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Polyconvex Potentials, Invertible Deformations, and Thermodynamically Consistent Formulation of the Nonlinear Elasticity Equations

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Abstract—It is shown that the nonstationary finite-deformation thermoelasticity equations in Lagrangian and Eulerian coordinates can be written in a thermodynamically consistent Godunov canonical form satisfying the Friedrichs hyperbolicity conditions, provided that the elastic potential is a convex function of entropy and of the minors of the elastic deformation Jacobian matrix. In other words, the elastic potential is assumed to be polyconvex in the sense of Ball. It is well known that Ball's approach to proving the existence and invertibility of stationary elastic deformations assumes that the elastic potential essentially depends on the second-order minors of the Jacobian matrix (i.e., on the cofactor matrix). However, elastic potentials constructed as approximations of rheological laws for actual materials generally do not satisfy this requirement. Instead, they may depend, for example, only on the first-order minors (i.e., the matrix elements) and on the Jacobian determinant. A method for constructing and regularizing polyconvex elastic potentials is proposed that does not require an explicit dependence on the cofactor matrix. It guarantees that the elastic deformations are quasi-isometries and preserves the Lame constants of the elastic material.

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Key words: elasticity equations, polyconvexity, entropy solutions, quasi-isometric mappings.

1. INTRODUCTION

The nonlinear finite-deformation elasticity theory has been, and remains, a source of difficult unsolved problems in modern analysis, geometry, and computational mathematics. In 1977, a major breakthrough was made in the theoretical study of stationary elasticity problems when Ball introduced the fundamental concept of a polyconvex elastic potential (i.e., a convex function of the minors of the deformation Jacobian matrix). As a result, not only existence theorems for the variational problem of constructing elastic deformations were proved [1], but it was also proved that they are Sobolev homeomorphisms [2]. The physical interpretation of polyconvexity remained unclear for a long time until it was shown in [3, 4] that, due to polyconvexity, geometric conservation laws for the minors of the Jacobian matrix can be included in an extended system of nonstationary elasticity equations and a system of first-order equations can be obtained that is symmetrizable and hyperbolic in the sense of Friedrichs. The results of [3, 4] provided a physical interpretation of Ball's condition. It was found that the well-posedness of the equations is ensured by adding an explicit dependence of the internal energy on additional variables minors of the Jacobian matrix. It should also be noted that the possibility of symmetrizing the nonlinear elasticity equations was first shown in [5] assuming that the internal energy is convex. The case of a polyconvex internal energy was studied in [6, 3] without considering the dependence of the general form on the minors of the Jacobian matrix. It is shown below that the thermoelasticity equations can also be written in a thermodynamically consistent Godunov canonical form [7, 8]) for the general polyconvex elastic potential. The existence of this form is critical both for the theoretical study of the equations and for their numerical solution, namely, for the implementation of the Godunov scheme.

From a theoretical point of view, there are several approaches to the study and approximation of nonstationary elasticity problems. In the first approach, the elastic deformation at every time step is determined by solving a convex minimization problem on an affine manifold given by a set of linearized geometric conservation laws that are satisfied by the deformation [9]. By using time step refinement, it can be proved that this semidiscrete solution converges to a measure-valued solution of the original system. Unfortunately, measure-valued solutions make no physical sense and cannot be viewed as elastic deformations. Moreover, the invertibility of the resulting mapping at any given time cannot be proved. A complete proof was constructed only in the one-dimensional case in [10], where it was also proved that the limiting solution is an entropy one. Another approach is based on Godunov's ideas [11] and makes use of piecewise constant approximations of variables and a Riemann solver at cell interfaces at each time step. There are strong grounds to believe that the well-known Godunov scheme is a method for deriving entropy solutions to systems of hyperbolic equations by passage to the limit, but the theoretical study of this approach still takes its first steps.

In fact, all the approaches to the discretization of the elasticity equations are heuristic, since the fundamental problem of approximating Sobolev homeomorphisms by piecewise affine homeomorphisms has not been solved. The most general result concerning this approximation was proved in [12]. It holds for two-dimensional deformations and cannot be applied even to stationary elasticity problems, since, as in the method of [9], the limit of the sequence of discrete invertible mappings may be outside the set of admissible deformations of the variational problem. To solve the approximation problem, one has to use all the variety of methods of irregular geometry and analysis, primarily, A.D. Aleksandrov and Yu.G. Reshetnyak's results concerning the approximation of irregular manifolds of bounded curvature (MBC) by polyhedral manifolds. The MBC theory deals primarily with intrinsic geometry problems, while in the class of extrinsic geometry objects inheriting the remarkable properties of MBC, of greatest interest are surfaces representable as the difference of convex functions (DC surfaces) [13] and mappings representable as the difference of convex functions (DC mappings) [14]. It can be assumed that the surfaces for which the support function can be represented as the difference of convex functions belong to the same class [15].

If a quasi-isometric mapping $u(x)$ belongs to the DC class and is almost isometric to its tangent cone at each point, we can expect that, for the inverse mapping $u^{-1}(x)$, a sequence of piecewise affine approximations $v_k(u)$ can be constructed such that the composition of $u(x)$ and $v_k(u)$ is a quasi-isometry whose equivalence constants converge to unity as $k \longrightarrow +\infty$. Thus, a "correct" piecewise affine approximation can be constructed for this class of homeomorphisms. However, to approximate elastic deformations by DC mappings, we need one more stage where the elastic potential is regularized so that admissible deformations are only quasi-isometries without losing polyconvexity and invariance under rotations in Lagrangian and Eulerian coordinates. It is shown below that Ball's results on the existence of solutions of the variational problem can be extended to "quasi-isometric" elastic potentials. Note that similar results for special polyconvex potentials were obtained in [16]. Whether quasi-isometric elastic deformations belong to the DC class remains an open question, since there is no constructive criterion for a mapping to be in the DC class. Thus, a fundamental unsolved problem is to find a measure of curvature of irregular surfaces and mappings such that its boundedness guarantees that they belong to the DC class.

It is well known that the linear elasticity equations admit various singular solutions. The classical linear elasticity theory is an approximation to nonlinear models in the case where the elastic deformation is an almost isometric mapping. The finite-deformation elasticity theory is a more adequate physical model, but it also admits various singular solutions. Singular solutions are associated with no actual elastic deformations, since, in the case of sufficiently large deformations and stresses, other physical effects have to be taken into account, primarily, the plasticity and fracture of the material. It can be concluded that the elasticity theory proper should refer to the study of quasi-isometric elastic deformations with admissible quasi-isometry constants chosen so large that the influence of regularization is negligible as compared to other physical effects.

Thus, the study of the elasticity equations consists of the following steps: (a) the elasticity equations are regularized so as to guarantee that elastic deformations are quasi-isometric mappings; (b) it is proved that a measure of curvature of these deformations is bounded and that they belong to the DC class; (c) an approximation of DC mappings by piecewise affine homeomorphisms is constructed; and (d) the limiting solution is proved to be an entropy solution of the elasticity equations. Each step requires the solution of complicated open problems in modern analysis and geometry. Some issues concerning correct polyhedral approximation of DC surfaces were addressed in [17]. Note that the approximation of Sobolev homeomorphisms by quasi-isometries with growing equivalence constants is a fairly natural approach, since a piecewise affine homeomorphism is always a quasi-isometry.

2. LEGENDRE TRANSFORMATION

Recall the definition and properties of the Legendre transformation.

Let a twice continuously differentiable function of d variables be given in $\mathbb{R}^d$:

$$v = v(x) = v(x_1, \ldots, x_d).$$

We introduce a new set of variables $p = p_1, \dots, p_d$ by using the transformation

$$p_i = \frac{\partial v}{\partial x_i}. \tag{2.1}$$

Assume that the Hessian (i.e., the determinant of the Hessian matrix) of v is nowhere zero. Then, by using Eq. (2.1), x_i can be locally expressed as functions of $p_1, \dots, p_d$.

Define the new function v^* as

$$v^* = x^{\mathrm{T}}p - v(x). \tag{2.2}$$

Expressing x in terms of p and substituting the result into (2.2) gives v^* expressed in terms of the new variables p_i:

$$v^* = v^*(p_1, \dots, p_d).$$

Consider an infinitesimal variation of v^* caused by arbitrary infinitesimal variations of p_i:

$$\delta v^* = \sum \frac{\partial v^*}{\partial p_i}\delta p_i = \sum (x_i \delta p_i + p_i \delta x_i) - \delta v = \sum \left(x_i \delta p_i + \left(p_i - \frac{\partial v}{\partial x_i} \right) \delta x_i \right).$$

Since v^* is a function of p, the variations of x_i have to be expressed in terms of the variations of p_i. However, this can be avoided since, by virtue of (2.1), the coefficients multiplying δx_i are zero. Then we immediately obtain

$$x_i = \frac{\partial v^*}{\partial p_i}. \tag{2.3}$$

This result reflects the dualism of the Legendre transformation, which can be expressed by the following scheme (see [18]):

	Old system	New system
Variables	$x_1, \dots, x_d$	$p_1, \dots, p_d$
Functions	$v = v(x_1, \dots, x_d)$	$v^* = v^*(p_1, \dots, p_d)$
	Transformation	
	$p_i = \dfrac{\partial v}{\partial x_i}$	$x_i = \dfrac{\partial v^*}{\partial p_i}$
	$v^* = x^{\mathrm{T}}p - v(x)$	$v = x^{\mathrm{T}}p - v^*(p)$
	$v^* = v^*(p_1, \dots, p_d)$	$v = v(x_1, \dots, x_d)$
	$H_{ij} = \dfrac{\partial^2 v}{\partial x_i \partial x_j}, \quad H = (H^*)^{-1}$	$H^*_{ij} = \dfrac{\partial^2 v^*}{\partial p_i \partial p_j}, \quad H^* = H^{-1}$

(2.4)

Thus, the new variables are the partial derivatives of the old function with respect to the old variables, while the old variables are the partial derivatives of the new functions with respect to the new variables. The Hessian matrices of the old and new functions are mutually inverse. The transformation defined by (2.4) is completely symmetric. Thus, the old and new systems are completely equivalent. Note that the above argument is not mathematically rigorous at least because there is no guarantee that the nonlinear systems (2.1) and (2.3) are uniquely solvable.

Now assume that $v(x)$ is convex (if the Hessian is nonzero, it is strictly convex). Then the equality $v^*(p) = x^{\mathrm{T}}p - v(x)$, where x is expressed in terms of p from $p_i = \partial v/\partial x_i$, can be formulated as the solution of the maximization problem

$$v^*(p) = \max_x \{x^{\mathrm{T}}p - v(x)\}.$$

Since $x^{\mathrm{T}}p - v(x)$ is a strictly convex function, its maximum is reached at a single stationary point where the gradient of the function vanishes, namely at $p_i = \partial v/\partial x_i$.

The dual function $v^*(p)$ is also strictly convex.

The Legendre transformation was extended to nonsmooth functions in [19]. Let $v(x) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a function such that it does not take $-\infty$ values and is not identically equal to $+\infty$ and its epigraph is a closed set. The Legendre–Young–Fenchel transformation of $v(x)$ is defined as

$$v^*(p) = \sup_x [x^{\mathrm{T}} p - v(x)].$$

Then, v^* is a convex function and its epigraph is a closed set. If $v(x)$ is convex, then $v^{**}(x) = v(x)$.

We say that $u(x)$ is a strictly convex regular barrier function if it is strictly convex, finite, twice continuously differentiable at interior points of an open convex domain d with a twice continuously differentiable boundary, and its Hessian matrix $u(x)$ is positive definite in Ω. Moreover, $u(x) \longrightarrow +\infty$ for $x \in \Omega$, $x \longrightarrow \partial\Omega$, and $u(x) = +\infty$ for all $x \notin \Omega$. Speaking about strictly convex functions, we will always assume that their Hessian matrices are strictly positive definite.

