHEDRA

Let us consider two-dimensional polyhedral surface in $\mathbb{R}^3$. For polyhedra intrinsic curvature is located at the vertices. For a vertex c shown in Fig. 8 it is equal to

$$K(c) = 2\pi - \Sigma\theta_i,$$

where θ_i is an angle of the i-th flat face incident to the vertex c.

In order to define extrinsic curvature of a polyhedron one can use concept of generalized support plane. Consider ball which touches polyhedral surface at the vertex c. Tangent plane to the ball at point c is called support plane at c. Unit normal vector to this plane defines nonunique normal vector at the point c. If c is the vertex of the convex cone K then directions of all normal vectors at c define dual cone K^*. Intersection of this dual cone with unit sphere centered at c defines convex spherical polygone $F_s(c)$ which is called spherical image of the point c (see [25]), while intersection of cone K^* with the tangent plane to this sphere at a certain internal point of F_s is planar convex polygone F_n which is called normal image of the point c.

Now consider the case of nonregular (or fan-like) vertex c. Consider simple contour γ around c shown in Fig. 9a. One can roll the tangent ball along curve γ. Unit normal to the rolling sphere at the touching point draw curve γ_c on the unit sphere which is called spherical image of the contour γ. Let us note that on the planar faces rolling ball can switch from inside to outside positions and vice versa. Obviously γ_c does not depend on the particular shape of curve γ as far as it intersects only faces incident to c.

In this example curve γ_c is self-intersecting. Extrinsic curvature of the vertex c is the sum of signed areas of simple subdomains Q^+ and Q^-, while absolute extrinsic curvature at c is the sum of absolute values of

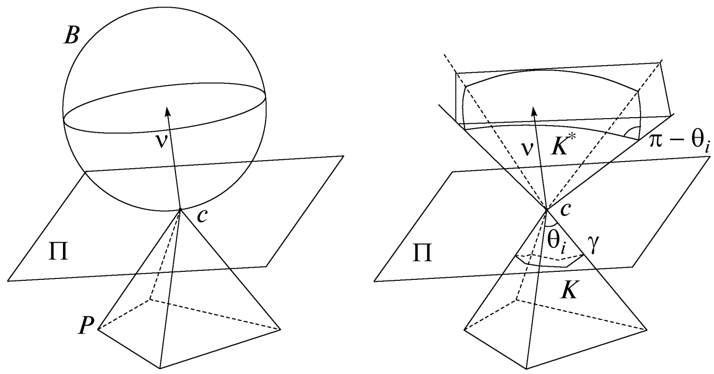

Fig. 8. Spherical image and normal image of the convex vertex.

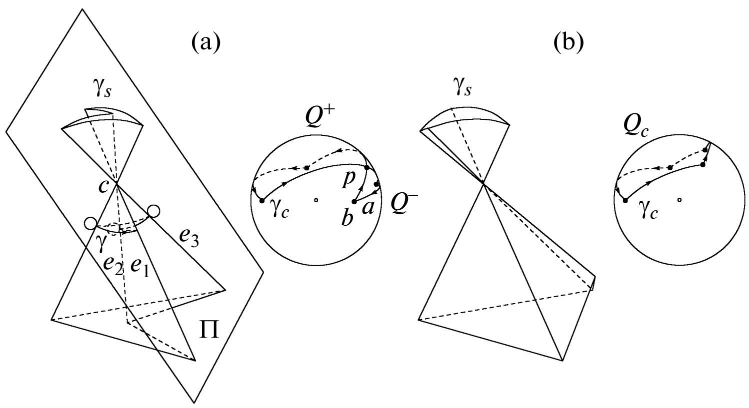

Fig. 9. (a) Spherical image of the fan vertex, (b) reflections preserve intrinsic curvature and lead to simple spherical image.

areas of Q^+ and Q^-. Let us consider a plane passing through edges e_1 and e_3 as shown in Fig. 9a. Reflecting edge e_2 relative to this plane, one can obtain convex cone shown in Fig. 9b. Obviously, such reflections do not change intrinsic curvature as well as keep sum of signed areas of elementary domains in spherical image intact.

6. EXTRINSIC DISCRETE CURVATURES BASED ON THE DUALITY PRINCIPLE

Consider 2D paraboloid

$$P = \left\{x_3 = u(x_1, x_2), u(x_1, x_2) = \frac{1}{2}(h_{11}x_1^2 + 2h_{12}x_1x_2 + h_{22}x_2^2)\right\}.$$

In the following we shall use upper index l to denote vectors from $\mathbb{R}^3$, while values without superscript l will denote their orthogonal projections onto plane $x_3 = 0$. Here "l" is abbreviation for "lifting" procedure, described in Section 2. Hence we will use notations

$$x^l = (x_1\ x_2\ x_3)^{\mathrm{T}}, \quad x = (x_1\ x_2)^{\mathrm{T}}.$$

It is convenient to write function u as $u(p) = \dfrac{1}{2}p^{\mathrm{T}}Hp$, where H is the shape operator matrix of the paraboloid P at the origin. It is assumed that symmetric matrix H is not singular.

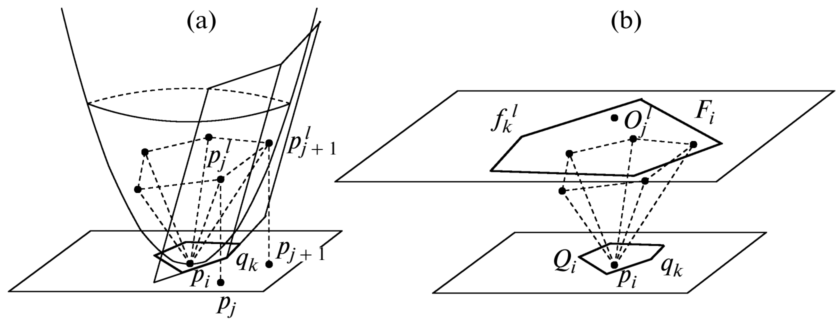

Fig. 10. (a) Polyhedral surface inscribed into elliptic paraboloid; (b) dual face and normal image of the vertex.

Consider fragments of the convex polyhedral surfaces P_h and P_h^* being polar with respect to the paraboloid P. Polarity relations mean in particular that each vertex q^l of P_h is dual to face Q of P_h^*, where the plane of this face is defined by equality

$$x_3 + (q^l)_3 = q^{\mathrm{T}} H x.$$

Polarity relations should be fully symmetric hence one can exchange P_h and P_h^*.

Figure 10a shows inscribed polyhedron P_h and superscribed polyhedron P_h^* which are particular cases of polar polyhedra.

Suppose that i-th vertex of P_h is given by

$$(p_i^l)^{\mathrm{T}} = (p_i^{\mathrm{T}}\ \delta_i), \quad p_i = 0.$$

Let us denote by $\mathcal{V}(p_i^l)$ the set of vertices of P_h belonging to edges, incident to p_i^l, while notation $\mathcal{V}(G)$ is used for the set of vertices of the face G.

Vertices p_j^l belonging to edges of P_h, incident to p_i^l are defined by relations

$$(p_j^l)^{\mathrm{T}} = (p_j^{\mathrm{T}}\ u(p_j) + \delta_j).$$

Values δ_j define deviation of the vertices of P_h from the surface of paraboloid.

The plane of the face, polar to the vertex p_j^l is defined by equality

$$x_3 + (p_j^l)_3 = p_j^{\mathrm{T}} H x, \tag{12}$$

while plane polar to p_i^l is given as

$$x_3 + \delta_i = 0. \tag{13}$$

Consider face Q_i of the polyhedron P_h^* polar to vertex p_i^l. One can construct normal image of the vertex p_i^l, namely the convex polygon F_i on the plane $x_3 = 1 + \delta_i$. The vertices of this polygon are intersections of rays passing through p_i^l and orthogonal to faces, incident to p_i^l, with the plane $x_3 = 1 + \delta_i$. Polygons Q_i and F_i are shown in Fig. 10b.

