Quasi-isometric surface parameterization

Article *in* Applied Numerical Mathematics · November 2005

DOI: 10.1016/j.apnum.2005.04.032

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SCIENCE DIRECT®
Applied Numerical Mathematics 55 (2005) 295–311
APPLIED NUMERICAL MATHEMATICS
www.elsevier.com/locate/apnum

Quasi-isometric surface parameterization

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Available online 1 June 2005

Abstract

Parameterization of surface is defined by a one-to-one mapping from a planar domain to the surface. Well established methods based on harmonic, conformal and quasi-conformal mappings may create parameterizations with singularities. Singularity-free parameterization technique is suggested based on the concept of quasi-isometric mappings. Well-posed variational formulations for quasi-isometric parameterizations are discussed based on existence theory for hyperelasticity. Distortion minimization, invariance and mesh independence are discussed with numerical examples.

Keywords: Surface parameterization; Bi-Lipschitz mappings; Polyconvex functional; Minimal distortion

1. Requirements for surface parameterizations

In order to find good solution to optimal surface parameterization problem one should formulate basic requirements for parameterizations.

- *Parameterization should be defined by one-to-one mapping.*
- *The distortion of parameterization should be minimal or bounded in a certain sense. Parameterization should not contain singularities.*
- *Mapping defining parameterization should be smooth when it is allowed by problem data.*
- *Parameterization should be constructed as unique and stable solution of a well-posed variational problem.*
- *Parameterization should depend on invariant properties of surfaces.*

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0168-9274/$30.00
doi:10.1016/j.apnum.2005.04.032

- *Parameterization for smooth and non-smooth surfaces should be constructed in unified way.*
- *The curvature measure of parameterization should be minimal or bounded in a certain sense.*
- *Discrete parameterization should be one-to-one, non-singular and should converge with mesh refinement to target parameterization.*

These requirements are discussed in detail in [12].

2. Distortion of parameterization

Let us consider bounded disk-like surface. For simplicity in what follows we will not strictly adhere to precise mathematical definitions and statements. Several rigorous statements are included in Appendix A. Suppose that this surface is defined via parametric representation

$$y(x_1, x_2) = \big(y_1(x_1, x_2),\, y_2(x_1, x_2),\, y_3(x_1, x_2)\big)^{\mathbb{T}}.$$

Hence $y(x)$ maps a certain domain $\Omega_x \in \mathbb{R}^2$ onto M and this mapping is one-to-one.

The notion of distortion is clear from intuitive point of view. One can draw any geometric figures inside Ω_x, such as circles, triangles, squares, map these figures onto surface M and quantify the changes in shape, size and area of these figures. We consider only such a distortion which changes the distance measured along the surface. The parameterization problem is in fact a problem of *intrinsic geometry*.

In order to introduce various notions related to distortion let us consider simpler case of planar mapping $x(\xi)$

$$x = \big(x_1(\xi_1, \xi_2),\, x_2(\xi_1, \xi_2)\big)^{\mathbb{T}}. \tag{1}$$

If not stated otherwise we will assume that mappings are smooth, all necessary derivatives make sense in a classical sense and domain boundaries are regular enough. Suppose that $x(\xi)$ maps Ω_ξ onto Ω_x. Consider the Jacobian matrix S of mapping (1)

$$S = \nabla_\xi x(\xi) = \left(\frac{\partial x}{\partial \xi_1}, \frac{\partial x}{\partial \xi_2}\right).$$

The distortion can now be described via algebraic properties of matrix S. Let us denote by σ_i the singular values of matrix S which are the square roots of eigenvalues of matrix $\overline{G} = S^{\mathrm{T}} S = |\bar{g}_{ij}|$. Matrix function $\overline{G}(\xi)$ is called metric tensor of deformation $x(\xi)$ due to relation

$$dl^2 = \bar{g}_{11}\, d\xi_1\, d\xi_1 + \bar{g}_{12}\, d\xi_1\, d\xi_2 + \bar{g}_{21}\, d\xi_1\, d\xi_2 + \bar{g}_{22}\, d\xi_2\, d\xi_2$$

between arc lengths in ξ and their images in x coordinates.

The change of area under mapping $x(\xi)$ is defined by $\det S = \sigma_1\sigma_2$. When $\sigma_1\sigma_2 = 1$ the mapping is called *equiareal* or *incompressible* and it does not change areas of any geometric figures.

When $\sigma_1 = \sigma_2 = \sigma$ the mapping is called *conformal*. For any conformal mapping tensor $\overline{G} = \sigma^2 I$. Any conformal mapping is locally shape preserving in a sense that it preserves intersection angle between any arcs.

The *shape distortion* is bounded when inequality

$$\frac{1}{L_0} \leqslant \frac{\sigma_1}{\sigma_2} \leqslant L_0, \tag{2}$$

holds for any $\xi \in \Omega_\xi$, where $L_0 \geqslant 1$ is a constant. The dimensionless shape distortion measure

$$\beta_l(S) = \sigma_{\max}(S)/\sigma_{\min}(S) \tag{3}$$

is called *linear dilatation* of the mapping. Planar mapping satisfying (2) is called *quasi-conformal*, and in the case of multiple dimensions it is called *mapping with bounded distortion* [22].

Both incompressible and conformal mappings admit singularities. For incompressible mappings it is possible that $\beta_l(S) \to \infty$, while for conformal and quasi-conformal mappings situations $\det S \to 0$ or $\det S \to \infty$ are possible.

When mapping is both conformal and incompressible it is called *isometric*. In this case $\overline{G} = I$ so mapping is length preserving.

The length distortion induced by the mapping is bounded when inequality

$$\frac{1}{L_1} \leqslant \sigma_i \leqslant L_1 \tag{4}$$

holds for any $\xi \in \Omega_\xi$ and constant $L_1 \geqslant 1$. In this case the mapping is called *quasi-isometric*, while L_1 is sometimes called the quasi-isometry constant.