3. CANONICAL FORM OF SYSTEMS OF HYPERBOLIC EQUATIONS AND ENTROPY SOLUTIONS

In what follows, summation over repeated indices is implied, while a subscript denotes the derivative with respect to the corresponding variable.

Consider the system of first-order differential equations

$$\frac{\partial y}{\partial t} + \frac{\partial F^j(y)}{\partial \xi_j} = 0, \tag{3.1}$$

where $y^{\mathrm{T}} = (y_1 \ldots y_d)$and $y_i = y_i(\xi_1, \xi_2, \xi_3, t)$.

We say that system (3.1) has an entropy pair (see, e.g., [20–22]) if there is a convex function $\Sigma : \mathbb{R}^d \longrightarrow \mathbb{R}$ and functions $Q^j : \mathbb{R}^d \longrightarrow \mathbb{R}$, $j = 1, 2, 3$ such that

$$\frac{\partial Q^j}{\partial y_k} = \frac{\partial (F^j)_l}{\partial y_k} \frac{\partial \Sigma}{\partial y_l}.$$

The entropy solution of system (3.1) is a function $y(\xi, t)$ that satisfies the differential inequality

$$\frac{\partial \Sigma(y)}{\partial t} + \frac{\partial Q^j(y)}{\partial \xi_j} \le 0. \tag{3.2}$$

A solution of system (3.1) for which (3.2) holds as an equality is called an isentropic solution. Obviously, any smooth solution of system (3.1) is isentropic.

Indeed,

$$\frac{\partial \Sigma(y)}{\partial t} + \frac{\partial Q^j(y)}{\partial \xi_j} = \frac{\partial \Sigma}{\partial y_l} \frac{\partial y_l}{\partial t} + \frac{\partial Q^j}{\partial y_k} \frac{\partial y_k}{\partial \xi_j} = \frac{\partial \Sigma}{\partial y_l}\left(\frac{\partial y_l}{\partial t} + \frac{\partial (F^j)_l}{\partial \xi_j}\right) = 0.$$

Entropy solutions are obtained by introducing viscosity into system (3.1) followed by passage to the inviscid limit [20]. This approach can be illustrated by the following unrigorous argument. After introducing viscous terms into system (3.1), the resulting viscous solution is denoted by y^ε. It is a solution of the system

$$\frac{\partial y^\varepsilon}{\partial t} + \frac{\partial F^j(y^\varepsilon)}{\partial \xi_j} = \varepsilon \Delta y^\varepsilon. \tag{3.3}$$

Substituting y^ε into (3.2) and using system (3.3) gives the equation

$$\frac{\partial\Sigma(y^\varepsilon)}{\partial t}+\frac{\partial Q^j(y^\varepsilon)}{\partial\xi_j} = \varepsilon\frac{\partial\Sigma(y^\varepsilon)}{\partial y_i^\varepsilon}\Delta y_i^\varepsilon, \tag{3.4}$$

since

$$\varepsilon\frac{\partial\Sigma(y^\varepsilon)}{\partial y_i^\varepsilon}\Delta y_i^\varepsilon = \varepsilon\frac{\partial}{\partial\xi_j}\frac{\partial\Sigma(y^\varepsilon)}{\partial y_i^\varepsilon}\frac{\partial y_i^\varepsilon}{\partial\xi_j}-\varepsilon\frac{\partial^2\Sigma(y^\varepsilon)}{\partial y_i^\varepsilon\partial y_k^\varepsilon}\frac{\partial y_i^\varepsilon}{\partial\xi_j}\frac{\partial y_k^\varepsilon}{\partial\xi_j}\le\varepsilon\frac{\partial}{\partial\xi_j}\frac{\partial\Sigma(y^\varepsilon)}{\partial y_i^\varepsilon}\frac{\partial y_i^\varepsilon}{\partial\xi_j}, \tag{3.5}$$

we get

$$\frac{\partial\Sigma(y^\varepsilon)}{\partial t}+\frac{\partial Q^j(y^\varepsilon)}{\partial\xi_j}\le\varepsilon\frac{\partial}{\partial\xi_j}\frac{\partial\Sigma(y^\varepsilon)}{\partial y_i^\varepsilon}\frac{\partial y_i^\varepsilon}{\partial\xi_j}. \tag{3.6}$$

If the limiting solution $y = y^{\varepsilon=0}$ obtained by passage to the limit as $\varepsilon \longrightarrow 0$ is sufficiently smooth, then the right-hand side of Eq. (3.4) tends to zero. As a result, inequality (3.2) becomes an equality. If the limiting solution is not smooth, then it is assumed and, for some classes of problems, proved (see [20]) that the right-hand side of expression (3.5) tends to zero. Thus, the limiting solution satisfies inequality (3.2).

In [7] (see also [8]) a method was proposed for choosing a vector of variables q and special potentials $\mathcal{L}(q)$ and $\mathcal{L}^j(q)$) such that many systems of equations in mathematical physics can be reduced to the canonical form

$$\frac{\partial\mathcal{L}_q}{\partial t}+\frac{\partial\mathcal{L}_q^j}{\partial\xi_j} = 0, \tag{3.7}$$

where $\mathcal{L}(q)$ is a strictly convex function and $\mathcal{L}_q$ denotes the partial derivative with respect to q. Moreover, the entropy solutions of system (3.7) satisfy the inequality

$$\frac{\partial(q^{\mathrm{T}}\mathcal{L}_q-\mathcal{L})}{\partial t}+\frac{\partial(q^{\mathrm{T}}\mathcal{L}_q^j-\mathcal{L}^j)}{\partial\xi_j}\le 0, \tag{3.8}$$

which becomes an equality for smooth solutions.

System (3.7) in nonconservative form is written as

$$\mathcal{L}_{q_kq_m}\frac{\partial q_m}{\partial t}+\mathcal{L}^j_{q_kq_m}\frac{\partial q_m}{\partial\xi_j} = 0. \tag{3.9}$$

System (3.9) is symmetric and hyperbolic in the sense of Friedrichs [23], since the matrix $\mathcal{L}_{q_kq_m}$ is symmetric and positive definite and the matrix $\mathcal{L}^j_{q_kq_m}$ is symmetric.

Let us show that Eq. (3.8) is the equation for an entropy pair. Adding viscosity to the system of equations

$$\frac{\partial\mathcal{L}_{q^\varepsilon}}{\partial t}+\frac{\partial\mathcal{L}^j_{q^\varepsilon}}{\partial\xi_j} = \varepsilon\Delta q^\varepsilon \tag{3.10}$$

and calculating the residual of differential operator (3.8) for the viscous solution q^ε gives

$$\frac{\partial(q^{\varepsilon\mathrm{T}}\mathcal{L}_{q^\varepsilon}-\mathcal{L})}{\partial t}+\frac{\partial(q^{\varepsilon\mathrm{T}}\mathcal{L}^j_{q^\varepsilon}-\mathcal{L}^j)}{\partial\xi_j} = \varepsilon q^{\varepsilon\mathrm{T}}\Delta q^\varepsilon = \varepsilon\frac{\partial}{\partial\xi_m}q_i^\varepsilon\frac{\partial q_i^\varepsilon}{\partial\xi_m}-\varepsilon\sum_k\left|\frac{\partial q^\varepsilon}{\partial\xi_m}\right|^2\le\varepsilon\frac{\partial}{\partial\xi_m}q_i^\varepsilon\frac{\partial q_i^\varepsilon}{\partial\xi_m}.$$

In [20] it was proposed to look for an entropy solution in the class of bounded measurable functions with a gradient belonging to the Lebesgue space L^1. Thus, if the viscous solution q^ε remains in this class as $\varepsilon \longrightarrow 0$, then the right-hand side of the entropy inequality tends to zero in the weak sense, so that inequality (3.8) holds. Note that this convergence has not been proved for a wide class of systems of hyperbolic equations, including gasdynamic and elasticity equations. The viscosity coefficient in Eq. (3.3) can be represented in the general tensor form $\varepsilon b_{ij}(q)$. It was shown in [24] that, even in the one-dimensional case, examples of hyperbolic equations can be constructed that have nonunique entropy solutions depending on b_{ij} despite the passage to the limit as $\varepsilon \longrightarrow +0$.

3.1. Potentials Based on Divergence-Free Vector Fields

Consider potentials $\mathcal{L}^j$ of special form that can be represented as

$$\mathcal{L}^j(q) = \mathcal{L}_0^j(q) + v_j^k(q)\phi_k(q), \tag{3.11}$$

where the functions v_j^k for $k = 1, 2, \ldots, K$ make up a divergence-free vector field v^k that satisfies the conservation law

$$\frac{\partial v_j^k}{\partial \xi_j} = 0. \tag{3.12}$$

Let $\tilde{\mathcal{L}}_q^j$ denote the following incomplete derivative of $\mathcal{L}^j$ with respect to q:

$$\tilde{\mathcal{L}}_q^j = \mathcal{L}_{0_q}^j + v_j^k\phi_{k_q}. \tag{3.13}$$

The generalized canonical form is defined as the system of equations

$$\frac{\partial \mathcal{L}_q}{\partial t} + \frac{\partial \tilde{\mathcal{L}}_q^j}{\partial \xi_j} = 0. \tag{3.14}$$

It is supplemented with the entropy inequality

$$\frac{\partial(q^{\mathrm{T}}\mathcal{L}_q - \mathcal{L})}{\partial t} + \frac{\partial(q^{\mathrm{T}}\tilde{\mathcal{L}}_q^j - \mathcal{L}^j)}{\partial \xi_j} \le 0, \tag{3.15}$$

which becomes an equality for smooth solutions. Apparently, this generalized canonical form was first proposed by E.I. Romensky in his difficult-to-access paper [6], which did not appear in a journal version, so that it was not known to the author and was suggested by the reviewer.

It should be noted that the additional conservation law (3.12) is a consequence of system (3.14); i.e., we have to look for such a weak solution of (3.14) that, if condition (3.12) holds initially, then it holds for any $t > 0$.

The nonconservative formulaton of system (3.14) looks like

$$\mathcal{L}_{q_k q_m}\frac{\partial q_m}{\partial t} + \tilde{\mathcal{L}}_{q_k q_m}^j\frac{\partial q_m}{\partial \xi_j} = 0, \tag{3.16}$$

where the incomplete derivatives $\tilde{\mathcal{L}}_{q_k q_m}^j$ are symmetric, since

$$\tilde{\mathcal{L}}_{q_k q_m}^j = \mathcal{L}_{0_{q_k q_m}}^j + v_j^l\phi_{l_{q_k q_m}}.$$

Both the Godunov canonical representation and its generalization (3.14) are constructed using the Legendre transformation, so that the following relations hold as applied to system (3.1):

$$\Sigma = \mathcal{L}^*, \quad \mathcal{L} = \Sigma^*, \quad F^j(y) = \tilde{\mathcal{L}}_q^j(q(y)), \quad q = \Sigma_y. \tag{3.17}$$

Thus, if we can find a set of potentials $\mathcal{L}$ and $\mathcal{L}^j$ for which equalities (3.17) hold, then system (3.14) coincides with the original system (3.1), which justifies the choice of using $\tilde{\mathcal{L}}_q^j$ in the definition of (3.14).