Consider vertex q_k^l of the face Q_i, polar to the face G_k which is incident to vertex p_i^l. Vector q_k^l is intersection of the planes polar to the vertices of G_k, thus using Eqs. (12) and (13) one obtains

$$-\delta_i + u(p_j) + \delta_j = p_j^{\mathrm{T}} H q_k, \quad \text{for} \quad j \in \mathcal{V}(G_k), \quad i \neq j.$$

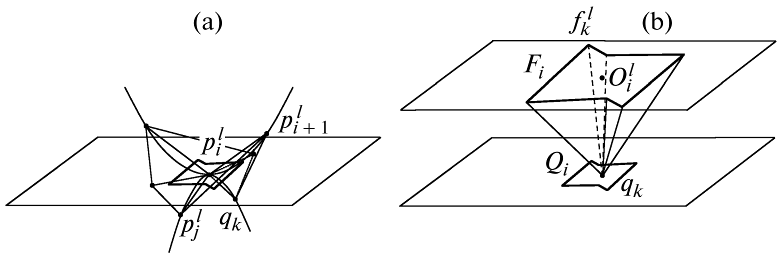

Fig. 11. (a) Polyhedral surface inscribed into hyperbolic paraboloid; (b) dual face and normal image of the vertex.

Now let us find vertex f^l_k of the polygon F_i. Its coordinate vector is the solution of the linear system

$$\begin{aligned} n^l(p^l_i)^{\mathrm{T}}(f^l_k - p^l_i) &= \left|n^l(p^l_i)\right|, \\ (p^l_j - p^l_i)^{\mathrm{T}}(f^l_k - p^l_i) &= 0, \quad j \in (\mathcal{V})(G_k), \quad i \neq j, \end{aligned} \tag{14}$$

where vector n^l is normal to face Q_i, which can be assigned to vertex p^l_i as well, and p^l_j are vertices of a face $G_k \in \mathrm{star}(p^l_i)$, i.e. incident to p^l_i. In our case $n^l(p^l_i) = (0\ 0\ 1)^{\mathrm{T}}$. From Eq. (14) it follows

$$(f^l_k)_3 = 1 + \delta_i$$

and

$$u(p_j) + \delta_j - \delta_i + p_j^{\mathrm{T}} f_k = 0, \quad j \in \mathcal{V}(G_k), \quad i \neq j.$$

Hence one obtains the following equality

$$p_j^{\mathrm{T}} H q_k = -p_j^{\mathrm{T}} f_k, \quad j \in \mathcal{V}(G_k), \quad i \neq j. \tag{15}$$

Since matrix with vectors p_j as a columns has full rank we get $f_k = -Hq_k$. As a result the following theorem is proved.

Theorem 3. (see [28]). *Polygons F_i and Q_i are affine equivalent, i.e. $Q_i = \phi^*_i(F_i)$ and jacobian matrix of the affine map ϕ^*_i coincides with $-H$, where H is the matrix of the shape operator of the paraboloid P at the origin.*

Let us note that deriving equality $f_k = -Hq_k$ we did not use the fact that matrix H is positive definite. Formally (15) holds for arbitrary matrix H. It is just required that matrix with vectors p_j as columns has the full rank. Thus duality principle for computation of curvature tensor can be applied in the case of hyperbolic paraboloid, shown in Fig. 11.

From the duality principle it follows that the face G_k of the polyhedron P_h, incident to p^l_i, corresponds to vertices q^l_k and f^l_k. If faces G_m and G_k have common edge, then vertices q^l_m and q^l_k should be connected by an edge as well. The same is true for vertices f^l_m and f^l_k. These arguments can be applied in the case when boundary of F_i and Q_i is self-intersecting closed polyline. In this case vertex p^l_i can be called conical or sharp vertex since polyhedron P_h provides poor local approximation to paraboloid P in the neighborhood of p^l_i.

6.1. Generalization to Multiple Dimensions

One can easily check that the proof of Theorem 3 does not use the fact that the paraboloid P is two-dimensional one. One can consider general case of d-dimensional paraboloid in $\mathbb{R}^{d+1}$. In this case $(x^l)^{\mathrm{T}} = (x^{\mathrm{T}}\ x_{d+1})$,

$x \in \mathbb{R}^d$, $(p_i^l)^{\mathrm{T}} = ((p_i)^{\mathrm{T}}\,(p_i^l)_{d+1})$. Face Q_i, dual to vertex p_i^l and normal image F_i of p_i^l are d-dimensional polyhedra. Paraboloid P is defined by $x_{d+1} = \frac{1}{2}x^{\mathrm{T}}Hx$. Polar plane for the vertex

$$(p_j^l)^{\mathrm{T}} = \left(p_j^{\mathrm{T}} \;\; \frac{1}{2}p_j^{\mathrm{T}}Hp_j + \delta_j\right)$$

looks like

$$(p_j^l)_{d+1} + x_{d+1} = x^{\mathrm{T}}Hp_j.$$

If in the coordinate frame x_i one assumes that $(p_i^l)^{\mathrm{T}} = (0 \;\; \delta_i)$, then $(d+1)$-th coordinate of the face Q_i is equal to $-\delta_i$ and equality (15) holds along with multidimensional counterpart of Theorem 3.

7. LOCAL POLARITY RELATIONS AND DISCRETE CURVATURES

Let us consider neighborhood of a regular point on two-dimensional DC surface M where second differential and tangent paraboloid are defined. One can choose cartesian frame x_i in such a way that surface M can be locally written as $x_3 = f(x_1, x_3)$, and

$$f(x_1, x_2) = u(x_1, x_2) + o(|x|^2), \quad h_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(0, 0),$$

which means that P is precisely the tangent paraboloid at certain point $\tilde{p}_i^l$ of the surface M. We will use notation f^l for the function $f^l(x^l) = x_3 - f(x_1, x_2)$. Surface M is defined by equality $f^l(x^l) = 0$.

Consider again a pair of polyhedral surfaces P_h and P_h^*. We assume that locally they are almost polar with respect to P in the following sense: let

$$(p_i^l)^{\mathrm{T}} = (o(h^2)\;\; \delta_i + o(h^2)) \tag{16}$$

and

$$(p_j^l)^{\mathrm{T}} = (p_j^{\mathrm{T}}\;\; \delta_j + u(p_j) + o(h^2)), \quad |p_j| = O(h),$$

where $j \in \mathcal{V}(G_k)$, $j \neq i$ be indices of the vertices of the face G_k of P_h, incident to p_i^l. Parameter h has the meaning of local characteristic edge length and simultaneously the size of the neighborhood of the regular point $\tilde{p}_i^l$ on the surface M.

Relation (16) means that in the coordinate frame $\{x_1, x_2, x_3\}$ point p_i^l lies almost on the same vertical line as the point $\tilde{p}_i^l$ and tangent plane to M at $\tilde{p}_i^l$ is horizontal.

The plane of the face Q_i polar to p_i^l is given by

$$x_3 + \delta_i + o(h^2) = 0, \tag{17}$$

while plane of the face Q_j, polar to p_j^l, is given by

$$x_3 + (p_j^l)_3 = p_j^{\mathrm{T}}Hx + o(h^2). \tag{18}$$

Relations (17), (18) make sense for $|x| \leqslant O(h)$. In practice matrix H is not known hence in order to approximately compute plane polar to point p_j^l one has to compute approximate orthogonal projection point $\tilde{p}_j^l \in M$, such that there exists scalar value α satisfying

$$\alpha(p_j^l - \tilde{p}_j^l) = \nabla f^l(\tilde{p}_j^l) + o(h)$$

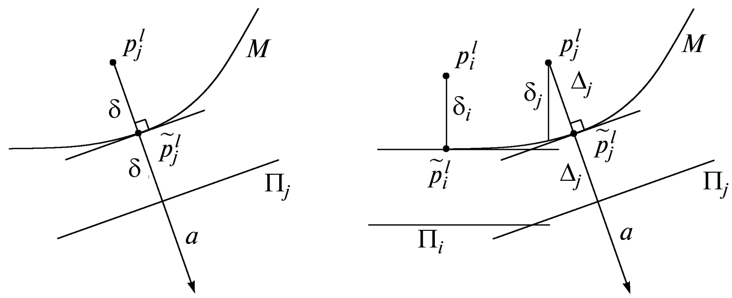

Fig. 12. Vertex and approximate polar plane.

which is illustrated in Fig. 12. Here ∇f^l denotes directional gradient at $\tilde{p}_j^l$. It is not unique, but difference between gradients in any direction is within $o(h)$.

