Optimal Triangulations: Existence, Approximation and Double Differentiation of P_1 finite element functions

Konstantin Lipnikov

Department of Mathematics, University of Houston, Houston, TX, 77204, USA, e-mail: lipnikov@math.uh.edu

Yuri Vassilevski

Institute of Numerical Mathematics, Russian Academy of Sciences, Gubkina 8, 119991, Moscow, RUSSIA, e-mail: vasilevs@dodo.inm.ras.ru

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Abstract

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1 Introduction

In last decade, it has been shown in a number of papers (see, for example, [1,2]) that triangles with obtuse and acute angles stretched along the direction of minimal second derivative of a solution may be the best elements for minimizing the interpolation error. As the result, an optimal adaptive mesh may frequently contain anisotropic elements, *i.e.*, elements with obtuse and acute angles. A theoretical analysis of anisotropic meshes is still the challenging problem for researchers. In this paper, we review several theoretical issues related to optimal (possibly anisotropic) triangulations. The results published in our previous papers [3,8] will be presented briefly.

The paper outline is as follows. In Section 2, we introduce a concept of an *optimal* triangulation and under certain assumptions prove its existence. In Section 3, we formulate the main property of optimal triangulations and give L_∞ error estimates for a linear interpolation operator. In Section 4, we introduce notation of *quasi-optimal* triangulations and show that they are approximations to the optimal one. The methodology used in this paper is based on the Hessian recovered from a discrete P_1 solution. In Section 5, we discuss a few methods for the Hessian recovery.

2 Existence of optimal triangulations

Let $\Omega \in \Re^2$ be a polygonal domain and Ω_h be its conformal partition into triangles,

$$\Omega_h = \bigcup_{i=1}^{\mathcal{N}(\Omega_h)} e_i,$$

where $\mathcal{N}(\Omega_h)$ is the number of elements in Ω_h. Let $C^k(D)$ be a space of functions with continuous in $D \subset \bar{\Omega}$ partial derivatives up to order k. Let $\|\cdot\|_{\infty,D}$ and $\|\cdot\|_{2,D}$ denote the norms on the spaces $L_\infty(D)$ and $C^2(D)$, respectively, and $\|\cdot\|_\infty \equiv \|\cdot\|_{\infty,\Omega}$. In addition, we shall use notation $P_1(\Omega_h)$ for the space of functions continuous in Ω and linear on each element of Ω_h. Furthermore, let $\mathcal{P}^h_{\Omega_h}\colon C^0(\bar{\Omega}) \to P_1(\Omega_h)$ be a projector on the discrete space $P_1(\Omega_h)$ and $\mathcal{I}^h_{\Omega_h}\colon C^0(\bar{\Omega}) \to P_1(\Omega_h)$ be the linear interpolation operator. We shall omit mesh related subscripts whenever it does not result in ambiguity.

Some theoretical results formulated in this paper are based on the assumption that the solution of a continuous second order boundary value problem belongs to $C^2(\bar{\Omega})$. However, constants in our error estimates are independent of the actual value of C^2-norm of the solution. Since $C^2(\bar{\Omega})$ is dense in $C^0(\bar{\Omega})$, one can try to analyze regularized problems with smooth solutions and get error estimates for the original problem by the density arguments. We shall address this challenging problem in the future papers.

Definition 1 *Let $u \in C^0(\bar{\Omega})$ and $\mathcal{P}^h_{\Omega_h}$ be given. The mesh $\Omega_h(N_T, u)$ consisting of at most N_T elements is called optimal if it is a solution of the optimization problem*

$$\Omega_h(N_T, u) = \arg \min_{\Omega_h : \mathcal{N}(\Omega_h) \le N_T} \|u - \mathcal{P}^h_{\Omega_h} u\|_\infty. \tag{1}$$

Another optimization problem can be formulated when we restrict the number of mesh nodes rather than the number of triangles. Let $\mathcal{M}(\Omega_h)$ denote the number of mesh nodes in Ω_h.

Definition 2 *Let $u \in C^0(\bar{\Omega})$ and $\mathcal{P}^h_{\Omega_h}$ be given. The mesh $\Omega_h(N_P, u)$ consisting of at most N_P nodes is called optimal if it is a solution of the optimization problem*

$$\Omega_h(N_P, u) = \arg \min_{\Omega_h : \mathcal{M}(\Omega_h) \le N_P} \|u - \mathcal{P}^h_{\Omega_h} u\|_\infty. \tag{2}$$

In a general case, the optimization problems (1) and (2) may be not well posed: the optimal triangulation may not exist. However, by definition, there exists an arbitrary close approximation of the optimal triangulation. Under certain conditions, the existence of the optimal triangulation may be proved. Since the number of triangles does not exceed the double number of nodes, the optimization problems (1) and (2) are equivalent.