Let us show that the existence of canonical representation (3.14) implies that the original system of equations can also be symmetrized. The nonconservative form of system (3.1) is

$$\frac{\partial y_m}{\partial t} + \tilde{F}_{m_{y_k}}^j\frac{\partial y_k}{\partial \xi_j} = 0, \tag{3.18}$$

where the incomplete derivative is given by

$$\tilde{F}^j_{m_{y_k}} = \tilde{\mathcal{L}}^j_{q_m q_l} \frac{\partial q_l}{\partial y_k} = \tilde{\mathcal{L}}^j_{q_m q_l} \frac{\partial^2 \Sigma}{\partial y_l \partial y_k}.$$

Thus, system (3.18) can be symmetrized if it is multiplied from the left by a symmetric positive definite matrix $\frac{\partial^2 \Sigma}{\partial y_l \partial y_k}$.

4. VARIATIONAL PRINCIPLE OF NONLINEAR ELASTICITY IN LAGRANGIAN COORDINATES

Let ξ_1, ξ_2, and ξ_3 be the Lagrangian coordinates of a material point, while x_1, x_2, and x_3 be its Eulerian coordinates. The mapping $x(\xi)$ defines a stationary deformation of an elastic body. The Jacobian matrix of $x(\xi)$ is denoted by C, where $c_{ij} = \partial x_i/\partial \xi_j$.

The elastic deformation $x(\xi)$ is the minimizing mapping of the (stored energy) functional

$$J(x) = \int_\Omega \tilde{\Phi}(C) d\xi,$$

where $\tilde{\Phi}(C)$ is the elastic potential and Ω is the domain defining the elastic body in Lagrangian coordinates.

The elastic potential of an isotropic material has the following properties:

(i) It is invariant and objective: $\tilde{\Phi}(UCV^{\mathrm{T}}) = \tilde{\Phi}(C)$, where U and V are arbitrary orthogonal matrices with a positive determinant.

(ii) The absolute minimum of $\tilde{\Phi}(C)$ is attained at $C = U$, where U is an arbitrary orthogonal matrix with a positive determinant.

A symmetric matrix $\mathcal{E}$ defined as

$$\mathcal{E} = \frac{1}{2}(C^{\mathrm{T}}C - I), \tag{4.1}$$

is called the Green–Saint-Venant strain tensor. It is well-known [25] that, if $\tilde{\Phi}(C)$ is a smooth function of the orthogonal invariants of $C^{\mathrm{T}}C$ and reaches its absolute minimum at $C = U$, where U is an orthogonal matrix with a positive determinant, then the following representation (Hooke's law') holds in the case of small deformations:

$$\tilde{\Phi}(C) = \tilde{\Phi}(I) + \frac{\lambda}{2}(\mathrm{tr}\mathcal{E})^2 + \mu \mathrm{tr}\mathcal{E}^2 + o(\|\mathcal{E}\|^2). \tag{4.2}$$

Here, μ and λ are called the Lame constants of the elastic material.

In fact, Ball's theorems on the existence of solutions to variational elasticity problems [1, 2] are generalizations of the Weierstrass theorem, which states that a continuous function given on a compact set reaches its minimum. Moreover, a class of admissible Sobolev mappings is defined and is assumed to be nonempty and the infimum of the stored strain energy is proved to be reached for a mapping of this class. In [2] a sufficient condition on the polyconvex elastic potential was obtained under which a stationary elastic deformation that minimizes the stored elastic energy and such that its boundary values coincide with those of certain homeomorphism is itself a homeomorphism.

Below, additional conditions on the elastic potential that follow from the technique used for proving the existence theorems are formulated in a simplified form:

(i) barrier property: $\tilde{\Phi}(C) \longrightarrow +\infty$ as $\det C \longrightarrow +0$ (this property is incompatible with convexity [25]);

(ii) polyconvexity: $\tilde{\Phi}(C)$ is a convex function of the minors of C; i.e., there exists a convex function Φ such that $\tilde{\Phi}(C) = \Phi(C, \det C, \mathrm{cof} C)$ (recall that the cofactor matrix $\mathrm{cof}C$ for $\det C \neq 0$ is defined as $C^{\mathrm{T}} \mathrm{cof} C = I \det C$);

(iii) certain growth conditions (coercivity).

The growth conditions can be roughly classified as follows:

(i) growth conditions that guarantee the ralidity of the existence theorem for the stationary elasticity equations, for example,

$$\tilde{\Phi}(C) \geq c_1(\|C\|^2 + \|\mathrm{cof}C\|^{3/2}) - c_2;$$

(ii) growth conditions that guarantee the continuity of the minimizing mapping, for example,

$$\tilde{\Phi}(C) \geq c_1\|C\|^p, \quad p > 3;$$

(iii) growth conditions that guarantee the invertibility of the minimizing mapping:

$$\tilde{\Phi}(C) \geq c_2 + c_1(\|C\|^p + \|\mathrm{cof}C\|^q + (\det C)^{-s}) \tag{4.3}$$

for all matrices C with a positive determinant, where $p > 3$, $q > 3$, $s > \dfrac{2q}{q-3}$, and $c_1 > 0$ is a constant.

It turns out that the elastic potentials of actual materials do not satisfy these conditions. For example, they are frequently independent of $\mathrm{cof}C$. We see that the theoretical growth conditions in the working range of deformations rely heavily on the admissibility of deformations near the singular points or for singular stresses, when the behavior of the material is not described by elasticity theory.

5. WAVE EQUATIONS OF NONLINEAR ACOUSTICS

As the first step in the construction of a canonical form of the elasticity equations, we consider the nonstationary wave equations of nonlinear acoustic. Assume that the elastic deformation depends on time t: $x(t, \xi)$. The nonlinear wave equations are derived from the variational Lagrange principle and are written as

$$\rho_0 \ddot{x}_i - \frac{\partial}{\partial \xi_j} \frac{\partial \tilde{\Phi}}{\partial(\partial x_i/\partial \xi_j)} = 0,$$

where ρ_0 is the initial density of the undeformed material, which is hereafter assumed to be in a constant. According to [1], each polyconvex function $\tilde{\Phi} : \mathbb{R}^{3\times 3} \longrightarrow \mathbb{R}$ is rank-one convex; i.e., it satisfies

$$\tilde{\Phi}(\lambda C_1 + (1-\lambda)C_2) \leq \lambda \tilde{\Phi}(C_1) + (1-\lambda)\tilde{\Phi}(C_2) \quad \text{for} \quad \mathrm{rank}(C_1 - C_2) \leq 1.$$

If $\tilde{\Phi}$ is twice continuously differentiable, then rank-one convexity is equivalent to the Hadamard–Legendre condition (ellipticity condition)

$$\frac{\partial^2 \tilde{\Phi}}{\partial c_{ij} \partial c_{km}} p_i p_k r_j r_m \geq 0 \quad \text{for all} \quad p, r \in \mathbb{R}^3. \tag{5.1}$$

Thus, the wave equations of nonlinear acoustics with a polyconvex potential are hyperbolic in the sense of Friedrichs.

5.1. Wave Equations of Nonlinear Acoustics in Extended Form

Let $u_i(t, \xi) = \partial x_i/\partial t$ be the velocity components. The nonlinear wave equations can be written as a system of first-order equations

$$\frac{\partial \rho_0 u_i}{\partial t} - \frac{\partial}{\partial \xi_j} \tilde{\Phi}_{c_{ij}} = 0,$$

$$\frac{\partial c_{ij}}{\partial t} - \frac{\partial u_i}{\partial \xi_j} = 0,$$

with the entropy function defined by the total energy

$$E = \frac{1}{2}\rho_0 u_i u_i + \tilde{\Phi}(C).$$

The resulting system can formally be symmetrized. However, for real materials, it is not hyperbolic in the sense of Friedrichs, since E is generally not convex.

6. EXTENDED NONLINEAR ELASTICITY EQUATIONS IN LAGRANGIAN COORDINATES

6.1. The Form of the Potential and the Choice of Extended Variables

For isotropic elastic materials, the elastic potential can always be written as a function of three independent invariants $\mathcal{I}_k$ of C^TC:

$$\tilde{\Phi}(C) = F(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3) = \tilde{F}(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3),$$

where $\mathcal{G}_k$ are the principal invariants; i.e.,

$$\mathcal{G}_1 = \mathrm{tr} C^T C, \quad \mathcal{G}_2 = \mathrm{tr}\,\mathrm{cof} C^T \mathrm{cof} C, \quad \mathcal{G}_3 = \det C.$$

The question then arises about variables in which the internal energy of the elastic material can be written as a convex function. Based on long-time studies, numerical computations, and comparisons with experimental data, Godunov has formulated the following statement: elastic potentials for actual materials have to be constructed using all three independent invariants, more specifically, $w = \det C$ and two independent invariants, while only 10 variables (C and $w = \det C$) are sufficient to be used as extended variables. This conclusion was drawn by analyzing impact deformations in metals.

On the other hand, the study of deformations in polymers and rubberlike materials can rely on somewhat different considerations, which are supported by analyzing published data [26]. Specifically, if the dependence on all three principal invariants is essential, then the total set of 19 extended variables C, $w = \det C$, and $A = \mathrm{cof} C$ apparently have to be used as unknowns in the extended system. On the other hand, if C and $w = \det C$ are sufficient to be used as unknowns in the extended system, then we can assume that the elastic potential is, in fact, described by only two invariants.

Let ξ_1, ξ_2, and ξ_3 be the Lagrangian coordinates of a material point, while x_1, x_2, and x_3 be its Eulerian coordinates. The mapping $x(t, \xi)$ defines a stationary deformation of the elastic body, and $u_i = \partial x_i/\partial t$ are the velocity components of the material point. The Jacobian matrix of $x(\cdot, \xi)$ is denoted by C, where $c_{ij} = \partial x_i/\partial \xi_j$. Let the determinant of this matrix be denoted by $w = \det C$ and A stand for the cofactor matrix $A = \mathrm{cof} C$. If $\det C \neq 0$, then

$$C^T \mathrm{cof} C = I \det C, \quad A = \det C\, C^{-T}.$$

The elasticity equations for u_i, the elements of C, and the total energy

$$E = \rho_0 \frac{u_i u_i}{2} + \tilde{\Phi}$$

are written as follows (see, e.g., [11]):

$$\begin{aligned} \frac{\partial \rho_0 u_i}{\partial t} - \frac{\partial \tilde{\Phi}_{c_{ij}}}{\partial \xi_j} &= 0, \\ \frac{\partial c_{ij}}{\partial t} - \frac{\partial u_i}{\partial \xi_j} &= 0, \end{aligned} \tag{6.1}$$

$$\frac{\partial}{\partial t}\left(\rho_0 \frac{u_i u_i}{2} + \tilde{\Phi}\right) - \frac{\partial}{\partial \xi_j} u_i \tilde{\Phi}_{c_{ij}} = 0, \tag{6.2}$$

where the elastic potential $\tilde{\Phi}(C, S)$ defines the equation of state of the material, S is the entropy, and ρ_0 is the initial density of the elastic material. More precisely, for irreversible processes, the term “potential” becomes inappropriate and the correct name of $\tilde{\Phi}(C, S)$ is the specific internal energy per unit Lagrangian volume. In what follows, the word “potential” is used only for brevity.

For smooth solutions of system (6.1), (6.2), we have the entropy conservation law

$$\frac{\partial S}{\partial t} = 0.$$

For nonsmooth solution, the entropy conservation law is replaced with the following selection rule for physically meaningful solutions:

$$\frac{\partial S}{\partial t} \geq 0.$$

The function $\tilde{\Phi}$ is assumed to be strictly polyconvex in the sense of Ball [1]; i.e., it can be written in as

$$\tilde{\Phi}(C, S) = \Phi(C, \det C, \operatorname{cof} C, S) = \Phi(C, w, A, S),$$

where $\Phi(\cdot, \cdot, \cdot, S)$ S is a regular strictly convex barrier function (see p. 1564).