Plane Π_j polar to point p_j^l is defined by equality

$$\nabla f^l(\tilde{p}_j^l)^{\mathrm{T}}(x^l - 2\tilde{p}_j^l + p_j^l) = 0. \tag{19}$$

Obviously it satisfies the polarity relation with respect to tangent paraboloid at the point $\tilde{p}_j^l$. Let us show that Π_j satisfies polarity relation with respect to the paraboloid P, i.e. relation (18) hold.

In the following we consider neighborhood $|x| \leqslant O(h)$ and assume that $p_j = O(h)$. Let us introduce vector $(a^l)^{\mathrm{T}} = (a^{\mathrm{T}}\, a_3)$ parallel to the difference $p_j^l - \tilde{p}_j^l$ and satisfying $a_3 = -1$. Then $a = O(h)$. Point $\tilde{p}_j^l$ is the solution of the following nonlinear equation

$$f^l(p_j^l + ta^l) = 0$$

or

$$(p_j)_3 + ta_3 - \frac{1}{2}(p_j + ta)^{\mathrm{T}}H(p_j + ta) + o(h^2) = 0.$$

Assuming that δ_j cannot be larger than $O(h)$ one obtains

$$t = \delta_j + o(h^2). \tag{20}$$

Substituting relation (20) into (19) one obtains

$$x_3 - p_j^{\mathrm{T}}Hx + \frac{1}{2}p_j^{\mathrm{T}}Hp_j + \delta_j = o(h^2) - ta^{\mathrm{T}}Hx - t^2a^{\mathrm{T}}Ha. \tag{21}$$

Obviously right hand side of relation (21) behaves like $o(h^2)$ thus it is shown that approximate polarity relation (18) holds. The case when P_h is inscribed polyhedron and P_h^* is the superscribed one is the particular case of the above derivation.

Denote again by F_i normal image of the vertex p_i^l. The vertex f_k of the polygon F_i can be computed via solution of linear system (14) resulting in equation

$$u(p_j) + \delta_j - \delta_i + p_j^{\mathrm{T}}f_k + o(h^2) = 0, \quad j \in \mathcal{V}(G_k), \quad i \neq j$$

while vertex q_k of the dual face Q_i is approximate solution of the intersection problem of the plane of Q_i (17) with the planes of faces Q_j defined by relations (18). As a result one obtains

$$u(p_j) + \delta_j - \delta_i = p_j^{\mathrm{T}}Hq_k + o(h^2)$$

and eventually

$$p_j^{\mathrm{T}}(Hq_k - f_k) = o(h^2). \tag{22}$$

(a) (b)

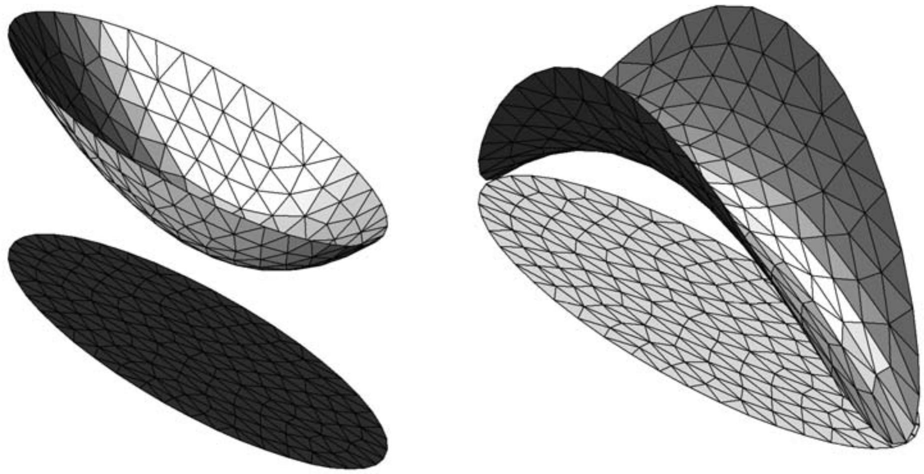

Fig. 13. Triangulations of elliptic and hyperbolic paraboloids and their projections onto horizontal plane.

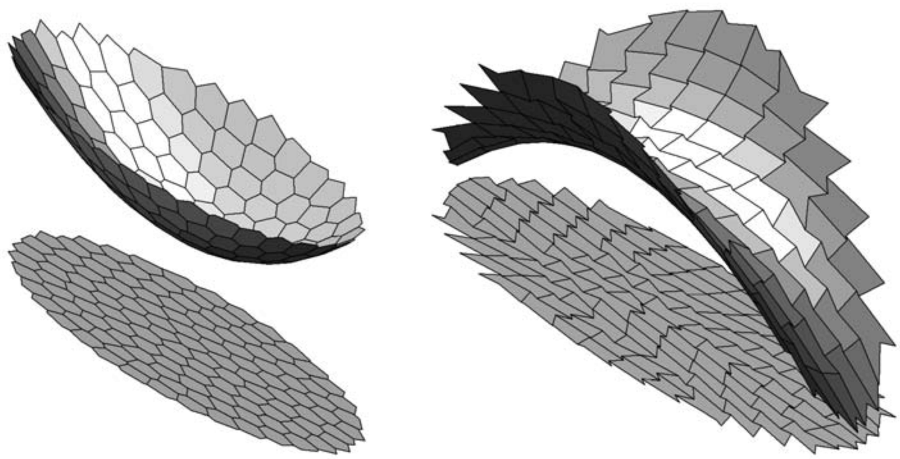

Fig. 14. Dual polyhedral surfaces and their projections.

Before coming to interpretation of relation (22) let us introduce notion of regularity for dual polyhedra.

Definitions 5. The vertex p_i^l of polyhedral surface P_h is called regular if its dual polygon Q_i and orthogonal projection of the normal image polygon F_i onto the plane of polygon Q_i are simple polygons and contain point p_i^l strictly inside.

Thus, one can say that regular vertex is not conical one since at such vertex tangent plane is uniquely defined. The same can be said about regular edge, i.e. the edge with both vertices being regular and both incident faces dual to regular vertices. We say that regular edge is not a sharp one since one can assign tangent plane to this edge where tangent basis vectors are parallel to edge itself and its dual edge.

On Fig. 13 there are shown fragments of triangulated polyhedral surfaces P_h inscribed into elliptic and hyperbolic paraboloid and their projection on the plane $x_3 = 0$. Dual polyhedral surfaces P_h^* and their projections are shown in Fig. 14. In the case of elliptic paraboloid all dual faces are convex polygons, while in the hyperbolic case dual faces Q_i are quadrilaterals with concave sides (edges), i.e. the turn of edges from the side of Q_i is nonpositive.

It should be noted that the dual polyhedral surface P_h^* in the case of elliptic paraboloid is precisely the Voronoi generatrice and projection of its faces onto horizontal plane defines partitioning of a plane into convex polygons being affine image of the Voronoi partitioning.

Obviously in the general case of DC surface dual face and corresponding normal image are no longer affine equivalent. However relation (22) leads us to conclusion that they are almost affine equivalent. In order to formalize this statement one has to introduce notion of shape regularity for faces of approximating polyhedra. We say that face Q_i is shape regular if its dual vertex p_i^l is regular and there exists triangulation of $\mathcal{T}_i^Q$ of Q_i where length l of any edge of triangulation satisfies inequality

$$Ch \leqslant l \leqslant h$$

for certain positive constant C, while the minimal angle of the triangulation is bounded from below by a constant not depending on h. Thus one can construct piecewise affine mapping $\phi_i^*: Q_i \longrightarrow F_i$ being linear on each triangle. Denote by $-A_{im}^*$ jacobian matrix of affine map of triangle $T_{im}^* \in \mathcal{T}_i^Q$ onto m-th triangle of $\mathcal{T}_i^F$, i.e.

$$A_{im}^* = -\nabla\phi_i^*\big|_{T_{im}^*}. \tag{23}$$

Now we are in the position to formulate the following theorem.

Theorem 4. (About local approximation of DG surfaces by polyhedra) *Consider neighborhood of the regular point c on the* DC *surface with the size O(h). Let regular vertex p_i^l of the polyhedral surface P_h and the plane of dual face Q_i be locally polar with respect to the tangent paraboloid P at point c. Suppose that regular vertices p_j^l belonging to faces of P_h adjacent to p_i^l are locally polar with respect to P to faces Q_j of P_h^*. Suppose that faces of P_h adjacent to p_i^l and face Q_i of* a P_h^* *are shape regular. Then with* $h \longrightarrow 0$

$$A_{im}^* = H + o(1),$$

where H is the curvature tensor at the point c.