Theorem 3 *Let $u \in C^0(\bar{\Omega})$ and $\|u - \mathcal{P}^h_{\Omega_h} u\|_\infty$ be a continuous functional of mesh coordinates. Furthermore, let the projector $\mathcal{P}^h_{\Omega_h}$ satisfy*

$$\|u - \mathcal{P}^h_{\Omega^2_h} u\|_\infty \leq \|u - \mathcal{P}^h_{\Omega^1_h} u\|_\infty$$

for any triangulation Ω^2_h being a hierarchical partitioning of triangulation Ω^1_h. Then, the optimization problem (2) has a solution.

Proof. Since $\|u - \mathcal{P}^h_{\Omega_h} u\|_\infty \geq 0$, there exists a sequence of triangulations $\{\Omega^k_h\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} \|u - \mathcal{P}^h_{\Omega^k_h} u\|_\infty = \inf_{\Omega_h :\, \mathcal{M}(\Omega_h) \leq N_P} \|u - \mathcal{P}^h_{\Omega_h} u\|_\infty. \tag{3}$$

The triangulation Ω^k_h may be represented by the set of nodes X^k_h and connectivity table (list of triangles with references to the nodes) T^k_h. Since the Cartesian product $\Omega \times \ldots \times \Omega$ of N_P compact and bounded sets is compact and bounded, the sequence X^k_h contains a convergent (in the product metric) subsequence. For the sake of simplicity we assume that $\{X^k_h\}_{k=1}^\infty$ is this subsequence. Let

$$X^\infty_h = \lim_{k\to\infty} X^k_h \tag{4}$$

and N^∞_P be the number of elements in the set X^∞_h. It is obvious that $N^\infty_P \leq N_P$.

Let x^k_j, $j = 1, \ldots, N^k_P$, be the elements (points in $\bar{\Omega}$) of X^k_h, $k = 1, 2, \ldots, \infty$ with minimum distance between the points δ^k. The convergence property (4) implies that for any small $\varepsilon > 0$ there exists k_ε such that any point x^k_i, $k \geq k_\varepsilon$, belongs to a disk $B(x^\infty_j, \varepsilon)$ of radius ε centered at a node x^∞_j.

Let α^∞ denote the minimum angle of all possible non-degenerated triangles with nodes from X^∞_h and $\varepsilon_m = 1/m$ where $m \geq m_0 > 0$ is an integer and m_0 is a sufficiently large integer providing $\delta^\infty \sin \alpha^\infty / 10 > \varepsilon_m$. Due to (3) there exists a conformal triangulation $\Omega^{k_m}_h = \{X^{k_m}_h, T^{k_m}_h\}$ whose nodes are in ε_m-vicinity of X^∞_h. Let the nodes from X^∞_h be counted similarly to $X^{k_m}_h$. Multiple nodes from $X^{k_m}_h$ in ε_m-vicinity of a node x^∞_j result in multiple counting x^∞_j. This allows us to define formally a triangulation $\hat{\Omega}^{k_m}_h = \{X^\infty_h, T^{k_m}_h\}$ which is an ε_m-perturbation of $\Omega_h^{k_m}$. Therefore, by the continuity assumption:

$$\|u - \mathcal{P}^h_{\hat{\Omega}_h^{k_m}} u\|_\infty \leq \|u - \mathcal{P}^h_{\Omega_h^{k_m}} u\|_\infty + C(u)\varepsilon_m$$