At the first stage, the analysis is restricted to the case where Φ does not explicitly depend on $A = \operatorname{cof} C$. Consider the extended system of equations that was considered, for example, in [11]:

$$\begin{gathered}
\frac{\partial \rho_0 u_i}{\partial t} - \frac{\partial}{\partial \xi_j}(a_{ij}\Phi_w + \Phi_{c_{ij}}) = 0, \\
\frac{\partial w}{\partial t} - a_{ij}\frac{\partial u_i}{\partial \xi_j} = 0, \\
\frac{\partial c_{ij}}{\partial t} - \frac{\partial u_i}{\partial \xi_j} = 0, \\
\frac{\partial}{\partial t}\left(\rho_0 \frac{u_i u_i}{2} + \Phi\right) - \frac{\partial}{\partial \xi_j} u_i (a_{ij}\Phi_w + \Phi_{c_{ij}}) = 0.
\end{gathered} \tag{6.3}$$

It is well known that all the minors of the Jacobian matrix are null Lagrangians; i.e., their integration over a domain is reduced to an integral over the domain boundary. Obviously, the columns a^i of A satisfy

$$a^i = \frac{1}{2}\left(\frac{\partial}{\partial \xi_j}(x \times c_k) - \frac{\partial}{\partial \xi_k}(x \times c_j)\right),$$

where c_k is the kth column of C and $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$. This equality implies the geometric conservation law

$$\frac{\partial a_{ij}}{\partial \xi_j} = 0, \tag{6.4}$$

which is known in elasticity theory as the Piola relationship. Equality (6.4) simply means that the divergence of an arbitrary constant vector field vanishes.

Thus, system (6.3) can be written in the divergence form

$$\begin{gathered}
\frac{\partial \rho_0 u_i}{\partial t} - \frac{\partial}{\partial \xi_j}(a_{ij}\Phi_w + \Phi_{c_{ij}}) = 0, \\
\frac{\partial w}{\partial t} - \frac{\partial}{\partial \xi_j} a_{ij} u_i = 0, \\
\frac{\partial c_{ij}}{\partial t} - \frac{\partial u_i}{\partial \xi_j} = 0, \\
\frac{\partial}{\partial t}\left(\rho_0 \frac{u_i u_i}{2} + \Phi\right) - \frac{\partial}{\partial \xi_j} u_i (a_{ij}\Phi_w + \Phi_{c_{ij}}) = 0.
\end{gathered} \tag{6.5}$$

The advection equation for c_{ij} implies

$$\frac{\partial}{\partial t}\beta_{jk} = 0, \quad \beta_{jk} = \frac{\partial c_{ij}}{\partial \xi_k} - \frac{\partial c_{ik}}{\partial \xi_j},$$

where β_{jk} are the components of the skew-symmetric Burgers tensor $\mathcal{B}$. Thus, if initially $\mathcal{B}(0, \xi) = 0$, then $\mathcal{B}(t, \xi) = 0$. This means that the given gradient field $c_{ij}(t, \xi)$ can be used to recover the deformation $x(t, \xi)$. This also implies the additional conservation law (6.4) and the validity of relation $w = \det C$ for $t > 0$ if it holds at $t = 0$.

7. CONSTRUCTION OF A CANONICAL REPRESENTATION IN LAGRANGIAN COORDINATES

To construct potentials and variables for transforming system (6.5) to the generalized canonical representation (3.14), we consider the total energy

$$E(u, C, w, S) = \rho_0\frac{u_iu_i}{2} + \Phi(C, w, S), \tag{7.1}$$

which is a strictly convex function of $3 + 9 + 1 + 1 = 14$ variables u, C, w, and S. Assume that $E_S > 0$. By writing formula (7.1) for the total energy in the form

$$\Psi(u, C, w, S) - E = 0,$$

where Ψ is a sufficiently smooth strictly convex function, the entropy S can be expressed from this equality as

$$S = -\Pi(u, C, w, E),$$

where Π is a strictly convex function. Indeed, joining u, C, an w in a single vector y of dimension $3 + 9 + 1 = 13$,

$$H_\Pi = \frac{1}{\Psi_S}\begin{pmatrix} I & S_y \\ 0 & S_E \end{pmatrix} H_\Psi \begin{pmatrix} I & S_y \\ 0 & S_E \end{pmatrix}^{\mathrm{T}},$$

where H_Π and H_Ψ denote the Hessian matrices of Π and Ψ, respectively.

We use as the primary variable $\tilde{u} = \rho_0 u$. The dual variables u^*, c^*_{ij}, w^*, and E^* are defined as

$$u^* = -\frac{\partial S}{\partial \tilde{u}},\; C^* = -\frac{\partial S}{\partial C},\quad w^* = -\frac{\partial S}{\partial w},\; E^* = -\frac{\partial S}{\partial E}.$$

The dual function $\mathcal{L} = (-S)^*$ is written as

$$\mathcal{L} = \tilde{u}_i u^*_i + c_{ij}c^*_{ij} + ww^* + EE^* + S.$$

The properties of the Legendre transformation imply that $\mathcal{L}$ is a convex function of the components of the vector q composed of $3 + 9 + 1 + 1 = 14$ variables u^*, c^*_{ij}, w^*, and E^*.

The potentials $\mathcal{L}^j$ are defined as

$$\mathcal{L}^j = \frac{u^*_i}{E^*}(a_{ij}w^* + c^*_{ij}).$$

This expression is derived using the relations, which follow from the expression for the total energy $E = \dfrac{\tilde{u}_i\tilde{u}_i}{2\rho_0} + \Phi(C, w, S)$. Thus,

$$\Phi_S S_C + \Phi_C = 0,\quad \rho_0\Phi_S S_{\tilde{u}} + \tilde{u} = 0,$$

$$\Phi_S S_E = 1,\quad \Phi_S S_w + \Phi_w = 0.$$

The incomplete gradient of $\mathcal{L}^j$ is given by

$$\Phi_S = -\frac{1}{E^*}, \quad \Phi_C = -\frac{C^*}{E^*},$$

$$u = -\frac{u^*}{E^*}, \quad \Phi_w = -\frac{w^*}{E^*}.$$

$$\tilde{\mathcal{L}}^j_{q_i} = \begin{pmatrix} \frac{1}{E^*}(a_{ij}w^* + c^*_{ij}) \\ \frac{u^*_i}{E^*}\delta_{ij} \\ \frac{u^*_i}{E^*}(a_{ij}) \\ -\frac{u^*_i}{E^{*2}}(a_{ij}w^* + c^*_{ij}) \end{pmatrix} = \begin{pmatrix} -(a_{ij}\Phi_w + \Phi_{c_{ij}}) \\ -u_i\delta_{ij} \\ -u_i a_{ij} \\ -u_i(a_{ij}\Phi_w + \Phi_{c_{ij}}) \end{pmatrix}.$$

Thus, the generalized canonical system (3.14) coincides with the original system (6.5), while the additional conservation law (3.15) is the entropy law

$$\frac{\partial S}{\partial t} \ge 0,$$

which, for smooth solutions, is replaced by the conservation law

$$\frac{\partial S}{\partial t} = 0.$$

As a result, the canonical system (3.14) admits the nonconservative formulation (3.16), which is symmetric and hyperbolic in the sense of Friedrichs.

7.1. General Polyconvex Elastic Potential

For completeness, consider the most general elastic potential

$$\Phi = \Phi(C, w, A, S).$$

Examples of rubberlike materials and polymers with such a potential were described in [25, 27–29]. In this case, the original vector of variables consists of $3 + 9 + 1 + 9 + 1 = 23$ variables u, C, w, A, and S, and the dual function $\mathcal{L} = (-S)^*$ is written as

$$\mathcal{L} = \tilde{u}_i u^*_i + c_{ij}c^*_{ij} + ww^* + a_{ij}a^*_{ij} + EE^* + S.$$

The dual variables have been augmented with the matrix

$$a^*_{ij} = \frac{\partial \mathcal{L}}{\partial a_{ij}},$$

which satisfies the obvious relation

$$\Phi_A = -\frac{A^*}{E^*}.$$

The potentials $\mathcal{L}^j$ are modified as follows:

$$\mathcal{L}^j = \frac{u_i^*}{E^*}(a_{ij}w^* + c_{ij}^* + d_{kmij}a_{km}^*),$$

where

$$d_{kmij} = \frac{\partial a_{km}}{\partial c_{ij}}.$$

To make the system of elasticity equations closed, we need an equation of motion for the matrix A, which is derived using the equality

$$a^i = c_j \times c_k,$$

where a^i is the i-th column of A; c_j is the j-th column of C; and (i, j, k) is a cyclic permutation of 1, 2, 3. Thus,

$$\frac{\partial a^i}{\partial t} = \frac{\partial c_j}{\partial t} \times c_k + c_j \times \frac{\partial c_k}{\partial t} = u_{\xi_j} \times c_k - u_{\xi_k} \times c_j, \tag{7.2}$$

which yields the closing equation in divergence form

$$\frac{\partial a^i}{\partial t} = \frac{\partial}{\partial \xi_j}(u \times c_k) - \frac{\partial}{\partial \xi_k}(u \times c_j). \tag{7.3}$$

Meanwhile, $\mathcal{L}$ and $\mathcal{L}^j$ generate the following closure equation of the form

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial a_{km}^*} + \frac{\partial}{\partial \xi_j}\left(\frac{u_i^*}{E^*}d_{kmij}\right) = 0,$$

or, in the old variables,

$$\frac{\partial}{\partial t}a_{km} - \frac{\partial}{\partial \xi_j}(u_i d_{kmij}) = 0, \tag{7.4}$$

which exactly coincides with Eq. (7.3). Let us prove this assertion and also show that d_{kmij} satisfies the conservation law

$$\frac{\partial}{\partial \xi_j}d_{kmij} = 0. \tag{7.5}$$

Indeed, the cofactor matrix $A = \mathrm{cof}C$ is defined by the equality (see, e.g., [25])

$$a_{km} = \frac{1}{2}\varepsilon_{lnk}\varepsilon_{pqm}c_{lp}c_{nq},$$

where the three-index array ε_{ijk}, which is known in elasticity as the orientation tensor, is defined as

$$\varepsilon_{ijk} = \begin{cases} 1, \quad \text{if } \{i, j, k\} \text{is an even permulation of } \{1, 2, 3\}, \\ (-1, \quad \text{if } \{i, j, k\}) \text{ is an even permulation of } \{1, 2, 3\}, \\ 0 \text{ if any two indices coincide.} \end{cases} \tag{7.6}$$

Then

$$d_{kmij} = \frac{\partial a_{km}}{\partial c_{ij}} = \frac{1}{2}\varepsilon_{lnk}\varepsilon_{pqm}(\delta_{il}\delta_{jp}c_{nq} + \delta_{in}\delta_{jq}c_{lp}) = \frac{1}{2}(\varepsilon_{ink}\varepsilon_{jqm}c_{nq} + \varepsilon_{lik}\varepsilon_{pjm}c_{lp}) = \varepsilon_{ink}\varepsilon_{jqm}c_{nq},$$

so that

$$\frac{\partial d_{kmij}}{\partial \xi_j} = \varepsilon_{ink}\varepsilon_{jqm}\frac{\partial^2 x_n}{\partial \xi_q \partial \xi_j} = 0,$$

since this expression involves the convolution, over two indices, of a symmetric and a skew-symmetric index array.