The above inequality can be written in the equivalent matrix form

$$\frac{1}{L_1^2}|d|^2 \leqslant d^{\mathrm{T}} \nabla_\xi x^{\mathrm{T}} \nabla_\xi x d \leqslant L_1^2 |d|^2, \tag{5}$$

where d is arbitrary vector.

Obviously (4) implies (2), where $L_0 = L_1^2$. The reverse is not true. However if the shape distortion is bounded and $\frac{1}{L_2} \leqslant \det S \leqslant L_2$, then (4) holds with constant $L_1 = \sqrt{L_0 L_2}$. Thus control of length distortion can be achieved via shape and area distortion control.

Now we come to conclusion that distortion is minimal when mapping $x(\xi)$ satisfies (4) with minimal possible quasi-isometry constant L_1.

2.1. Distortion versus curvature

Another intuitive visual requirement for minimally distorting mappings is that any straight line segment should be mapped on straight line segment. In this sense any affine mapping is optimal. However the length distortion introduced above is in fact the distortion measure of affine mapping being local approximation of smooth mapping. Thus curvature distortion is effect of smaller order of magnitude compared to length distortion. But it does not make curvature distortion less important. Tentatively one can formulate the curvature minimization criteria in the following way: among the mappings satisfying (4) with the same constant L_1, the optimal one is that which maps any straight segment onto arc having curvature measure minimal in a certain sense, or its deviation from straight segment is minimal.

This notion can be clarified using simple example when unit square is mapped onto quadrilateral with straight edges. For this problem obviously the length distortion bound L_1 cannot be better than $\max(\sigma, \frac{1}{\sigma})$, where σ is any singular value in the corner, and maximum is taken over all corners. Hence there are lots of mappings which have the same distortion bounds.

One well-known example is *bilinear mapping* (see Fig. 1)

$$x_i = x_i^{00}(1-\xi_1)(1-\xi_2) + x_i^{01}(1-\xi_1)\xi_2 + x_i^{10}\xi_1(1-\xi_2) + x_i^{11}\xi_1\xi_2, \tag{6}$$

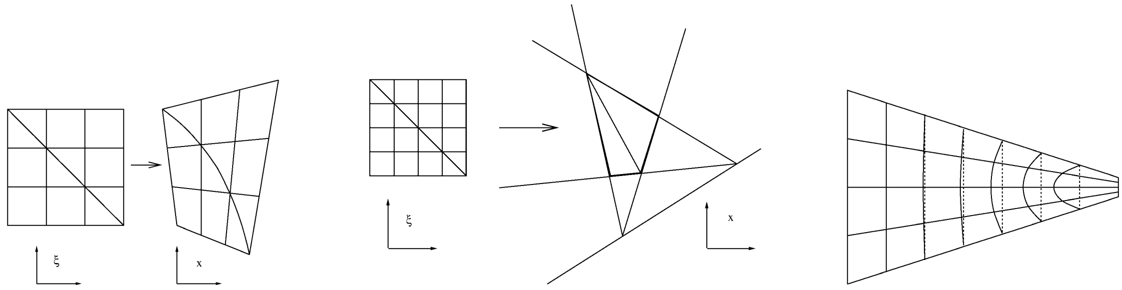


Fig. 1. (a) Bilinear mapping, (b) projective mapping, (c) bubbling effect.

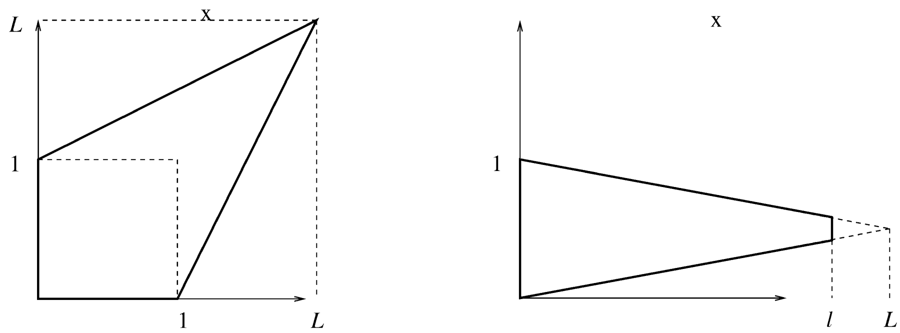


Fig. 2. Examples of bad behaviour of projective mapping.

where x_i^{jk} are coordinates of the corners. This mapping maps segments $\xi_1 = \text{const}$ and $\xi_2 = \text{const}$ onto straight segments, but segments which are not vertical or horizontal are mapped onto curved arc segments.

Another solution to the same problem is provided by *projective mapping* [14]

$$x_1 = \frac{a_{11}\xi_1 + a_{12}\xi_2 + a_{13}}{a_{31}\xi_1 + a_{32}\xi_2 + a_{33}}, \qquad x_2 = \frac{a_{21}\xi_1 + a_{22}\xi_2 + a_{23}}{a_{31}\xi_1 + a_{32}\xi_2 + a_{33}}. \tag{7}$$

The layout of this mapping is illustrated in Fig. 1(b). Basic advantage of projective mapping is that it maps any straight segment on straight segment. Moreover, inverse projective mapping is in turn projective. Unfortunately projective mappings tend to contain boundary and internal layers so linearity preservation is attained at the expense of quasi-isometry constants. Simple examples of bad conditioning of projective mappings are presented below.

Projective mappings for quadrilaterals shown in Fig. 2 are defined by following relations

$$\text{(a)}\quad \begin{cases} x = \frac{L\xi_1}{(1-L)(\xi_1+\xi_2)+2L-1}, \\ y = \frac{L\xi_2}{(1-L)(\xi_1+\xi_2)+2L-1}, \end{cases}$$

$$\text{(b)}\quad \begin{cases} x = \frac{l\xi_1}{\frac{l}{L}\xi_1+1-\frac{l}{L}}, \\ y = \frac{\frac{l}{2L}\xi_1+(1-\frac{l}{L})\xi_2}{\frac{l}{L}\xi_1+1-\frac{l}{L}}. \end{cases}$$

In the case (a) for bilinear mapping the global condition number $\kappa_g = \max_\xi \sigma / \min_\xi \sigma$ behaves as $O(L)$, while for projective mapping $\kappa_g = O(L^2)$. In the case (b) when $L = 1/\varepsilon$ and $l = (1-\varepsilon)L$ the global condition number of bilinear mapping is $O(1/\varepsilon^2)$, while for projective mapping the asymptotics is $O(1/\varepsilon^3)$.