Theorem 4 immediately follows from (22). It is important that formulation of this theorem is completely symmetric in a sense that one can exchange P_h and P_h^*. Of course in practice P_h is triangulated surface, while P_h^* has general polygonal faces which is illustrated in Fig. 14. Dual formulation of Theorem 4 in this case simply says that mapping $\phi_k: G_k \longrightarrow B_k$ of triangular face G_k of P_h dual to vertex q_k^l to normal image B_k of vertex q_k^l is affine and its gradient converges to matrix H of tangent paraboloid at point $\tilde{q}_k^l$. Here $\tilde{q}_k^l$ denotes approximate orthogonal projection of point q_k^l onto DC surface M.

Let us remark that matrices A_{im}^* in general are not symmetric. Hence in order to compute principal curvatures one have to use singular values of these matrices instead of eigenvalues, and the singular value decomposition (SVD) of A_{im}^* should be used for computation of approximate principal directions.

It is possible to prove local approximation Theorem 4 under relaxed shape regularity requirements for faces of surfaces P_h and P_h^*. To this end let us consider local frame $\{x_1, x_2, x_3\}$ where tangent paraboloid is written as $x_3 = \frac{1}{2}x^{\mathrm{T}}Hx$ and consider affine transformation

$$y = Ux, \quad y_3 = x_3,$$

such that $\det U = 1$ and matrix $H' = U^{\mathrm{T}}HU$ is well conditioned in a sense that there exists constant C such that eigenvalues of matrix H' satisfy the following inequality

$$\frac{1}{C} \leqslant \frac{|\lambda_1(H')|}{|\lambda_2(H')|} \leqslant C.$$

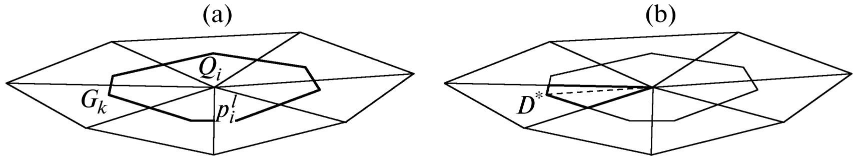

Fig. 15. Construction of piecewise-affine homeomorphism $w_h : P_h^* \longrightarrow P_h$.

Since entries of the matrix H are bounded from above, entries of the matrix U^{-1} are bounded from above as well. Then the following statement hold.

Remark. Suppose that the conditions of the local approximation Theorem 4 hold in transformed coordinate frame y_i. Then local approximation of curvature is attained in original coordinate frame x_i as well.

Geometric meaning of this remark is simple: when matrix H is anisotropic, say with one eigenvalue being $O(1)$ and another one being arbitrary small, then faces of the polyhedral approximations can be sharply elongated along direction of minimal eigenvalue of matrix H. Parameter h here has the meaning of the characteristic edge length in the transformed coordinates y_i.

7.1. Curvature Deviation Measures

In practice it may happen that only one polyhedral approximating surface P_h is available while surface M is not known. In this case one have to reconstruct dual polyhedral surface P_h^* using certain apriori information about surface M. Let us consider the case when P_h is inscribed polyhedron and M is twice continuously differentiable surface. Then faces of the dual surface P_h^* are defined by the planes passing through vertices of P_h. Hence unknowns in the P_h^* recontruction problem are normal vectors at the vertices of P_h defining approximate tangent planes. Of course one can use well known methods for computing approximate normals to polyhedral surfaces which guarantee that deviation from exact surface normal is within $o(h)$. This condition along with regularity and shape regularity conditions guarantees convergence of discrete curvatures to exact ones.

On the other hand one can attain the same objective via reformulating reconstruction of the dual surface P_h^* as the optimization problem for a certain curvature deviation measure. Denote by ϕ_h and ϕ_h^* piecewise affine mappings coinciding with ϕ_k on G_k and with ϕ_i^* on Q_i, respectively. We will use notations $\nabla\phi_h$ and $\nabla\phi_h^*$ for piecewise constant functions which coincide with jacobian matrices of mappings ϕ_h and ϕ_h^* where the jacobians are well defined.

The formulation of the optimization problem is very simple: surface P_h provides piecewise affine mapping ϕ_h, while surface P_h^* provides mapping ϕ_h^*. Since both mappings are supposed to approximate the same spherical mapping ϕ, dual surface P_h^* providing minimal value for certain deviation measure between ϕ_h and ϕ_h^* guarantees that $\nabla\phi_h$ and $\nabla\phi_h^*$ converge to $\nabla\phi$.

In order to compare mappings ϕ_h and ϕ_h^* one have to introduce piecewise affine homeomorphism ω_h which maps P_h onto P_h^*. Homeomorphism can be constructed face by face using projection and intersection of triangulations of general shape faces.

In the case of convex surface M construction of such a homeomorphism is especially simple.

On Fig. 15a it is shown fragment of polyhedral surface, with direction of view being orthogonal to the plane of polygon Q_i. Projection of the face G_k onto the plane of dual face Q_i has nonempty intersection with Q_i, namely polygon D^*, shown in Fig. 15b. If vertices of P_h adjacent to p_i^l and vertices of the face Q_i are regular then domain D^* consists of two triangles from $\mathcal{T}_i^Q$. Preimage of D^* is the quadrilateral D belonging to the face G_k also consisting of two triangles from $\mathcal{T}_k^G$. Thus mapping $w_h : D^* \longrightarrow D$ is affine for each triangle from these pairs.

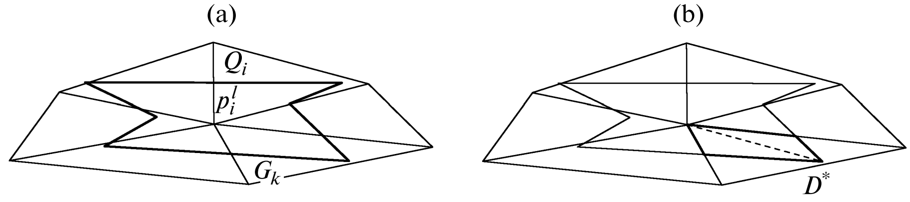

Fig. 16. Construction of piecewise-affine homeomorphism $w_h: P_h^* \longrightarrow P_h$, non-convex case.

On Fig. 16 it is shown fragment of the saddle surface. In this case dual face Q_i is nonconvex and it is too restrictive to require that Q_i is star-shaped with respect to p_i^l hence one should use intersection of general triangulations.

As a result the curvature deviation measure can be introduced as follows

$$\delta(p) = \left\|\nabla\phi_h(p) - \nabla\phi_h^*(w_h^{-1}(p))\right\|,$$

where p is a point lying on P_h, and function $\delta(p)$ is piecewise constant. Here $\|\cdot\|$ means the Frobenius matrix norm.

Theorem 5. *Suppose that polyhedral surface P_h is inscribed into twice continuosly differentiable surface M and H is the matrix of the tangent paraboloid at the vertex p_i^l. Suppose that all vertices of faces of P_h incident to p_i^l are regular and their dual faces are shape regular. Let* faces *of P_h incident to p_i^l be shape regular and vertices of P_h^* dual to these faces be regular. If with $h \longrightarrow 0$ relation*

$$\delta(p) = o(1) \tag{24}$$

holds for all $p \in \mathrm{star}(p_i^l)$ (i.e. for all faces of P_h incident to vertex incident to vertex p_i^l) then

$$\nabla\phi_h^*|_{Q_i} = H + o(1).$$

Proof. One can consider the paraboloid P_i and the set of paraboloids P_k^* corresponding to vertices q_k^l of the face Q_i. With each paraboloid one can associate coordinate frame. Suppose that P_i is defined as $x_3 = u(x_1, x_2)$, while P_k^* is defined as $y_3 = v_k(y_1, y_2)$. Face Q_i and vertex p_i^l are locally polar with respect to P_i, while vertex q_k^l and face G^k (incident to p_i^l) are locally polar with respect to P_k^*. Relation (24) means that

$$u(x_1, x_2) = \tilde{v}_k^*(x_1, x_2) + o(|x|^2), \tag{25}$$

where $\tilde{v}_k^*$ is height function for paraboloid P_k^* in the coordinate frame x_i. Thus face G^k and vertex q_k^l are locally polar with respect to P_i. Since relation (24) holds for all $p \in \mathrm{star}(p_i^l)$ we obtain that (25) holds for vertex p_j^l and its associated coordinate frame, hence P_i can be considered as tangent paraboloid at p_i^l and Theorem 4 can be used to finalize the proof.