Let us show that Eqs. (7.3) and (7.4) coincide. Relation (7.2) can be written as

$$\frac{\partial a^i}{\partial t} = \varepsilon_{ijk}(u_{\xi_j} \times c_k).$$

In turn,

$$(u_{\xi_j} \times c_k)_l = \varepsilon_{lqm} u_{q\xi_j} c_{mk}.$$

Thus, we obtain

$$\frac{\partial a_{li}}{\partial t} = \varepsilon_{ijk}\varepsilon_{lqm} u_{q\xi_j} c_{mk} = d_{liqj} u_{q\xi_j},$$

which, by virtue of conservation law (7.5), coincides with (7.4).

It can be shown that, if the conservation law (7.5) holds at $t = 0$, then it holds for $t > 0$.

7.2. Polyconvexity, Rank-One Convexity, and Hyperbolicity of One-Dimensional Equations

Since each convex function $f(C)$ is rank-one convex, the function $f(C + \delta C)$ is convex with respect to the matrix δC of rank 1. An example of a rank-one matrix is one with a single nonzero column.

Consider the elasticity equations assuming that an elastic deformation can be represented as the sum of a general mapping $x(\xi_1, \xi_2, \xi_3, t)$ and a correction δx, which is a function of only one Lagrangian coordinate, for example, $\delta x = \delta x(\xi_1, t)$. In this case, the Jacobian matrix of the mapping $x + \delta x$ is written as $C + \delta C$, where δC is a matrix of rank 1 and the polyconvex elastic potential $\tilde{\Phi}(C + \delta C, S)$ is a convex function of δC. Thus, the one-dimensional elasticity equations in terms of δC and S for δx and S admit a Godunov canonical representation and are hyperbolic in the sense of Friedrichs. It is well-known that Godunov's scheme for the elasticity equations is implemented via the solution of a series of one-dimensional Riemann problems at grid cell interfaces. As a result, we see that the Godunov scheme can be implemented without using the extended system, which includes the minors of C. Note that the correction can be a function of the single variable $w_1\xi_1 + w_2\xi_2 + w_3\xi_3$, where w_i are arbitrary constants. It follows that Godunov scheme can be applied not only on rectangular cells in ξ_i but also on triangular or polygonal (polyhedral) cells when the direction along which the Riemann problem is solved is arbitrary.

8. SYMMETRIZATION IN EULER VARIABLES

8.1. Nonlinear Elasticity Equations

The solution of the elasticity equations in the Eulerian formulation is the mapping $\xi(x, t)$. Thus, to write the equations in terms of x_k and t, it is convenient to use the inverse B of the matrix C. The unknowns are the density $\rho = \det B\rho_0$ and the adjugate matrix $R = \rho_0 \operatorname{cof} B^{\mathrm{T}} = \rho C$. These quantities are conveniently represented in a table:

$$B = C^{-1}, \quad \rho = \det B\rho_0 = \frac{\rho_0}{\det C} = \frac{\rho_0}{w} = \frac{1}{V},$$

$$R = \rho_0 \operatorname{cof} B^{\mathrm{T}}, \quad R = \rho C.$$

In Eulerian variables, the elastic potential $\tilde{\Phi}(C, S) = \Phi(C, \det C, \operatorname{cof} C, S) = \Phi(C, w, A, S)$ is replaced with a function $\Theta(R, \rho, B, S)$, where

$$\Theta(R, \rho, B, \tilde{S}) = \rho\Phi\left(\frac{1}{\rho}R, \frac{1}{\rho}, \frac{1}{\rho}B^{\mathrm{T}}, \frac{\tilde{S}}{\rho}\right), \tag{8.1}$$

and $\tilde{S} = \rho S$ is the entropy per unit Eulerian volume. Since Φ is strictly convex, Θ is strictly convex as well (see, for example, [1]).

Note that the minors of C are dimensionless variables, so formally the internal energy in Eulerian coordinates should also be written in terms of density-dimensionless variables as follows:

$$\Theta = \frac{\rho}{\rho_0}\Phi\left(\frac{\rho_0}{\rho}R, \frac{\rho_0}{\rho}, \frac{\rho_0}{\rho}B^{\mathrm{T}}, \frac{\rho_0\tilde{S}}{\rho}\right).$$

However, following the notation in [3, 8], the dependence on ρ_0 in this formula is not explicitly indicated. Note that extended forms of the elasticity equations were also considered in [30].

The elastic stress tensor $\mathcal{S}$ with components σ_{ik} is given by

$$\sigma_{ik} = \rho c_{kj}\left(\Phi_V(\mathrm{cof}C)_{ij} + \Phi_{c_{ij}} + \Phi_{a_{lm}}\frac{\partial a_{lm}}{\partial c_{ij}}\right).$$

Since

$$\Theta_{r_{ij}} = \Phi_{c_{ij}}, \quad \Phi_V = -\rho^2\Phi_\rho, \quad \Theta_{b_{ij}} = \Phi_{a_{ji}},$$

we obtain

$$\sigma_{ik} = -\delta_{ik}\rho^2\Phi_\rho + r_{kj}\Theta_{r_{ij}} + z_{kipm}\Theta_{b_{mp}},$$

where

$$z_{kipm} = \rho c_{kj}d_{pmij} = \rho c_{kj}\frac{\partial a_{pm}}{\partial c_{ij}}. \tag{8.2}$$

The elasticity equations in Cartesian coordinates x_k can be written as follows (see, e.g., [8]):

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_i u_k - \sigma_{ik}) = 0, \tag{8.3}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k) = 0, \tag{8.4}$$

$$\frac{\partial \rho c_{ij}}{\partial t} + \frac{\partial}{\partial x_k}(u_k\rho c_{ij} - u_i\rho c_{kj}) = 0, \tag{8.5}$$

$$\frac{\partial b_{ij}}{\partial t} + \frac{\partial}{\partial x_j}(u_l b_{il}) = 0, \tag{8.6}$$

$$\frac{\partial}{\partial t}\left(\Theta + \frac{1}{2}\rho u_i u_i\right) + \frac{\partial}{\partial x_k}\left(\frac{1}{2}\rho u_k u_i u_i + u_k\Theta - u_i\sigma_{ik}\right) = 0. \tag{8.7}$$

This system is supplemented with the entropy law

$$\frac{\partial}{\partial t}(\rho S) + \frac{\partial}{\partial x_k}(\rho S u_k) \ge 0,$$

which becomes an equality for smooth solutions.

At the first stage of the analysis, we assume that the elastic potential is independent of S. In this case, the role of mathematical entropy is played by the total energy per unit Eulerian volume, i.e., by the quantity

$$\mathcal{E} = \Theta(R, \rho, B) + \frac{1}{2}\rho u_i u_i.$$

In this model, the total energy dissipates at the discontinuities. Apparently, the model has no mechanical meaning, but is instructive from a mathematical point of view. The argument presented below is close to that used in [8].

To solve the symmetrization problem and construct potentials, we introduce the set of primary unknowns

$$f_i = \rho u_i, \rho, \quad r_{ij} = \rho c_{ij}, b_{ij}.$$

The total energy $\mathcal{E}$ is written as a convex function of these arguments:

$$\mathcal{E} = \Theta(R, \rho, B) + \frac{1}{2}\frac{f_i f_i}{\rho}.$$

The elasticity equations with these unknowns are given by

$$\frac{\partial f_i}{\partial t} + \frac{\partial}{\partial x_k}\left(\frac{f_i f_k}{\rho} + \delta_{ik}\rho^2\Phi_\rho - r_{ij}\Theta_{r_{jk}} - z_{kipm}\Theta_{b_{mp}} = 0\right), \tag{8.8}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(f_k) = 0, \tag{8.9}$$

$$\frac{\partial r_{ij}}{\partial t} + \frac{\partial}{\partial x_k}(u_k r_{ij} - u_i r_{kj}) = 0, \tag{8.10}$$

$$\frac{\partial b_{ij}}{\partial t} + \frac{\partial}{\partial x_j}(u_l b_{il}) = 0. \tag{8.11}$$

The "entropy" equation is written as

$$\frac{\partial}{\partial t}\mathcal{E} + \frac{\partial}{\partial x_k}(u_k\mathcal{E} - u_i\sigma_{ik}) \le 0.$$

As the basic potential, we use the Legendre transform of $\mathcal{E}$ with respect to f_i, ρ, r_{ij}, and b_{ij}; i.e.,

$$\mathcal{L} = f_i f_i^* + \rho\rho^* + r_{ij}r_{ij}^* + b_{ij}b_{ij}^* - \mathcal{E}, \tag{8.12}$$

where, as usual, the dual variables are the derivatives of the original potential $\mathcal{E}$ with respect to the original variables:

$$\rho^* = \Theta_\rho - \frac{1}{2}\frac{f_i f_i}{\rho^2} = \Phi - V\Phi_V - c_{ij}\Phi_{c_{ij}} - a_{ij}\Phi_{a_{ij}} - \frac{1}{2}u_i u_i,$$

$$f_i^* = \frac{f_i}{\rho} = u_i, \quad r_{ij}^* = \Theta_{r_{ij}} = \Phi_{c_{ij}}, \quad b_{ij}^* = \Theta_{b_{ij}} = \Phi_{a_{ji}},$$

As a result, the final formula for ρ^* is

$$\rho^* = \Phi + \rho\Phi_\rho - \frac{1}{\rho}r_{ij}r_{ij}^* - \frac{1}{\rho}b_{ij}b_{ij}^* - \frac{1}{2}u_i u_i.$$

Substituting the formulas for the dual variables into (8.12) yields

$$\mathcal{L} = \rho\Theta_\rho - \Theta + r_{ij}r_{ij}^* + b_{ij}b_{ij}^* = \rho^2\Phi_\rho.$$

The potentials $\mathcal{L}^k$ are given by

$$\mathcal{L}^k = f_k^*\mathcal{L} - f_i^* r_{kj} r_{ij}^* - f_i^* z_{kipm} b_{mp}^*.$$

Here, r_{kj} and z_{kipm} play the role of divergence-free vector fields in (3.11), since, according to the Appendix, they satisfy

$$\frac{\partial \rho_{kj}}{\partial x_k} = 0, \quad \frac{\partial z_{kipm}}{\partial x_k} = 0. \tag{8.13}$$

If the additional conservation laws (8.13) hold at $t = 0$, then they hold for $t > 0$.

Since the divergence-free fields r_{kj} and z_{kipm} do not need to be differentiated, the incomplete partial derivatives of $\mathcal{L}^k$ with respect to the dual variables are written as

$$\frac{\partial\tilde{\mathcal{L}}^k}{\partial\rho^*} = \rho f_k^* = \rho u_k, \quad \frac{\partial\tilde{\mathcal{L}}^k}{\partial r_{ij}^*} = f_k^* r_{ij} - f_i^* r_{kj} = u_k r_{ij} - u_i r_{kj},$$

$$\frac{\partial\tilde{\mathcal{L}}^k}{\partial f_i^*} = \delta_{ik}\mathcal{L} + f_k^* f_i - r_{kj} r_{ij}^* - z_{kipm} b_{mp}^* = \rho u_i u_k + \delta_{ik}\rho^2\Phi_\rho - r_{ij}\Theta_{r_{jk}} - z_{kipm}\Theta_{b_{mp}}, \tag{8.14}$$

$$\frac{\partial\tilde{\mathcal{L}}^k}{\partial b_{ij}^*} = f_k^* b_{ij} - z_{klji} f_l^* = \delta_{jk} u_l b_{il}.$$

The derivation of the last equality can by found in the Appendixs.