When length of one side of quadrilateral tends to zero it is possible that mapping with the same length distortion bounds as bilinear mapping has coordinates lines in the shape of bubbles, shown in Fig. 1(c). Genesis of this bubbling phenomenon is very simple: for bilinear mapping shape distortion measure is good and relatively uniformly distributed, while determinant of Jacobian matrix varies very sharply. Requiring that determinant variations are smoothed out at the expense of shape distortion distribution, one can obtain exactly this bubbling effect.

These examples lead to the following idea for construction of optimal quasi-isometric parameterization of curvilinear quadrilateral [14]. Namely, optimal parameterization, i.e. mapping from curvilinear quadrangle onto unit square should be constructed as composition of two mappings. First one maps initial domain onto quadrilateral with straight edges, and second mapping is the projective (or bilinear) mapping. This approach can sharply diminish bubbling phenomena.

3. Mapping smoothness criteria

The smoothness of mappings was one of the natural requirements in construction of curvilinear grids since smoothness of coordinate transformation directly affects the accuracy of finite difference approximations of governing equations in transformed coordinates. Similar experience is observed in geometric modeling, when, for example, “sleek” high quality illumination of surfaces requires smooth and non-distorted parameterizations.

Common solution for smoothness problem is to construct mappings using PDEs satisfying ellipticity conditions. The simplest example is the system of Laplace equations

$$-\Delta x_1 = 0, \qquad -\Delta x_2 = 0, \qquad \Delta = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2}. \tag{8}$$

It is well known that solution of system (8) possesses infinite number of derivatives in a classical sense. The mapping satisfying equations (8) is called a *harmonic mapping*. Any conformal mapping also satisfies (8). The difference is in boundary conditions. Harmonic mapping can be constructed when on all boundary of Ω_ξ the deformation is prescribed, while for conformal mapping according to Riemann theorem correspondence between only 3 points on $\partial\Omega_\xi$ and $\partial\Omega_x$ can be prescribed.

Eqs. (8) are in fact Euler–Lagrange equations for the Dirichlet functional

$$\int_{\Omega_\xi} \frac{1}{2} \operatorname{tr}\left(\nabla_\xi x^{\mathrm{T}} \nabla_\xi x\right) d\xi, \tag{9}$$

where tr denote trace of square matrix—the sum of diagonal elements.

4. Variational methods

Variational methods are widely used for grid generation and surface parameterization [24,7,15,20,21, 18,14,16].

There are at least two basic advantages of variational methods over PDE solutions for construction of mappings. The first one is purely mathematical. Quite standard situation for mapping construction problems is that variational problem for minimization of certain functional can be well-posed, while for Euler–Lagrange equations of this functional well-posedness cannot be proved [4,5]. Another consideration is related to construction of efficient iterative methods. Experience of author allows to conclude that methods based only on Euler–Lagrange equations of functional which do not use functional values are not robust for highly non-linear stiff problems.

We start formulation of variational principle for mapping construction using three basic ideas:

(a) variational problem should be well-posed,
(b) length distortion of minimizing mapping should be bounded in the sense of (4), and
(c) minimizing mappings should be one-to-one.

Harmonic and conformal mappings can satisfy these requirements only for special sets of boundary conditions. A conformal mapping is one-to-one but in order to guarantee quasi-isometry one have to impose quite restrictive compatibility conditions on Ω_ξ and Ω_x [14]. Harmonic mapping is one-to-one when Ω_x is convex domain due to Rado theorem. However the mapping can contain various singularities when a corner is mapped onto smooth fragment of boundary (or vice versa). This phenomenon is called in mechanics stress concentration.

Let us show that conditions (a)–(c) essentially do not allow much freedom in constructing variational functionals.

Variational principle is restricted to the following

$$\int_{\Omega_\xi} f(\nabla_\xi x)\, d\xi, \tag{10}$$

where $f(\nabla_\xi x)$ is a certain distortion measure. Since distortion minimization problem is formulated as direction-invariant problem, we have to seek $f(S)$, $S = \nabla_\xi x$ as a function of $\det S$ and $\mathrm{tr}(S^{\mathrm{T}} S)$ which are orthogonal invariants of matrix $S^{\mathrm{T}} S$. Another natural requirement is that function $f(S)$ is smooth. The simplest shape distortion measure is given by

$$f_s(S) = \frac{\frac{1}{2}\,\mathrm{tr}(S^{\mathrm{T}} S)}{\det S}. \tag{11}$$

This distortion measure is essentially the integrand of Winslow functional [24] which can be derived by exchanging dependent and independent variables in Dirichlet functional (9). f_s can be understood as a mean arithmetic to mean geometric ratio in terms of eigenvalues of $S^{\mathrm{T}} S$ and as such is naturally generalized to multiple directions. Function f_s is dimensionless so its inverse defines a shape quality measure which is *fair* when applied to triangular elements.

On the other hand, $f_s(S)$ is related to the condition number of S in Frobenius norm

$$\|S\|_F \left\|S^{-1}\right\|_F = \sqrt{\mathrm{tr}\left(S^{\mathrm{T}} S\right) \mathrm{tr}\left(S^{-\mathrm{T}} S^{-1}\right)} = 2 f_s(S).$$

It is possible to use other matrix conditions numbers but (11) remains the simplest one. The idea of shape distortion via condition number naturally generalizes for multiple dimensions. Below it will be shown why it is not so good idea for construction of mappings.