where $C(u)$ depends only on u. However, $\hat{\Omega}_h^{k_m}$ is not always a conformal triangulation since some of the triangles from $T_h^{k_m}$ may degenerate (have zero area) or even tangle. The assumption $\delta^\infty \sin\alpha^\infty/10 > \varepsilon_m$ implies that each tangled triangle may intersect with only one neighboring triangle. We modify $T_h^{k_m}$ within two steps in order to get a conformal triangulation $\tilde{\Omega}_h^{k_m}$ with the nodes X_h^∞. First, we eliminate from $T_h^{k_m}$ all degenerated triangles. It will not change the norm of the error. Hanging nodes left by triangles degenerated into mesh edges are transformed to nodes of a conformal mesh by a hierarchical partition of neighboring-to-degenerated triangles without insertion of additional nodes. The partition can not increase the norm of the error. At the second step, we split all the overlapping neighbors of the tangled triangles into subtriangles without insertion of additional nodes. The tangled triangles constitute a subset of the set of subtriangles and may be eliminated from $T_h^{k_m}$ in order to get a conformal mesh $\tilde{\Omega}_h^{k_m}$. The partition and the elimination can not increase the norm of the error. By the theorem assumptions

$$\|u - \mathcal{P}^h_{\tilde{\Omega}_h^{k_m}} u\|_\infty \leq \|u - \mathcal{P}^h_{\hat{\Omega}_h^{k_m}} u\|_\infty \leq \|u - \mathcal{P}^h_{\Omega_h^{k_m}} u\|_\infty + C(u)\varepsilon_m$$

Therefore,

$$\lim_{m\to\infty} \|u - \mathcal{P}^h_{\tilde{\Omega}_h^{k_m}} u\|_\infty \leq \lim_{m\to\infty} \|u - \mathcal{P}^h_{\Omega_h^{k_m}} u\|_\infty = \inf_{\Omega_h:\,\mathcal{M}(\Omega_h)\leq N_P} \|u - \mathcal{P}^h_{\Omega_h} u\|_\infty$$

and

$$\lim_{m\to\infty} \|u - \mathcal{P}^h_{\tilde{\Omega}_h^{k_m}} u\|_\infty = \inf_{\Omega_h:\,\mathcal{M}(\Omega_h)\leq N_P} \|u - \mathcal{P}^h_{\Omega_h} u\|_\infty.$$

Since the total number of the conformal connectivity tables for fixed X_h^∞ is finite, there exists a conformal triangulation Ω_h^∞ with nodes X_h^∞ minimizing $\|u - \mathcal{P}^h_{\tilde{\Omega}_h} u\|_\infty$ such that

$$\|u - \mathcal{P}^h_{\Omega_h^\infty} u\|_\infty = \inf_{\Omega_h:\,\mathcal{M}(\Omega_h)\leq N_P} \|u - \mathcal{P}^h_{\Omega_h} u\|_\infty.$$

The theorem is proved.

Note that the interpolation operator $\mathcal{I}^h_{\Omega_h}$ satisfies assumptions of Theorem 3. Therefore the optimization problem (2) with $\mathcal{P}^h_{\Omega_h} = \mathcal{I}^h_{\Omega_h}$ has a solution which is not necessary unique.

3 Error estimates for optimal meshes

Let function $u \in C^2(\bar{\Omega})$ have a non-singular Hessian $\boldsymbol{H}(x) = \{H_{ps}(x)\}_{p,s=1}^2$, *i.e.*, $det\boldsymbol{H}(x) \neq 0$ for $\forall x \in \bar{\Omega}$. Since the Hessian is symmetric, the spectral decomposition of $\boldsymbol{H}$ is possible for any $x \in \bar{\Omega}$,

$$\boldsymbol{H} = W^t \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} W,$$

where W is an orthonormal matrix and $|\lambda_1| < |\lambda_2|$. It is clear that $\lambda_1 \neq 0$ and

$$|\boldsymbol{H}| = W^t \begin{pmatrix} |\lambda_1| & 0 \\ 0 & |\lambda_2| \end{pmatrix} W$$

defines the continuous metric on Ω. Let $|\Omega|_{|\boldsymbol{H}|}$ be the volume of Ω in this metric. Then, the following *a priory* error estimates for the P_1 interpolation operator hold.

Theorem 4 *Let $N_T > 0$, $u \in C^2(\bar{\Omega})$ and $|\boldsymbol{H}|$ be a metric generated by the Hessian of u. Furthermore, let $\tilde{\Omega}_h$ be the optimal mesh, and for any element $\tilde{e} \in \tilde{\Omega}_h$ the following estimate holds:*

$$||H_{ps} - H_{e,ps}||_{\infty,e} < q\,|\lambda_1(\boldsymbol{H}_e)|/2, \quad 0 < q < 1, \quad p,s = 1,2, \tag{5}$$

where q is a constant, $x_e = \arg\max_{x\in e} |\det(\boldsymbol{H}(x))|$, and $\boldsymbol{H}_e = \boldsymbol{H}(x_e)$. Then

$$C_1(q)\frac{|\Omega|_{|\boldsymbol{H}|}}{N_T} \leq \|u - \mathcal{I}^h_{\tilde{\Omega}_h} u\|_\infty \leq C_2(q)\frac{|\Omega|_{|\boldsymbol{H}|}}{N_T}, \tag{6}$$

where $C_1(q), C_2(q)$ depend only on q.