Let us compute the expression

$$\tilde{\mathcal{L}}^k_{\rho^*}\rho^* + \tilde{\mathcal{L}}^k_{f_i^*} f_i^* + \tilde{\mathcal{L}}^k_{r_{ij}^*} r_{ij}^* + \tilde{\mathcal{L}}^k_{b_{ij}^*} b_{ij}^* - \mathcal{L}^k.$$

It can be written as

$$u_k\rho^*\rho + \rho u_k u_i u_i + \rho u_k \rho\Phi_\rho - u_i r_{ij} r_{jk}^* + u_k(r_{ij} r_{ij}^* + b_{ij} b_{ij}^*) - u_k(u_i r_{kj} r_{ij}^* + u_i z_{kipm} b_{mp}^*)$$
$$- u_k\rho^2\Phi_\rho + u_i r_{kj} r_{ij}^* + u_i z_{kipm} b_{mp}^* = \rho u_k\Big(\Phi + \rho\Phi_\rho - \frac{1}{\rho}(r_{ij} r_{ij}^* + b_{ij} b_{ij}^*) - \frac{1}{2}u_i u_i\Big)$$
$$+ \rho u_k u_i u_i - u_i(r_{ij} r_{jk}^* + z_{kipm} b_{mp}^*) u_k (r_{ij} r_{ij}^* + b_{ij} b_{ij}^*)$$
$$= \frac{1}{2}\rho u_k u_i u_i + u_k\Theta - u_i(-\delta_{ik}\rho^2\Phi_\rho + r_{ij} r_{jk}^* + z_{kipm} b_{mp}^*) = \frac{1}{2}\rho u_k u_i u_i + u_k\Theta - u_i\sigma_{ik}.$$

Thus, we have proved, for the vector q of the dual variables f_i^*, ρ^*, r_{ij}^*, and b_{ij}^*, elasticity equations (8.8)–(8.11) can be written in the generalized canonical form (3.14):

$$\frac{\partial\mathcal{L}_q}{\partial t} + \frac{\partial\tilde{\mathcal{L}}^k_q}{\partial x_k} = 0, \tag{8.15}$$

This is supplemented with entropy inequality (3.15):

$$\frac{\partial(q^{\mathrm{T}}\mathcal{L}_q - \mathcal{L})}{\partial t} + \frac{\partial(q^{\mathrm{T}}\tilde{\mathcal{L}}^k_q - \mathcal{L}^k)}{\partial x_k} \le 0, \tag{8.16}$$

which becomes an equality for smooth solutions. Since the potential $\mathcal{L}$ is strictly convex, the resulting system of equations is hyperbolic in the sense of Friedrichs.

If the elastic potential Θ is independent of B, then Eq. (8.11) and the dual variable b_{ij}^* are dropped.

8.2. Symmetrization of the Thermoelasticity Equations

Now consider the complete system of thermoelasticity equations in the variables $f_i = \rho u_i$, ρ, $r_{ij} = \rho c_{ij}$, $\mathcal{E}$, and b_{ij}:

$$\frac{\partial f_i}{\partial t} + \frac{\partial}{\partial x_k}\Big(\frac{1}{\rho}f_i f_k - \sigma_{ik}\Big) = 0,$$

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_k}(f_k) = 0,$$

$$\frac{\partial r_{ij}}{\partial t} + \frac{\partial}{\partial x_k}(u_k r_{ij} - u_i r_{kj}) = 0,$$

$$\frac{\partial b_{ij}}{\partial t} + \frac{\partial}{\partial x_j}(u_l b_{il}) = 0, \tag{8.17}$$

$$\frac{\partial}{\partial t}(\mathscr{E}) + \frac{\partial}{\partial x_k}(u_k\mathscr{E} - u_i\sigma_{ik}) = 0.$$

This system is supplemented with the entropy inequality

$$\frac{\partial}{\partial t}(\tilde{S}) + \frac{\partial}{\partial x_k}(\tilde{S}u_k) \ge 0,$$

which becomes an equality for smooth solutions. Here, $\tilde{S} = \rho S$ is the entropy per unit Eulerian volume.

The total energy $\mathscr{E}$ is a strictly convex function of ρ, r_{ij}, b_{ij}, f_i, and $\tilde{S}$. Moreover, $\mathscr{E}_{\tilde{S}} > 0$. Therefore, $-\tilde{S}$ can be represented as a strictly convex function of ρ, R, B, f_i, and $\mathscr{E}$ (see the exposition in Section 7).

We are interested primarily in the case where $\mathscr{E}$ is independent of b_{ij}, and the general presentation is of interest from a mathematical point of view. Overall, a canonical representation is constructed following [8, 11]. As a potential $\mathscr{L}$, we use the Legendre transformation of $-\tilde{S}$ with respect to ρ, r_{ij}, b_{ij}, f_i, and $\mathscr{E}$:

$$\mathscr{L} = \rho\rho^* + r_{ij}r_{ij}^* + b_{ij}b_{ij}^* + f_i f_i^* + \mathscr{E}\mathscr{E}^* + \tilde{S}, \tag{8.18}$$

while the potentials $\mathscr{L}^k$ are defined as

$$\mathscr{L}^k = \frac{1}{\mathscr{E}^*}(-f_k^*\mathscr{L} + f_i^* r_{kj} r_{ij}^* + f_i^* z_{kipm} b_{mp}^*). \tag{8.19}$$

To prove that $\mathscr{L}$ and $\mathscr{L}^k$ are potentials for the thermoelasticity equations, we consider the following equation for the implicit function $\tilde{S}$:

$$\frac{f_i f_i}{2\rho} + \Theta(R, \rho, B, \tilde{S}) - \mathscr{E} = 0.$$

The chain rule yields

$$\mathscr{E}^* = -\tilde{S}_{\mathscr{E}} = -\frac{1}{\Theta_{\tilde{S}}} = -\frac{1}{\Phi_S} = -\frac{1}{T},$$

where T is the temperature,

$$\rho^* = -\tilde{S}_\rho = -\mathscr{E}^*\left(\Theta_\rho - \frac{f_i f_i}{2\rho^2}\right) = -\mathscr{E}^*\left(\Phi + \rho\Phi_\rho - \frac{1}{\rho}(\Theta_{r_{ij}} r_{ij} + \Theta_{b_{ij}} b_{ij}) + \frac{1}{\rho}\tilde{S}\frac{1}{\mathscr{E}^*} - \frac{1}{2}u_i u_i\right),$$

$$f_i^* = -\tilde{S}_{f_i} = -\frac{f_i\mathscr{E}^*}{\rho},$$

and

$$r_{ij}^* = -\tilde{S}_{r_{ij}} = -\mathscr{E}^*\Theta_{r_{ij}} = -\mathscr{E}^*\Phi_{c_{ij}},$$

$$b_{ij}^* = -\tilde{S}_{b_{ij}} = -\mathscr{E}^*\Theta_{b_{ij}} = -\mathscr{E}^*\Phi_{a_{ji}}.$$

Using these expressions for the dual variables, we find $\mathscr{L}$:

$$\mathscr{L} = -\mathscr{E}^*\left(\rho\Phi + \rho^2\Phi_\rho - \Theta_{r_{ij}} r_{ij} - \Theta_{b_{ij}} b_{ij} - \frac{1}{2}\rho u_i u_i\right) - \tilde{S}$$

$$-\mathscr{E}^*\rho u_i u_i + \mathscr{E}^*\left(\rho\Phi + \frac{1}{2}\rho u_i u_i\right) - \mathscr{E}^* r_{ij}\Theta_{r_{ij}} - \mathscr{E}^* b_{ij}\Theta_{b_{ij}} + \tilde{S}.$$

Finally,

$$\mathcal{L} = -\mathcal{E}^*\rho^2\Phi_\rho = -\frac{\Phi_V}{T}.$$

The incomplete partial derivatives of $\mathcal{L}^k$ with respect to the dual variables are computed again taking into account that r_{kj} and z_{kipm} define divergence-free vector fields, which do not need to be differentiated. As a result, we obtain

$$\frac{\partial\tilde{\mathcal{L}}^k}{\partial\rho^*} = -\frac{1}{\mathcal{E}^*}f_k^*\rho = u_k,$$

$$\frac{\partial\tilde{\mathcal{L}}^k}{\partial r_{ij}^*} = \frac{1}{\mathcal{E}^*}(f_k^* r_{ij} + f_i^* r_{kj}) = (u_k r_{ij} - u_i r_{kj}),$$

$$\frac{\partial\tilde{\mathcal{L}}^k}{\partial b_{ij}^*} = \frac{1}{\mathcal{E}^*}(-f_k^* b_{ij} + f_l^* z_{klji}) = \delta_{jk} u_l b_{il},$$

$$\frac{\partial\tilde{\mathcal{L}}^k}{\partial f_i^*} = \frac{1}{\mathcal{E}^*}(-\delta_{ik}\mathcal{L} - f_k^* f_i + r_{kj} r_{ij}^* + z_{kipm} b_{mp}^*) = \rho u_k u_i + \delta_{ik}\rho^2\Phi_\rho - r_{kj}\Theta_{r_{ij}} - z_{kipm}\Theta_{b_{mp}} = \rho u_k u_i - \sigma_{ki},$$

$$\frac{\partial\tilde{\mathcal{L}}^k}{\partial\mathcal{E}^*} = -\frac{1}{\mathcal{E}^{*2}}(-f_k^*\mathcal{L} + f_i^* r_{kj} r_{ij}^* + f_i^* z_{kipm} b_{mp}^*) + -\frac{1}{\mathcal{E}^*} f_k^*\mathcal{E}$$

$$= u_k\mathcal{E} - u_i(-\delta_{ik}\rho^2\Phi_\rho + r_{kj}\Theta_{r_{ij}} + z_{kipm}\Theta_{b_{mp}}) = u_k\mathcal{E} - u_i\sigma_{ik}.$$

The verification is completed by calculating the function

$$\frac{\partial\tilde{\mathcal{L}}^k}{\partial\rho^*}\rho^* + \frac{\partial\tilde{\mathcal{L}}^k}{\partial r_{ij}^*} r_{ij}^* + \frac{\partial\tilde{\mathcal{L}}^k}{\partial f_{ij}^*} f_{ij}^* + \frac{\partial\tilde{\mathcal{L}}^k}{\partial b_{ij}^*} b_{ij}^* + \frac{\partial\tilde{\mathcal{L}}^k}{\partial\mathcal{E}^*}\mathcal{E}^* - \mathcal{L}^k = -u_k\tilde{S}.$$

Thus, we have shown that, with the use of generating potentials (8.18) and (8.19), the system of thermoelasticity equations can be reduced to the thermodynamically consistent canonical form (8.15), (8.16). Moreover, the strict convexity of $\mathcal{L}$ guarantees that the system is hyperbolic in the sense of Friedrichs.

If the potential is independent of B, then Eq. (8.17) is dropped from the system; b_{ij}, from the vector of primary variables; b_{ij}^*, from the dual variables; and the dependence of b_{ij}^*, from the generating potentials.