The choices for area distortion measures are essentially not very rich. Area variations are uniformly bounded when, for example, function $\max(\det S/\bar{v}, \bar{v}/\det S)$ is uniformly founded from above. Here constant $\bar{v} > 0$ is "target" value of $\det S$. One can set $\bar{v} = 1$ using global scaling. Keeping in mind smoothness constraint we get

$$f_a(S) = \frac{1}{2}\left(\frac{\det S}{\bar{v}} + \frac{\bar{v}}{\det S}\right). \tag{12}$$

When $f_a(S)$ is bounded from above then $\det S/\bar{v}$ is uniformly bounded from below and from above. The idea of such area (volume) distortion measures is not new. It is known in hyperelasticity at least from 1970s [4].

Now for any function $f(S) = f_0(f_s(S), f_a(S))$, where f_0 is increasing in both variables and $f_0(1,1) = 1$ the inequality $f(S) < C$ will imply (4).

4.1. *Polyconvexity and barrier property*

The particular choice of $f(S)$ should be based on *polyconvexity criteria* [4] which was suggested as one of the sufficient conditions of well-posedness of variational problems (10).

Matrix function is called polyconvex when it can be written as a convex function of matrix minors (subdeterminants). In the planar case the function $f(S): \mathbb{R}^{2\times 2} \to \mathbb{R}$ is called polyconvex iff there exists convex function $g: \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ such that

$$f(S) = g(S, \det S).$$

Both $f_s(S)$ and $f_a(S)$ are polyconvex, hence linear combination

$$f_1(S) = (1-\theta) f_s(S) + \theta f_a(S), \quad 0 < \theta < 1 \tag{13}$$

is polyconvex as well. This is generally not true for product of shape and area measures! The matrix condition numbers in 3-D generally are not polyconvex as well [19].

The theory of well-posed variational problems in hyperelasticity is based on the concept of the *set of admissible deformations* which essentially includes mapping regularity condition in terms of Sobolev spaces along with *pointwise polyconvex constraint*. Generic polyconvex constraint can be written as follows [4,8]

$$L(S, \det S) \leqslant 0, \tag{14}$$

where function $L(\cdot,\cdot)$ is convex.

Let us rewrite the set $f_1(S) = (1-\theta) f_s(S) + \theta f_a(S) \leqslant \frac{1}{t}$ in the following way

$$\det S \geqslant t(1-\theta)\frac{1}{2}\operatorname{tr}\left(S^{\mathrm{T}} S\right) + t\theta\frac{1}{2}\left(\frac{(\det S)^2}{\bar{v}} + \bar{v}\right), \quad 0 < t \leqslant 1. \tag{15}$$

The boundary of this set in 5-D space with coordinates being elements of S and $\det S$ is ellipsoid . Thus constraint (15) defines polyconvex bounded set. If $\bar{v} = 1$, then any mapping satisfying (15) will satisfy also (4) with constant [10]

$$L_1^2 = \left(c_2 + \sqrt{c_2^2 - 1}\right)\left(c_1 + \sqrt{c_1^2 - 1}\right), \quad c_1 = 1 + \frac{1-t}{\theta t},\ c_2 = 1 + \frac{1-t}{(1-\theta)t}.$$

We can define now integrand f of functional (10) as

$$f(S) = \begin{cases} f_1(S), & \text{if } \det S > 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

We see that integrand has an *infinite barrier* on the boundary of the set of orientation-preserving mappings in a sense that $f(S) \to +\infty$ when $\det S \to +0$. This is another important ingredient in existence theory [4]. As a result minimum can be obtained only in the class of orientation-preserving mappings. The idea of barrier functionals was introduced into grid generation community in [18].

Numerical evidences suggest that in many practical cases the above functional allows to obtain satisfactory results. However it is still not satisfactory from the theoretical point of view. First of all the *growth conditions* for $f_1(S)$ are not strong enough for Ball existence theory to be applicable. It means that situations are possible when minimizing mapping either do not exist or is not continuous [3]. Another problem is that minimum can be attained on singular mapping. In this case $f(S)$ can become infinite at some points, but total value of functional attains minimum. This fact is overlooked in many papers on grid generation and geometric modeling, when the title of the method is unfounded from the theoretical point of view [16].

4.2. *Functional for construction of quasi-isometric mappings*

In order to resolve these problems constraint (15) is directly incorporated into integrand as follows

$$f(S) = f_2(S) = \begin{cases} (1-t)\frac{\phi(S,\det S)}{\det S - t\phi(S,\det S)}, & \text{if } \det S - t\phi(S,\det S) > 0, \\ +\infty, & \text{otherwise,} \end{cases} \tag{16}$$

where

$$\phi(S, J) = (1-\theta)\frac{1}{2}\operatorname{tr}\bigl(S^{\mathrm{T}}S\bigr) + \theta\frac{1}{2}\left(\frac{J^2}{\bar{v}} + \bar{v}\right).$$

Obviously

$$f_2(S) = (1-t)\frac{f_1(S)}{1 - tf_1(S)}.$$

The polyconvexity of $f_2(S)$ was proven in [11]. It can be easily seen that $f_2(S)$ by itself is dimensionless length distortion measure and its inverse is fair quality measure. The fact that $f_2(S)$ is dimensionless is very important since it allows to solve minimization problem for (10), (16) with different boundary conditions. It is possible to find positions of free boundaries as a result of functional minimization.

5. Surface parameterization

As preliminary step for derivation of functionals in the case of mapping between surfaces the composition of mappings is considered.

For convenience of notations let us rewrite functional (10) as follows:

$$\int_{\Omega_\eta} f(\nabla_\eta y)\, d\eta. \tag{17}$$

The mapping $y(\eta)$ is represented as a composition of mappings $\xi(\eta)$, $x(\xi)$ and $y(x)$, or as $y(x(\xi(\eta)))$. The invertible mappings $y(x)$ and $\eta(\xi)$ are specified while the function $x(\xi)$ is the new unknown solution.