One can consider dual formulation of Theorem 5. Suppose that for the same surface M circumscribed polyhedron P_h^* is given while the only available information about P_h is that its vertices belong to faces of P_h^*. If relation (24) holds for $p \in \mathrm{star}\ q_k^l$ then one can show that P_h is approximately inscribed polyhedron and with $h \longrightarrow 0$

$$\nabla\phi_h|_{G_k} = A(\tilde{q}_k^l) + o(1),$$

where $A(\tilde{q}_k^l)$ is curvature tensor of M at the point $\tilde{q}_k^l$ being approximate orthogonal projection of q_k^l onto M.

8. DISCRETE APPROXIMATIONS TO BENDING ENERGIES

Consider regular surface M and functional

$$E_g(M) = \int_M g(A)d\sigma, \quad g(A) \geqslant |\det A|, \tag{26}$$

where $d\sigma$ is the surface area element, $A \in \mathbb{R}^{2\times 2}$ is the shape operator matrix or curvature tensor and $g(A)$ is certain curvature density measure. Functional $E_g(M)$ is called bending energy of the surface. If energy $E_g(M)$ is bounded, then absolute Gauss curvature of the surface is bounded as well.

Well known example of bending energy is given by the mean total quadratic curvature measure

$$E_2(M) = \frac{1}{2}\int_M \mathrm{tr}(A^{\mathrm{T}}A)d\sigma. \tag{27}$$

Absolute minimum of this functional is attained when a surface homeomorphic to sphere is precisely the sphere. Bending energy (27) takes finite values for the surfaces of finite bending (see [26]). For disk-like domain Ω on such a surface one can find local parameterization $x(\xi) : Q \longrightarrow \Omega$, $Q \subset \mathbb{R}^2$ in the Sobolev class $W_2^2(Q)$. On the other hand, mean quadratic curvature measure is not suitable for description of non-regular surfaces since it is not defined for polyhedra. In other words it takes infinite value for polyhedral surface.

If bending energy E_g majorates absolute curvature and remains bounded for refined sequence of polyhedra then one can expect that the limiting surface for this sequence will be surface of bounded curvature. One can consider the following curvature measure which makes sense for polyhedra:

$$E_1(M) = \int_M \{[\mathrm{tr}(A^{\mathrm{T}}A)]^{1/2} + |\det A|\}d\sigma. \tag{28}$$

Discrete counterpart of the energy E_1 can be easily constructed for convex surfaces replacing sharp edges with strips of cylindrical surfaces with radius r and rounding conical vertices using ball with radius r. Going to the limit $r \longrightarrow 0$ one can obtain the following discrete counterpart of energy (28)

$$E_1(P_h) = \sum_e l_e|\phi_e| + \sum_i |\mathrm{area}(\psi(c_i))|,$$

where l_e is the length of the edge e, ϕ_e is the angle between normals to faces adjacent to e, while $\psi(c_i)$ denotes spherical image of the internal vertex c_i of P_h and $|\mathrm{area}(\psi(c_i))|$ essentially means total variation of the are a of spherical mapping.

Duality-based discretization of the bending energy leads to quite different discrete counterparts of E_g, Since in this case two polyhedral surfaces P_h and P_h^* are available, one can construct two discrete bending energies: bending energy for dual polyhedron P_h^*

$$E_g(P_h^*) = \sum_i g(A)|_{Q_i}\mathrm{area}(Q_i), \tag{29}$$

where

$$A|_{Q_i} = \nabla\phi_h^*|_{Q_i}$$

is gradient of piecewise-affine discrete approximation to spherical mapping on the face Q_i, while

$$E_g(P_h) = \sum_k g(A)|_{G_k}\mathrm{area}(G_k), \tag{30}$$

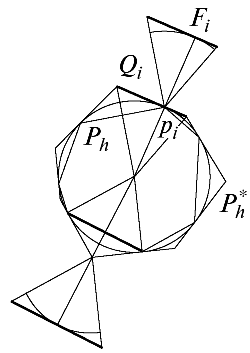

Fig. 17. For unit sphere dual face and normal image are congruent.

where

$$A|_{G_k} = \nabla \phi_h|_{G_k}$$

is gradient of piecewise-affine discrete approximation to spherical mapping on the face G_k. When conditions of the local approximation Theorem 4 hold for all vertices of P_h and P_h^*, and surface M is regular enough then both discrete energies converge to exact bending energy $E_g(M)$.

Nice property of the duality based discretization is that it provides exact values of curvatures for polyhedron, inscribed into sphere. Consider polyhedra inscribed and circumscribed around unit d-dimensional sphere in $\mathbb{R}^{d+1}$. On Fig. 17 it is shown simplest case of a unit circle ($d = 1$).

One can easily see that in the 1d case the face Q_i, dual to the vertex p_i^l of the inscribed polyhedron P_h is congruent to the normal image of vertex p_i^l. Hence discrete shape operator matrix is equal to minus identity matrix. It is also obvious that the same congruence property holds in the d-dimensional case as well.

In 2004 A.I. Bobenko (see [27]) introduced discrete Willmore energy $W(P_h)$ (called also conformal energy) being invariant to 3D Mebius transforms. This discrete energy is supposed to approximate exact Willmore energy

$$W(M) = \frac{1}{2}\int_M (k_1 - k_2)^2 d\sigma = \int_M \left[\frac{1}{2}\operatorname{tr}(A^{\mathrm{T}}A) - \det A\right] d\sigma,$$

where k_i are principal curvatures. In [27] it was proven that $W(P_h) = 0$ when P_h is convex polyhedron inscribed into sphere.

The same property obviously holds for duality-based energy

$$E(P_h^*) = \sum_i \left[\frac{1}{2}\operatorname{tr}(A^{\mathrm{T}}A) - \det A\right]\Bigg|_{Q_i} \operatorname{area}(Q_i) \tag{31}$$

since matrix A on each dual face Q_i is equal to $-I$. Obviously equality $E(P_h) = 0$ holds as well since matrix A on the face G_k is equal to unity matrix multiplied by scalar converging to 1 with $h \longrightarrow +0$.

Main differences with conformal energy are that the discrete duality-based energy properties are the same for d-dimensional sphere in $\mathbb{R}^{d+1}$, and terms in discrete Willmore energy (31) converge to, the same terms in exact Willmore energy without any assumptions on polyhedra P_h and P_h^* beside those guaranteeing area convergence. This is not the case for conformal energy which converges only for special polyhedra.

9. DUALITY BASED CURVATURES MEASURES FOR NONSMOOTH SURFACES AND DISCRETE CURVATURES

As it was mentioned above, the set of conical points M_c on DC surface M is at most countable. Moreover, the number of conical points with absolute extrinsic curvature above certain threshold ε (*essential* conical points) is finite in every bounded domain on M. The set M_e of sharp edges on DC surfaces consists of at most countable set of rectifiable curves. Cutting off sharp edges where dihedral angle is larger than $\pi - \varepsilon$ leads to *essential* sharp edges made up of the finite set of curves with the bounded variation of turn.

Measure of the spherical image $\mu(\psi)$ for DC surface M is fully additive function of Borel sets with bounded variation (see [24, 25]). In other words it is a nonpositive definite measure with bounded variation (charge distribution). Classical Lebesque decomposition (see, e.g., [21, 22])

$$\mu(\psi(M)) = C(M) + S(M) + D(M) \tag{32}$$

says that $\mu(\psi(M))$ can be decomposed into absolutely continuous part $C(M)$, which takes zero value for any Borel set of zero measure, singular part $S(M)$ defined on the sets of zero measure, and discrete part $D(S)$ defined on the countable point set. Fortunately, for DC surfaces Lebesque decomposition can be written explicitly. Namely, for any Borel set $B \subset M$

$$D(B) = \sum_{c_i \in B} \text{area}(\psi(c_i)),$$

where c_i denotes i-th conical vertex and area $(\psi(c_i))$ is measure of its spherical image. Singular charges are given by

$$S(B) = \sum_{\gamma_i \cap B} \text{area}(\psi(\gamma_i \cap B)),$$

where γ_i is i-th curve of the set of sharp edges and $\psi(\gamma)$ is spherical image of the sharp edge γ_i.