The proof may be found in [3,8]. A straightforward corollary from this theorem is that for a projector $\mathcal{P}^h_{\tilde{\Omega}_h}$ satisfying

$$\|u - \mathcal{P}^h_{\tilde{\Omega}_h} u\|_\infty \leq \hat{C}\|u - \mathcal{I}^h_{\tilde{\Omega}_h} u\|_\infty \tag{7}$$

the inequality (6) implies

$$\|u - \mathcal{P}^h_{\tilde{\Omega}_h} u\|_\infty \leq \hat{C}C_2(q)\frac{|\Omega|_{|\boldsymbol{H}|}}{N_T}. \tag{8}$$

It is pertinent to note that error estimates (6) are in perfect agreement with Tichomirov's result [4]: for any discrete space V_h and $\Omega \in \Re^2$

$$\inf_{V_h:\, \dim V_h \le N_T} \ \sup_{\|u\|_{2,\,\Omega}=1} \ \inf_{v_h \in V_h} \|u - v_h\|_\infty \simeq N_T^{-1}.$$

4 Quasi-optimal meshes as approximations to the optimal mesh

Since the exact solution is unknown, the error $\|u - \mathcal{P}^h_{\Omega_h} u\|_\infty$ can not be estimated. Therefore, the optimization problem (1) has to be replaced by another optimization problem whose solution at least approaches the solution of (1). To this end, we introduce concepts of a mesh quality and a mesh quasi-optimality.

Let $Q(\Omega_h)$ be an easily computed quantitative characteristic of mesh Ω_h such that $0 < Q(\Omega_h) \le 1$. We shall use the definition of $Q(\Omega_h)$ proposed in [6]. Let a predefined number of elements N_T be given, $\boldsymbol{G}(x) = \{G_{ps}(x)\}^2_{p,s=1}$, $x \in \Re^2$, be a continuous metric in Ω_h, and $x_e \in e$ be a point in triangle e where $|\det(G(x))|$ attains its maximal value. We set $\boldsymbol{G}_e = \boldsymbol{G}(x_e)$ and define the area of this triangle and the length of its edge $\vec{l_e} \in \Re^2$ (in metric $\boldsymbol{G}$) by

$$|e|_G = |e|(\det(\boldsymbol{G}_e))^{1/2} \quad \text{and} \quad |\vec{l_e}|_G = (\boldsymbol{G}_e \vec{l_e},\, \vec{l_e})^{1/2},$$

respectively, where $|e|$ is the triangle area in the Cartesian coordinate system. Denote the sum of lengths of edges of triangle e measured in metric $\boldsymbol{G}$ by $|\partial\, e|_G$. Let $|\Omega_h|_G$ be the total area of the computational domain measured in metric $\boldsymbol{G}$,

$$|\Omega_h|_G = \sum_{e \in \Omega_h} |e|_G.$$

Following [6], we define $Q(\Omega_h)$ as follows:

$$Q(\Omega_h) = \min_{e \in \Omega_h} Q(e) \quad \text{with} \quad Q(e) = 6\sqrt{2}\frac{|e|_G}{|\partial\, e|^2_G} F\left(\frac{|\partial\, e|_G}{3h^*}\right) \tag{9}$$

where function $F(\cdot)$ and the average length of a triangle edge h^* (in metric $\boldsymbol{G}$) are given by

$$F(x) = \left(\min\left\{x,\, \frac{1}{x}\right\}\left(2 - \min\left\{x,\, \frac{1}{x}\right\}\right)\right)^3 \quad \text{and} \quad h^* = \sqrt{\frac{4|\Omega_h|_G}{\sqrt{3}N_T}},$$

respectively. Hereafter we shall use notation $Q(\boldsymbol{G},\, N_T,\, \Omega_h)$ instead of $Q(\Omega_h)$ to emphasize its dependence on the metric $\boldsymbol{G}$ and the predefined number of elements N_T. It is easy to check that $0 < Q(\boldsymbol{G},\, N_T,\, \Omega_h) \le 1$ and the maximal value is attained when all mesh elements are equilateral (in metric $\boldsymbol{G}$) triangles with the edge length h^*. We refer to $Q(\boldsymbol{G},\, N_T,\, \Omega_h)$ as the *mesh quality* with respect to the metric $\boldsymbol{G}$ and the number of elements N_T.