We assume that $y(x)$ and $\eta(\xi)$ are quasi-isometric mappings, but possibly with large constants L_1. Using the notations

$$H = \nabla_\xi \eta, \qquad S = \nabla_\xi x, \qquad Q = \nabla_x y, \qquad T = \nabla_\eta y, \qquad J = \det T,$$

we get

$$T = QSH^{-1}, \qquad J = \frac{\det Q \det S}{\det H}, \qquad d\eta = \det H\, d\xi,$$

and the functional (17) is simply rewritten as

$$\int\limits_{\Omega_\xi} f\left(Q\nabla_\xi x H^{-1}\right) \det H\, d\xi, \tag{18}$$

where $\eta(\Omega_\eta) = \Omega_\xi$. Function $f_2(T)$ depends only on the orthogonal invariants of the matrix $T^{\mathbb{T}}T$. Using equalities

$$T^{\mathbb{T}}T = H^{-\mathbb{T}}S^{\mathbb{T}}GSH^{-1}, \qquad G(x) = Q^{\mathbb{T}}Q, \qquad \widetilde{H}(\xi) = H^{\mathbb{T}}H,$$

and the fact that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, we get that f_2 can be written via the orthogonal invariants of the matrix

$$W = S^{\mathbb{T}}GS\widetilde{H}^{-1}$$

as follows

$$f_2(W) = (1-t)\frac{\tilde{\phi}(W)}{(\det W)^{1/2} - t\tilde{\phi}(W)}, \qquad \tilde{\phi}(W) = (1-\theta)\frac{1}{2}\operatorname{tr}(W) + \theta\frac{1}{2}\left(\frac{\det W}{\bar{v}} + \bar{v}\right). \tag{19}$$

So finally the functional (18), (19) can be written in the following way

$$\int\limits_{\Omega_\xi} f_2(W)(\det \widetilde{H})^{1/2}\, d\xi. \tag{20}$$

This functional constructs quasi-isometric mapping between manifolds with metrics $\widetilde{H}(\xi)$ and $G(x)$ respectively.

In [10] it was suggested special procedure for maximization of t. It was found also that best results in terms of length distortion are obtained when $\theta = 4/5$.

The inequality (5) for such mappings takes the following form

$$\frac{1}{L_1^2} d^{\mathrm{T}}\widetilde{H}d \leqslant d^{\mathrm{T}}\nabla_\xi x^{\mathrm{T}} G \nabla_\xi x d \leqslant L_1^2 d^{\mathrm{T}}\widetilde{H}d, \tag{21}$$

where d is arbitrary vector.

5.1. *Surface flattening as variational problem*

Let us show that surface flattening can be formulated as variational problem (20). Suppose that a surface is defined in 3D coordinates η via parameterized representation $\eta(\xi)$. This essentially means that initial flattening is already available (see Fig. 3).

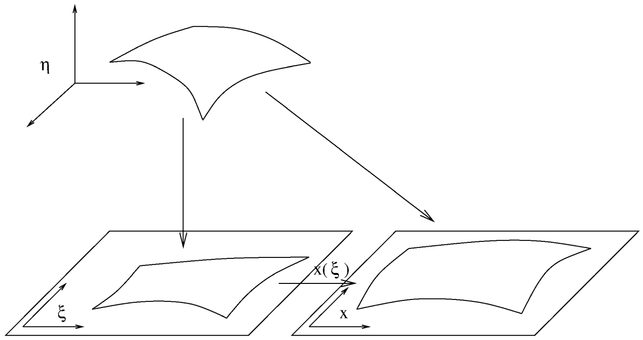


Fig. 3. Optimization of surface flattening.

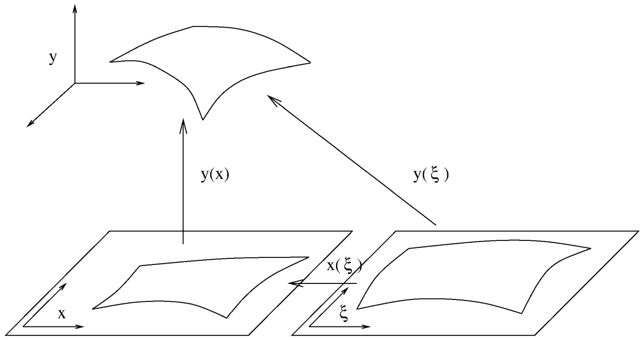


Fig. 4. Optimization of surface parameterization.

Optimal quasi-isometric flattening is defined by mapping from η to x coordinates on the plane. Thus one have to compute the mapping $x(\xi)$ which defines the deformation of initial flattening into optimal one. Since $\nabla_x \eta = \nabla_\xi \eta \nabla_\xi x^{-1}$, it is obvious that optimality condition for $x(\xi)$ is given by (21) with $G = I$ and $\widetilde{H}(\xi) = \nabla_\xi \eta^{\mathrm{T}} \nabla_\xi \eta$. Optimal mapping can be constructed via minimization of (20).

5.2. Construction of parameterization as variational problem

Similar reasoning can be applied to construction of optimal surface parameterization. Suppose that initial parameterization is given by $y(x)$ while optimal parameterization is defined by $y(\xi)$. Again planar deformation $x(\xi)$ should be found (see Fig. 4).

Since $\nabla_\xi y = \nabla_x y \nabla_\xi x$, it is also obvious that optimality condition for $x(\xi)$ is given by (21) with $\widetilde{H} = I$ and $G(x) = \nabla_x y^{\mathrm{T}} \nabla_x y$.

The above derivation suggests that general formulation when both G and $\widetilde{H}$ are present corresponds to the mapping of one surface onto another.

5.3. Invariant functionals

One can introduce invariant functionals in many ways. It was already mentioned above that *direction-invariance* (*isotropy*) *principle* leads to integrands being the functions of orthogonal invariants of metric tensor of deformation. One can impose requirement that functional is *conformally invariant*, i.e. it is invariant with respect to conformal transformation of either the surface it maps onto, or of mapped surface.