9. EXAMPLES OF POLYCONVEX ELASTIC POTENTIALS AND THEIR BEHAVIOR IN THE LIMIT OF SMALL DEFORMATIONS

To describe processes in metals at high pressures, an elastic potential was proposed in [11] that can be represented as the sum of a "gasdynamic" component and an "elastic" component. The gasdynamic component is derived from the assumption that the medium satisfies the two-term equation of state (see [31])

$$E(p, \rho) = \frac{p - c_0^2(\rho - \rho_0)}{(\gamma - 1)\rho}, \tag{9.1}$$

where E is the internal energy per unit of mass, p is the pressure, ρ is the density of the medium, ρ_0 is the initial density, γ is the adiabatic index (set equal to 4 for metals), and c_0 is the speed of sound in the medium. The pressure is expressed in terms of the density and entropy via the relation

$$p(\rho, S) = \frac{\rho_0 c_0^2}{\gamma}\left(s(S)\left(\frac{\rho}{\rho_0}\right)^\gamma - 1\right), \tag{9.2}$$

where $s(S)$ is the entropy function given by

$$s(S) = e^{S/c_V},$$

and c_V is the specific heat capacity at constant volume. Substituting (9.2) into (9.1) yields the following expression for the internal energy per unit volume up to a constant (see [11]):

$$\tilde{\Phi}_g(C, S) = \rho_0 E = \frac{\rho_0 c_0^2}{\gamma}\left(\frac{s(S)}{(\gamma-1)(\det C)^{\gamma-1}} + \det C\right), \tag{9.3}$$

where $V = 1/\rho$ denotes the specific volume and

$$\det C = \rho_0/\rho = \rho_0 V.$$

Since the potential $\tilde{\Phi}_g(C)$ ignores the shear stresses in the medium, in [11] it was proposed to add an "elastic" component that can be treated as a measure of shear stresses:

$$\tilde{\Psi}_e(C) = \frac{1}{3}\rho_0 c_1^2[(k_1-k_2)^2 + (k_2-k_3)^2 + (k_3-k_1)^2] = \rho_0 c_1^2\left[\operatorname{tr}C^{\mathrm{T}}C - \frac{1}{3}(\operatorname{tr}\sqrt{C^{\mathrm{T}}C})^2\right], \tag{9.4}$$

where $k_i(C)$ denotes the i-th singular value of C. For the total elastic potential to be polyconvex, it was assumed in [11] that the shape distortion is bounded; i.e., k_i/k_j is bounded from above for arbitrary i and j. Meanwhile, it is easy to construct shear stress measures that satisfy the polyconvexity condition and approximate (9.4), for example,

$$\tilde{\Phi}_e(C) = \frac{3}{2}\rho_0 c_1^2\left(\frac{1}{3}\operatorname{tr}C^{\mathrm{T}}C - (\det C)^{2/3}\right). \tag{9.5}$$

To analyze function (9.5), we assume that $k_i = k + \delta_i$, where $||\delta_i| \le \delta$ and δ is small as compared with 1 and k. Then

$$\frac{1}{3}\operatorname{tr}C^{\mathrm{T}}C = \frac{1}{3}\left(\sum k_i^2\right) = k^2 + \frac{2}{3}k\sum\delta_i + \frac{1}{3}\sum\delta_i^2,$$

$$(\det C)^{2/3} = k^2 + \frac{2}{3}k\sum\delta_i + \frac{2}{3}(\delta_1\delta_2 + \delta_2\delta_3 + \delta_3\delta_1) - \frac{1}{9}\sum\delta_i^2 + O(\delta^3/k) + O(\delta^4/k^2).$$

Thus,

$$\begin{aligned}\frac{3}{2}\left(\frac{1}{3}\operatorname{tr}C^{\mathrm{T}}C - (\det C)^{2/3}\right) &= \frac{1}{3}[(\delta_1-\delta_2)^2 + (\delta_2-\delta_3)^2 + (\delta_3-\delta_1)^2] + O(\delta^3/k)\\ &= \frac{1}{3}[(k_1-k_2)^2 + (k_2-k_3)^2 + (k_3-k_1)^2] + O(\delta^3/k).\end{aligned} \tag{9.6}$$

As a result, in the quadratic approximation, $\tilde{\Psi}_e(C)$ and $\tilde{\Phi}_e(C)$ coincide and remain close for strong compression and tension of the material.

The resulting elastic potential $\tilde{\Phi}(C, S) = \Phi(C, \det C, S)$ can be written as

$$\Phi(C, w, S) = \frac{\rho_0 c_0^2}{\gamma}\left(\frac{s(S)}{\gamma-1}\frac{1}{w^{\gamma-1}} + w\right) + \frac{3}{2}\rho_0 c_1^2\left(\frac{1}{3}\operatorname{tr}C^{\mathrm{T}}C - w^{2/3}\right). \tag{9.7}$$

The function $\Phi(C, w, S)$ is convex and can be expressed as a function of the principal invariants of $C^{\mathrm{T}}C$.

Since $\Phi_S > 0$, the potential Φ can be used to derive a canonical system of Godunov equations. Thus, a slight modification of the elastic potential from [11] allows us not only to make it convex for all arguments but also to avoid the singular value decomposition in the computation of the potential and its derivatives and in the approximate solution of the Riemann problem.

9.1. Behavior of the Potential under Small Deformations

Consider the behavior of the potential $\Phi(C, w, S)$ in the limit of small deformations. The natural state of the material corresponds to an orthogonal matrix C with a positive determinant, so that $C^{\mathrm{T}}C = I$, $w =$

$\det C = 1$, and $S = 0$. Again, the notation $\mathscr{E} = \frac{1}{2}(C^{\mathrm{T}}C - I)$ is used for the Green–Saint-Venant tensor. In the case of small deformations, we have expansion (4.2) (Hooke's law)

$$\tilde{\Phi}(C) = \tilde{\Phi}(I) + \frac{\lambda}{2}(\mathrm{tr}\mathscr{E})^2 + \mu\mathrm{tr}\mathscr{E}^2 + o(\|\mathscr{E}\|^2),$$

where μ and λ are the Lame constants. Obviously, when $S = 0$, function (9.7) reaches its absolute minimum at $C = U$ and $w = \det C = 1$. Thus, it admints representation (4.2). The Lame constants are computed using the well-known expansion formulas for the principal invariants of $C^{\mathrm{T}}C$ near the natural state (see, e.g., [25]):

$$\mathrm{tr}C^{\mathrm{T}}C = 3 + 2\mathrm{tr}\mathscr{E},$$

$$f(\det C) = f(1) + f'(1)\left(\mathrm{tr}\mathscr{E} + \frac{1}{2}(\mathrm{tr}\mathscr{E})^2 - \mathrm{tr}\mathscr{E}^2\right) + \frac{1}{2}f''(1)(\mathrm{tr}\mathscr{E})^2 + O(\|\mathscr{E}\|^3).$$

Here, $f(\cdot)$ denotes a sufficiently smooth function of one variable and f' and f'' are its first and second derivatives. We get

$$\frac{1}{\gamma-1}\frac{1}{w^{\gamma-1}} + w = \frac{\gamma}{\gamma-1} + \frac{\gamma}{2}(\mathrm{tr}\mathscr{E})^2 + O(\|\mathscr{E}\|^3)$$

and

$$\frac{3}{2}\left(\frac{1}{3}\mathrm{tr}C^{\mathrm{T}}C - w^{\frac{2}{3}}\right) = \mathrm{tr}\mathscr{E}^2 - \frac{1}{3}(\mathrm{tr}\mathscr{E})^2 + O(\|\mathscr{E}\|^3).$$

Substituting these relations into (9.7) results in

$$\Phi(C, w, 0) = \Phi(I, 1, 0) + \rho_0 c_1^2\mathrm{tr}\mathscr{E}^2 + \frac{1}{2}(\mathrm{tr}\mathscr{E})^2\left(\rho_0 c_0^2 - \frac{2}{3}\rho_0 c_1^2\right) + O(\|\mathscr{E}\|^3).$$

Thus, the equalities

$$\mu = \rho_0 c_1^2$$

and

$$\lambda = \rho_0\left(c_0^2 - \frac{2}{3}c_1^2\right)$$

can be used to determine c_0 and c_1 given the Lame constants λ and μ of the elastic material. Note that the bulk modulus κ is given by

$$\kappa = \rho_0 c_0^2.$$

Different versions of general polyconvex elastic potentials have been proposed for the simulation of rubberlike materials in elasticity theory. For example, the Ogden potential [28] can be represented as

$$\tilde{\Phi}_{\mathrm{O}}(C) = A(C) + B(\mathrm{cof}C) + \Gamma(\det C), \tag{9.8}$$

where A, B, and Γ are convex functions. In [25], it was shown how to relate the Lame constants to the unknown constants of the elastic potential given by (9.8). Another example in a similar class is the Hadamard elastic potential [25].

The use of the cofactor matrix as a variable in the elasticity equations substantially complicates the equations themselves and the solution method. For this reason, in most cases, experimental data are approximated so as to avoid the dependence on $\mathrm{cof}C$. Nevertheless, this dependence has to be used in certain cases. An example is the Blatz–Ko potential [32], which describes the behavior of foamed rubberlike materials. It can be written as

$$\tilde{\Phi}_{\mathrm{BK}}(C) = \frac{\mu}{2}\left(\frac{1}{k_1^2} + \frac{1}{k_2^2} + \frac{1}{k_3^2} - 3\right) + \mu(k_1k_2k_3 - 1),$$

or

$$\tilde{\Phi}_{\mathrm{BK}}(C) = \frac{\mu}{2}\left(\frac{\operatorname{tr}(\operatorname{cof}C^{\mathrm{T}}\operatorname{cof}C)}{\det C^2} - 3\right) + \mu(\det C - 1).$$

Since

$$\operatorname{tr}(\operatorname{cof}C^{\mathrm{T}}\operatorname{cof}C) = 3 + 4\operatorname{tr}\mathscr{E} + 2(\operatorname{tr}\mathscr{E})^2 - 2\operatorname{tr}\mathscr{E}^2 + o(\|\mathscr{E}\|^2)$$

and

$$\frac{1}{\det C^2} = 1 - 2\operatorname{tr}\mathscr{E} + 2(\operatorname{tr}\mathscr{E})^2 + 2\operatorname{tr}\mathscr{E}^2 + o(\|\mathscr{E}\|^2),$$

we obtain

$$\tilde{\Phi}_{\mathrm{BK}}(C) = \mu\operatorname{tr}\mathscr{E}^2 - \frac{\mu}{2}(\operatorname{tr}\mathscr{E})^2 + o(\|\mathscr{E}\|^2).$$

It is well known that if the ratio k_i/k_j is bounded above by a constant, then the Blatz–Ko potential satisfies the Hadamard–Legendre condition. It is unclear whether the Blatz–Ko potential is polyconvex.

It is well known (see, e.g., [33, 29, 26]) that the representation of elastic potentials as the sum of a term responsible for material compression/expansion (a measure of volume distortion or the dilatation potential), which depends only on $\det C$, and a term responsible for shape distortions (distortion potential), which is a symmetric function of the singular values of the matrix $(\det C)^{-1/3}C$, is a popular method for approximating experimental rheological data for various materials. For example, this class of elastic potentials was used in [34] to describe deformations of biological soft tissues, specifically, of elastic deformations of arteries.

A similar approach was used in [16, 35] to develop a method for constructing invertible mappings with prescribed properties. Specifically, a potential was constructed from geometric considerations as a convex linear combination of a volume distortion measure and a shape distortion measure:

$$W(C) = (1-\theta)\frac{\left(\frac{1}{3}\operatorname{tr}C^{\mathrm{T}}C\right)^{3/2}}{\det C} + \theta\frac{1}{2}\left(\det C + \frac{1}{\det C}\right), \quad 0 < \theta < 1. \tag{9.9}$$

The volume and shape distortion measures can be expanded

$$\frac{\left(\frac{1}{3}\operatorname{tr}C^{\mathrm{T}}C\right)^{3/2}}{\det C} = 1 + \operatorname{tr}\mathscr{E}^2 - \frac{1}{3}(\operatorname{tr}\mathscr{E})^2 + O(\|\mathscr{E}\|^3),$$

$$\frac{1}{2}\left(\det C + \frac{1}{\det C}\right) = 1 + \frac{1}{2}(\operatorname{tr}\mathscr{E})^2 + O(\|\mathscr{E}\|^3),$$

where the matrix $\mathscr{E}$ is defined in (4.1). Thus, the Lame constants for a hypothetical elastic material with the potential $W(C)$ are

$$\mu = (1-\theta), \quad \lambda = \frac{5\theta}{3} - \frac{2}{3}, \quad \kappa = \theta.$$

Moreover, the dimensionless shape distortion measure can also be treated as a measure of shear stresses in deformations. It has an expansion similar to (9.6), more precisely,

$$\frac{\left(\frac{1}{3}\operatorname{tr}C^{\mathrm{T}}C\right)^{3/2}}{\det C} = 1 + \frac{1}{3k^2}[(k_1-k_2)^2 + (k_2-k_3)^2 + (k_3-k_1)^2] + O(\delta^3/k^3), \tag{9.10}$$

where $k_i(C)$ is the ith singular value of C. Thus, in comparison with the standard quadratic shear stress measure (9.4), the shear stresses in (9.10) arising under strong compression of the material are rather severely penalized.