Definition 5 *Let $\boldsymbol{G}$ be a continuous metric and N_T be a given integer. The mesh Ω_h is called $\boldsymbol{G}$-quasi-optimal if there is a fixed positive constant Q_0 such that $Q_0 = O(1)$ and*

$$Q(\boldsymbol{G},\, N_T,\, \Omega_h) > Q_0.$$

Definition 6 *Let $u \in C^2(\bar{\Omega})$ and $|\boldsymbol{H}|$ be a metric generated by the Hessian of u. For the given function u and a given integer N_T, the mesh $\Omega_h(N_T, u)$ is called quasi-optimal if it is $|\boldsymbol{H}|$-quasi-optimal.*

A quasi-optimal mesh satisfying $Q(\boldsymbol{H},\, N_T,\, \Omega_h) = 1$ may not exist because of restrictions imposed by the boundary of Ω. Fixing $Q_0 < 1$ relaxes the above constraint. On the other hand, due to $Q(\boldsymbol{H},\, N_T,\, \Omega_h) < 1$, the number of mesh elements $\mathcal{N}(\Omega_h)$ in the $|\boldsymbol{H}|$-quasi-optimal mesh may differ from N_T but approaches it when $Q_0 \to 1$.

The quasi-optimal meshes (QOMs) have been studied in [3,8]. It turns out that in certain cases the QOM is an approximate solution of optimization problem (1).

Theorem 7 *Let $N_T > 0$, $u \in C^2(\bar{\Omega})$ and $|\boldsymbol{H}|$ be a metric generated by the Hessian of u. Furthermore, let $\Omega_h(N_T,\, u)$ and $\tilde{\Omega}_h(N_T,\, u)$ be the quasi-optimal and optimal meshes, respectively, and $e^* \in \Omega_h$ be the element where $\|u - \mathcal{I}^h_{\Omega_h} u\|_\infty$ is attained. Let for both any element $\tilde{e} \in \tilde{\Omega}_h$ and the element $e^* \in \Omega_h$ the following estimate holds:*

$$\|H_{ps} - H_{e,ps}\|_{\infty,e} < q\,|\lambda_1(\boldsymbol{H}_e)|/2, \quad 0 < q < 1, \quad p, s = 1, 2, \tag{10}$$

where q is a constant, $x_e = \arg\max_{x\in e} |\det(\boldsymbol{H}(x))|)$, and $\boldsymbol{H}_e = \boldsymbol{H}(x_e)$. Then

$$\|u - \mathcal{I}^h_{\Omega_h} u\|_\infty \le C(Q_0, q)\, \|u - \mathcal{I}^h_{\tilde{\Omega}_h} u\|_\infty, \tag{11}$$

where $C(Q_0, q)$ is a constant depending only on q and Q_0 from Definition 5.

The proof may be found in [3]. A straightforward corollary from this theorem is that for the projector $\mathcal{P}^h_{\Omega_h}$ satisfying (7), formulae (6) and (11) imply

$$\|u - \mathcal{P}^h_{\Omega_h} u\|_\infty \le \hat{C} C_2(q) C(Q_0, q) \frac{|\Omega|_{|\boldsymbol{H}|}}{N_T}. \tag{12}$$

5 Double differentiation on optimal and quasi-optimal meshes

Obviously, the Hessian $\boldsymbol{H}(x)$ is an unknown function. In computations we use its approximation $\boldsymbol{H}^h$ recovered from the discrete solution $\mathcal{P}^h_{\Omega_h} u$. In the following, we describe briefly Hessian recovery algorithms [5,6] and advocate the replacement of $\boldsymbol{H}(x)$ by its discrete counterpart $\boldsymbol{H}^h$.