Another natural requirement for functionals is that they are *surface invariants* themselves, meaning that functional does not depend on particular surface parameterization. As a result Euler–Lagrange equations of the functional will also be invariant with respect to surface parameterization. Well-known examples of surface invariants are harmonic functional and Beltrami equations.

Let us show that functional (19), (20) is surface invariant. Consider first the optimal parameterization problem. Suppose that beside $y(x)$ we have another parameterization of the same surface $y(x')$. Then $G' = \nabla_{x'} y^{\mathrm{T}} \nabla_{x'} y$. Since $\nabla_{x'} x^{-1} \nabla_\xi x = \nabla_\xi x'$ we get

$$W' = \nabla_\xi x'^{\mathrm{T}} G' \nabla_\xi x' = \nabla_\xi x^{\mathrm{T}} \nabla_{x'} x^{-\mathrm{T}} G' \nabla_{x'} x^{-1} \nabla_\xi x = \nabla_\xi x^{\mathrm{T}} G \nabla_\xi x = W,$$

which proves the invariance.

Let us consider now optimal flattening problem. Suppose that two initial flattenings are given: $\eta(\xi)$ and $\eta(\xi')$. Then $\widetilde{H}' = \nabla_{\xi'} \eta^{\mathrm{T}} \nabla_{\xi'} \eta = \nabla_{\xi'} \xi^{\mathrm{T}} \widetilde{H} \nabla_{\xi'} \xi$, $\nabla_{\xi'} \eta = \nabla_\xi \eta \nabla_{\xi'} \xi$ and $\nabla_{\xi'} x = \nabla_\xi x \nabla_{\xi'} x$. As a result

$$\left(\det \widetilde{H}\right)^{1/2} d\xi = (\det \widetilde{H}')^{1/2} d\xi'$$

and

$$W' = \nabla_{\xi'} x^{\mathrm{T}} \nabla_{\xi'} x \widetilde{H}'^{-1} = \nabla_{\xi'} \xi^{\mathrm{T}} W \nabla_{\xi'} \xi^{-\mathrm{T}}.$$

Since $\operatorname{tr} W' = \operatorname{tr} W$ and $\det W' = \det W$ the invariance is proved.

The important implication of such invariance is that one does not need global initial flattening. Instead as it is common in the theory of manifolds one can define a chart: finite set of overlapping subdomains on the surface and for each subdomain separate one-to-one quasi-isometric flat projection. It is assumed that when these subdomains overlap there is one-to-one quasi-isometric transformation rule between local coordinates. Then optimization problem is invariant with respect to the choice of this chart.

Is the above invariance property an absolute requirement for any variational method in geometric modeling and grid generation? The answer is definitely no. Let us consider the following functionals:

$$J_i = \int\limits_{\Omega_\xi} \frac{1}{2} \frac{\operatorname{tr}(H^{-\mathrm{T}} \nabla_\xi x^{\mathrm{T}} \nabla_\xi x H^{-1})}{\det(\nabla_\xi x H^{-1})} \det H \, d\xi, \tag{22}$$

$$J_n = \int\limits_{\Omega_\xi} \frac{1}{2} \frac{\operatorname{tr}(H^{-\mathrm{T}} \nabla_\xi x^{\mathrm{T}} \nabla_\xi x H^{-1})}{\det(\nabla_\xi x H^{-1})} \, d\xi. \tag{23}$$

Functional (22) was essentially suggested in [15]. When $x(\xi)$ is smooth one-to-one mapping without singularities, then functional (22) is harmonic functional with respect to $\xi(x)$ [9], thus bearing all advantages and drawbacks of harmonic mapping technique.

Functional (23) was suggested for flattening of triangulated surfaces in [16], and for grid orthogonalization near boundaries in [17,10]. For discrete flattening problem matrix H locally can be Jacobian matrix of the mapping of equilateral planar triangle onto surface triangle isometrically projected on the plane, while $\nabla_\xi x$ is the Jacobian matrix of the mapping of equilateral triangle onto triangle of optimized triangular grid of flat projection. Functional (23) is not invariant with respect to surface parameterization! Does it mean that distortion estimates of flattening will deteriorate? Not necessarily. In the case of triangulated surface discrete functional is a sum of contributions from triangles. For small triangles

$\det H$ is relatively small value. It means that compared to "reference" functional (22), the contributions to functional (23) from small triangles are multiplied by large weight $1/\det H$. Thus shape distortion for small triangles is less likely to happen compared to large triangles on the same surface. As a result overall distortion is triangulation dependent. Major drawback of non-invariant formulation is that different triangulations may lead to different flattening even with grid refinement. Suppose that the same surface is approximated by two converging sequences of triangulations. Then flat projections can converge to different geometric figures or convergence can be absent. This drawback is alleviated by the fact that both functionals attain absolute minimum on isometric mapping when developable surfaces are considered. For invariant functional one can expect convergence with grid refinement.

Another non-trivial example is related to grid orthogonalization near boundary. From above arguments it follows that when one constructs structured grids with condensation near boundaries via composition with given one-dimensional mapping then functional (22) will simply result in one-dimensional redistribution of coordinate lines in resulting curvilinear grid. Whereas using functional (23) will result in orthogonal grid near boundary since shape distortion is less for smaller cells [17,10]. Thus the result of lack of invariance is *the change of curvature of coordinate lines*!

We can conclude that invariant functionals are generally preferable for grid generation and geometric modeling. However compatibility of invariant functionals with local orthogonality or local alignment control is questionable.