Absolutely continuous measure C can be obtained from $\mu(\psi)$ by assigning zero measure to all sharp edges and conical vertices, or equivalently, for any Borel set B one should exclude its intersection with the M_e and M_c, which is correct operation since M_e and M_c are Borel sets as well. Radon–Nikodim differentiation theorem for absolutely continuous measures allows to obtain the following representation for C

$$C(M) = \int_M K d\sigma,$$

where K is measurable function and $d\sigma$ is standard surface measure differential. Function K is nothing else but the Gauss curvature of the DC surface M which as it was mentioned above is defined and finite almost everywhere on M.

In the conventional approach spherical image for polyhedra is concentrated at the vertices. It means that by definition area of the spherical image is discrete measure. Hence approximation of DC surfaces by polyhedra implies approximation of general nonpositive measure with bounded variation by discrete measure. Of course such approximation is possible only in a weak sense. Precise formulation of weak convergence results can be found in [15, 24, 25]. Essential drawback of weak convergence is that one cannot expect pointwise convergence of curvature even for smooth enough surface M.

Thus natural formulation of the approximation problem should imply the construction of the sequence of polyhedral surfaces P_k such that on each surface measure of spherical image $\mu(\psi_k)$ for P_k is defined in such a way that

$$\mu(\psi_k(P_k)) = C_k(P_k) + S_k(P_k) + D_k(P_k) \tag{33}$$

and with $k \longrightarrow +\infty$

$$C_k(P_k) \longrightarrow C(M), \quad S_k(P_k) \longrightarrow S(M), \quad D_k(P_k) \longrightarrow D(M). \tag{34}$$

Moreover

$$C_k(P_k) = \int_{P_k} K_k d\sigma,$$

where K_k has the meaning of Gauss curvature of polyhedral surface P_k and function K_k converges to K at least in the sense of measure. Notion of convergence (34) requires clarification. Using the idea of duality, one can formulate the following concept of regular convergence in the sense of (34). We consider a pair of surfaces P_k and P_k^* which converge to DC surface M in a pointwise manner, and M lies between P_k and P_k^*. A pair of bilipschitz mappings $\beta_k : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ and $\beta_k^* : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ exists such that

$$P_k = \beta_k(M), \quad P_k^* = \beta_k^*(M),$$

and equivalence constants of mappings β_k, β_k^* converge to 1 when $k \longrightarrow \infty$. Moreover β_k and β_k^* map essential conical points on M onto conical points (vertices) on P_k and P_k^* while essential edges of M are mapped onto sharp edges of P_k and P_k^*. It is convenient to use 2d mappings

$$\tilde{\beta}_k = \beta_k|_M, \quad \tilde{\beta}_k^* = \beta_k^*|_M.$$

Images of any Borel set $B \subset M$ under these mappings are Borel sets $B_k \subset P_k$ and $B_k^* \subset P_k^*$. Convergence of absolutely continuous part means that

$$K_k(\beta_k(p))\det\nabla\tilde{\beta}_k(p) \longrightarrow K(p)$$

and

$$K_k^*(\beta_k^*(p))\det\nabla\tilde{\beta}_k^*(p) \longrightarrow K(p)$$

in the sense of measure. Positive and negative part of K_k and K_k^* should converge to positive and negative part of K as well. Convergence of the discrete part means that areas of spherical images of sharp vertices in B_k and B_k^* converge to areas of spherical images of their preimages in B. Positive and negative part of these areas should converge separately. And at last, area of spherical image of an arbitrary Borel fragment of the essential edge curve on M should be approximated by areas of its images on P_k and P_k^*. In order to ensure existence of mappings β_k, β_k^* one should require strengthened condition of local almost isometry of DC surface to tangent cone compared to the original Alexandrov formulation.

Precise statement of the approximation theorem and rigorous proof is beyond the scope of this paper. Here we just illustrate the concept of such "regular" convergence considering DC surface M glued from finite set of elementary patches M_i, where M_i is either regular convex surface or saddle surface, or developable surface. It is assumed that tangent paraboloid exists for each internal point of M_i and for boundary points as well meaning existence of directional second differential. The boundary curve ∂M_i should be regular enough. In order to construct polyhedral approximations one should glue polyhedral surfaces P_k and P_k^* from elementary patches cut from a) convex polyhedral surface, b) simple saddle polyhedral surface, or c) polyhedral developable surface.

Figure 18 shows developable surface glued from elementary cylindrical and conical patches.

In this case elementary patches can be chosen in such a way that they lay on convex surfaces. Figure 19 shows spherical mapping for the simple case of the side surface of the cone. Spherical image of the cone consists of the image of the summit which in this case is circle on the sphere, while image of the side surface is nothing else but the boundary of this circle.

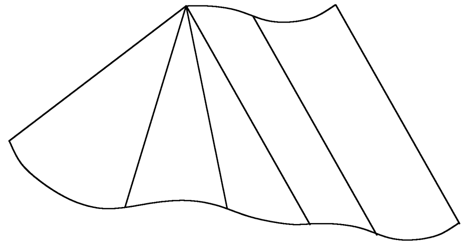

Fig. 18. Developable surface glued from elementary patches.

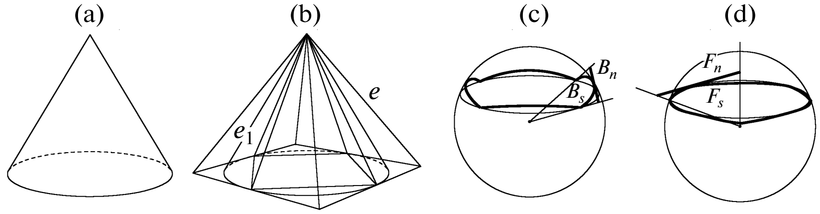

Fig. 19. Dual polyhedral approximation for side surface of the circular cone.

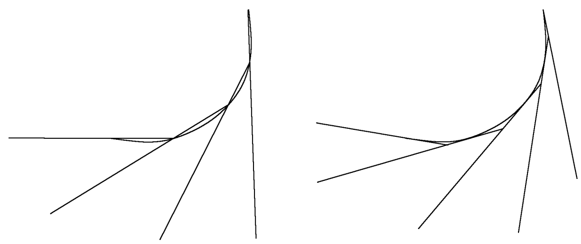

Fig. 20. Developable torse surface.

In the case of conical surface instead of polarity relation "point–plane" one should consider polarity relation "plane–straight line," both of them passing through the summit of the cone. For cylinder the summit of the cone should be set infinitely far on the axis. The counterpart of tangent paraboloid in this case in a tangent circular cone. Thus local polarity relation can be introduced in the vicinity of the straight line being the contact line for the tangent cone. Curvature at the contact line is deduced from total angle at the vertex of the tangent cone. Figure 19 shows simplest case of the dual approximants - just inscribed and superscribed piecewise-linear cones P_h and P_h^*, Spherical image of the edge e of P_h^* is the edge B_s of the convex spherical polygone inscribed into circle on the sphere. Normal image of the edge e is the straight segment B_n as shown in Fig. 19c. Thus discrete spherical mapping ϕ_h maps face of P_h dual to e onto B_n. Spherical image of the edge e_1 of P_h is the edge F_s of the convex spherical polygon superscribed around circle. Normal image of e_1 is straight segment F_n shown in Fig. 19d. Again, discrete spherical mapping ϕ_h^* maps face of P_h^* dual to e_1 onto F_n. Of course, in both cases area of spherical image is equal to zero which is natural for developable surfaces. Spherical image of the sharp vertex of P_h converges to spherical image of the cone summit. The same is true for P_h^*.

The most complicated case of developable surface is the torse surface shown in Fig. 20.

Torse surface is generated by straight lines which are tangent to a spatial curve with nonzero torsion. For torse surface one can define tangent cone as well. In order to define dual polyhedra one should construct a pair of approximately polar polylines for spatial curve which allows to approximate its curvature and torsion. Details of such an approximation are beyond the scope of the paper. Each of the pair generates polyhedral torse surface as shown in Fig. 20.

Let us consider now gluing of the elementary patches. Simplest case is a curve on smooth surface M which separates convex and saddle subdomains as shown in Fig. 21 and Fig. 22. In this case primal polyhedron P_k is split into subdomains and every full face of P_k belongs to the certain subdomain. Image of separating line is present in P_k as a set of full edges.