Let $u^h = \mathcal{P}^h_{\Omega_h} u$ be a discrete function from $P_1(\Omega_h)$. The discrete Hessian $\boldsymbol{H}^h = \{H^h_{ps}\}^2_{p,s=1}$, $H^h_{ps} \in P_1(\Omega_h)$, is defined as follows. In interior mesh node a_i, the Hessian entries $H^h_{ps}(a_i)$, $p, s = 1, 2$, are defined by

$$\int_{\sigma_i} H^h_{ps}(a_i) v^h \,\mathrm{d}x = -\int_{\sigma_i} \frac{\partial u^h}{\partial x_p}\frac{\partial v^h}{\partial x_s}\,\mathrm{d}x \qquad \forall v^h \in P_1(\sigma_i),\ v^h = 0 \ \text{ on } \ \partial\sigma_i, \tag{13}$$

where σ_i is the union of triangles sharing the node a_i (superelement). At boundary node a_i, values of $H^h_{ps}(a_i)$, $p, s = 1, 2$, are weighted extrapolations from the neighboring interior values [9]:

$$H^h_{ps}(a_i) = \frac{\displaystyle\int_{\sigma_i} \varphi(a_i)\, \mathring{H}^h_{ps}\,\mathrm{d}x}{\displaystyle\int_{\sigma_i} \varphi(a_i)\left(\sum_{a_j \notin \partial\Omega_h} \varphi(a_j)\right)\mathrm{d}x}, \tag{14}$$

where $\varphi(a_i)$ denotes the nodal basis function from $P_1(\Omega_h)$ and $\mathring{H}^h_{ps}$ stands for the finite elements function defined by (13) and vanishing on $\partial\Omega_h$.

Theorem 8 *Let $N_T > 0$, $u \in C^2(\bar{\Omega})$, $u^h = \mathcal{P}^h_{\Omega_h} u$, $\boldsymbol{H}$ be the Hessian of u, and $\boldsymbol{H}^h$ be the discrete Hessian recovered from u^h according to (13)–(14). Furthermore, let for any superelement $\sigma \in \Omega_h$ associated with a mesh node a the following estimates hold:*

$$\|H_{ps} - H_{\sigma,ps}\|_{\infty,\sigma} < \delta, \tag{15}$$

$$|H^h_{ps}(a) - H_{\sigma,ps}| < \varepsilon, \tag{16}$$

where $\boldsymbol{H}_\sigma = \boldsymbol{H}(x_\sigma)$ and $x_\sigma = \arg\max_{x\in\sigma} |\det(\boldsymbol{H}(x))|)$. Then for ε and δ sufficiently small with respect the minimal eigenvalue of $|\boldsymbol{H}_\sigma|$, the $|\boldsymbol{H}^h|$-quasi-optimal mesh Ω_h ($Q(|\boldsymbol{H}^h|, N_T, \Omega_h) \geq Q_0$) is also $|\boldsymbol{H}|$-quasi-optimal:

$$Q(|\boldsymbol{H}|, N_T, \Omega_h) \geq C\, Q_0$$

with constant C independent of N_T and $\|u\|_{2,\Omega}$.

The proof may be found in [3]. The theorem states that under certain assumptions, the sufficient condition of $|\boldsymbol{H}|$-quasi-optimality is $|\boldsymbol{H}^h|$-quasi-optimality. The assumption (15) implies small variations of the Hessian on any superelement σ and assumption (16) is the requirement of nodal-wise approximation for the Hessian. The latter assumption does not always hold true in practice, since it is implicative of a small gradient error for u^h. The small gradient error is not typical for the functions with singularities. In order to resolve the problem of the discrete Hessian recovery for non-smooth functions, we suggest another definition of the discrete Hessian which satisfy (16) in a weaker norm [10].

The alternative definition of the discrete Hessian is based on the following formula:

$$\int_\sigma H_{ps} v dx = \int_\sigma u \frac{\partial^2 v}{\partial x_s \partial x_p} dx - \int_{\partial\sigma} u \frac{\partial v}{\partial x_s} n_p dt, \ \forall v \in C^2(\sigma), \ v = 0 \text{ on } \partial\sigma, \quad (17)$$

where $p, s = 1, 2$. Representation (17) has an important advantage over (13). It defines the Hessian through the function value excluding function derivatives. Its main drawback is higher smoothness of the test functions which imposes restrictions on the shape of σ, and, as a consequence, on the triangulation, in the case of a discrete Hessian recovery.