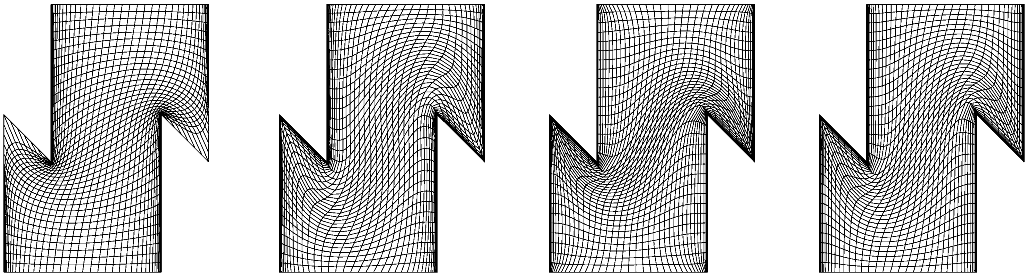
5.4. Minimal curvature parameterization

Similar to the planar case one can define optimal mapping as the one having minimal constant L_1 in (21) and simultaneously minimizing certain curvature measure. Parameterization is curvature-free when it maps any straight line segment on the plane onto segment of shortest arc (geodesic segment) on the surface. Such parameterization exists for surfaces of constant curvature, such as sphere and Lobachevski plane. This *Schur parameterization* was used in [14] for construction of quasi-isometric mappings. The idea of the method suggested by S.K. Godunov is to map conformally a curvilinear quadrilateral on the surface onto (unknown!) quadrilateral on the surface of constant curvature with sides being geodesic segments. This quadrilateral is mapped onto plane using inverse Schur mapping. The resulting quadrilateral with straight sides is parameterized using projective mapping. The final parameterization is constructed as composition of 3 mappings. Under certain restrictions this technique allows to construct quasi-isometric parameterizations [14]. Unfortunately as was mentioned above its quasi-isometry constant is far from optimal, but in some sense this is an attempt to construct parameterization with minimal curvature.

This deep and fruitful idea in principle admits another realization. One can replace conformal mapping technique by variational method described above which can be applied for much wider class of surfaces and domains. The additional mapping onto surface of constant curvature and Schur parameterization can serve as a means for decreasing curvature of parameterization.

6. Numerical examples

First numerical example illustrates behaviour of various functionals for domain with non-smooth boundaries. 41×41 structured mesh is constructed. Case (a) in Fig. 5 corresponds to functional (22).



(a) (b) (c) (d)

Fig. 5. Planar structured mesh test case. Effect of one-dimensional change of variables.

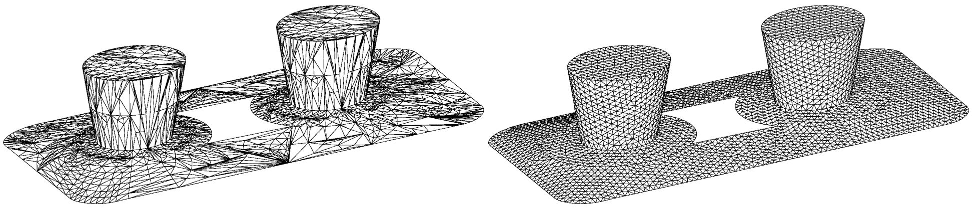


Fig. 6. Hat-like surface and different meshes.

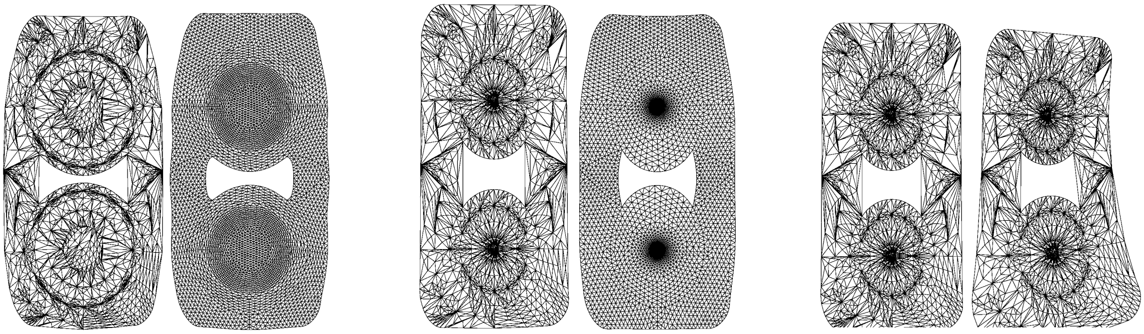
Matrix $H(\xi)$ is introduced here via additional one-dimensional mapping in order to provide grid condensation near left and right boundaries. Global condition number κ_g for case (a) is equal to $2.7 \cdot 10^3$, which corresponds to the presence of singularities. In the case (b) functional (20) is used, $\kappa_g = 7.3$. Case (c) corresponds to functional (23), here $\kappa_g = 21$ and grid is orthogonal near boundary. Case (d) is constructed using (20), where $\tilde{\phi}$ is replaced by $\tilde{\phi}/\det H$. Here $\kappa_g = 16$.

It should be noted that cases (c) and (d) are comparable in quality, but mesh step near boundary for case (c) is 3 times larger than expected, while quasi-isometric technique provides target mesh step along with orthogonality.

Another test case is related to flattening procedure for “double hat” surface shown in Fig. 6. Bad quality mesh (Fig. 6 (left)) and a sequence of high quality triangulations ranging from 2526 to 150471 triangles are considered (Fig. 6 (right)).

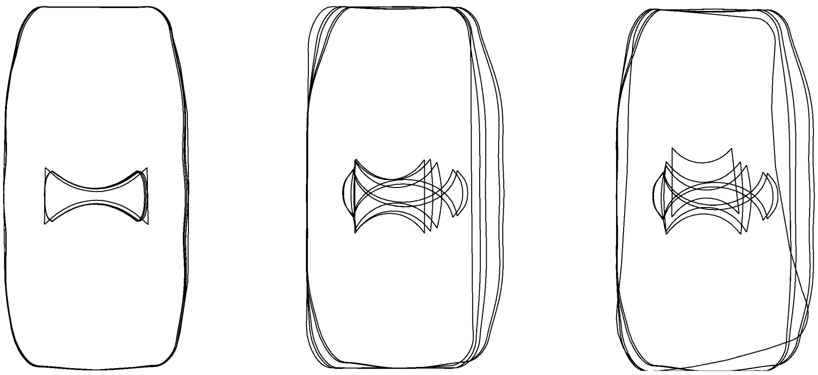
It was found that invariant quasi-isometric projection based on functional (20) is mesh-independent starting from very coarse bad quality meshes, which is shown in Fig. 7(a). Here $\min\sigma = 0.5$, $\max\sigma = 2.3$. Invariant functional (22) is more sensitive to mesh quality and size of cells which is shown Fig. 7(b). In this case on the average $\min\sigma = 0.12, \max\sigma = 2.3$. Area distortion under this flattening is very large. Non-invariant functional (23) leads to the loss of symmetry of projection, which is shown in Fig. 7(c), even though distortion is somewhat smaller compared to that for (22).