Gauss curvature at the separating line is equal to zero. Behavior of spherical mapping for neighborhood of the point lying on the separating line was illustrated in Fig. 6 and Fig. 7. This behavior is reproduced in polyhedral case as well. Face Q of the dual polyhedron P_k^* is split by separating line into subdomains Q^+ and Q^-. Discrete spherical mapping maps $Q^{\pm}$ onto $F^{\pm}$, where normal image F^+ has positive area, while F^- has negative area. One can easily show that

$$\text{area}(F^{\pm})/\text{area}(Q^{\pm}) \longrightarrow 0 \quad \text{when} \quad k \longrightarrow \infty.$$

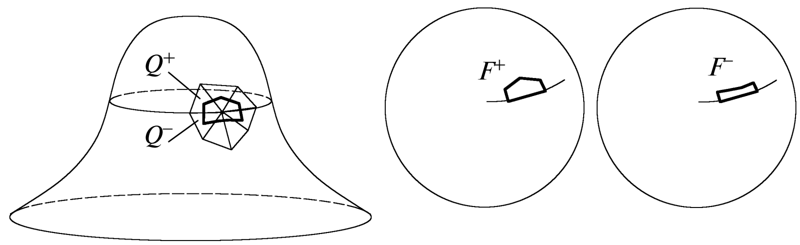

Fig. 21. Dual face and normal image for vertex on separating line for bell-like surface.

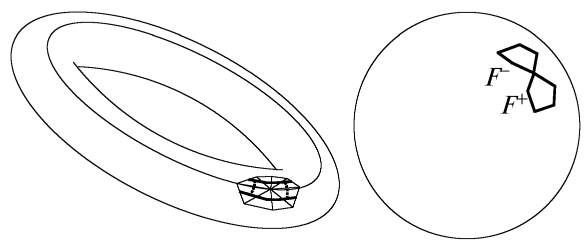

Fig. 22. Dual face and normal image for vertex on separating line for torus.

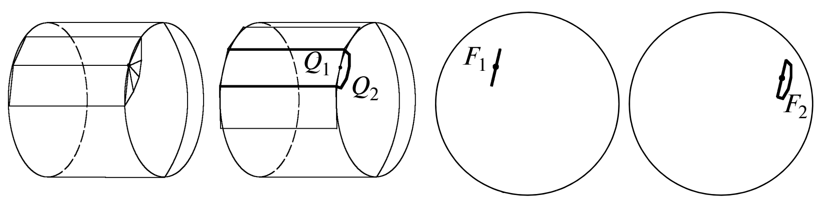

Fig. 23. Dual face and normal image for vertex lying on separating line being sharp edge.

Similar property holds for face G_m of P_k adjacent to separating line. If B_m is its normal image, then

$$\text{area}(B_m)/\text{area}(G_m) \longrightarrow 0 \quad \text{when} \quad k \longrightarrow \infty.$$

On the other hand, discrete spherical mappings $\phi_h^{*\pm} : Q^{\pm} \longrightarrow F^{\pm}$ and $\phi_h : G_m \longrightarrow B_m$ allow to approximate principal curvatures and principal directions near separating line. In the more general case when several separating lines intersect in the same point, dual face Q for this vertex should be split into subdomains, while full faces of P_k belong to subdomains. Similar analysis can be applied in the case when gluing results in the smooth surface but second derivatives across separating line are not available.

More complicated case arises when separating line is a sharp edge as shown in Fig. 23.

Here full faces of approximant P_k belong to regular subdomains of P_k, while "dual face" of the vertex of P_k lying on separating line is glued from several subfaces with different normals. In the example shown in Fig. 23 cylinder is glued to a spherical cap. Hence image of subface Q_1 is just a straight segment F_1, while image of subface Q_2 under discrete spherical mapping is convex polygone F_2. This consruction allows to approximate one-sided limits of principal curvatures and principal directions at both sides of the sharp edge.

Construction of spherical image for sharp edges is illustrated in Fig. 24.

Spherical image of the sharp edge e of the polyhedron P_k is convex spherical quadrilateral F_e shown in Fig. 24a. Vertices of this quadrilateral are defined by normals to four subfaces of dual polyhedron P_k^* which are adjacent to two vertices of the edge e. Spherical image of sharp edge e^* of P_k^* is convex spherical quadrilateral F_e^* shown in Fig. 24b. As before, vertices of this quadrilateral are defined by normals to four

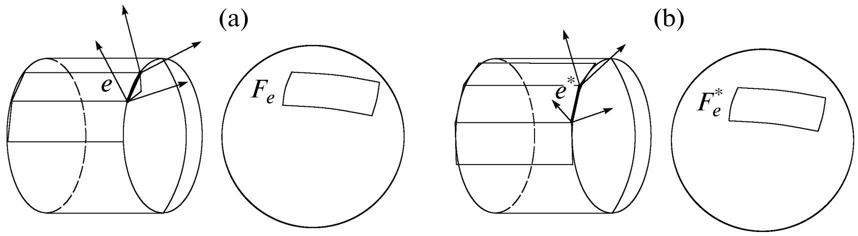

Fig. 24. Spherical image for sharp edges of polyhedra.

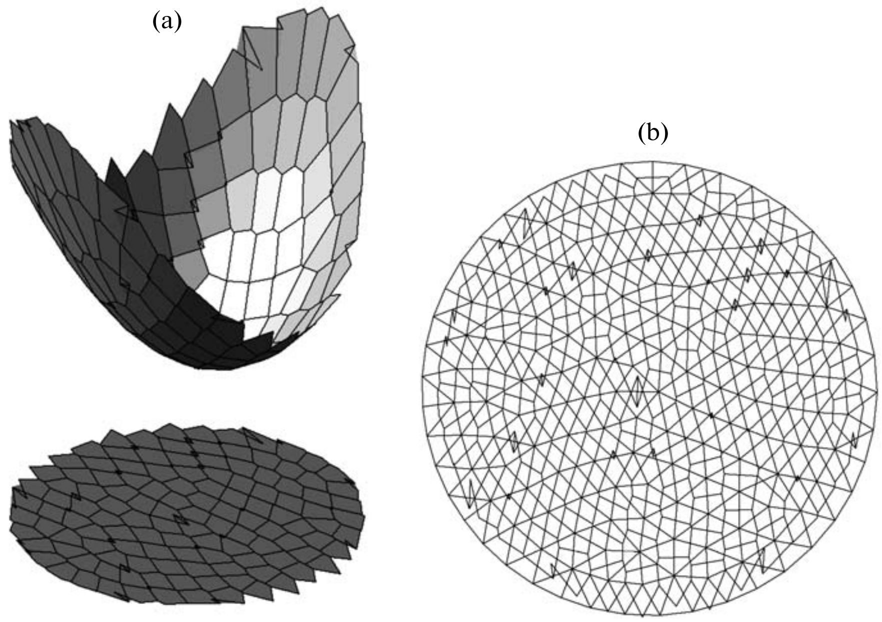

Fig. 25. (a) Dual polyhedron with errors, (b) projection of surface triangulation and dual surface.

faces of polyhedron P_k which are adjacent to two vertices of the edge e. Thus singular component of the area of spherical image is defined both for P_k and P_k^*. Both $S_k(P_k)$ and $S_k(P_k^*)$ converge to $S(M)$.

10. OPTIMIZATION OF POLYHEDRAL SURFACES

Polyhedral surface inscribed into regular convex or simple saddle surface can contain nonregular vertices even in the case when face normals converge to exact normals of the surface with refinement of the polyhedral surface. An example of nonregular triangulation P_h inscribed into elliptic paraboloid P

$$x_3 = 2x_1^2 + \frac{1}{5}x_2^2,$$

is shown in Fig. 25. One can see from Fig. 25a that certain "faces" of the dual surface P_h^* are selfintersecting, while triangulation P_h consists of well shaped triangles.