Definition 9 *A triangulation Ω_h satisfies the condition* $\mathbf{A}$*, if for any interior superelement σ_i there exists an affine mapping $\mathcal{F}_i = \mathcal{S}_i \circ \mathcal{R}_i$ s.t. $\mathcal{F}_i(\sigma_i)$ is a shape regular superelement with the diameter 1. Here, $\mathcal{S}_i$ and $\mathcal{R}_i$ denote scaling and rotation matrices, respectively.*

We notice that not all triangulations satisfy condition $\mathbf{A}$. A two-dimensional mesh with two anisotropic neighboring triangles whose axes of stretching are orthogonal, is a simple example. Adaptive triangulations, however, do satisfy condition $\mathbf{A}$. The condition $\mathbf{A}$ does not imply shape regularity for any triangle. Rather, it requires a local similarity of triangle shapes. Thus, adaptive anisotropic meshes satisfy the condition $\mathbf{A}$.

Let a_i be an interior node of a mesh Ω_h and σ_i be a corresponding superelement. Let $\hat{B}_i$ be the largest ball centered at $\mathcal{F}_i(a_i)$ and inscribed in $\mathcal{F}_i(\sigma_i)$. Due to the shape regularity of $\mathcal{F}_i(\sigma_i)$, the radius $\hat{R}_i$ of $\hat{B}_i$ is $O(1)$. Introducing the polar coordinates with the origin at $\mathcal{F}_i(a_i)$, we define a smooth function $\hat{v}_i = 1 - r^2/\hat{R}_i^2$ on $\hat{B}_i$. A span of functions $v = \alpha\mathcal{F}_i^{-1}(\hat{v}_i)$, $\alpha \in \Re^1$, define a space of local test functions V_i. We notice $v \in V_i$ implies that its support $B_i = \mathcal{F}_i^{-1}(\hat{B}_i)$ satisfies $|B_i| \gtrsim |\sigma_i|$ and $v \in C^2(B_i)$, $v = 0$ on ∂B_i.

Now, we are ready to recover the discrete Hessian $H^h_{ps} \in P_1(\Omega_h)$ in interior mesh node a_i. The Hessian entries $H^h_{ps}(a_i)$, $p, s = 1, 2$, are defined by

$$\int\limits_{B_i} H^h_{ps}(a_i) v^h \partial x = \int\limits_{B_i} u^h \frac{\partial^2 v^h}{\partial x_s \partial x_p} \partial x - \int\limits_{\partial B_i} u^h \frac{\partial v}{\partial x_s} n_p \partial t, \quad \forall v^h \in V_i. \tag{18}$$

There exist triangulations where not all components of the Hessian may be recovered by the Green's formula (17). Therefore, at the boundary nodes, the values of the discrete Hessian H^h are the weighted extrapolations given by (14).

As we already mentioned, the condition **A** is a natural restriction to the shape of the superelement σ_i. In order to establish a convergence of the discrete Hessian to the differential one, we have to impose additional restrictions on the mesh triangles. We recall that according to the condition **A**, for any superelement σ_i there exists a pair of operators (rotation $\mathcal{R}_i$ and scaling $\mathcal{S}_i$) whose combination transforms the superelement into a shape regular one. Therefore, for any triangle $\Delta \subset \Omega^h$ (superelement $\sigma \subset \Omega^h$) there exists a rotation operator $\mathcal{R}_\Delta$ ($\mathcal{R}_\sigma$) such that the image $\Delta_\mathcal{R} = \mathcal{R}_\Delta(\Delta)$ ($\sigma_\mathcal{R} = \mathcal{R}_\sigma(\sigma)$) may be scaled along the coordinate axes to get a shape regular element. For the rotated simplex $\Delta_\mathcal{R}$ it is natural to introduce its sizes, $h_{\mathcal{R},k} = \max\limits_{x,y \in \Delta_\mathcal{R}} |(x)_k - (y)_k|$, $k = 1, 2$. The same definition is applicable to the rotated superelement $\sigma_\mathcal{R}$. We notice that the rotation operator does not affect the best P_1 approximation of the function u:

$$\overline{u_\mathcal{R}} = \mathcal{R}_\Delta(\bar{u}), \quad u_\mathcal{R} = u(\mathcal{R}_\Delta(x)).$$

We recall that the best P_1 approximation is defined as

$$\int\limits_\Delta (u - \bar{u}) dx = 0, \quad \int\limits_\Delta \partial^\alpha (u - \bar{u}) dx = 0, \quad \forall\ |\alpha| = 1, \tag{19}$$

where $\alpha = \{\alpha_1, \alpha_2\}$ is the multiindex, $\alpha_k = 0, 1$.