Mesh convergence is illustrated in Fig. 8, where outer contours of flat projections for all meshes are drawn. In the case (a), corresponding to functional (20), convergence is quite fast. In the case (b) (functional (22)) differences between finest meshes are still visible, and the projection becomes self-overlapping for finest meshes. Technique which protects against overlapping [11] was not applied here. In the case (c) (functional (23)) results for fine grids are identical to case (b), since surface triangles areas are almost constant.



(a) (b) (c)

Fig. 7. Flat projections. Mesh independence test.



(a) (b) (c)

Fig. 8. Boundary of flat projection. Mesh refinement results.

7. Conclusions

Quasi-isometric mapping technique allows to construct surface parameterizations with very small length distortion. Mesh independence study results are most favorable for suggested functional.

Appendix A

Let $\Omega \subset \mathbb{R}^n$ be bounded connected strongly Lipschitz domain. Let $u(x): \Omega \to \mathbb{R}^n$ be a spatial mapping.

The function $u(x)$ belongs to feasible set $\tilde{A}$ when

$$u(x) \in W^{1,p}(\Omega; \mathbb{R}^n), \quad p > n, \tag{A.1}$$

$$w(\nabla u H^{-1}) = \det(\nabla u H^{-1}) - t\phi(\nabla u H^{-1}, \det(\nabla u H^{-1})) > 0 \tag{A.2}$$

almost everywhere in Ω. Here $W^{1,p}$ denotes standard Sobolev space consisting of functions belonging to $L^p(\Omega)$ along with their gradients and $\phi:\mathbb{R}^{n\times n}\times\mathbb{R}\to\mathbb{R}$ is given by

$$\phi(S,\eta)=\theta\left(\frac{1}{n}\operatorname{tr}\left(S^{\mathrm{T}}S\right)\right)^{n/2}+\frac{1}{2}(1-\theta)\left(\bar{v}+\frac{1}{\bar{v}}\eta^2\right), \tag{A.3}$$

and $0<\theta<1$, $0<t<1$, $\bar{v}>0$ are given constants. Function $H(x):\Omega\to\mathbb{R}^{n\times n}$ belongs to $L^\infty(\Omega)$, $\det H(x)>0$ almost everywhere in Ω, and singular values of $H(x)$ are a.e. uniformly bounded from below and from above.

Mapping $u(x)$ is sought as the minimizer of the functional $J(u)$:

$$J(u)=\int_\Omega f_2(x,\nabla u)\,dx, \tag{A.4}$$

where

$$f_2(x,\nabla u)=f\left(\nabla u H^{-1}(x)\right)\det H(x) \tag{A.5}$$

and $f:\mathbb{R}^{n\times n}\to[0,\infty]$ is given by

$$f(S)=\begin{cases}(1-t)\frac{\phi(S,\det S)}{\det S-t\phi(S,\det S)}, & \text{if } \det S-t\phi(S,\det S)>0,\\ +\infty & \text{otherwise.}\end{cases} \tag{A.6}$$

The set of admissible deformations is augmented by boundary conditions. Three sets of boundary conditions are considered. Let $\partial\Omega=\Gamma_1\cup\Gamma_2$ and $u(\Gamma_1)=u_0(\Gamma_1)$, where u_0 is given continuous function. We consider the following cases: (a) $\Gamma_1=\partial\Omega$, (b) Γ_1 is open subset of $\partial\Omega$ with positive measure and (c) $\Gamma_1=\emptyset$. In the case (c) the admissible set is augmented by constraint

$$\int_\Omega u\,dx=e,$$

where $e\in\mathbb{R}^n$ is constant vector.

The following theorems are formulated below without proof.

Theorem A.1. *Suppose there exist $v\in\tilde{A}$ such that $J(v)<+\infty$, then there exist $\bar{u}\in\tilde{A}$ such that*

$$J(\bar{u})=\inf_{v\in\tilde{A}}J(v).$$

Proof of this theorem follows the idea of the proof of Theorem 7.14 from [4].

For the optimal flattening problem existence of element $v\in\tilde{A}$ providing finite value of functional, or in other words existence of initial quasi-isometric flattening can be proved for non-smooth surfaces which are essentially manifolds of bounded curvature (MBC) [1,23] under certain constraints on negative and positive parts of curvature [2,6].

Theorem A.2. *Let $u_0:\overline{\Omega}\to\mathbb{R}^n$ be one-to-one continuous mapping and $u_0(\Omega)$ is bounded connected strongly Lipschitz domain. If $u(\partial\Omega)=u_0(\partial\Omega)$ and other conditions of Theorem* A.1 *hold, then minimizing mapping is one-to-one bi-Lipschitz mapping.*

This theorem is direct consequence of J. Ball inverse function theorem [5] and embedding theorems for Sobolev spaces.

Theorem A.3. *Suppose that the set of admissible deformations $\tilde{A}$ is defined by* (A.1), (A.2) *and inequality*

$$\int_{\Omega} \det \nabla u \, dx \leqslant \operatorname{vol} u(\Omega),$$

where vol *denotes volume of domain. If other conditions of Theorem* A.1 *hold, then minimizing mapping exists and is one-to-one almost everywhere.*

The proof of this theorem is similar to the proof of Theorem 7.9-1 from [8].

Detailed proofs of above theorems can be found in [13].

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