The following regularization of elastic potential (9.9) was proposed in [16, 35] to construct quasi-isometric mappings:

$$W_\alpha(C) = \begin{cases} \dfrac{(\alpha-1)W(I)W(C)}{\alpha W(I)-W(C)} & \text{if} \quad \alpha W(I)-W(C)>0, \quad 1<\alpha<+\infty. \\ +\infty \quad \text{if} \quad \alpha W(I)-W(C)\le 0, \end{cases} \tag{9.11}$$

This potential takes finite values only if

$$W(C) < \alpha W(I). \tag{9.12}$$

If λ and μ are the Lame constants of $W(C)$, then the Lame constants of $W_\alpha(C)$ are obtained in the form

$$\lambda_\alpha = \frac{\alpha}{\alpha-1}\lambda, \quad \mu_\alpha = \frac{\alpha}{\alpha-1}\mu.$$

Thus, as $\alpha \longrightarrow +\infty$, the Lame constants of the regularized potential converge to the original values, whereas a decrease in α can be interpreted as a stiffening effect.

Note that, if the regularized potential is defined as

$$\tilde{W}_\alpha(C) = \frac{(\alpha-1)^2}{\alpha}\frac{W(I)W(C)}{\alpha W(I)-W(C)}, \quad 1<\alpha<+\infty, \tag{9.13}$$

then the Lame constants are invariant under the transformation $W(C) \longrightarrow \tilde{W}_\alpha(C)$ for $1 \le \alpha < +\infty$. It is easy to see that this transformation is polyconvexity-preserving.

In [16] it was proved that the stationary elastic deformation for potential (9.11) is a quasi-isometry. Note that regularization (9.11) with large α can be applied to a fairly arbitrary elastic potential, say, to (9.7). This assertion can be stated as a theorem.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Consider the stored energy functional $J_\alpha(x)$ written as

$$J_\alpha(x) = \int_\Omega W_\alpha(\nabla_\xi x)d\xi, \tag{9.14}$$

where $\nabla_\xi x = C$ is the Jacobian matrix of the mapping $x(\xi)$ and $W_\alpha(C)$ is the function defined in (9.11).

The set of admissible deformations $\mathcal{A}_\alpha$ is defined as

$$\mathcal{A}_\alpha = \{y \in W_p^1(\Omega), p>3, W(\nabla_\xi y) < \alpha W(I) \text{ almost everywhere in } \Omega\}, \tag{9.15}$$

where $W_p^1(\Omega)$ is the Sobolev space consisting of generalized functions that belong to the Lebesgue space $L^p(\Omega)$ together with their gradients. The function $W(C)$ is assumed to have the following properties.

Property 1. $W(C)$ is a continuous convex function.

Property 2. $W(U) = W(I) > 0$ is the absolute minimum of W, where U is an arbitrary orthogonal matrix with a positive determinant.

Property 3. The inequality $W(C) < \alpha W(I)$ implies that there exist continuous monotonically increasing locally bounded functions $\phi_1(\alpha)$, $\phi_2(\alpha)$ such that $\phi_i(1) = 1$, $\phi_i(+\infty) = +\infty$, and

$$\det C > \frac{1}{\phi_1(\alpha)},$$

$$\frac{1}{\phi_2(\alpha)} < k_i(C) < \phi_2(\alpha).$$

Property 4. There exists a continuous monotonically increasing locally bounded function $\phi_3(\alpha)$ such that, if the matrix C does not satisfy $(\phi_3(\alpha))^{-1} < k_i(C) < \phi_3(\alpha)$, then $W(C) \ge \alpha W(I)$.

Theorem 1. *Suppose that* $W(C)$ *satisfies Conditions* 1–4. Let $\Omega, \Omega_1 \subset \mathbb{R}^3$ *be bounded Lipschitz domains such that there exists a quasi-isometric mapping* $y_0(\xi) : \overline{\Omega} \longrightarrow \overline{\Omega}_1$ *satisfying* $y_0 \in \mathcal{A}_{\alpha_0}$ *and* $J_{\alpha_0}(y_0) < +\infty$ *for some* $1 < \alpha_0 < +\infty$.

Then there exists $x^*(\xi) \in \mathcal{A}_{\alpha_0}$ *such that*

$$J_{\alpha_0}(x^*) = \inf_{y \in \mathcal{A}_{\alpha_0},\, y|_{\partial\Omega} = y_0|_{\partial\Omega}} J_{\alpha_0}(y) \tag{9.16}$$

and $x^*(\xi) : \overline{\Omega} \longrightarrow \overline{\Omega}_1$: *is a quasi-isometric mapping.*

Proof. By using Ball's direct method [1] modified according to [16], we can show the existence of a mapping $x^*(\xi)$ satisfying (9.16). The sufficient injectivity condition [2] implies that $x^*(\xi) : \overline{\Omega} \longrightarrow \overline{\Omega}_1$ is a homeomorphism. The Sobolev embedding theorems imply that $x^*(\xi)$ is a Lipschitz function in Ω. Consider the composition of mappings $y_0^{-1} \circ x^* : \Omega \longrightarrow \Omega$. The mapping $z = y_0^{-1} \circ x^*$ is Lipschitz in Ω and is the identity mapping on $\partial\Omega$. Thus, it can be extended by identity to $\mathbb{R}^3/\Omega$. It follows that $x^*(\xi)$ is a Lipschitz mapping in $\overline{\Omega}$. Applying the same argument to x^{*-1}, we conclude that $x^*(\xi) : \overline{\Omega} \longrightarrow \overline{\Omega}_1$ is a quasi-isometric mapping.

The regularization $W(C) \longrightarrow W_\alpha(C)$ can be interpreted as a method for approximating elastic deformations by a sequence of quasi-isometric deformations with (potentially) growing quasi-isometry constants. Moreover, in contrast to the Ball existence theorems, dependence on the cofactor matrix $\mathrm{cof}C$ does not need to be introduced into the elastic potential if it does not follow from experimental data. On the other hand, the parameter α can be chosen so large that $W_\alpha(C)$ approximates the same experimental data as $W(C)$ within the measurement error.

It was shown in [36] (see also [37]) that, if the elastic potential $\tilde{\Phi}$ satisfies the inequality

$$\mathrm{tr}\left(C^{\mathrm{T}}\frac{\partial\tilde{\Phi}}{\partial C}\right) \le a\tilde{\Phi}(C) + b \tag{9.17}$$

with constants $a > 0$ and b, then the nonstationary elasticity equations have a classical solution that exists for only a finite time $t < T$. More precisely, for a bounded elastic body B with free boundaries, there is an initial deformation $x(0, \xi)$ and an initial velocity field $\frac{\partial x}{\partial t}(0, \xi)$ such that the integral

$$G(t) = \int_B \rho_0 |x(t, \xi)|^2 d\xi$$

becomes infinite in a finite time. Thus, the elastic material blows up in a finite time. Inequality (9.17) is not very restrictive. For example, it is satisfied by potentials (9.3), (9.4), and (9.7). Moreover, examples of polyconvex potentials can be constructed that satisfy both growth condition (4.3) and inequality (9.17).

Obviously, regularization (9.11) makes the blow-up effect impossible.

The natural question arises as to whether regularization (9.11) at $\alpha \longrightarrow +\infty$ can be regarded as a method for choosing invertible deformations that are local minima of stored energy functions with sufficiently general elastic potentials. Can this approach be used to prove the invertibility of deformations that solve the nonstationary elasticity equations? Clearly, to prove the well-posedness of the weak variational formulation of the Euler–Lagrange equations for $x^*(\xi)$ and to study the well-posedness of the nonstationary formulation, we need to introduce a variation into $\mathcal{A}_\alpha$ that leaves the functional J_α finite. This task will be addressed elsewhere.

CONCLUSIONS

A method for regularizing polyconvex elastic potentials was proposed that guarantees that the stationary elastic deformation is a quasi-isometric mapping. It was shown that this regularization does not change the Lame material constants. A thermodynamically consistent Godunov canonical form of the nonstationary finite-deformation thermoelasticity equations in Lagrangian and Eulerian coordinates was constructed. It was shown that the polyconvexity of an elastic potential implies that it is hyperbolic in the sense of Friedrichs.

APPENDIX

Let us show that formula (8.13) holds:

$$\frac{\partial r_{kj}}{\partial x_k} = 0, \quad \frac{\partial z_{kipm}}{\partial x_k} = 0. \tag{A.1}$$

The fact that r_{kj} are divergence-free is obvious, since r_{kj} make up the adjugate matrix for the Jacobian matrix B of the mapping $\xi(x)$. Since z_{kipm} are given by (8.2), i.e.,

$$z_{kipm} = \rho c_{kj} d_{pmij},$$

and since the field d_{pmij} is divergence-free in Lagrangian variables,

$$\frac{\partial d_{kmij}}{\partial \xi_j} = 0,$$

the last relation can be written in Cartesian (Eulerian) variables exactly in the same manner as the second relation in (A.1).

Now let us verify formula (8.14):

$$f_k^* b_{ij} - z_{klji} f_l^* = \delta_{jk} u_l b_{il}.$$

In fact, it is necessary to prove that

$$z_{klji} = \delta_{kl} b_{ij} - \delta_{jk} b_{il}, \quad \text{where} \quad z_{klji} = \rho c_{kr} \frac{\partial a_{ji}}{\partial c_{lr}}.$$

Since $a_{ji} = \frac{1}{2}\varepsilon_{qmj}\varepsilon_{pni}c_{qp}c_{mn}$, we have

$$\frac{\partial a_{ji}}{\partial c_{lr}} = \frac{1}{2}\varepsilon_{lmj}\varepsilon_{rni}c_{mn} + \frac{1}{2}\varepsilon_{qlj}\varepsilon_{pri}c_{qp}$$

and

$$c_{kr}\frac{\partial a_{ji}}{\partial c_{lr}} = \varepsilon_{lmj}\varepsilon_{rni}c_{mn}c_{kr}.$$

On the other hand,

$$\delta_{kl}a_{ji} - \delta_{jk}a_{li} = \frac{1}{2}(\varepsilon_{qdj}\varepsilon_{mni}\delta_{kl} - \varepsilon_{mnl}\varepsilon_{qdi}\delta_{jk})c_{mq}c_{nd}.$$

Since $\delta_{kl} = \delta_{kn}\delta_{ln}$, $\delta_{jk} = \delta_{jn}\delta_{nk}$, we obtain

$$\frac{1}{2}(\varepsilon_{qdj}\varepsilon_{mli} + \varepsilon_{mlj}\varepsilon_{qdi})c_{mq}c_{kd} = \varepsilon_{mli}\varepsilon_{qdi}c_{mq}c_{kd} = \varepsilon_{lmj}\varepsilon_{rni}c_{mn}c_{kr},$$

which completes the proof.

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