Definition 10 *For a given function $u \in W^{1,p}(\Omega)$, a triangle from a triangulation satisfying the condition **A**, satisfies the condition **B**, if there exists a constant $C_B > 0$ such that*

$$\max_{|\alpha|=1} h^\alpha_\mathcal{R} \|\partial^\alpha (u_\mathcal{R} - \overline{u_\mathcal{R}})\|_{L_p(\Delta_\mathcal{R})} \le C_B \min_{|\alpha|=1} h^\alpha_\mathcal{R} \|\partial^\alpha (u_\mathcal{R} - \overline{u_\mathcal{R}})\|_{L_p(\Delta_\mathcal{R})}, \tag{20}$$

where

$$h^\alpha_\mathcal{R} := \prod_{k=1}^d h^{\alpha_k}_{\mathcal{R},k}.$$

The condition **B** requires isotropic distribution of the gradient error due to the best P_1 approximation (19) of a function $u \in W^{1,p}(\Omega)$. It implies neither minimal nor maximum angle conditions. Rather, it requires the triangle Δ to be adapted with respect to the local behavior of the function u.

Definition 11 *A triangle Δ from a triangulation Ω^h satisfying the condition* **A** *and a function $u \in W^{1,p}(\Omega)$ satisfy the condition* **C**, *if there exist a constant $C_C > 0$ such that*

$$h_{\mathcal{R}}^{\alpha} \|\partial^{\alpha}(u_{\mathcal{R}} - \overline{u_{\mathcal{R}}})\|_{L_p(\Delta_{\mathcal{R}})} \leq C_C h_{\mathcal{R}}^{\alpha\beta}, \quad |\alpha| = 1, \, \beta > 0. \tag{21}$$

The condition **C** requires convergence of the best P_1 approximation $\bar{u}$ towards the function u on Δ. Actually, it implies a higher than $W^{1,p}(\Omega)$ smoothness of u. We do not specify a space of smooth functions here since the appropriate class varies widely depending on the application.

Both conditions **B** and **C** assume a certain relationship between the mesh and the function. The triangles of the mesh must provide an isotropic distribution of the error (20) due to the best P_1 approximation as well as its convergence towards u (21) if u possesses a little additional smoothness. It is clear that not all the meshes complying with the condition **A**, satisfy conditions **B** and **C**. However, meshes adapted to the function do match both **B** and **C**.

Theorem 12 *Let a function $u \in W^{1,p}(\Omega) \cap W^{2,1}(\Omega)$, $p > d$, and an interior superelement σ_i satisfying the Conditions* **A, B, C** *be given. Furthermore, let on σ_i the differential Hessian H deviates slightly from its mean value:*

$$\|H_{ps} - \bar{H}_{ps}\|_{L_1(\sigma_i)} \leq \delta, \quad p, s = 1, 2. \tag{22}$$

Then, the discrete Hessian recovered by (18) from the piece-wise linear interpolant $\mathcal{I}_{\sigma_i}^h u$ converges to the differential Hessian:

$$\|H_{ps} - H_{ps}^h\|_{L_1(B_i)} \lesssim \delta + C_B C_C |\sigma_i|^{1-1/p} \min_{k=1,\ldots,d} h_{\mathcal{R}_i,k}^{\beta-1}. \tag{23}$$

Moreover, the following estimate holds

$$\|H_{ps} - H_{ps}^h\|_{L_1(B_i)} \lesssim \delta + C_B C_C \min_{k=1,\ldots,d} h_{\mathcal{R}_i,k}^{\beta-1/p}. \tag{24}$$

The proof may be found in [10]. We remark that the estimate (24) implies the local convergence of the discrete Hessian in the weak norm as the number of triangles in the mesh tends to infinity. Indeed, for any triangulation adapted to a function with a non-singular Hessian $\max_{\sigma_i} \min_{k=1,\ldots,d} h_{\mathcal{R}_i,k} \to 0$ as $\mathcal{N}(\Omega_h) \to \infty$. The importance of the theorem is that it is the first (to our knowledge) result where the *local* convergence of the recovered Hessian is shown on an anisotropic meshes and for functions with singularities.

Conclusions

Several theoretical issues related to optimal triangulations have been reviewed. The collection of the results give an insight into the asymptotic properties of the optimal and quasi-optimal triangulations as well as the recovery of the discrete Hessian.

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