Projections of surface triangulation P_h and dual surface P_h^* onto the plane $x_3 = 0$ are shown in Fig. 25b. Here direction of maximal curvature of P is horizontal. It is clear that non-convex edges of P_h lead to dual edges with wrong orientation and hence to self-intersection for boundaries of faces dual to vertices of these primal edges. In order to eliminate "non-convex" edges in P_h, one can apply "edge flip" operation which considers a pair of adjacent triangle as a single quadrilateral, deletes existing diagonal, and creates another one. As applied to above example, edge flip operation can completely eliminate incoming edges and self-

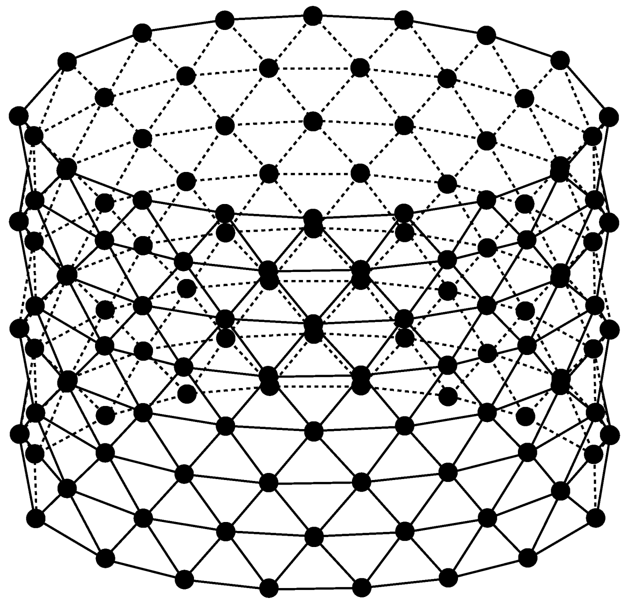

Fig. 26. Schwarz lantern: triangulation of the cylindrical surface.

intersecting faces in P_h^* and make all vertices of P_h regular, even though it may lead to triangles with small angles.

Thus one naturally comes to the optimization problem for approximating polyhedra. Let us note that a number of canonical optimization problems for polyhedra can be posed. In [30] it is considered minimization of Hausdorff distance from convex dual polyhedral surface P_k and P_k^* to convex surface M as well as optimization of convex polyhedral surfaces with respect to other error measures. It should be noted that construction of polyhedral approximations which guarantee componentwise convergence for Lebesque decomposition of the area of spherical image is closely related to construction of best approximants in the sense of Hausdoff distance.

Here we consider optimization problem for construction of "least jagged" polyhedral approximations. Intuitively, the idea of optimization is very simple: polyhedral surfaces should not contain excessive wrinkles. Mathematical formalization of this idea can be formulated as the following principle: *total variation of the area of spherical image of polyhedral surface should be minimal.* Similar formulation of the same optimization principle is that *the difference between absolute extrinsic curvature and absolute intrinsic discrete curvature should be minimal.* In order to obtain optimization problems in the closed form, one can, for example, fix the vertices of triangulated polyhedral surface P_k and optimization procedure will be reduced to edge flips.

Figure 26 shows the so-called Schwarz lantern, namely, the polyhedron inscribed into circular cylinder. This lantern can be constructed by subdividing the side surface of the cylinder using planes orthogonal to its axis into m equal parts. Regular n-sided polygon should be inscribed into each circular cross section. This polygon is rotated by π/n when passing to the next cross-section. As a result the side surface of the discretized cylinder consists of isosceles triangles. When $m, n \longrightarrow \infty$ this triangulation converges to the surface of cylinder pointwisely, but the limit of the sum of the triangle areas is $2\pi R\left(H^2 + \frac{\kappa}{4}\pi^4 R^2\right)^{1/2}$, where κ is the limit of the ratio n/m^2 when $n \longrightarrow \infty$, provided that it exists (see [29]). Here R, H denote radius and height of the cylinder, respectively. Obviously, absolute extrinsic curvature of the lantern is unbounded in the refinement limit. Applying edge flips which minimize difference between absolute exrinsic curvature and absolute intrinsic discrete curvature of the side surface inevitably leads to developable surface.

One can consider important particular case: optimization of the cone.

The neighborhood of nonregular vertex p of a surface triangulation can be described as a "fan," i.e. as a cone K^+ with wrinkles. Normal image Σ^+ of the side surface of this fan is self-intersecting contour, which is shown in Fig. 27a. Normal image of the vertex of the cone is multivalued one. If cone K^+ belongs to a

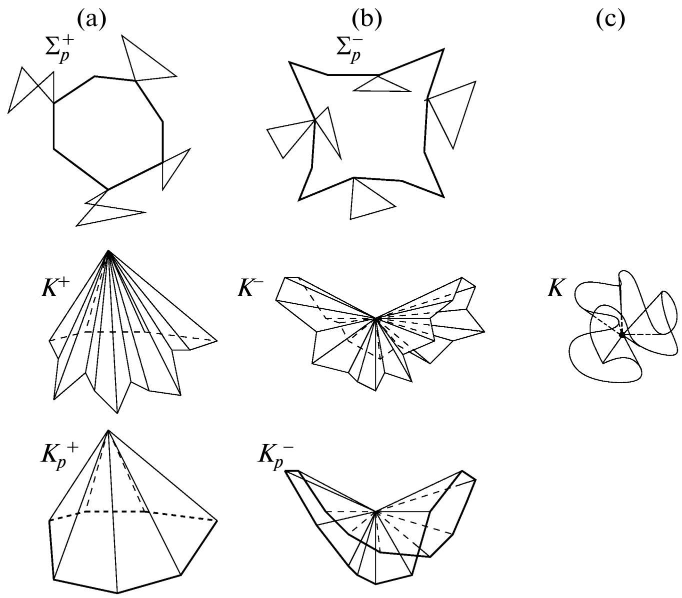

Fig. 27. Filtering of edges and principal component of normal image.

certain half-space, then one can construct its convex envelope – cone K_p^+. Normal image of the convex cone K_p^+ is convex polygon Σ_p^+ drawn in Fig. 27a by bold lines. We will call Σ_p^+ by *principal component* of the normal image. Principal component can be constructed for saddle point as well, which is shown in Fig. 27b.

In order to construct principal component one need to eliminate edges from K^- until one obtains canonical saddle K_p^- with a normal image being quadrilateral with the sides of nonpositive turn.

Let us note that the principal component in the convex case is unique since it is defined by unique convex envelope cone. In the case of the saddle cone in order to choose principal component one have to consider the set of the cone edges which leads to simple saddle cone with minimal area of spherical image.

Generic cone with undefined normal image is shown in Fig. 27c. Neighborhood of the vertex of this cone is not DC surface.

When normal image F_i is self-intersecting, its boundary can be decomposed into simple closed arcs bounding simple polygons F_{ik} with different orientations. Then the area of the normal image of p_i is given by

$$N_i = \sum_k s_k \text{area}(F_{ik}),$$

where $s_k \in \{-1, 1\}$ is orientation of the i-th contour. Total variation of the normal image area is given by

$$V_i = \sum_k \text{area}(F_{ik}).$$

Target function for optimization of polyhedral surface P_k is just the sum of differences

$$\delta_a(P_k) = \sum_i (V_i - |N_i|). \tag{35}$$

When normal image is a simple polygon, obviously $V_i = |N_i|$, otherwise $V_i > |N_i|$ thus absolute minimum of the function $\delta(P_k)$ is attained when all vertices of P_k are either summits of convex cones or summits of simple saddle cones. Such a solution can be attained when, for example, P_k is boundary of convex polyhedron. To some extent above optimality principle means minimization of absolute extrinsic curvature but unlike [31], this minimization is applied only at the nomregular vertices of polyhedron.

It should be noted that a certain degree of ambiguity is present in the definition of normal image since one has to assume that normal vector at the vertex p_i is known. Of course this ambiguity can be eliminated using spherical image F_i^s instead of normal image F_i when deriving δ_a. However formulation with the normal image does make sense since in this case one can optimize both primal polyhedron P_k and dual polyhedron P_k^*, minimizing for both surfaces difference between absolute extrinsic and absolute intrinsic curvature and simultaneously minimizing certain curvature deviation measure between P_k and P_k^*.

Two basic operations of optimization procedure are edge flips and edge filtering around vertices in order to extract principal component of the normal image.

11. CONCLUSIONS

One can conclude that duality principle allows to construct polyhedral approximations to surfaces representable as a difference of convex functions (Alexandrov surfaces) in such a way that discrete curvatures converge to curvature of the surface. This method can be applied in the general case of d-dimensional surface in $\mathbb{R}^{d+1}$.

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